CHAPTER ONE
MATRICES

Matrices: which are rectangular arrays of numbers or functions (elements). Matrices are important because they let us express a large amount of data and functions in an organized and simplified form.

For Example:

$$A=\begin{bmatrix}1 & 0 & -2\\ 4 & 1 & 3\end{bmatrix} \quad , \quad B=\begin{bmatrix}e^{-x} & 2x^2\\ e^{6x} & 4x\end{bmatrix} \quad , \quad C=\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}$$

Whereas: *A* is a matrix of 2 rows and *3* columns, we call it A_{2*3} .

***In general** A_{n*m} is a matrix that has *n*-rows and *m*-columns. The element in *ith*-row and *jth*-column of matrix *A* can be expressed by a_{ij} for example $a_{23}=3$, $a_{12}=0$.

Properties of the Matrices

1 – Equality: If *A* and *B* are two matrices, we can say A=B if and only if they have same elements in the same position.

Ex:

$$A=\begin{bmatrix}2 & 5\\ 6 & 7\end{bmatrix}, \quad B=\begin{bmatrix}2 & 5\\ 6 & 7\end{bmatrix} \qquad A=B$$

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2 – Adding and Subtraction: If *A* and *B* are two matrices can be added or subtracted if they have the same number of rows and columns.

Ex:

$$A=\begin{bmatrix}2 & 1\\ 3 & 0\\ 4 & -1\end{bmatrix} \quad , \quad B=\begin{bmatrix}1 & -1\\ 2 & 3\\ 0 & 1\end{bmatrix}$$

$$A+B=\begin{bmatrix}2+1 & 1-1\\ 3+2 & 0+3\\ 4+0 & -1+1\end{bmatrix}=\begin{bmatrix}3 & 0\\ 5 & 3\\ 4 & 0\end{bmatrix} \quad , \quad A-B=\begin{bmatrix}2-1 & 1+1\\ 3-2 & 0-3\\ 4-0 & -1-1\end{bmatrix}=\begin{bmatrix}1 & 2\\ 1 & -3\\ 4 & -2\end{bmatrix}$$

3- Multiplication:

A – To multiply a matrix A by constant c, we multiply each element of A by c.

Ex:

$$A=\begin{bmatrix}4 & -2\\ 3 & 1\end{bmatrix},\ c=2$$

$$A*c=\begin{bmatrix}4*2 & -2*2\\ 3*2 & 1*2\end{bmatrix}=\begin{bmatrix}8 & -4\\ 6 & 2\end{bmatrix}$$

B– To multiply a matrix A by matrix B, the number of columns in A must be equal to the number of rows in B. we multiply row by column.

Ex: Find A*B, B*A

$$A=\begin{bmatrix}1 & 3 & -1\\ 0 & 1 & 0\end{bmatrix}_{2*3},\quad B=\begin{bmatrix}1 & 0\\ 0 & 2\\ -1 & 3\end{bmatrix}_{3*2}$$

$$A*B=\begin{bmatrix}(1*1)+(3*0)+(-1*-1) & (1*0)+(3*2)+(-1*3)\\ (0*1)+(1*0)+(0*-1) & (0*0)+(1*2)+(0*3)\end{bmatrix}=\begin{bmatrix}2 & 3\\ 0 & 2\end{bmatrix}$$

$$B*A=\begin{bmatrix}1+0 & 3+0 & -1+0\\ 0+0 & 0+2 & 0+0\\ -1+0 & -3+3 & 1+0\end{bmatrix}=\begin{bmatrix}1 & 3 & -1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{bmatrix}$$

Note: In general $A*B\neq B*A$

4- Square Matrix: It is the matrix that has same number of rows and columns A_{n*n}

Ex:

$$A_{3*3}=\begin{bmatrix}2 & 5 & -1\\ 0 & 4 & 3\\ 8 & 7 & 2\end{bmatrix}$$

5- Unity Matrix (*I*): It is a matrix in which the element of main diagonal is equal one and the other elements are zero.

$$I=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

6- **Zero Matrix**: It is a matrix in which all the elements are equal zero.

$$A=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

7- $A(BC)=(AB)C$ where *A*,*B* and *C* are matrices.

8- $AB \neq BA$

9- $(kA)B=k(AB)$ where *A* and *B* are matrices. *k* is constant.

Ex: *Show that* $(kA)B=k(AB)$ *if*:

$$A=\begin{bmatrix}4&2\\1&8\end{bmatrix},\quad B=\begin{bmatrix}2&1\\-2&4\end{bmatrix},\quad k=2$$

$$kA=\begin{bmatrix}8&4\\2&16\end{bmatrix}\quad\Rightarrow(kA)B=\begin{bmatrix}8&24\\-28&66\end{bmatrix}$$

$$AB=\begin{bmatrix}4&12\\-14&33\end{bmatrix}\quad\Rightarrow k(AB)=\begin{bmatrix}8&24\\-28&66\end{bmatrix}$$

10- $(A+B)C=AC+BC$ Where *A*, *B* and *C* are matrices

Ex: Show that $(A+B)C=AC+BC$ *if*:

$$A=\begin{bmatrix}4&2\\1&8\end{bmatrix},\quad B=\begin{bmatrix}2&1\\-2&4\end{bmatrix},\quad C=\begin{bmatrix}2&1\\1&0\end{bmatrix}$$

$$A+B=\begin{bmatrix}6&3\\-1&12\end{bmatrix}\quad\Rightarrow\quad(A+B)C=\begin{bmatrix}15&6\\10&-1\end{bmatrix}$$

$$AC=\begin{bmatrix}10&4\\10&1\end{bmatrix},\quad BC=\begin{bmatrix}5&2\\0&-2\end{bmatrix}$$

$$AC+BC=\begin{bmatrix}15&6\\10&-1\end{bmatrix}$$

HW1: Find A*B and B*A if:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -7 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 4 \\ 2 & 1 & 4 \end{bmatrix}$$

HW2: Find A*B and B*A if:

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -5 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \\ -1 & -3 \end{bmatrix}$$

Determinants

For each square matrix, there is a number called determinant and denoted by:

$$det.A \quad or \ \ |A|$$

- If $A = [-2]$,
  then $|A| = |-2| = -2$
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,
  then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- In general, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
  Then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

***Ex1*:** Find the det.A if:

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

Ex2: Find the $|A|$if:

$$A = \begin{bmatrix} 2 & 0 & 5 \\ 3 & -1 & 2 \\ 4 & -2 & 3 \end{bmatrix}$$

Ex3: Find the $|A|$ if:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$$

Properties of Determinant:

1- If two rows or columns are identical then the determinant is zero.
2- If row or column multiplied by a constant then the determinant will multiply same constant.
3- If the elements of the matrix above or below the main diagonal is equal to zero then the determinant is the product of the element of the main diagonal.
4- If all the elements of row or column of a square matrix is equal to zero then the determinant is zero.
5- If two rows or columns are interchanged then the determinant just change its sign.

Rank of the Matrix

It is the order of the highest square matrix with determinant does not equal zero. If A is a square matrix (n*n) it has Rank=n if and only if:

$|A| \neq 0$, Then the rank =n.

Ex1: Find the rank of A if:

$$A = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 3 & 0 \\ -2 & 4 & 0 \end{bmatrix}$$

Ex2: Find the rank of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Ex3: Find the rank of the following matrix:

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -1 & 7 & 4 & 9 \\ 7 & -7 & 0 & -5 \end{bmatrix}$$

Inverse of the Matrix

To find the inverse of the square matrix A^{-1}:

1) Find the determinant of A.
2) Write down the minors of A (min. A).
3) Write down the cofactor matrix of A (cof. A).
4) Find the adjoint matrix (adj. A) by transposing cof. A.
5) $A^{-1} = \frac{adj\ A}{|A|}$.

<u>Ex1</u>: If $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ findA^{-1}.

<u>Ex2</u>: If $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$ findA^{-1}.

<u>Ex3</u>: If $A = \begin{bmatrix} 2 & 3 & -4 \\ 3 & -2 & 5 \\ 1 & 4 & -3 \end{bmatrix}$ find A^{-1}.

Solving Equations using Matrices

If we have (n) equations with (m) variables for example

$a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

These equations can be written in matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A * X = B$

There are three methods to solve these equations using matrices:

1) Cramer's Rule.
2) Inverse of the matrix.
3) Gaussian Elimination Method

1- **Cramer's Rule**: which include that

$x = \frac{|A_x|}{|A|}$, $y = \frac{|A_y|}{|A|}$, $z = \frac{|A_z|}{|A|}$

Note: $|A| \neq 0$

Where

$$A_x = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_y = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

$$A_z = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Ex1: Using Cramer's rule solve the following equations:

$2x - 3y + 4z = -19$

$6x + 4y - 2z = 8$

$x + 5y + 4z = 23$

Ex2: Using Cramer's rule solve the following equations:

$x + 2y - z = 3$

$2x - y + z = 4$

$3x + y = 7$

HW1: Solve the following linear system equations using Cramer's rule:

$x - y + z = 4$

$2x + y + z = 7$

$2z - x - 2y + 1 = 0$

2- Inverse of the Matrix Method

If the linear system equations in matrix form is given by:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A * X = B$$

$$A^{-1}A * X = A^{-1}B$$

$$I * X = A^{-1}B$$

$$\boxed{X = A^{-1}B}$$

Ex1: Solve the following system equations using inverse of the matrix method.

$$x + 2y = 1$$

$$x - y = 4$$

HW2: Solve the following system equations using inverse of the matrix method.

$$x + 2y + 2z = 5$$

$$3x - 2y + z = -6$$

$$2x + y - z = -1$$

3- Gauss Elimination Method

1) Arrange the equations according to the unknown variables.
2) Write the augmented Matrix $\widetilde{A} = [A:B]$.
3) Eliminate x_1 from second and third equations.
4) Eliminate x_2 from third equation.
5) Find x_3, x_2 and x_1 using back substitution.

Ex1: Solve the linear system equations

$$x_2 - x_1 + 2x_3 = 2$$

$$x_3 + 3x_1 - x_2 = 6$$

$$3x_2 - x_1 + 4x_3 = 4$$

Ex2: Solve the following equations using Gauss elimination method:

$$3x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + x_2 + x_3 = 0$$

$$6x_1 + 2x_2 + 4x_3 = 6$$

Ex3: Solve the following equations using Gauss elimination method:

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$

$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$

$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

HW2: For the following current equations, find the values of $\boldsymbol{I_1, I_2}$ *and* $\boldsymbol{I_3}$.

$$I_2 = I_3 + I_1$$

$$10I_2 + 20I_1 = 80$$

$$25I_3 + 10I_2 = 90$$

3. Use matrices to solve: x + 2y + 3z = 5

2x – 3y – z = 3

-3x + 4y + 5z = 3

1. Find the general solution to the system of equations given by

$$-3x_1 + 6x_2 - x_3 + x_4 = -7,$$
$$x_1 - 2x_2 + 2x_3 + 3x_4 = -1,$$
$$2x_1 - 4x_2 + 5x_3 + 8x_4 = -4.$$

1. Using Gaussian elimination with back substitution, solve the following two systems of equations:

(a)

$$3x_1 - 7x_2 - 2x_3 = -7,$$
$$-3x_1 + 5x_2 + x_3 = 5,$$
$$6x_1 - 4x_2 = 2.$$

(b)

$$x_1 - 2x_2 + 3x_3 = 1,$$
$$-x_1 + 3x_2 - x_3 = -1,$$
$$2x_1 - 5x_2 + 5x_3 = 1.$$

HW: Using Gauss Elimination method solve the following equations

$-2x_3 - 7x_2 + 3x_1 = -7$

$5x_2 + x_3 - 3x_1 = 5$

$6x_1 - 4x_2 = 2$

HW: Using Gauss Elimination method solve the following equations

$3x_3 - 2x_2 + x_1 = 1$

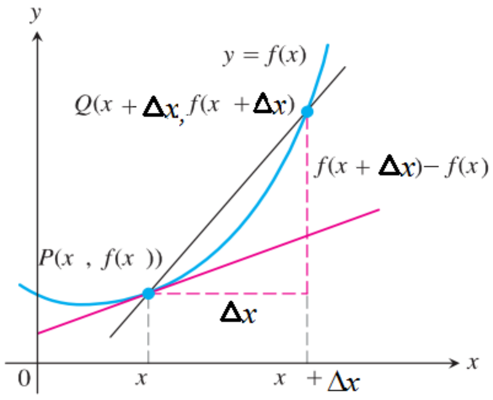
$3x_2 - x_3 - x_1 = -1$

$2x_1 - 5x_2 + 5x_3 = 1$

Differentiation:

It helps us to understand how functional relationships change such as the position or speed of moving object as function of time, the changing slope of curve being traversed by a point moving along it.

Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by:



$$\boxed{y' = \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

Ex1 **:** If $y = x^2$ find $\frac{dy}{dx}$ using limits?

Sol: $f(x) = x^2, \quad f(x+\Delta x) = (x+\Delta x)^2$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} (2x+\Delta x) = 2x$$

Ex2: If $y = \sqrt{x}$ find y' using limits?

Sol: $f(x) = \sqrt{x}, \quad f(x+\Delta x) = \sqrt{x+\Delta x}$

$$y' = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} * \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$$

$$y' = \lim_{\Delta x \to 0} \frac{(x+\Delta x) + \sqrt{x}\sqrt{x+\Delta x} - \sqrt{x}\sqrt{x+\Delta x} - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{x+\Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$y' = \lim_{\Delta x \to 0} \frac{1}{(\sqrt{x+\Delta x} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Differentiation Rules:

1) $\frac{d(c)}{dx} = zero \qquad (\text{where : c is a constant})$.

2) $\frac{d\left(cx^n\right)}{dx} = cn\, x^{n-1}$

3) $\frac{d}{dx}\left(f(x)\right)^n = n\left(f(x)\right)^{n-1}\frac{df(x)}{dx}$

4) $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

5) $\frac{d}{dx}\left(f(x) * g(x)\right) = f(x)\frac{d}{dx} g(x) + g(x)\frac{d}{dx} f(x)$

6) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{(g(x))^2}$

7) $\frac{d^2}{dx^2} f(x) = y''(x), \qquad \frac{d^2}{dx^2} f(x) = \text{high order derivative}$

Ex1 : If $y(x)=6x^3+5x+10$ find y'?

Sol: $y'=18x^2+5+0$

Ex2: find $\frac{dy}{dx}$ if $y(x)=\frac{1}{x^4}+x^{-3/4}$?

Sol: $y(x)=x^{-4}+x^{-3/4}$

$$\frac{dy}{dx}=-4x^{-5}+(-\frac{3}{4}x^{-7/4})$$

Ex3: find $\frac{dy}{dx}$ if $y(x)=(x^2+1)(x^3+3)$?

Sol:

$$\frac{dy}{dx}=(x^2+1)*3x^2+(x^3+3)*2x$$

$$\frac{dy}{dx}=3x^4+3x^2+2x^4+6x$$

$$\frac{dy}{dx}=5x^4+3x^2+6x$$

Ex4: find $\frac{dy}{dt}$ if $y(t)=\frac{t^2-1}{t^3+1}$?

Sol:

$$\frac{dy}{dt}=\frac{(t^3+1)2t-(t^2-1)3t^2}{(t^3+1)^2}$$

$$\frac{dy}{dt}=\frac{2t^4+2t-3t^4+3t^2}{(t^3+1)^2}$$

$$\frac{dy}{dt}=\frac{-t^4+3t^2+2t}{(t^3+1)^2}$$

HW: 3.2: p169

Differentiation of Trigonometric Functions:

Derivative $f'(u)$, $u=g(\theta)$
$\frac{d}{d\theta}(\sin u)=\cos u\frac{du}{d\theta}$
$\frac{d}{d\theta}(\cos u)=-\sin u\frac{du}{d\theta}$
$\frac{d}{d\theta}(\tan u)=\sec^2 u\frac{du}{d\theta}$
$\frac{d}{d\theta}(\cot u)=-\csc^2 u\frac{du}{d\theta}$
$\frac{d}{d\theta}(\sec u)=\sec u\ \tan u\frac{du}{d\theta}$
$\frac{d}{d\theta}(\csc u)=-\csc u\ \ \cot u\frac{du}{d\theta}$

Ex1: Proof that $\frac{d}{d\theta}(\tan\theta)=\sec^2\theta$.

Sol: $\tan\theta=\frac{\sin\theta}{\cos\theta}$

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right)=\frac{\cos\theta\cos\theta-(-\sin\theta\sin\theta)}{\cos^2\theta}=\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}=\sec^2\theta$$

Ex2: find $\frac{dy}{dx}$ if $y(x)=\sin^2 x^2$.

Sol:

$$\frac{dy}{dx}=2\sin x^2\left(\cos x^2\right)2x$$

$$\frac{dy}{dx}=4x\sin x^2\cos x^2$$

Ex3: find $\frac{dy}{dx}$ if $y(x)=x^2\sin x$.

Sol:

$$\frac{dy}{dx}=x^2\cos x+2x\sin x$$

Ex4: find $\frac{dy}{dx}$ if $y(x)=\frac{\cos x}{1-\sin x}$.

Sol:

$$\frac{dy}{dx}=\frac{-(1-\sin x)\sin x-\cos x(0-\cos x)}{(1-\sin x)^2}$$

$$\frac{dy}{dx}=\frac{-\sin x+\sin^2 x+\cos^2 x}{(1-\sin x)^2}$$

$$\frac{dy}{dx}=\frac{(1-\sin x)}{(1-\sin x)^2}=\frac{1}{1-\sin x}$$

Ex5: find y'' if $y(x)=\sec x$.

Sol:

$$y'=\sec x\tan x$$

$$y''=\sec x\sec^2 x+\tan x\sec x\tan x$$

$$y''=\sec^3 x+\sec x\tan^2 x$$

HW: 3.4: p188

The Chain Rule:

It is used for composite function. If $y=f(u)$ and $u=g(x)$ then:

$$\frac{dy}{dx}=\frac{dy}{du}\bullet\frac{du}{dx}$$

Where $\frac{dy}{du}$ is evaluated *u=g(x)*.

EXAMPLE 1

The function $y=(3x^2+1)^2$

is the composite of $y=u^2$ and $u=3x^2+1$. Calculating derivatives, we see that

$$\frac{dy}{du}=2u=2(3x^2+1)$$

$$\frac{du}{dx}=6x$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=2(3x^2+1)\cdot 6x$$
$$=36x^3+12x.$$

Calculating the derivative from the expanded formula, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(9x^4+6x^2+1\right)$$
$$=36x^3+12x.$$

EXAMPLE 2

$x(t)=\cos(t^2+1)$. Find the dx/dt.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x=\cos(u)$ and $u=t^2+1$. We have

$$\frac{dx}{du}=-\sin(u) \qquad x=\cos(u)$$

$$\frac{du}{dt}=2t. \qquad u=t^2+1$$

By the Chain Rule,

$$\frac{dx}{dt}=\frac{dx}{du}\cdot\frac{du}{dt}$$
$$=-\sin(u)\cdot 2t \qquad \frac{dx}{du}\text{ evaluated at } u$$
$$=-\sin(t^2+1)\cdot 2t$$
$$=-2t\sin(t^2+1).$$

EXAMPLE 3

The function $y(t) = \tan(5 - \sin 2t)$.

is the composite of $y = \tan u$ and $u = 5 - \sin 2t$

$$\frac{dy}{du} = \sec^2(u) = \sec^2(5 - \sin 2t)$$

$$\frac{du}{dt} = -2(\cos 2t)$$

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = \sec^2(5 - \sin 2t) \cdot -2(\cos 2t)$$

HW: section 3.5: p 201

Implicit Differentiation:

$$x^2 + y^2 - 25 = 0, \qquad y^2 - x = 0, \qquad \text{or} \quad x^3 + y^3 - 9xy = 0.$$

The above eq. we cannot put in the form of $y = f(x)$ to differentiate it in usual way, so we use implicit differentiation.

Steps of Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Ex1: find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin xy$?

Sol:

$$\frac{d}{dx}(y^2) = 2x + \cos xy \frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + \cos xy\left(x\frac{dy}{dx} + y\right)$$

$$2y\frac{dy}{dx} - x\cos xy\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx}(2y - x\cos xy) = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

Ex2: find $\frac{dy}{dx}$ if $y^3 + 3x^2y^2 + xy + x^2 = 2$?

Sol:

$$3y^2\frac{dy}{dx} + 3\left[2x^2y\frac{dy}{dx} + 2y^2x\right] + \left[x\frac{dy}{dx} + y\right] + 2x = 0$$

$$\frac{dy}{dx}\left[3y^2 + 6x^2y + x\right] + 6y^2x + y + 2x = 0$$

$$\frac{dy}{dx} = -\frac{6y^2x + y + 2x}{3y^2 + 6x^2y + x}$$

Ex3: find the slope of the tangent to the curve $x^2 + xy + y^2 = 7$ at (1,2)?

Sol:

$$2x + \left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[x + 2y] + 2x + y = 0$$

$\frac{dy}{dx} = -\frac{2x+y}{[x+2y]}$ Slope of the tangent

At point (1,2), $\frac{dy}{dx} = -\frac{2(1)+(2)}{[1+2(2)]} = -\frac{4}{5}$

Partial Derivative:

If z=f(x, y) then

$\frac{\partial z}{\partial x} = z_x$: represents the partial derivative of z with respect to x.

$\frac{\partial z}{\partial y} = z_y$: represents the partial derivative of z with respect to y.

Ex1: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if

$$f(x,y) = x^2 + 3xy + y - 1$$

Ex2: Verify that $f_{xy} = f_{yx}$ if $f(x,y) = x\sin y + y\sin x + xy$

Ex3: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y) = y\sin xy$

Applications of Derivative:

Ex1 : A body is moving on a coordinate line with $s(t) = t^2 - 3t + 2$ in meter and t in sec where 0 < t < 2 find:

1- The body displacement and average velocity for the given time interval.

2- Find the speed and the acceleration of the body at end points of the interval.

Sol:

$Displacement = \Delta s = s(0) - s(2)$

$s(0) = 2 \qquad\qquad s(2) = 0$

$\Delta s = 2 - 0 = 2m$

$$avg.velocity = \frac{\Delta s}{\Delta t} = \frac{2}{2} = 1\ m/sec$$

To find the speed:

$v(t) = \frac{ds(t)}{dt} = 2t - 3$

When $t = 0,\ v(0) = |(2 * 0) - 3| = 3\ m/\sec$

$t = 2,\ v(2) = |(2 * 2) - 3| = 1\ m/\sec$

To find the acceleration

$$a(t) = \frac{dv(t)}{dt} = 2$$

When $t = 0,\ a(0) = 2\ m^2/\sec$

$t = 2,\ a(2) = 2\ m^2/\sec$

Ex2: find the slope of the tangent to the curve $x^2 + xy + y^2 = 7$ at (1,2)?

Sol:

$$2x + \left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[x + 2y] + 2x + y = 0$$

$$\frac{dy}{dx} = -\frac{2x + y}{[x + 2y]}$$ Slope of the tangent

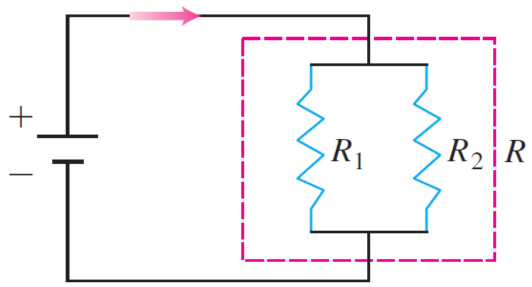
At point (1,2), $\frac{dy}{dx} = -\frac{2(1) + (2)}{[1 + 2(2)]} = -\frac{4}{5}$

Ex3: Find the tangent line and the normal line to the curve $\frac{x-y}{x-2y} = 2\ at\ (3,1)$.

Ex4

For the following electrical circuit, if R1 is decreasing at the rate of 1Ω/sec and R2 is increasing at the rate of 0.5 Ω/sec. What is the changing rate of R when R1=75Ω and R2=50Ω.

Sol:



$\frac{dR_1}{dt} = -1\ \Omega/sec$, $\frac{dR_2}{dt} = 0.5\ \Omega/sec$

$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$$\frac{-1}{R^2}\frac{dR}{dt} = \frac{-1}{{R_1}^2}\frac{dR_1}{dt} - \frac{1}{{R_2}^2}\frac{dR_2}{dt}$$

At R1=75Ω and R2=50Ω

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{75} + \frac{1}{50}$$

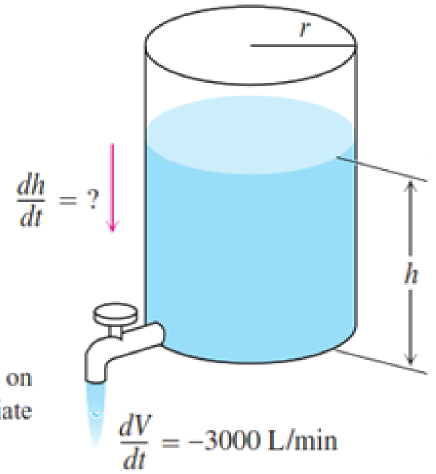
$$R = 30\Omega$$

$$\frac{-1}{(30)^2}\frac{dR}{dt} = \frac{1}{(75)^2} - \frac{0.5}{(50)^2} = \frac{1}{5625} - \frac{0.5}{2500}$$

$$\frac{dR}{dt} = 0.02\ \Omega/sec$$

Ex3:

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min? If the raduis of the cylindrical tank is 1 m.



Solution

$$\frac{dV}{dt} = -3000.$$ We pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.

We are asked to find $\frac{dh}{dt}$. How fast will the fluid level drop?

To find dh/dt, we first write an equation that relates h to V. The equation depends on the units chosen for V, r, and h. With V in liters and r and h in meters, the appropriate equation for the cylinder's volume is

$$V = \pi r^2 h$$

Since V and h are differentiable functions of t, we can differentiate both sides of the equation $V = \pi r^2 h$ with respect to t to get an equation that relates dh/dt to dV/dt:

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$ r is a constant.

We substitute the known value $dV/dt = \frac{-3000}{1000} = -3\ \text{m}^3/\text{min}$

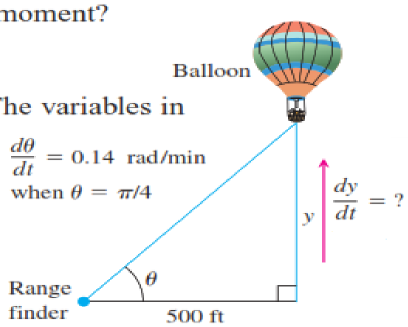
$$-3 = \pi r^2 \frac{dh}{dt}.$$

$$\frac{dh}{dt} = -\frac{3}{\pi r^2}.$$

If $r = 1$ m: $\frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95$ m/min

Ex4:

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?



Solution We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.43). The variables in the picture are
   θ = the angle in radians the range finder makes with the ground.
   y = the height in feet of the balloon.
2. *Write down the additional numerical information.*
$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \quad \text{when} \quad \theta = \frac{\pi}{4}$$
3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$.
4. *Write an equation that relates the variables y and θ.*
$$\frac{y}{500} = \tan\theta \quad \text{or} \quad y = 500\tan\theta$$
5. *Differentiate with respect to t.* The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).
$$\frac{dy}{dt} = 500(\sec^2\theta)\frac{d\theta}{dt}$$
6. *Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.*

$$\frac{dy}{dt} = 500(\sqrt{2})^2(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min.

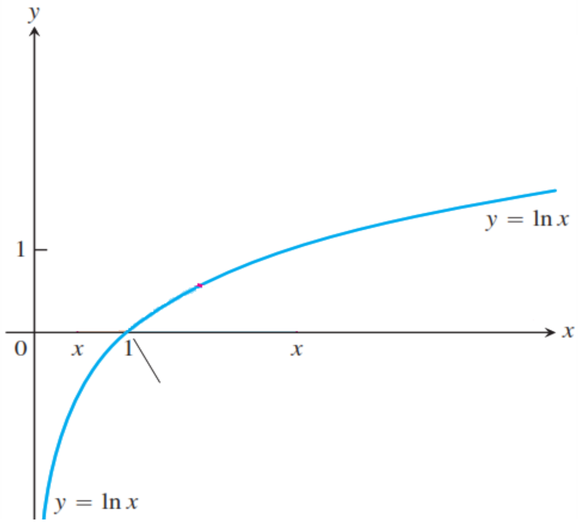
This chapter is study exponential, natural logarithmic, trigonometric, and hyperbolic function.

Natural Log.	General Log
$\log_e x = \ln x$	$\log_a x$
Inverse of Natural Log.	**Inverse** General Log
$e^x = \exp(x)$	a^x

1- Natural Logarithms

The natural logarithm of a positive number (x), written as *ln x*, is the value of an integral.

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$



If $x>1$, then *ln x* is the area under the curve $y = 1/t$ from $t=1$ to $t=x$. For $0<x<1$ *ln x* gives the negative of the area under the curve from x to1. The function is not defined for $x<0$.

x	$\ln x$
0	undefined
0.05	−3.00
0.5	−0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

1- Derivative: $$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0$$

2- Integral: $$\int \frac{1}{u}\,du = \ln|u| + C.$$

Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:*	$\ln ax = \ln a + \ln x$	
2. *Quotient Rule:*	$\ln \frac{a}{x} = \ln a - \ln x$	
3. *Reciprocal Rule:*	$\ln \frac{1}{x} = -\ln x$	Rule 2 with $a = 1$
4. *Power Rule:*	$\ln x^r = r \ln x$	r rational

Ex1 **:**Use the properties of Natural Log. to simplify the following expression:

1) $\ln sin\theta - \ln\left(\frac{sin\theta}{5}\right) = \ln\left(\frac{sin\theta}{\frac{sin\theta}{5}}\right) = \ln 5$.

2) $\ln\sec\theta + \ln\cos\theta = \ln(\sec\theta\cos\theta) = \ln 1 = 0$.

3) $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right) = \ln\left(\frac{3x^2-9x}{3x}\right) = \ln(x-3)$.

4) $\ln(8x+4) - \ln 2^2 = \ln(8x+4) - \ln 4 = \ln\left(\frac{8x+4}{4}\right) = \ln(2x+1)$.

5) $\frac{1}{2}\ln(4t^4) - \ln 2 = \ln\sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln\left(\frac{2t^2}{2}\right) = \ln t^2$.

6) $3\ln\sqrt[3]{t^2-1} - \ln(t+1) = 3\ln(t^2-1)^{\frac{1}{3}} - \ln(t+1) = 3\left(\frac{1}{3}\right)\ln(t^2-1) - \ln(t+1) =$

$\ln\left(\frac{(t+1)(t-1)}{(t+1)}\right) = \ln(t-1)$.

Ex2: Find $\frac{dy}{dx}$:

(a) $\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}$

(b) $\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}$.

(C) Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned} \ln y &= \ln\frac{(x^2+1)(x+3)^{1/2}}{x-1} \\ &= \ln((x^2+1)(x+3)^{1/2}) - \ln(x-1) && \text{Rule 2} \\ &= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) && \text{Rule 1} \\ &= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). && \text{Rule 3} \end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Ex3: Find the integral:

(a) $\int_0^2 \frac{2x}{x^2-5}dx = \ln\left|x^2-5\right|\Big]_0^2 = \ln(1)-\ln(5) = -\ln 5$

(b) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3+2\sin\theta}d\theta = 2\left[\ln\left|3+2\sin\theta\right|\Big]_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\right] = 2\left[\ln(3+2\sin(\frac{\pi}{2}))-\ln(3+2\sin(-\frac{\pi}{2}))\right]$

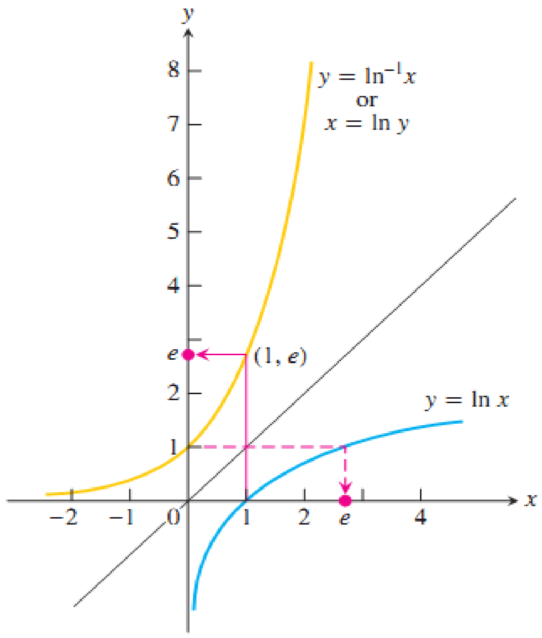
$= 2\left[\ln(5)-\ln(1)\right] = 2\ln 5$

(c) $\int \tan x dx = \int \frac{\sin x}{\cos x}dx = -\ln\left|\cos x\right| + c = \ln\frac{1}{\left|\cos x\right|} + c = \ln\left|\sec x\right| + c$

HW: section 7.2

2- Exponential

$$e^x = \exp(x) = \ln^{-1}(x)$$



Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$
5. $e^{\ln x} = x$ (all $x > 0$)
6. $\ln(e^x) = x$ (all x)

1- Derivative $$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

2- Integration $$\int e^u \, du = e^u + C.$$

Ex1 **:** Put the following expressions in form of *y=f(x)*:

1 - $\ln(y-1) - \ln 2 = x + \ln x$

Sol:

$$\ln(y-1) - \ln 2 - \ln x = x$$
$$\ln\left(\frac{y-1}{2x}\right) = x$$
$$e^{\ln\left(\frac{y-1}{2x}\right)} = e^x$$
$$y - 1 = 2xe^x$$
$$y(x) = 2xe^x + 1$$

2- $\ln(y^2-1) - \ln(y+1) = \ln(\sin x)$

Sol:

$$\ln\left(\frac{y^2-1}{y+1}\right) = \ln(\sin x) \quad \Rightarrow \quad \ln(y-1) = \ln(\sin x)$$
$$e^{\ln(y-1)} = e^{\ln(\sin x)} \qquad \Rightarrow y - 1 = \sin x$$
$$y = \sin x + 1$$

Ex2: Find $\frac{dy}{dx}$:

(a) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$

(b) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$

(c) $y = (1+2x)e^{-2x}$

$$y' = (1+2x)e^{-2x}\frac{d}{dx}(-2x) + 2e^{-2x}$$
$$y' = -2(1+2x)e^{-2x} + 2e^{-2x} = -4xe^{-2x}$$

Ex3: Find the integral:

(a) $\int_0^{\ln 2} e^{3x}dx = \frac{1}{3}\int_0^{\ln 2} 3 \bullet e^{3x}dx = \frac{1}{3}e^{3x}\Big|_0^{\ln 2}$

$$= \frac{1}{3}\left[e^{3\ln 2} - e^{3\ln 0}\right]$$

(a) $\displaystyle\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du$

$\displaystyle = \frac{1}{3}\int_0^{\ln 8} e^u\,du$

$\displaystyle = \frac{1}{3}e^u\Big]_0^{\ln 8}$

$\displaystyle = \frac{1}{3}(8-1) = \frac{7}{3}$

$u = 3x, \quad \frac{1}{3}du = dx, \quad u(0) = 0,$
$u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

(b) $\displaystyle\int_{\pi/4}^{\pi/2}\left(1+e^{\cot\theta}\right)\csc^2\theta\,d\theta$

$\displaystyle = \int_{\pi/4}^{\pi/2}\csc^2\theta\,d\theta - \int_1^0 e^u\,du$

$= [-\cot\theta]_{\pi/4}^{\pi/2} - [e^u]_1^0$

$= \left[-\cot\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{4}\right)\right] - \left(e^0 - e^1\right)$

$= (0+1) - (1-e) = e$

Let $u = \cot\theta$
$du = -\csc^2\theta\,d\theta;\ \theta = \frac{\pi}{4}$
$u = 1,\ \theta = \frac{\pi}{2} \Rightarrow u = 0;$

HW: section 7.3

3- General Logarithm

General Log	Inverse of general Log:
1- Derivative $\frac{d}{dx}\log_a u = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}$ **2- Integral** $\int \log_a u\,du = \int \frac{\ln u}{\ln a}du$	**1- Derivative** $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$ **2- Integral** $\int a^u du = \frac{a^u}{\ln a} + C$

Ex: Find $\frac{dy}{dx}$ if

$$y = \log_2\left(\frac{x^2e^2}{2\sqrt{x+1}}\right)$$

(a) $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$

(b) $\int \frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int\frac{\ln x}{x}dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

(c) $y = 2^{\sin 3t} \Rightarrow$

$$\frac{dy}{dt} = (2^{\sin 3t}\ln 2)(\cos 3t)(3)$$
$$= (3\cos 3t)(2^{\sin 3t})(\ln 2)$$

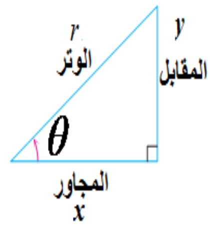
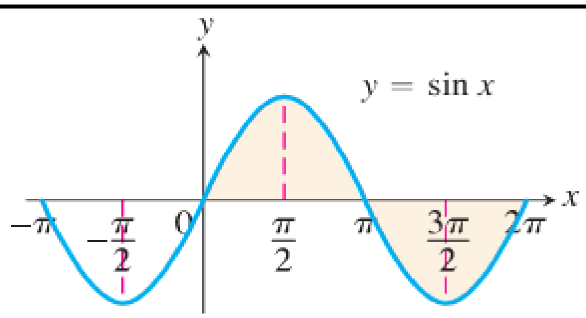
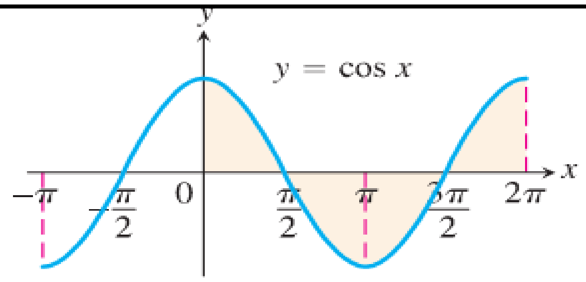
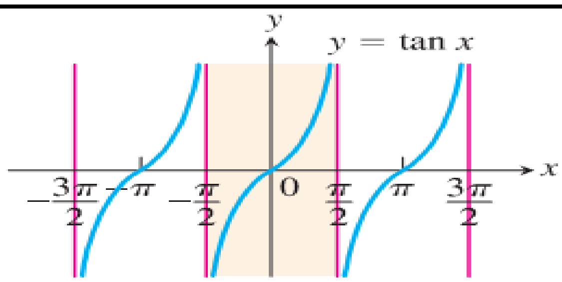
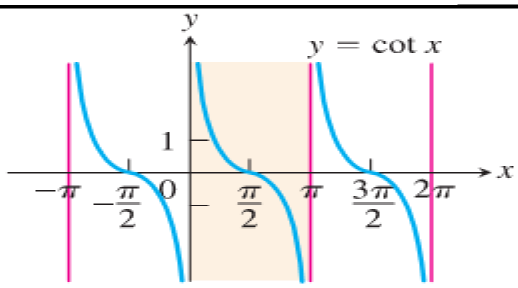
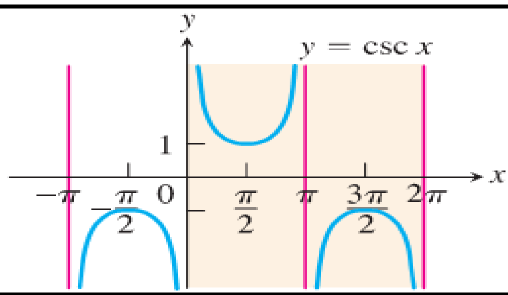
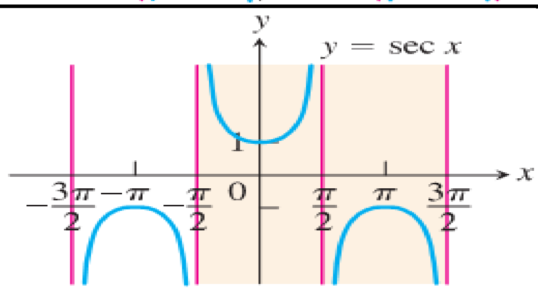
(d) $\int_0^{\pi/2} 7^{\cos t}\sin t\,dt$

$$= -\int_1^0 7^u\,du = \left[-\frac{7^u}{\ln 7}\right]_1^0$$
$$= \left(\frac{-1}{\ln 7}\right)(7^0 - 7) = \frac{6}{\ln 7}$$

Let $u = \cos t \Rightarrow du = -\sin t\,dt$
$t = 0 \Rightarrow u = 1, t = \frac{\pi}{2} \Rightarrow u = 0$

HW: section 7.4

3- **Trigonometric Functions** :

<table>
<tr><th>Function</th><th>Plot</th></tr>
<tr><td>1- $\sin\theta = \dfrac{y}{r}$
</td><td></td></tr>
<tr><td>2- $\cos\theta = \dfrac{x}{r}$</td><td></td></tr>
<tr><td>3- $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$</td><td></td></tr>
<tr><td>4- $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{x}{y}$</td><td></td></tr>
<tr><td>5- $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$</td><td></td></tr>
<tr><td>6- $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{x}{r}$</td><td></td></tr>
</table>

Some Important Rule of Trigonometric Functions:

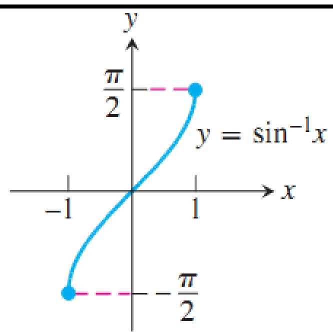
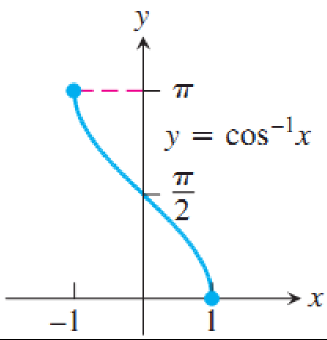
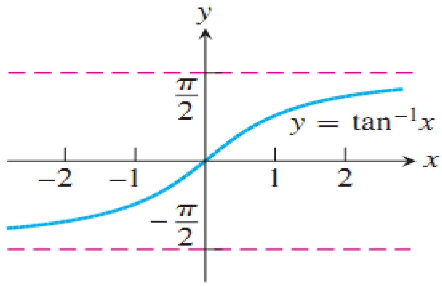
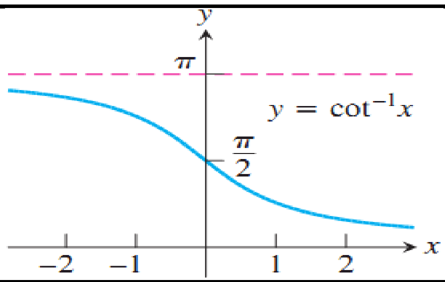
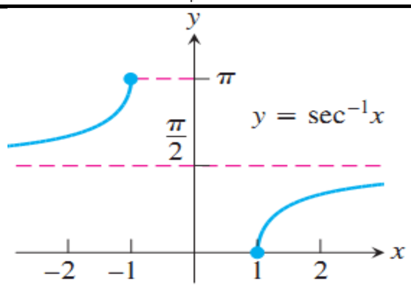
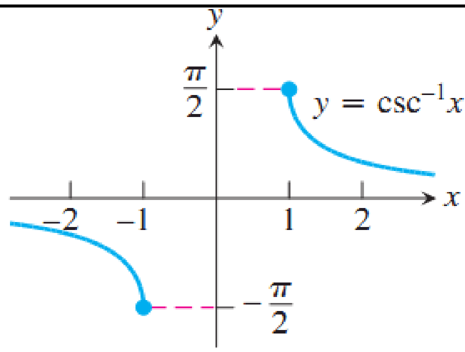
Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

1) $\cos^2\theta + \sin^2\theta = 1$	6) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
2) $1 + \tan^2\theta = \sec^2\theta$	7) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
3) $1 + \cot^2\theta = \csc^2\theta$	8) $\cos(A \mp B) = \cos A\cos B \pm \sin A\sin B$
4) $\cos 2\theta = \cos^2\theta - \sin^2\theta$	9) $\sin(A \mp B) = \sin A\cos B \mp \cos A\sin B$
5) $\sin 2\theta = 2\sin\theta\cos\theta$	

Differentiation and integration of Trigonometric Functions:

Derivative $f'(u)$, $u=g(\theta)$	Integration
1) $\dfrac{d}{d\theta}(\sin u) = \cos u\dfrac{du}{d\theta}$	$\int \sin u\, du = -\cos u + c$
2) $\dfrac{d}{d\theta}(\cos u) = -\sin u\dfrac{du}{d\theta}$	$\int \cos u\, du = \sin u + c$
3) $\dfrac{d}{d\theta}(\tan u) = \sec^2 u\dfrac{du}{d\theta}$	$\int \tan u\, du = -\ln\lvert\cos u\rvert + c$
4) $\dfrac{d}{d\theta}(\cot u) = -\csc^2 u\dfrac{du}{d\theta}$	$\int \cot u\, du = \ln\lvert\sin u\rvert + c$
5) $\dfrac{d}{d\theta}(\sec u) = \sec u\ \tan u\dfrac{du}{d\theta}$	$\int \sec u\, du = \ln\lvert\sec u + \tan u\rvert + c$
6) $\dfrac{d}{d\theta}(\csc u) = -\csc u\ \tan u\dfrac{du}{d\theta}$	$\int \csc u\, du = \ln\lvert\csc u - \cot u\rvert + c$

4- Inverse of Trigonometric Function:

Function $f(x)$	Plot
1- $y=\sin^{-1}x$	
2- $y=\cos^{-1}x$	
3- $y=\tan^{-1}x$	
4- $y=\cot^{-1}x$	
5- $y=\sec^{-1}x$	
6- $y=\csc^{-1}x$	

Derivative of the Inverse Trigonometric Functions:

Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

Some important rule:

$\sin(\sin^{-1} x) = x$ $\sin^{-1}(\sin x) = x$ $\cos(\cos^{-1} x) = x$ $\cos^{-1}(\cos x) = x$	$\cos^{-1} x + \sin^{-1} x = \pi/2$ $\cot^{-1} x + \tan^{-1} x = \pi/2$ $\csc^{-1} x + \sec^{-1} x = \pi/2$

Ex1: prove $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$

Sol :

$\sin y = x$ — $y = \sin^{-1} x \Leftrightarrow \sin y = x$

$\dfrac{d}{dx}(\sin y) = 1$ — Derivative of both sides with respect to x

$\cos y \dfrac{dy}{dx} = 1$ — Chain Rule

$\dfrac{dy}{dx} = \dfrac{1}{\cos y}$ — We can divide because $\cos y > 0$ for $-\pi/2 < y < \pi/2$.

$= \dfrac{1}{\sqrt{1-\sin^2 y}}$ — $\cos y = \sqrt{1-\sin^2 y}$

$= \dfrac{1}{\sqrt{1-x^2}}$

Ex2:

A particle moves along the x-axis so that its position at any time $t \geq 0$ is $x(t) = \tan^{-1}\sqrt{t}$. What is the velocity of the particle when $t = 16$?

Solution

$$v(t) = \frac{d}{dt}\tan^{-1}\sqrt{t} = \frac{1}{1 + (\sqrt{t})^2} \cdot \frac{d}{dt}\sqrt{t} = \frac{1}{1 + t} \cdot \frac{1}{2\sqrt{t}}$$

When $t = 16$, the velocity is

$$v(16) = \frac{1}{1 + 16} \cdot \frac{1}{2\sqrt{16}} = \frac{1}{136}.$$

Ex3: find $\frac{d}{dx}\sec^{-1}(5x^4)$

$$\begin{aligned} \frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4) \\ &= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \\ &= \frac{4}{x\sqrt{25x^8 - 1}} \end{aligned}$$

Integral of the inverse trigonometric functions

Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

Ex4: find $\int \frac{dx}{\sqrt{3-4x^2}}$

Sol:

$$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$$

$a = \sqrt{3}$, $u = 2x$, and $du/2 = dx$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

Ex5: find $\int \frac{dx}{\sqrt{e^{2x}-6}}$

Solution

$$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{du/u}{\sqrt{u^2-a^2}}$$

$u = e^x$, $du = e^x\,dx$,
$dx = du/e^x = du/u$,
$a = \sqrt{6}$

$$= \int \frac{du}{u\sqrt{u^2-a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

Ex6: evaluate $\int_2^4 \frac{2dx}{x^2-6x+10}$

$$\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{1+(x^2-6x+9)}$$

$$= 2\int_2^4 \frac{dx}{1+(x-3)^2} = 2[\tan^{-1}(x-3)]_2^4$$

$$= 2\left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

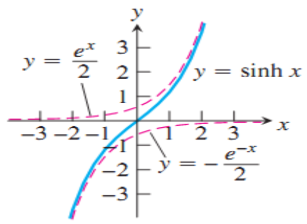
$$= 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$$

The Hyperbolic Functions:

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and they are important in applications. For instance, they are used in problems such as computing the tension in a cable suspended by its two ends, as in an electric transmission line. They also play an important role in finding solutions to differential equations. In this section, we give a brief introduction to hyperbolic functions, their graphs, how their derivatives are calculated, and why they appear as important antiderivatives.

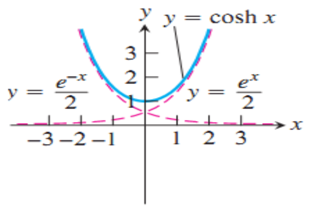
The basic hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(a)

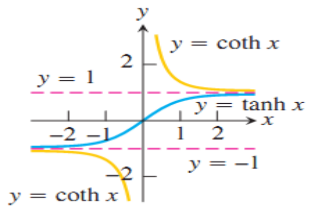
$$\cosh x = \frac{e^x + e^{-x}}{2}$$



(b)

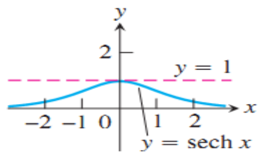
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



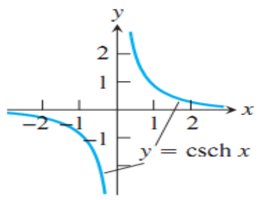
(c)

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(d)

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



(e)

Some Important rule of hyperbolic function

Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh^2 x + \sinh^2 x = \cosh 2x$$
$$\sinh 2x = 2\sinh x \cosh x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$
$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$
$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$
$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$
$$\int \cosh u \, du = \sinh u + C$$
$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$
$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$
$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$
$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

Ex1: Solve

(a) $y = \ln(\sinh z)$

$$\frac{dy}{dz} = \frac{1}{\sinh z}\cosh z = \coth z$$

(b) $\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1=t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$

$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

(c) $y = (x^2+1)\operatorname{sech}(\ln x)$

$$= (x^2+1)\left(\frac{2}{e^{\ln x}+e^{-\ln x}}\right)$$

$$= (x^2+1)\left(\frac{2}{x+x^{-1}}\right)$$

$$= (x^2+1)\left(\frac{2x}{x^2+1}\right)$$

$$= 2x \Rightarrow \frac{dy}{dx} = 2$$

Ex2: Solve :

(a) $\int \coth 5x\,dx = \int \frac{\cosh 5x}{\sinh 5x}\,dx$

$$= \frac{1}{5}\ln|\sinh 5x| + C$$

(b) $\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x\,\frac{e^x - e^{-x}}{2}\,dx$

$$= \int_0^{\ln 2}(2e^{2x}-2)\,dx$$

$$= \left[e^{2x}-2x\right]_0^{\ln 2} = (e^{2\ln 2}-2\ln 2)-(1-0)$$

$$= 4 - 2\ln 2 - 1$$

$$\approx 1.6137$$

Derivative and Integral of Inverse Hyperbolic Function:

Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \qquad u \neq 0$$

Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

Integrals leading to inverse hyperbolic functions

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & \text{if } u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

Ex1: simplify the hyperbolic function $y = 2\cosh(\ln x)$

$$y = 2\frac{e^{\ln x} + e^{-\ln x}}{2} = x + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$$

Ex2: simplify the hyperbolic function

$$y = \ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$$

$$y = \ln\left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right) + \ln\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$

$$y = \ln\left(\frac{2e^x}{2}\right) + \ln\left(\frac{2e^{-x}}{2}\right) = \ln(e^x) + \ln(e^{-x})$$

$$y = x - \frac{1}{x}$$

Ex3: simplify the hyperbolic function

$$y = (\sinh x + \cosh x)^4$$

$$y = \left(\frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}\right)^4$$

$$y = \left(\frac{2e^x}{2}\right)^4 = e^{4x}$$

Ex4: Find $\frac{dy}{dt}$ if $y = 2\sqrt{t}\,\tanh\sqrt{t}$

$$\frac{dy}{dt} = 2\sqrt{t}\,\frac{\operatorname{sech}^2\sqrt{t}}{2\sqrt{t}} + \frac{2\tanh\sqrt{t}}{\sqrt{t}} = \operatorname{sech}^2\sqrt{t} + \frac{\tanh\sqrt{t}}{\sqrt{t}}$$

Ex5: Find $\frac{dy}{dx}$ if $y = \sinh^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x}}.\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1+x)}}$$

Ex6: Find $\frac{dy}{d\theta}$ if $y = (1-\theta)\tanh^{-1}\theta$

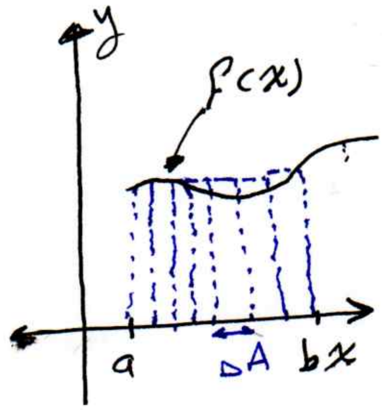
$$\frac{dy}{d\theta} = (1-\theta)\frac{1}{1-\theta^2} - \tanh^{-1}\theta$$

$$\frac{dy}{d\theta} = \frac{(1-\theta)}{(1-\theta)(1+\theta)} - \tanh^{-1}\theta$$

$$\frac{dy}{d\theta} = \frac{1}{(1+\theta)} - \tanh^{-1}\theta$$

Integration: (Anti- Derivative)

Let $f(x)$ is a function or real variable x over interval $[a,b]$ as shown in figure, then the area under the curve or (avarage of the function) is:



$$F(x) = \int_a^b f(x)\,dx$$

There are two type of integration:

① Infinite integral:

$$\int f(x)\,dx = F(x) + C$$

② Finite integral:

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Some properties of Integration:

① $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ ③ $\int_a^a f(x)\,dx = 0$

② $\int_a^b f(x) \mp g(x)\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx$

Integration:

① $\int du = u + c$ where $u = g(x)$

② $\int u^n du = \frac{u^{n+1}}{n+1} + c$ where $n \neq -1$

Ex1: find the integral of:

$$\int 5x - x^2 + 2 \, dx$$

$$= \frac{5x^2}{2} - \frac{x^3}{3} + 2x + c$$

Ex2: Solve the following integral:

$$\int \sqrt{2x+1} \, dx$$

$$= \int (2x+1)^{\frac{1}{2}} * \frac{2}{2} \, dx = \frac{1}{2} \int 2(2x+1)^{\frac{1}{2}} \, dx$$

$$= \frac{1}{2} \frac{(2x+1)^{3/2}}{3/2} + c = \frac{(2x+1)^{3/2}}{3} + c$$

Ex3: Find $y(x)$ for the following expression:

$$\frac{dy}{dx} = 3x^2$$

$$\int dy = \int 3x^2 \, dx \quad \text{(integrat both sides)}$$

$$y = \frac{3x^3}{3} + c = x^3 + c$$

Ex 4: Find $y(x)$ if $\frac{dy}{dx} = x\sqrt{1+x^2}$ at $x=0$ and $y=-3$.

Sol:

$$dy = x\sqrt{1+x^2}\, dx$$

$$\int dy = \int x(1+x^2)^{\frac{1}{2}} dx * \frac{2}{2}$$

$$y = \frac{1}{2}\frac{(1+x^2)^{3/2}}{3/2} + c = \frac{(1+x^2)^{3/2}}{3} + c$$

when: $y=-3$ & $x=0$ (to find the value of c):

$$-3 = \frac{(1+0)^{3/2}}{3} + c \Longrightarrow C = \frac{-10}{3}$$

then $y(x) = (1+x^2)^{3/2} - \frac{10}{3}$

Ex5: find the curve whose slope is $3x^2$ and passes through the point $P(1,-1)$

Sol.. $\frac{dy}{dx} = 3x^2$

$$\int dy = \int 3x^2\, dx$$

$$y = \frac{\cancel{3}x^3}{\cancel{3}} + c$$

$$y = x^3 + c$$

when $y=-1$, $x=1$ (at point P)

$$-1 = 1 + c \Longrightarrow c = -2$$

then the curve is $y = x^3 - 2$

Ex6: Evaluate the following integral:

$$\int \frac{3r}{\sqrt{1-r^2}}\,dr$$

sol.: $\int 3r\,(1-r^2)^{-\frac{1}{2}}\,dr \quad * \frac{-2}{-2}$

$$= \frac{-3}{2}\int -2r\,(1-r^2)^{-\frac{1}{2}}\,dr$$

$$= \frac{-3}{2}\,\frac{(1-r^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -3\sqrt{1-r^2} + c$$

Integration of Trigonometric Functions:

where $u = f(x)$

1. $\int \sin u\,du = -\cos u + c$

2. $\int \cos u\,du = \sin u + c$

3. $\int \sec^2 u\,du = \tan u + c$

4. $\int \csc^2 u\,du = -\cot u + c$

5. $\int \sec u \tan u\,du = \sec u + c$

6. $\int \csc u \cot u\,du = -\csc u + c$

Ex1 $\int \tan u \, du = \int \frac{\sin u}{\cos u} du$

$= -\ln|\cos u| + c = \ln|\sec u| + c$

Ex2: $\int \cot u \, du = \int \frac{\cos u}{\sin u} du$

$= \ln|\sin u| + c$

$= -\ln|\csc u| + c$

Ex3: find the $\int \cos 2t \, dt$

Sol: $\int \cos 2t \, dt * \frac{2}{2}$

$= \frac{1}{2}\int 2\cos 2t \, dt = \frac{1}{2}\sin 2t + c$

Ex4: $\int \frac{\cos 2x}{\sin^3 2x} dx$

Sol: $\int \cos 2x \sin^{-3} 2x \, dx * \frac{2}{2}$

$= \frac{1}{2} \frac{\sin^{-2} 2x}{-2} + c = \frac{-1}{4\sin^2 2x} + c$

Ex5: $\int \frac{1+\cos 2x}{\sin^2 2x} dx$

Sol: $\int \frac{1}{\sin^2 2x} dx + \int \frac{\cos 2x}{\sin^2 2x} dx = \int \csc^2 2x \, dx + \int \frac{\cos 2x}{\sin^2 2x} dx$

$= -\frac{1}{2}\cot(2x) + \frac{1}{2}\frac{\sin^{-1} 2x}{-1} + c$

$= -\frac{1}{2}\cot(2x) - \frac{1}{2}\csc(2x) + c$

Ex6: $\int \sin(3x)\cos(7x)\,dx$

ملاحظة: يجب ان تكون الزوايا متساوية لكي نعتبر احدهما مشتقة للاخر لهذا في هكذا حالة نلجأ الى القوانين *

Sol: $\int \sin(3x)\cos(7x)\,dx$

$$= \frac{1}{2}\left[\int \left(\sin(3+7)x + \sin(7-3)x\right) dx\right]$$

$$= \frac{1}{2}\left[\int \sin 10x\,dx + \int \sin 4x\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{10}\cos 10x - \frac{1}{4}\cos 4x + c\right]$$

* **Note:**

① $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$

② $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$

③ $\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$

Integration of Exponent Functions:

$$① \int e^u \, du = e^u + c$$

$$② \int a^u \, du = \frac{a^u}{\ln a} + c$$

where $u = f(x)$

Integration of Logarithmic Functions:

$$① \int \frac{1}{u} \, du = \ln|u| + c$$

$$② \int \log_a u \, du = \int \frac{\ln u}{\ln a} \, du$$

where $u = f(x)$

Ex1: $\int_0^2 \frac{2x}{x^2-5} \, dx = \left[\ln|x^2-5| \right]_0^2 = \ln(1) - \ln(5)$

$= -\ln 5$

Ex2: $\int (1+e^{\cot\theta})\csc^2\theta\, d\theta$

Sol: $= \int \csc^2\theta\, d\theta + \int e^{\cot\theta}\csc^2\theta\, d\theta$

$= -\cot\theta - e^{\cot\theta} + C$

Ex3: $\int 5^{\cos t} \cdot \sin t\, dt$

$= -\left[\frac{1}{\ln 5} 5^{\cos t}\right] + C$

Ex4: $\int \frac{\log_2 x}{x}\, dx$

Sol: $= \int \frac{\ln x}{x \ln 2}\, dx$

$= \frac{1}{\ln 2}\int \frac{1}{x}\ln x\, dx = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C$

$= \frac{(\ln x)^2}{2\ln 2} + C$

Some Integrals produce inverse of trigonometric functions:

① $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)+c$$

② $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$$

③ $$\int \frac{du}{|u|\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|+c$$

where $u = f(x)$

Methods of Integration:

① Power of trigonometric functions: sec (8.4)

(A) power of $\sin^n x$ & $\cos^n x$:

when n = odd no. $(\sin^2 x + \cos^2 x = 1)$

Ex1 $\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx$

$= \int (1 - \cos^2 x) \sin x \, dx$

$= \int \sin x \, dx - \int \cos^2 x \sin x \, dx$

$= -\cos x + \dfrac{\cos^3 x}{3} + c$

Ex2: $\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx$

$= \int (1 - \sin^2 x)^2 \cos x \, dx$

$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$

$= \int \cos x \, dx - \int 2\sin^2 x \cos x \, dx + \int \sin^4 x \cos x \, dx$

$= \sin x - 2\dfrac{\sin^3 x}{3} + \dfrac{\sin^5 x}{5} + c$

Ex 3: $\int \sin^7 x\,dx = \int \sin^6 x \sin x\,dx = \int (\sin^2 x)^3 \sin x\,dx$

$= \int (1-\cos^2 x)^3 \sin x\,dx$

$= \int (1-2\cos^2 x + \cos^4 x)(1-\cos^2 x) \sin x\,dx$

$= \int (1-3\cos^2 x + 3\cos^4 x - \cos^6 x) \sin x\,dx$

$= \int \sin x\,dx - 3\int \cos^2 x \sin x\,dx + 3\int \cos^4 x \sin x\,dx - \int \cos^6 x \sin x\,dx$

$= -\cos x + \cancel{3}\dfrac{\cos^3 x}{\cancel{3}} - \dfrac{3\cos^5 x}{5} + \dfrac{\cos^7 x}{7} + c$

Ex 4: $\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos^2 x \cos x\,dx$

$= \int \sin^3 x (1-\sin^2 x) \cos x\,dx$

$= \int \sin^3 x \cos x\,dx - \int \sin^5 x \cos x\,dx$

$= \dfrac{\sin^4 x}{4} - \dfrac{\sin^6 x}{6} + c$

H.W:

1. $\int \cos^7 x\,dx$
2. $\int \sin^2 x \cos^3 x\,dx$
3. $\int \cos^2 x \sin^3 x\,dx$

When n = even No. $\rightarrow$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

Ex1: $\int \sin^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)\, dx$

$= \int \frac{1}{2} dx - \frac{1}{2}\int \cos 2x \, dx * \frac{2}{2}$ ← ضعف الزاوية

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

Ex2: $\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx$

$= \int \frac{1}{4}(1+\cos 2x)^2 dx$

$= \frac{1}{4}\left[\int 1\, dx + \int 2\cos 2x \, dx + \int \cos^2 2x \, dx\right]$

$= \frac{1}{4}\left[x + \sin 2x + \frac{1}{2}\int (1+\cos 4x)\, dx\right]$

$= \frac{1}{4}\left[x + \sin 2x + \frac{1}{2}x + \frac{1}{2}\frac{\sin 4x}{4}\right] + C$

$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C$

Ex3: $\int \sin^2 x \cos^2 x \, dx$

$$= \int \frac{1}{2}(1-\cos 2x) \cdot \frac{1}{2}(1+\cos 2x)\, dx$$

$$= \frac{1}{4}\int (1-\cos^2 2x)\, dx$$

$$= \frac{1}{4}\int \sin^2 2x \, dx = \frac{1}{4}\int \frac{1}{2}(1-\cos 4x)\, dx$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x \, dx$$

$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$$

H.w:

① $\int \sin^4 x \, dx$

② $\int \sin^2 x \cos^4 x \, dx$

(B) power of $\tan^n x$ & $\cot^n x$:-

ملاحظة مهمة في نهاية الورقة

$\rightarrow \tan^2 x = \sec^2 x - 1$

$\rightarrow \cot^2 x = \csc^2 x - 1$

Ex1: $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$= \int \sec^2 x\, dx - \int dx = \tan x - x + c$

Ex2: $\int \cot^2 x\, dx = \int (\csc^2 x - 1)\, dx$

$= \int \csc^2 x\, dx - \int dx = -\cot x - x + c$

Ex3: $\int \tan^3 x\, dx = \int \tan x \tan^2 x\, dx$

$= \int \tan x (\sec^2 x - 1)\, dx$

$= \int \tan x \sec^2 x\, dx - \int \tan x\, dx$

$= \int \tan x \sec^2 x\, dx - \int \frac{\sin x}{\cos x}\, dx$

$= \frac{\tan^2 x}{2} + \ln|\cos x| + c$

ملاحظة: مهما كانت قيمة n فردية او زوجية نسحب $\tan^2 x$ او $\cot^2 x$ وثم نطبق عليها قوانين

① $\tan^2 x = \sec^2 x - 1$

② $\cot^2 x = \csc^2 x - 1$

وثم نبسط واجراء عملية التكامل.

<u>Ex 4:</u> $\int \cot^4 x\,dx = \int \cot^2 x \cot^2 x\,dx$

$= \int \cot^2 x\,(\csc^2 x - 1)\,dx$

$= \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx$

$= -\dfrac{\cot^3 x}{3} - \int (\csc^2 x - 1)\,dx$

$= \dfrac{-\cot^3 x}{3} + \cot x + x + c$

<u>Ex 5:</u> $\int \tan^5 x\,dx = \int \tan^3 x \tan^2 x\,dx$

$= \int \tan^3 x\,(\sec^2 x - 1)\,dx = \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx$

$= \dfrac{\tan^4 x}{4} - \int \tan x \underbrace{\tan^2 x}_{(\sec^2 x - 1)}\,dx$

$= \dfrac{\tan^4 x}{4} - \int \tan \sec^2 x\,dx - \int \tan x\,dx$

$= \dfrac{\tan^4 x}{4} - \dfrac{\tan^2 x}{2} + \ln|\cos x| + c$

(C) Power of $\sec^n x$ & $\csc^n x$

ملاحظة:

① في حالة n زوجي نسمي $\sec^2 x$ او $\csc^2 x$ حسب السؤال و ثم نطبق القوانين ادناه على المقدار المتبقي وكما موضح في الامثلة

$$\boxed{\begin{aligned} &① \ \sec^2 x = \tan^2 x + 1 \\ &② \ \csc^2 x = \cot^2 x + 1 \end{aligned}}$$

② في حالة n فردي لا يمكن التكامل بالطريقة اعلاه وانما يتم باستخدام طريقة (Integration by parts) والتي سيتم شرحها لاحقاً.

Ex1: $\int \sec^2 x \, dx = \tan x + c$

$\sec^2 x = \tan^2 x + 1$

Ex2: $\int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$

$= \int (\tan^2 x + 1) \sec^2 x \, dx = \int \tan^2 x \sec^2 x \, dx + \int \sec^2 x \, dx$

$= \dfrac{\tan^3 x}{3} + \tan x + c$

Ex3: $\int \csc^6 x\,dx = \int \underline{\csc^4 x}\,\csc^2 x\,dx$

$= \int (\csc^2 x)^2 \csc^2 x\,dx$

$= \int (\cot^2 x + 1)^2 \csc^2 x\,dx$

$= \int (\cot^4 x + 2\cot^2 x + 1)\csc^2 x\,dx$

$= \int \cot^4 x \csc^2 x\,dx + 2\int \cot^2 x \csc^2 x\,dx + \int \csc^2 x\,dx$

$= \dfrac{-\cot^5 x}{5} - \dfrac{2\cot^3 x}{3} - \cot x + c$

H.W:

① $\int \csc^4 x\,dx$

② $\int \tan x \sec^3 x\,dx$

③ $\int \cot^6 x\,dx$

② Trigonometric Substitution: (sec. 8.5 المصدر)

تستخدم هذه الطريقة اذا احتوى التكامل على احدى الصيغ او ~~الحط~~ الحالات ادناه:

Case 1 $\sqrt{a^2-u^2}$ In general $(a^2-u^2)^{m/n}$	Let $u=a\sin\theta$
Case 2 $\sqrt{u^2-a^2}$ In general $(u^2-a^2)^{m/n}$	Let $u=a\sec\theta$
Case 3 $\sqrt{u^2+a^2}$ In general $(u^2+a^2)^{m/n}$	Let $u=a\tan\theta$

* where $u=f(x)$

a: is a positive constant

m & n: is an integer No.

Steps of this Method:

① Assume the function $f(\theta)$

② Find the derivative of assumed function

③ substitute the assumed and its derivative in the given integral

④ simplify the integral, then integrate it.

⑤ convert the result of integration from $f(\theta) \rightarrow f(x)$

Ex1: $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

$a = 3 , u = x$

let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

نعوض عن x و dx في السؤال

$$= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\sqrt{9(\underbrace{1-\sin^2\theta}_{\cos^2\theta})}} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{\sqrt{9\cos^2\theta}} = \int \frac{9\sin^2\theta \cdot \cancel{3}\cancel{\cos\theta}}{\cancel{3}\cancel{\cos\theta}}\, d\theta$$

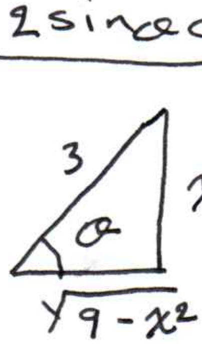
$$= \int 9\sin^2\theta\, d\theta = \frac{9}{2}\int [1-\cos 2\theta]\, d\theta$$

قانون

$* \sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{9}{2}\left[\theta - \sin\theta\cos\theta\right] + c$$



$$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3}\right] + c$$

Ex2: $\displaystyle\int \frac{x^3}{\sqrt{x^2+16}} \cdot dx$

$a = 4,\ u = x$

let $x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$$= \int \frac{64\tan^3\theta}{\sqrt{16\tan^2\theta + 16}} \cdot 4\sec^2\theta\, d\theta$$

$$= \int \frac{64\tan^3\theta * 4\sec^2\theta}{\sqrt{16\underbrace{(\tan^2\theta+1)}_{\sec^2\theta}}}\, d\theta = \int \frac{64\tan^3\theta * \cancel{4}\sec^{\cancel{2}}\theta}{\cancel{4}\,\cancel{\sec}\theta}\, d\theta$$

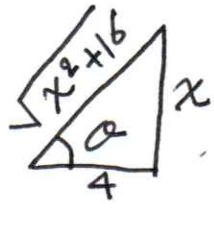
$$= \int 64\tan^3\theta\sec\theta\, d\theta = 64\int \tan^2\theta(\tan\theta\sec\theta)\, d\theta$$

$$= 64\int (\sec^2\theta - 1)(\tan\theta\sec\theta)\, d\theta$$

$$= 64\left[\int \sec^2\theta(\tan\theta\sec\theta)\, d\theta - \int \tan\theta\sec\theta\, d\theta\right]$$

$$= 64\,\frac{\sec^3\theta}{3} - 64\sec\theta + c$$

$$= \frac{64}{3}\left(\frac{\sqrt{x^2+16}}{4}\right)^3 - 64\left(\frac{\sqrt{x^2+16}}{4}\right) + c$$



Ex3: $\displaystyle\int \frac{x^2}{(x^2-1)^{5/2}}\,dx$ $\qquad a=1,\ u=x$

Let $x=\sec\theta$

$dx=\sec\theta\tan\theta\,d\theta$

$$=\int \frac{\sec^2\theta}{(\underbrace{\sec^2\theta-1}_{\tan^2\theta})^{\frac{5}{2}}}\cdot\sec\theta\tan\theta\,d\theta$$

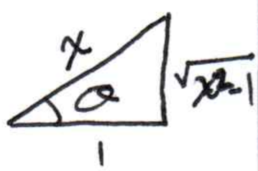
$$=\int \frac{\sec^3\theta\ \tan\theta}{(\tan^2\theta)^{\frac{5}{2}}}\,d\theta=\int \frac{\sec^3\theta\ \tan\theta}{(\tan\theta)^{\cancel{2}\cdot\frac{5}{\cancel{2}}}}\cdot d\theta$$

$$=\int \frac{\sec^3\theta\ \cancel{\tan\theta}}{\tan^{\cancel{5}\,4}\theta}\,d\theta=\int\frac{\sec^3\theta}{\tan^4\theta}\,d\theta$$

$$=\int \frac{1}{\cancel{\cos^3\theta}}\cdot\frac{\overset{\cos\theta}{\cancel{\cos^4\theta}}}{\sin^4\theta}\,d\theta=\int\frac{\cos\theta}{\sin^4\theta}\,d\theta$$

$$=\int \csc^3\theta\,\cot\theta\,d\theta=\int\csc^2\theta(\csc\theta\cot\theta)\,d\theta$$

$$=-\frac{\csc^3\theta}{3}+C$$



$$=-\frac{1}{3}\left[\frac{x}{\sqrt{x^2-1}}\right]^3+C$$

$\csc\theta=\dfrac{1}{\sin\theta}=\dfrac{x}{\sqrt{x^2-1}}$

Ex4: $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

$a=3$, $x=u$

Let $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

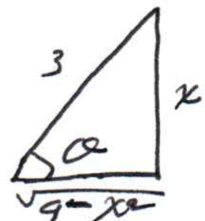
$$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{9(1-\sin^2\theta)}}{9\sin^2\theta} \cdot 3\cos\theta\, d\theta = \int \frac{\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta}\, \cancel{3}\cos\theta\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= \int \csc^2\theta\, d\theta - \int d\theta$$



$\sin\theta = \frac{x}{3}$

$$= -\cot\theta - \theta + c$$

$$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\frac{x}{3} + c$$

Ex 5: show that $\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$

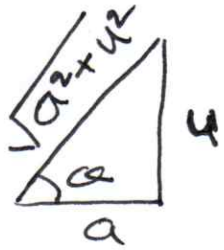
let $u = a\tan\theta$

$du = a\sec^2\theta\, d\theta$

$$= \int \frac{1}{a^2\tan^2\theta + a^2} \cdot a\sec^2\theta\, d\theta$$

$$= \int \frac{\cancel{a}\sec^2\theta}{a^{\cancel{2}}(\underbrace{\tan^2\theta + 1}_{\sec^2\theta})}\, d\theta = \int \frac{\sec^{\cancel{2}}\theta}{a\sec^{\cancel{2}}\theta}\, d\theta$$

$$= \int \frac{1}{a}\, d\theta = \frac{1}{a}\theta + c$$



$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

H.W: solve the following integrals:

① $\int \frac{(1-x^2)^{3/2}}{x^6}\, dx$

② $\int \frac{8}{(4x^2+1)^2}\, dx$

③ $\int \frac{\sqrt{y^2-25}}{y^3}\, dy$

③ Integration By Parts: (المصدر sec. 8.2)

$\int \ln x \, dx$, $\int \tan^{-1} x \, dx$, $\int e^{x} \cos x \, dx$

$\int x \sin x \, dx$, $\int x^2 \sin x^2 \, dx$, $\int \sec^3 x \, dx$

The above equations we can not integrate it using normal ways, therefore we use integration by parts to solve them.

ملاحظة: هذه الطريقة تنص على ان نقسم التكامل الى جزئين او دالتين الاولى سهلة الاشتقاق وتسمى (u) والثانية سهلة التكامل وتسمى (dv) ونطبق القانون التالي:

$$\boxed{\int u \, dv = uv - \int v \, du}$$

The priority of choosing (u) is according to this sequence:
اولوية اختيار u حسب التسلسل التالي:

1- Ln, Inverse of trigonometric functions.

2- x^n.

3- trigonometeric functions.

4- $e^{(x)}$.

Ex1: $\int \ln x \, dx$

Sol: Let $u = \ln x \Longrightarrow du = \frac{1}{x} dx$

$dv = dx \Longrightarrow v = x$

$\int u \, dv = uv - \int v \, du$

$\therefore \int \ln x \, dx = x \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} dx$

$\int \ln x \, dx = x \ln x - x + C$

Ex2: $\int \tan^{-1} x \, dx$

Sol: Let $u = \tan^{-1} x \Longrightarrow du = \frac{1}{x^2+1} dx$

$dv = dx \Longrightarrow v = x$

$\int u \, dv = uv - \int v \, du$

$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{x^2+1} dx$

$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln |x^2+1| + C$

Ex3: $\int x e^{x} dx$

sol: Let $u = x \Longrightarrow du = dx$

$dv = e^{x} dx \Longrightarrow v = e^{x}$

$\int u\,dv = uv - \int v\,du$

$\int x e^{x} dx = x e^{x} - \int e^{x} dx$

$\int x e^{x} dx = x e^{x} - e^{x} + c$

Ex4: $\int x^{3} e^{x^{2}} dx = \int x^{2} x e^{x^{2}} dx$

sol: Let $u = x^{2} \Longrightarrow du = 2x\,dx$

$dv = x e^{x^{2}} dx \Longrightarrow v = \int x e^{x^{2}} \ast \frac{2}{2} dx = \frac{1}{2} e^{x^{2}}$

$\int u\,dv = uv - \int v\,du$

$\int x^{3} e^{x^{2}} dx = \frac{1}{2} x^{2} e^{x^{2}} - \int \frac{1}{2} e^{x^{2}} \cdot 2x\,dx$

$\int x^{3} e^{x^{2}} dx = \frac{1}{2} x^{2} e^{x^{2}} - \frac{1}{2} e^{x^{2}} + c$

<u>Ex5</u>: $\int x^3 \cos x^2 dx = \int x^2 x \cos x^2 dx$

<u>sol</u>: Let $u = x^2 \Longrightarrow du = 2x dx$

$dv = x\cos x^2 dx \Longrightarrow v = \int x\cos x^2 dx = \frac{1}{2}\sin x^2$

$\int u dv = uv - \int v du$

$\int x^3 \cos x^2 dx = \frac{1}{2}x^2 \sin x^2 - \int \frac{1}{2}\sin x^2 \cdot 2x dx$

$\int x^3 \cos x^2 dx = \frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + C$

<u>Speical Cases of integration by parts:</u>

1- $\int x^n \cdot$ trigonometric dx

$\int x^n \cdot e^x \, dx$

<u>Ex6</u>: $\int x \sin x \, dx$

<u>sol</u>: Let $u = x \Longrightarrow du = dx$

$dv = \sin x dx \Longrightarrow v = -\cos x$

$\int u dv = uv - \int v du$

$\int x \sin x dx = -x\cos x - \int -\cos x dx$

$\int x \sin x dx = -x\cos x + \sin x + C$

Ex7: $\int x^2 \sin x\, dx$

sol/

Let $u = x^2 \longrightarrow du = 2x\, dx$

$dv = \sin x\, dx \longrightarrow v = \int \sin x\, dx = -\cos x$

$\int u\,dv = uv - \int v\,du$

$\int u\,dv = -x^2 \cos x - \int -\cos x \cdot 2x\, dx$

$= -x^2 \cos x + 2 \underbrace{\int x \cos x\, dx}_{u_1 \cdot dv_1}$ ------ (1)

Let $u_1 = x \longrightarrow du_1 = dx$

$dv_1 = \cos x\, dx \longrightarrow v_1 = \sin x$

$\int x \cos x\, dx = x \sin x - \int \sin x\, dx$

$= x \sin x + \cos x + c_1$

Then substitute in eq. (1)

$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + c$

Ex 8: $\int x^5 \cos x \, dx$

Tabular method

sol:

u		dv
		$\cos x$
x^5	$+$	$\sin x$
$5x^4$	$-$	$-\cos x$
$20x^3$	$+$	$-\sin x$
$60x^2$	$-$	$\cos x$
$120x$	$+$	$\sin x$
120	$-$	$-\cos x$
0		

$$\int x^5 \cos x \, dx = x^5 \sin x + 5x^4 \cos x - 20x^3 \sin x - 60x^2 \cos x + 120x \sin x + 120 \cos x + C$$

* ملاحظة: شروط هذه الطريقة

① u قابلة للإشتقاق الى الصفر.

② dv قابلة للتكامل الدوري مثل $\sin x$ او $\cos x$ او e^x

② $\int e^x \cdot \underset{\text{function}}{\overset{(\sin x/\cos x)}{\text{trigonometric}}}\, dx$ تكامل دوري

Ex9: $\int e^x \cdot \sin x \, dx$

sol.: let $u = \sin x \Longrightarrow du = \cos x \, dx$

$dv = e^x dx \Longrightarrow v = e^x$

$\int u\,dv = uv - \int v\,du$

$\int e^x \sin x \, dx = e^x \sin x - \underbrace{\int e^x \cos x \, dx}_{u_1 \quad dv_1}$

let $u_1 = \cos x \Longrightarrow du_1 = -\sin x \, dx$

$dv_1 = e^x dx \Longrightarrow v_1 = e^x$

$\int e^x \cdot \sin x \, dx = e^x \sin x - [u_1 v_1 - \int v_1 \, du_1]$

$\int e^x \cdot \sin x \, dx = e^x \sin x - \cos x + \int e^x (-\sin x) \, dx$

$\int e^x \cdot \sin x \, dx = e^x \sin x - \cos x \cdot e^x - \underline{\int e^x \sin x \, dx}$

$\int e^x \cdot \sin x \, dx + \int e^x \sin x \, dx = e^x \sin x - \cos x \, e^x + C$

$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$

$\int e^x \sin x \, dx = \frac{1}{2} [e^x \sin x - e^x \cos x + c]$

(3) $\int \sec^n x\,dx$, $\csc^n x$ n: odd No.

<u>EX10:</u> $\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$

<u>sol:</u>

Let $u = \sec x \Longrightarrow du = \sec x \tan x\,dx$

$dv = \sec^2 x\,dx \Longrightarrow v = \tan x$

$$\int u\,dv = uv - \int v\,du$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan x \cdot \sec x \tan x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

$$\int \sec^3 x\,dx = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx * \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + c$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x| + c\right]$$

Ex11: $\int x^3 \ln x \, dx$

Let $u = \ln x \Longrightarrow du = \frac{1}{x} dx$

$dv = x^3 \Longrightarrow v = \frac{x^4}{4}$

$\int u\,dv = uv - \int v\,du$

$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \int \frac{x^{\cancel{4}\,3}}{4} \cdot \frac{1}{\cancel{x}} dx$

$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$

Ex12: $\int x^2 e^x \, dx$

sol:

u		dv
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

$\int x^2 e^x = x^2 e^x - 2x e^x + 2e^x + c$

H.W: Integrate the following

① $\int \csc^3 x \, dx$

② $\int t^2 e^{4t} \, dt$

③ $\int x \sec^2 x \, dx$

Integration Using Partial Fraction Method — 8.3

$$\int \frac{h(x)}{g(x)}\,dx$$

* في حالة الدوال الكسرية إذا كانت درجة البسط $h(x)$ أكبر أو مساوية من درجة المقام $g(x)$ نقسم ثم نكامل. وكما موضح في الأمثلة أدناه. المصدر (sec. 8.3)

Ex1: Evaluate the integral:

$$\int \frac{x^2+3x+5}{x+2}\,dx$$

$$= \int (x+1)\,dx + \int \frac{3}{x+2}\,dx$$

$$= \frac{x^2}{2} + x + 3\,\mathrm{Ln}\,|x+2| + C$$

$$\begin{array}{r|l} & x+1 \\ \hline x+2 & x^2+3x+5 \\ \text{بالطرح} & \mp x^2 \mp 2x \\ \hline & x+5 \\ & \mp x \mp 2 \\ \hline & 3 \end{array}$$

نتوقف درجة البسط أقل من درجة المقام.

Ex2:
$$\int \frac{3x^3+5x^2-2x+4}{x^2+1}\,dx$$

$$= \int (3x+5)\,dx + \int \frac{-5x-1}{x^2+1}\,dx$$

$$= \frac{3x^2}{2} + 5x - \int \frac{5x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$= \frac{3x^2}{2} + 5x - \frac{5}{2}\,\mathrm{Ln}\,|x^2+1| + \tan^{-1}x + C$$

$$\begin{array}{r|l} & 3x+5 \\ \hline x^2+1 & 3x^3+5x^2-2x+4 \\ \text{بالطرح} & \mp 3x^3 \mp 0 \mp 3x \\ \hline & 0 \quad 5x^2-5x+4 \\ \text{بالطرح} & \mp 5x^2 \mp 0 \mp 5 \\ \hline & 0 \quad -5x-1 \end{array}$$

نتوقف لأن درجة باقي القسمة أصبحت أقل من المقام

Ex3: Find the $\int \ln(x-1)\,dx$

Let $u = \ln(x-1) \longrightarrow du = \dfrac{1}{|x-1|}\,dx$

$dv = dx \longrightarrow v = x$

$\int u\,dv = uv - \int v\,du$

$$\begin{array}{r|l} & \quad 1 \\ \cline{2-2} x-1 & \quad x \\ \text{بالطرح} & \overset{-}{}x \overset{+}{-} 1 \\ \cline{2-2} & \quad 0 \; +1 \end{array}$$

$$\int \ln(x-1)\,dx = x\ln(x-1) - \int \frac{x}{x-1}\,dx$$

$$\int \ln(x-1)\,dx = x\ln(x-1) - \int 1\,dx - \int \frac{1}{x-1}\,dx$$

$$\int \ln(x-1)\,dx = x\ln(x-1) - x - \ln|x-1| + c$$

② If the degree of $h(x)$ is less than the degree $g(x)$ then the function $\frac{h(x)}{g(x)}$ can be seprated into partial function.

① If $$\frac{h(x)}{(x-r_1)(x-r_2)} \Rightarrow \frac{A}{(x-r_1)} + \frac{B}{(x-r_2)}$$

② If $$\frac{h(x)}{(x-r_1)^2(x-r_2)} \Rightarrow \frac{A}{(x-r_1)} + \frac{B}{(x-r_1)^2} + \frac{C}{(x-r_2)}$$

③ If $$\frac{h(x)}{(x^2+px+q)(x-r_1)} \Rightarrow \frac{Ax+B}{(x^2+px+q)} + \frac{C}{(x-r_1)}$$

Ex1: $$\int \frac{5x+1}{(2x-1)(x+1)}\,dx$$

Sol:

$$\frac{5x+1}{(2x-1)(x+1)} = \frac{A}{(2x-1)} + \frac{B}{(x+1)}$$

$$\frac{5x+1}{\cancel{(2x-1)(x+1)}} = \frac{A(x+1)+B(2x-1)}{\cancel{(2x-1)(x+1)}}$$

$$5x+1 = Ax + A + 2Bx - B$$

$$5x+1 = (A+2B)x + (A-B)$$

$A + 2B = 5$ ---------- ①

بالطرح

$\overset{-}{A} \overset{+}{-} B = \overset{-}{1}$ ---------- ②

$3B = 4 \Longrightarrow \boxed{B = \frac{4}{3}}$

نعوض ب ①

$A + 2\left(\frac{4}{3}\right) = 5$

$A = 5 - \frac{8}{3} = \frac{15-8}{3} \Longrightarrow \boxed{A = \frac{7}{3}}$

$$\int \frac{5x+1}{(2x-1)(x+1)}\,dx = \int \frac{A}{(2x-1)}\,dx + \int \frac{B}{(x+1)}\,dx$$

$$= \frac{7}{3}\int \frac{1}{2x-1}\,dx \ast\frac{2}{2} + \frac{4}{3}\int \frac{1}{x+1}\,dx$$

$$= \frac{7}{6}\,\text{Ln}\,|2x-1| + \frac{4}{3}\,\text{Ln}\,|x+1| + C$$

ملاحظة:

درجة البسط اقل من درجة داخل ~~...~~ قوس المقام بواحد ~~...~~ حيث

① A ← درجة "0"

② $Ax + B$ ← درجة "1"

③ $Ax^2 + Bx + C$ ← درجة "2"

Ex2: $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

sol:

$$=\int \frac{x^2+2x-1}{x(2x^2+3x-2)}\,dx = \int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx$$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$\frac{x^2+2x-1}{\cancel{x(2x-1)(x+2)}} = \frac{A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)}{\cancel{x(2x-1)(x+2)}}$$

$$x^2+2x-1 = 2Ax^2 + 4Ax - Ax - 2A + Bx^2 + 2Bx + 2Cx^2 - Cx$$

$$x^2+2x-1 = (2A+B+2C)x^2 + (3A+2B-C)x - 2A$$

$$-2A = -1 \Longrightarrow \boxed{A = \frac{1}{2}}$$

$$2A+B+2C = 1 \quad ----\text{(1)}$$

$$3A+2B-C = 2 \quad ----\text{(2)}$$

$$B+2C = 0$$

$$2B - C = \frac{1}{2} \quad *]\,2$$

$$B+2C=0$$

$$4B-2C=1$$ طرح

$$5B = 1 \Longrightarrow \boxed{B = \frac{1}{5}} \rightarrow \frac{1}{5}+2C=0 \Longrightarrow \boxed{C = -\frac{1}{10}}$$

$$\int \frac{x^2+2x-1}{x(2x-1)(x+2)}dx = \int \frac{A}{x}dx + \int \frac{B}{2x-1}dx + \int \frac{C}{x+2}dx$$

$$= \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{5}\int \frac{1}{2x-1}dx \ast \frac{2}{2} - \frac{1}{10}\int \frac{1}{x+2}dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

EX3: $\int \frac{6x+7}{(x+2)^2}dx$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2}$$

$$6x+7 = A(x+2)+B$$

$$6x+7 = Ax+2A+B$$

$$\boxed{A=6}$$

$$2A+B=7,\ 2\ast 6+B=7 \Longrightarrow \boxed{B=-5}$$

$$\int \frac{6x+7}{(x+2)^2}dx = \int \frac{A}{x+2}dx + \int \frac{B}{(x+2)^2}dx$$

$$= \int \frac{6}{x+2}dx + \int \frac{-5}{(x+2)^2}dx$$

$$= 6\ln|x+2| - 5\int (x+2)^{-2}dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$

Ex 4: $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$$

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{(Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)}{(x^2+1)(x-1)^2}$$

$$-2x+4 = (Ax+B)(x^2-2x+1) + C(x^3-x^2+x-1) + Dx^2+D$$

$$-2x+4 = Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 - Cx^2 + Cx - C + Dx^2 + D$$

$$-2x+4 = (A+C)x^3 + (-2A - C + \overset{+B}{D})x^2 + (A-2B+C)x + (B-C+D)$$

$A + C = 0$ ---- ①

$-2A - C + \overset{+B}{D} = 0$ ---- ②

$A - 2B + C = -2$ ---- ③

$B - C + D = 4$ ---- ④

نعوض ④ في ② ينتج

$-2A = -4$ $\boxed{A = 2}$

from eq ① $\boxed{C = -2}$

from eq ③ $2 - 2B - 2 = -2$

$\boxed{B = 1}$

from eq ④

$1 + 2 + D = 4 \Rightarrow \boxed{D = 1}$

$$\therefore \int \frac{-2x+4}{(x^2+1)(x-1)^2} = \int \frac{Ax+B}{x^2+1}dx + \int \frac{C}{x-1}dx + \int \frac{D}{(x-1)^2}dx$$

$$= \int \frac{2x+1}{x^2+1}dx + \int \frac{-2}{x-1}dx + \int \frac{1}{(x-1)^2}dx$$

$$= \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx - \int \frac{2}{x-1}dx + \int \frac{1}{(x-1)^2}dx$$

$$= \operatorname{Ln}|x^2+1| + \tan^{-1}x - 2\operatorname{Ln}|x-1| - \frac{1}{x-1} + C$$

H.W 4: solve the following Integral

① $\int \frac{y}{y^2-2y-3}\,dy$

② $\int \frac{16x^3}{4x^2-4x+1}\,dx$

Applications of Definite Integrals

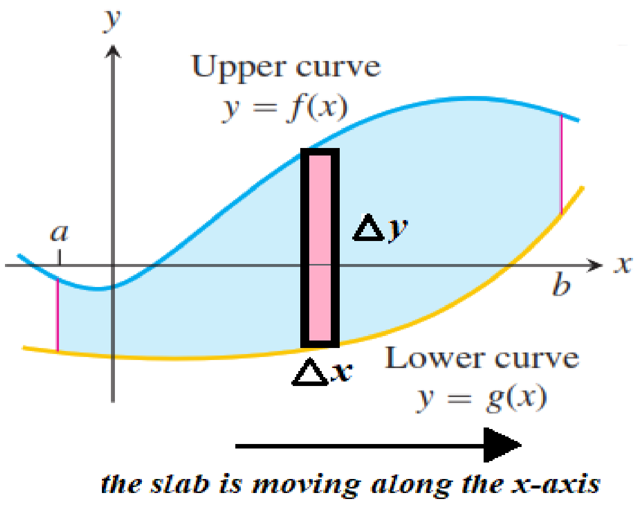
1- Area Between Curves: (sec. 5-6)

The area between curves can be found by two methods:

A) When the Slab is Moving Along the *x*-Axis:

We choose a rectangular slab and find the area of this slab, then find the total area by integrating the area of the slab with respect to *x*-axis over a given period.

$$Area\ of\ the\ slab\ As = \Delta x * \Delta y$$



Because the slab is moving along the *x*-axis then:

$$\Delta x = dx$$

$$\Delta y = upper\ curve - lower\ curve = f(x) - g(x)$$

Then the total area bounded by the two curves is:

$$\boxed{A = \int_a^b f(x) - g(x)\, dx}$$

B) When the Slab is Moving Along the *y*-Axis:

$$Area\ of\ the\ slab\ As = \Delta x * \Delta y$$

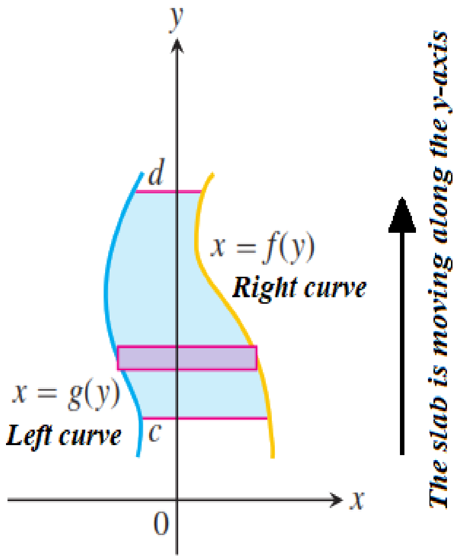
Because the slab is moving along the *y*-axis then:

$$\Delta y = dy$$

$$\Delta x = Right\ curve - Left\ curve = f(y) - g(y)$$

Then the total area bounded by the two curves is:

$$\boxed{A = \int_c^d f(y) - g(y)\, dy}$$



Ex1: Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

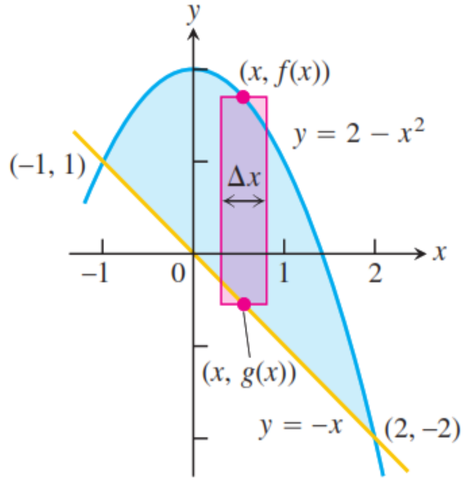
Sol: At first, we sketch the two curves.

x	$y = 2 - x^2$
-2	-2
-1	1
0	2
1	1
2	-2

x	$y = -x$
-2	2
-1	1
0	0
1	-1
2	-2

To find the intersection point between two curves:

$$2 - x^2 = -x \qquad \text{Equate } f(x) \text{ and } g(x).$$
$$x^2 - x - 2 = 0 \qquad \text{Rewrite.}$$
$$(x + 1)(x - 2) = 0 \qquad \text{Factor.}$$
$$x = -1, \quad x = 2. \qquad \text{Solve.}$$



The slab is moving from $x = -1$ to $x = 2$.

The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

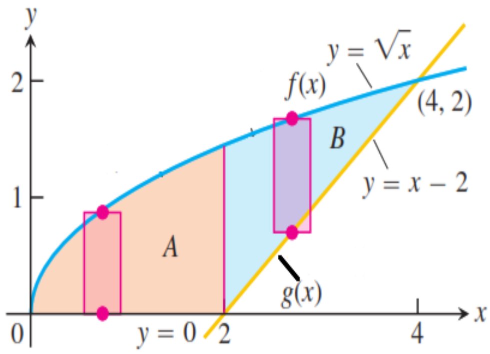
$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx$$
$$= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

Ex2: Find the area of the region in the 1st quadrant that is bounded by $y = \sqrt{x}$ and the line $y = x - 2$.

Sol: At first we sketch the region:

X	$y = \sqrt{x}$
-4	$\sqrt{-4}$ تهمل
-1	$\sqrt{-1}$ تهمل
0	0
1	± 1
4	± 2

x	$y = x - 2$
0	-2
2	0
-4	-6
-1	-3
1	-1
4	2



For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Method 2: If we choose a slab moving along y-axis then:

$$A = \int_c^d f(y) - g(y)\,dy$$

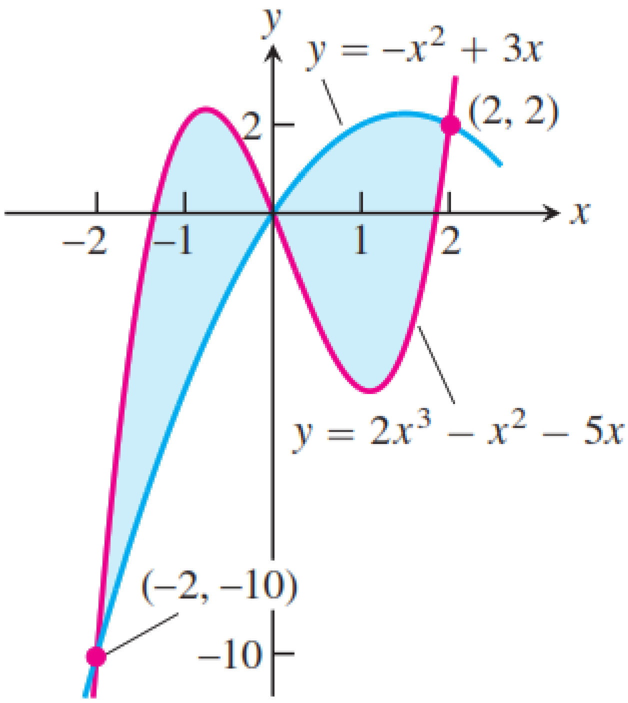
$$A = \int_0^2 (y + 2) - y^2\,dy$$

$$A = \left[\frac{y^2}{2} + 2y + \frac{y^3}{3}\right]_0^2 = \frac{10}{3}$$

HW:

1) Find the area of the region enclosed by $x - y^2 = 0$ and $x + 2y^2 = 3$ sketch the region.

2) Find the total area of the shaded region in the figure below:



2-Volumes of Revolution (Sec. 6.1)
A- Disk Method:

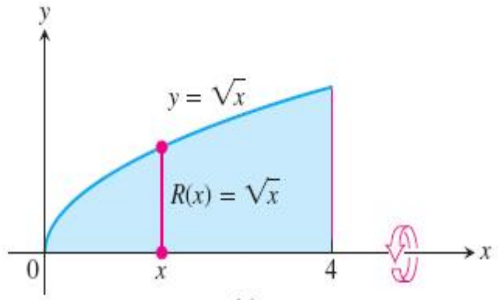
The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution. To find the volume of a solid like the one shown in figure below, we need only observe that the cross sectional area (***A***) is the area of a disk of radius ***(R)***, then the area of the disk is:

$$A = \pi[Radius]^2 = \pi[R]^2$$

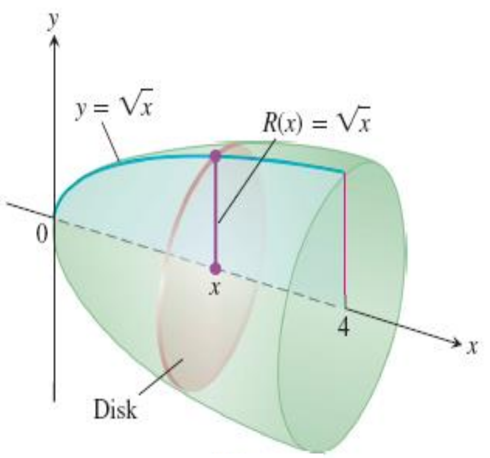
Then the volume is:

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx \qquad If\ the\ Rotation\ about\ the\ x-axis$$

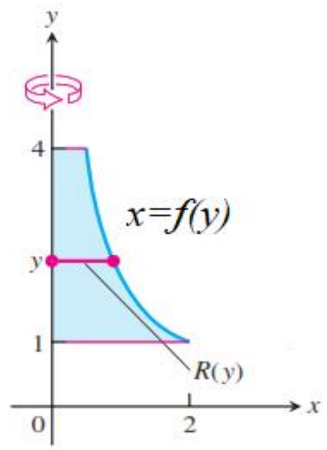
$$V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy \qquad If\ the\ Rotation\ about\ the\ y-axis$$



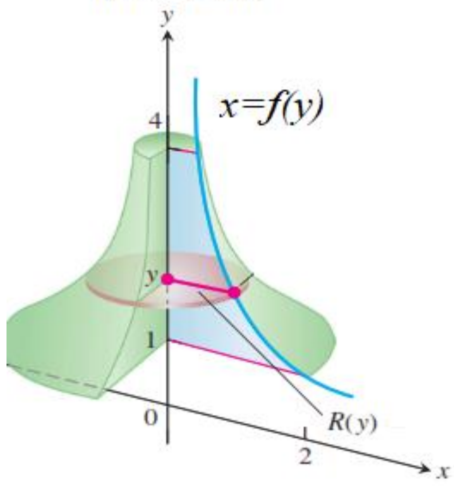
(2D- Plot)



(3D- Plot)



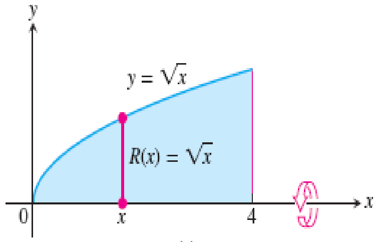
(2D- Plot)



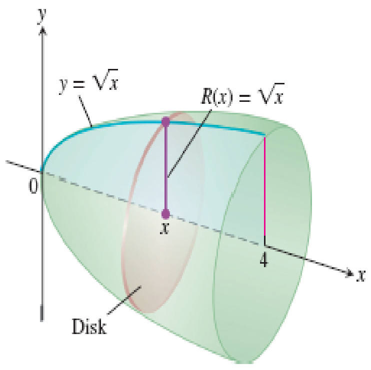
(3D- Plot)

EXAMPLE 1: A Solid of Revolution (Rotation About the x-Axis)

Find the region between the curve $y = \sqrt{x}, 0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



(a)



(b)

Solution We draw figures showing the region, a typical radius, and the generated solid The volume is

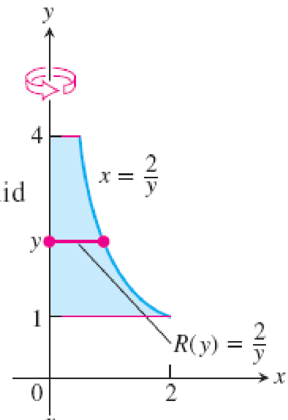
$$V = \int_a^b \pi[R(x)]^2\,dx \qquad R(x) = \sqrt{x}$$

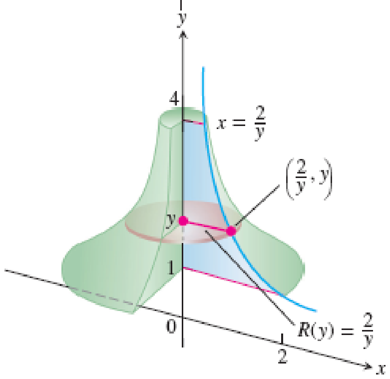
$$= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx$$

$$= \pi\int_0^4 x\,dx = \pi\frac{x^2}{2}\Big]_0^4 = \pi\frac{(4)^2}{2} = 8\pi.$$

EXAMPLE 2 Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.





Solution We draw figures showing the region, a typical radius, and the generated solid . The volume is

$$V = \int_1^4 \pi[R(y)]^2\,dy$$

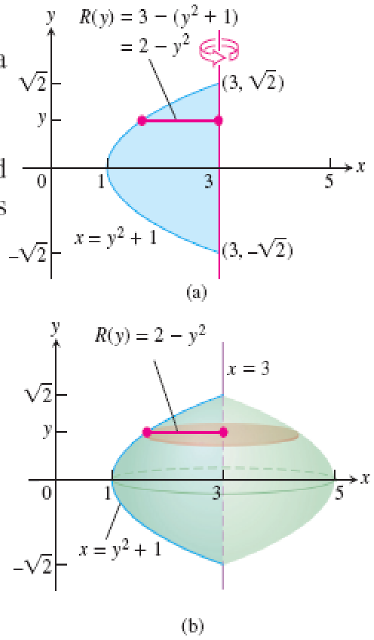
$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy$$

$$= \pi\int_1^4 \frac{4}{y^2}\,dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right]$$

$$= 3\pi.$$

EXAMPLE 3 Rotation About a Vertical Axis

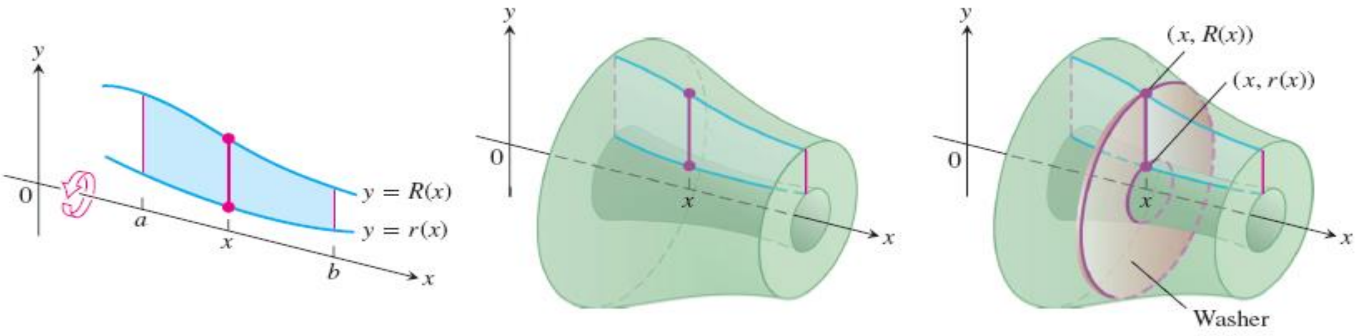
Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.



Solution We draw figures showing the region, a typical radius, and the generated solid. Note that the cross-sections are perpendicular to the line $x = 3$. The volume is

$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 \, dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 \, dy \qquad \begin{aligned} R(y) &= 3 - (y^2 + 1) \\ &= 2 - y^2 \end{aligned} \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] \, dy \\ &= \pi \left[4y - \frac{4}{3} y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{64\pi\sqrt{2}}{15}. \end{aligned}$$

B- Washer Method



If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it as shown in the above figure. The cross section perpendicular to the axis of revolution are washers instead of disk. The radius of a typical washer are:

Outer radius: R

Inner radius: r

Then the washer's area is:

$$A = Outer\ Area - Inner\ Area$$

$$A = \pi[R]^2 - \pi[r]^2 = \pi[R^2 - r^2]$$

Then the total volume is:

$$V = \int_a^b A(x)\,dx = \pi \int_a^b ([R(x)]^2 - [r(x)]^2)\ dx \qquad If\ the\ Rotation\ about\ the\ \boldsymbol{x} - axis$$

$$V = \int_c^d A(y)\,dy = \pi \int_c^d ([R(y)]^2 - [r(y)]^2)\ dy \qquad If\ the\ Rotation\ about\ the\ \boldsymbol{y} - axis$$

EXAMPLE 1 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.
   These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

$$\text{Outer radius:}\quad R(x) = -x + 3$$

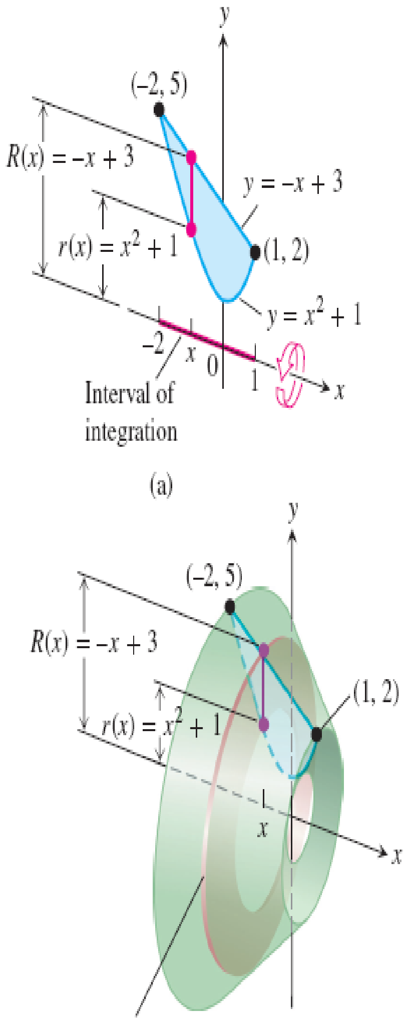
$$\text{Inner radius:}\quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ x = -2, &\quad x = 1 \end{aligned}$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \\ &= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\,dx \qquad \text{Values from Steps 2 and 3} \\ &= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx \\ &= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \end{aligned}$$



(a)

Washer cross section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

(b)

EXAMPLE 2 A Washer Cross-Section (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

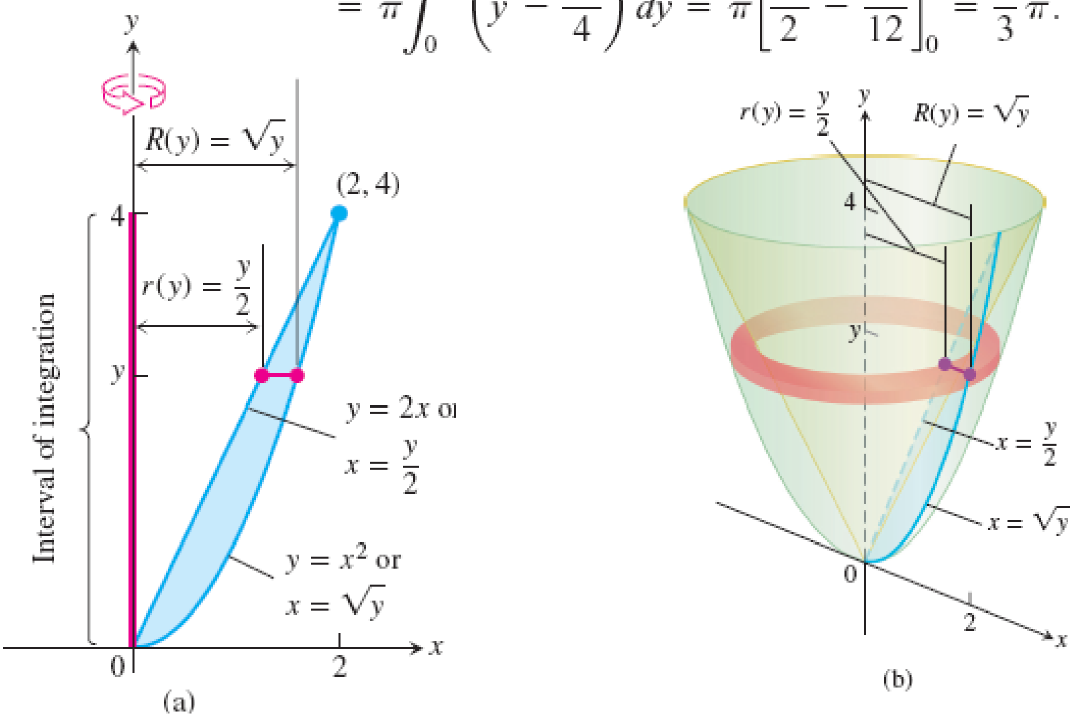
Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$V = \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy$$

$$= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy$$

$$= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi.$$



H.W:

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ *and* $x = 0$ about the:

1) x-axis.

2) y-axis.

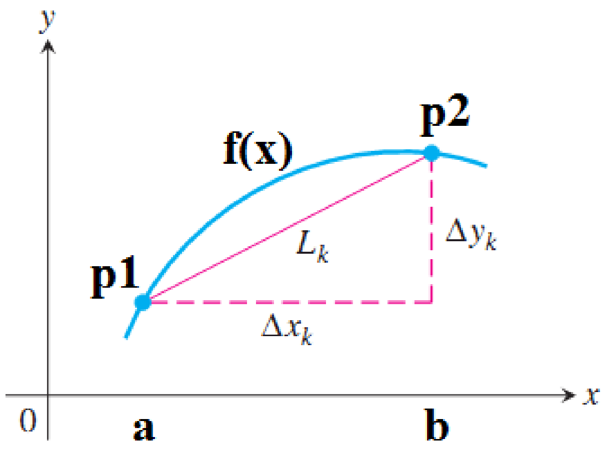
3) The line $y = 2$

4) The line $x = 4$.

Note: sketch the region and write only the equations with indicating the limits of integration that find the volume without solving the integrals.

3- Length of The Curve (sec. 6.3):

To find the length of the curve from *P1* to *P2* we use these formulas:



1- Formula for the Length of $y = f(x)$, $a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

2- Formula for the Length of $x = g(y)$, $c \le y \le d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

EXAMPLE 1 Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + 8x}\, dx$$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6}.$$

EXAMPLE 2 Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

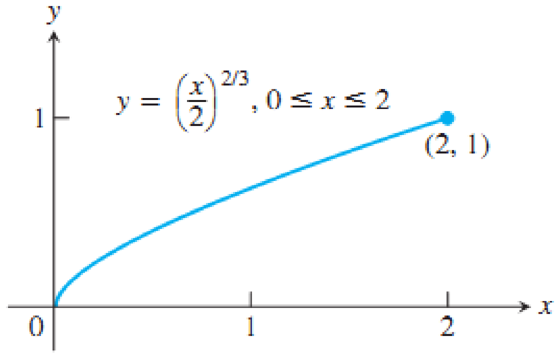
is not defined at $x = 0$, so we cannot find the curve's length with Equation (1).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

$$x = 2y^{3/2}.$$



The derivative is continuous on $y = 0$ to $y = 1$

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy$$

$$= \frac{1}{9}\cdot\frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

EXAMPLE 3 Find the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$ from y=1 to y=3.

Sol:

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16y^4}$$

$$\Rightarrow L = \int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\, dy$$

$$= \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\, dy$$

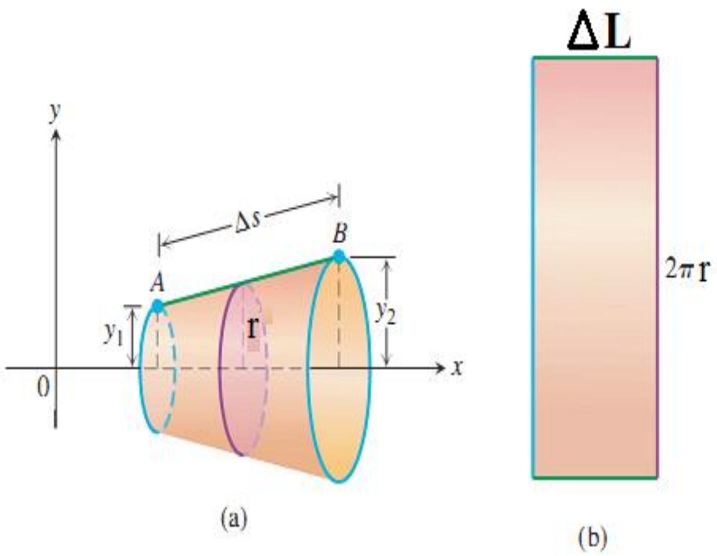
$$= \int_1^3 \sqrt{\left(y^2 + \frac{1}{4y^2}\right)^2}\, dy = \int_1^3 \left(y^2 + \frac{1}{4y^2}\right) dy$$

$$= \left[\frac{y^3}{3} - \frac{y^{-1}}{4}\right]_1^3 = \left(\frac{27}{3} - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}$$

$$= 9 + \frac{(-1 - 4 + 3)}{12} = 9 + \frac{(-2)}{12} = \frac{53}{6}$$

4- Surface Area of Revelution:
(sec. 6.5) للأطلاع

To find the surface area of the curve we choose washer with raduis r and thickness of ΔL then



$$S = \int circumference \text{ of the circle} * \Delta L$$

$$S = \int 2\pi\, r * \Delta L$$

1- Surface Area for Revolution About the x-Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

2- Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

EXAMPLE 1 Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about the x-axis.

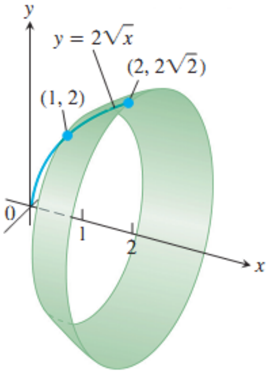
Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

with $a = 1$, $b = 2$, $y = 2\sqrt{x}$, $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$



With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$

EXAMPLE 2 The line segment $x = 1 - y$, $0 \le y \le 1$, is revolved about the y-axis to generate the cone in Figure. Find its lateral surface area (which excludes the base area).

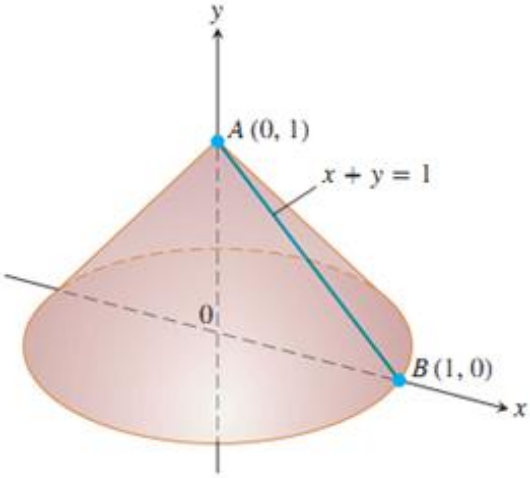
Solution

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1 - y)\sqrt{2}\, dy$$

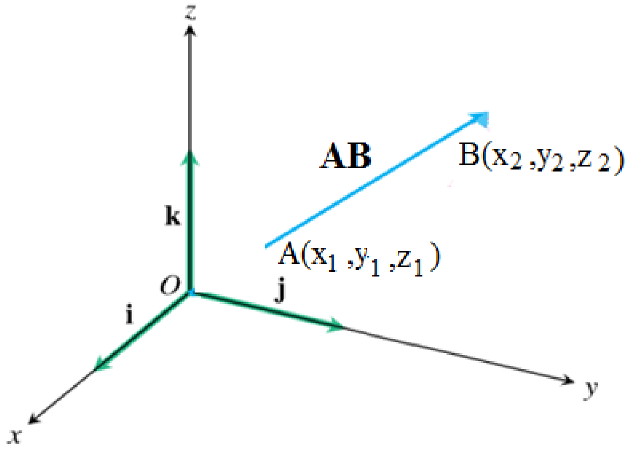
$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right) = \pi\sqrt{2}.$$



Vectors

Some of the things we measure are determined simply by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is.

Vector in Space:



$$\overrightarrow{AB}=(x_2-x_1)i+(y_2-y_1)j+(z_2-z_1)k$$

$$\overrightarrow{AB}=xi+yj+zk$$

Then the length (magnitude) of the vector is:

$$\left|\overrightarrow{AB}\right|=\sqrt{x^2+y^2+z^2}$$

Where as

i : is a unit vector in the direction of x.
j : is a unit vector in the direction of y.
k : is a unit vector in the direction of z.
Note: Two vectors are equal if they have the same length and direction.

Unit Vector:

It is a vector whose length is equal to the one unit of length along the coordinate axis.

$$U_{AB}=\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$$

Ex: let *A (-3, 4, 1)* and *B (-5, 2, 2)* two points in the space, find:

1- The vector $\overrightarrow{AB}$.
2- Length of $\overrightarrow{AB}$.
3- Unit vector of $\overrightarrow{AB}$.

Sol:

$$\overrightarrow{AB}=(x_2-x_1)i+(y_2-y_1)j+(z_2-z_1)k=(-5+3)i+(2-4)j+(2-1)k$$

$$\overrightarrow{AB}=-2i-2j+k$$

$$\left|\overrightarrow{AB}\right|=\sqrt{x^2+y^2+z^2}=\sqrt{4+4+1}=\sqrt{9}=3$$

$$U_{AB}=\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}=\frac{-2i-2j+k}{3}=\frac{-2}{3}i-\frac{2}{3}j+\frac{1}{3}k$$

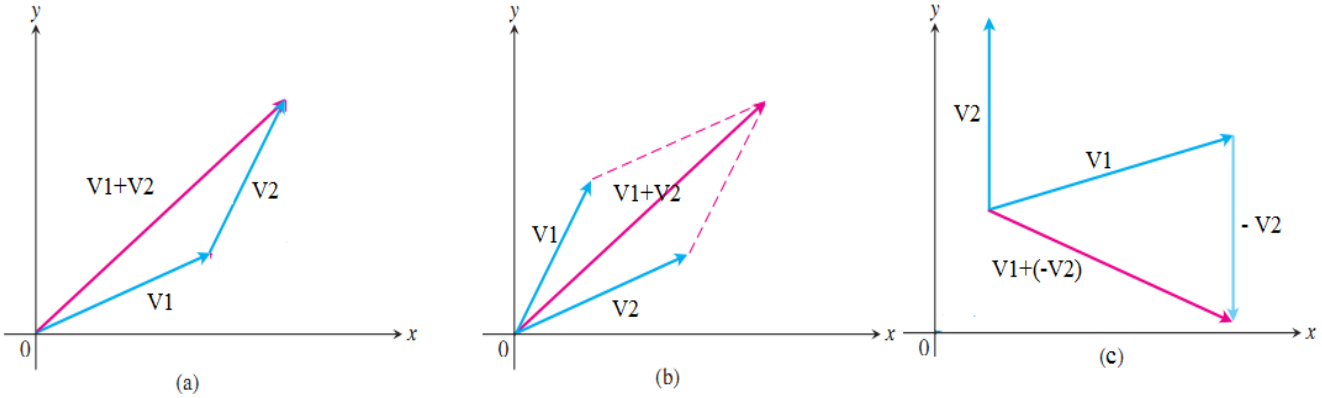
<u>Addition and Subtraction of Vectors</u>: let

$\vec{V}_1 = x_1 i + y_1 j + z_1 k$

$\vec{V}_2 = x_2 i + y_2 j + z_2 k$

$\vec{V}_1 + \vec{V}_2 = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$

$\overrightarrow{V_1} - \overrightarrow{V_2} = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k$



<u>Multiply Vectors with constant</u>:

When we multiply a vector with a constant, that is mean changing in the length of the vector (scaling vector length). Let C is a constant and the vector $\vec{V}$ is:

$\vec{V} = xi + yj + zk$

Then

$C * \vec{V} = (C * x)i + (C * y)j + (C * z)k$

Ex: let $\vec{V}_1 = -i + 3j + k$ and $\vec{V}_2 = 4i + 7j$ find:

1- $\vec{V}_1 + \vec{V}_2$. 2- $\vec{V}_1 - \vec{V}_2$. 3- $\left|\frac{1}{2}\vec{V}_1\right|$.

$\vec{V}_1 + \vec{V}_2 = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k = 3i + 10j + k$

$\overrightarrow{V_1} - \overrightarrow{V_2} = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k = -5i - 4j + k$

$$\left|\frac{1}{2}\overrightarrow{V_1}\right| = \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{11}{4}} = \frac{1}{2}\sqrt{11}$$

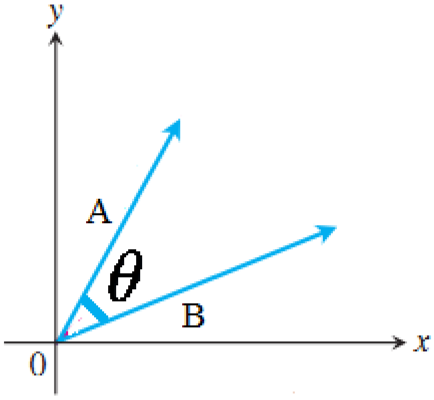
1-Dot (Scalar) Product:

If we want to measure the angle between two vectors we apply the dot product. Also we apply it to find the projection of one vector onto another. Then let we have two vectors:

$$\vec{A} = a_1 i + a_2 j + a_3 k$$
$$\vec{B} = b_1 i + b_2 j + b_3 k$$

The $\vec{A} \bullet \vec{B}$ is called the dot (scalar) product of $\vec{A}$ & $\vec{B}$ and given by:

$$\vec{A} \bullet \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{A}||\vec{B}|\cos\theta$$



$$\theta = \cos^{-1}\left(\frac{\vec{A} \bullet \vec{B}}{|\vec{A}||\vec{B}|}\right)$$, Where θ is the angle between two vectors

Note:

1) $i \bullet i = j \bullet j = k \bullet k = 1$
2) $i \bullet j = j \bullet k = k \bullet i = 0$
3) $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

$if \ \vec{A} \bullet \vec{B} = 0 \quad \therefore \ \vec{A} \perp \vec{B}$ because $\cos 90 = 0$

$if \ \vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}| \quad \therefore \ \vec{A} // \vec{B}$ because $\cos 0 = 1$

Ex: Find the angle between $\vec{A} = i - 2j - 2k$ and $\vec{B} = 6i + 3j + 2k$.

Sol:

$$\theta = \cos^{-1}\left(\frac{\vec{A} \bullet \vec{B}}{|\vec{A}||\vec{B}|}\right)$$

$$\vec{A} \bullet \vec{B} = (1*6) + (-2*3) + (-2*2) = -4$$

$$|\vec{A}| = \sqrt{a_1^{\,2} + b_1^{\,2} + c_1^{\,2}} = \sqrt{1+4+4} = 3$$

$$|\vec{B}| = \sqrt{a_2^{\,2} + b_2^{\,2} + c_2^{\,2}} = \sqrt{36+9+4} = 7$$

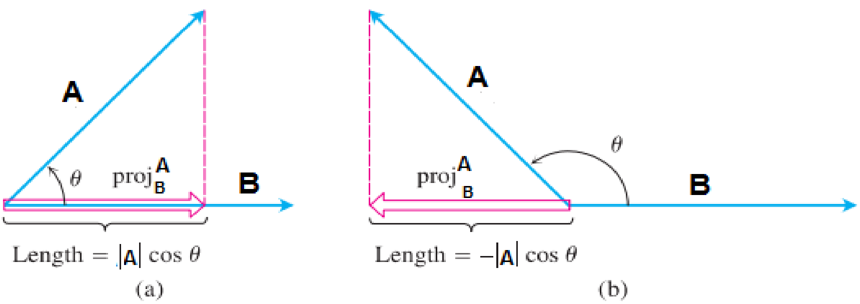
$$\theta = \cos^{-1}\left(\frac{\vec{A} \bullet \vec{B}}{|\vec{A}||\vec{B}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) = 100.98^0$$

Vector Projection:

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

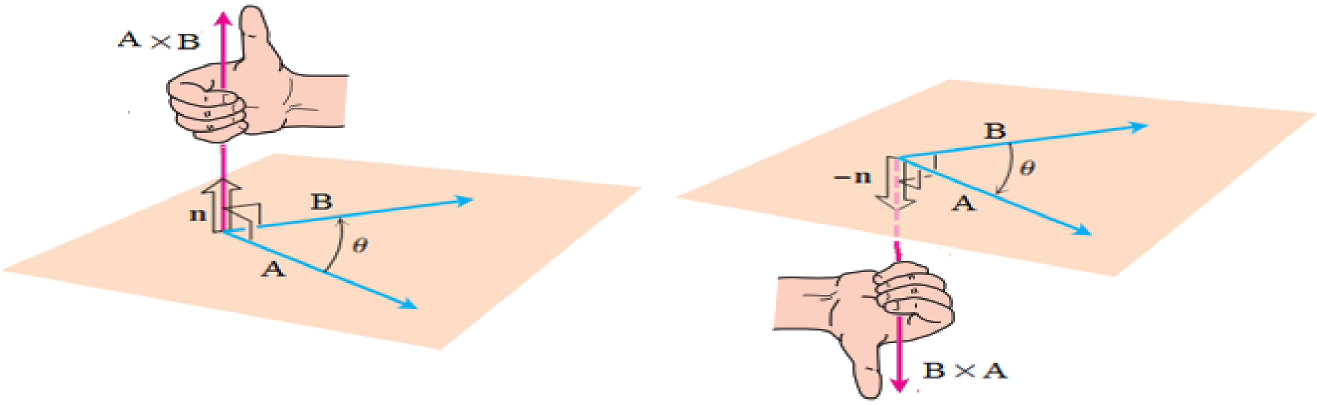
$$|\vec{A}|\cos\theta = \frac{\vec{A} \bullet \vec{B}}{|\vec{B}|}$$

$$\mathrm{Proj}_B^A = \left(\frac{\vec{A} \bullet \vec{B}}{|\vec{B}|}\right)$$



2- <u>Cross Product</u>:

When we apply the cross product onto two vectors we will get a new vector normal to these vectors. Also it gives us information about the area of the parallelogram which contains the vectors.



If we have two vectors:

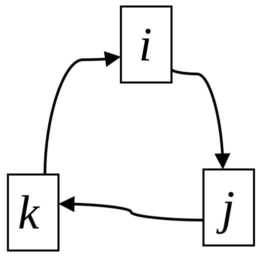
$$\vec{A} = a_1 i + a_2 j + a_3 k$$

$$\vec{B} = b_1 i + b_2 j + b_3 k$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$\vec{A}\times\vec{B} = n\left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

Note:



1− $if \quad \vec{A}//\vec{B}$ then $\sin\theta = 0 \quad \vec{A}\times\vec{B} = 0$

2− $i\times i = j\times j = k\times k = 0$

3- $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$

4− $(\vec{A}\times\vec{B})\bullet\vec{C} = (\vec{B}\times\vec{C})\bullet\vec{A} = (\vec{C}\times\vec{A})\bullet\vec{B}$

Ex: Find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

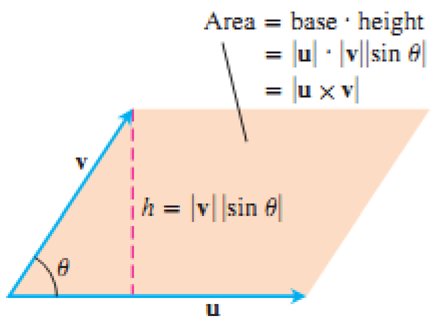
Solution

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

Note: $\left|\vec{u}\times\vec{v}\right|$ represents the area of the parallelogram



The parallelogram
determined by **u** and **v**.

Ex: 1-Find the normal vector to the plane which contains points A (1,-1,0), B(2,1,-1), C(-1,1,2).

2- Find the area of the parallelogram contains the points.
3- Find the normal unit vector to the plane.

Solution

$$\overrightarrow{AB}=(2-1)i+(1-(-1))j+(-1-0)k=i+2j-k$$
$$\overrightarrow{AC}=(-1-1)i+(1-(-1))j+(2-0)k=-2i+2j+2k$$

Then the normal vector to the plane is:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}=6i+6k$$

$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{36+36}=\sqrt{72}=6\sqrt{2}$ Area of the parallelogram contains these vectors.

$n=\dfrac{\overrightarrow{AB}\times\overrightarrow{AC}}{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}=\dfrac{6}{6\sqrt{2}}i+\dfrac{6}{6\sqrt{2}}k=\dfrac{1}{\sqrt{2}}i+\dfrac{1}{\sqrt{2}}k$ normal unit vector to the plane

H.w: Determine if the two vectors are orthogonal or parallel or not?

1- $\vec{A}=6i+6k$ and $\vec{B}=-2i+2j+2k$

2- $\vec{A}=3i-2j+\mathrm{k}$ and $\vec{B}=2j+4k$

3- $\vec{A}=6i+3j+2\mathrm{k}$ and $\vec{B}=12i+6j+4k$

3-**Parametric Equations of Lines in Space**: (للاطلاع)

In the plane, a point and a number giving the slope of the line determine a line. In space a line is determined by a point and a vector giving the direction of the line. Suppose that L is a line passing through a point $p_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{V} = v_1 i + v_2 j + v_3 k$. Then L is a set of all points $p(x, y, z)$ for which $\overrightarrow{PP_0}$ is parallel to $\vec{V}$. Then $\overrightarrow{PP_0} = t\vec{V}$ for some scalar t. The value of t depends on the location of the point P along the line and its domain $(-\infty,\infty)$. The expanded form of the equation $\overrightarrow{PP_0} = t\vec{V}$ is:

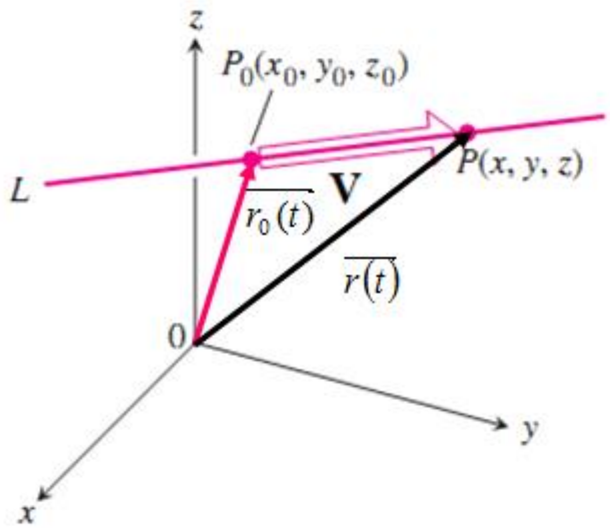
$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(v_1 i + v_2 j + v_3 k)$$

$$x\,i + y\,j + z\,k = x_0\,i + y_0\,j + z_0\,k + t(v_1 i + v_2 j + v_3 k)$$

Then the vector equation for the line L passes through the point $p_0(x_0, y_0, z_0)$ and parallel to $\vec{V}$ is:

$$\overrightarrow{r(t)} = \overrightarrow{r_0} + t\vec{V}$$

Where



$\overrightarrow{r(t)} = x\,i + y\,j + z\,k$ position vector of $p(x, y, z)$.

$\overrightarrow{r_0(t)} = x_0\,i + y_0\,j + z_0\,k$ position vector of $p(x_0, y_0, z_0)$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

***Ex1*:** Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

***Ex2*:** Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

The vector $\overrightarrow{PQ} = (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k}$
$= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$

is parallel to the line, with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

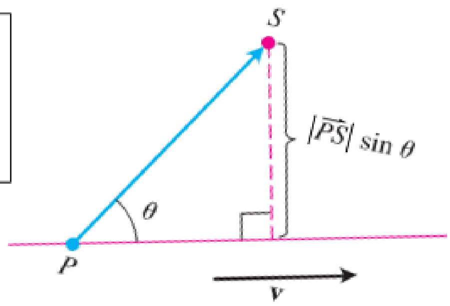
These equations serve as well as the first; they simply place you at a different point on the line for a given value of t. Notice that parametrizations are not unique.

4-The Distance from a Point to a Line in Space (للاطلاع)

To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$,

Distance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$



***Ex*:** Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \qquad x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

5- Plane Equation in space:

Suppose M is a plane passes through the point P_0 (x_0, y_0, z_0). Also M plane is a set of points $P\ (x, y, z)$. And $\vec{N}$ is a vector normal to the M plane. Then:

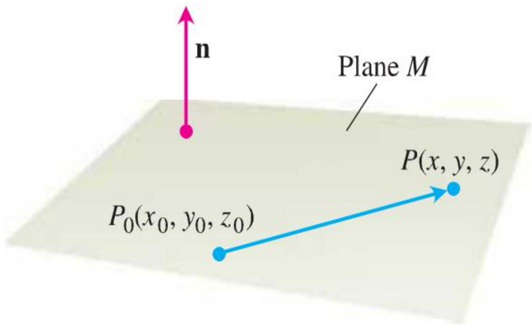
$$\overrightarrow{P_0P} = (x - x_0)i + (y - y_0)j + (z - z_0)k$$

$$\vec{N} = Ai + Bj + Ck$$

$$\overrightarrow{P_0P} \perp \vec{N}$$

$$\overrightarrow{P_0P} \bullet \vec{N} = |\overrightarrow{P_0P}||\vec{N}| \cos 90 = 0$$

$$\overrightarrow{P_0P} \bullet \vec{N} = (x - x_0)A + (y - y_0)B + (z - z_0)C = 0$$



$$\boxed{Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D}$$ Plane equation in space.

Where D is a constant.

***Ex₁*:** Find the equation for the plane passes through P_0 *(-3, 0, 7)* and perpendicular to $\vec{N} = 5i + 2j - k$.

Sol:

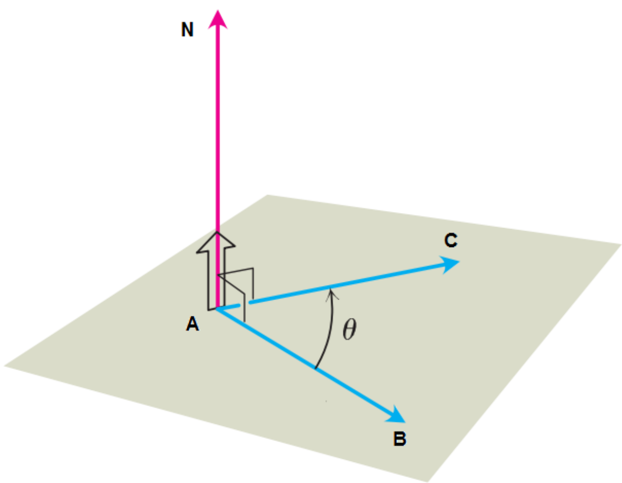
$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

$$5x + 2y - z = -15 + 0 - 7$$

$$5x + 2y - z = -22$$

$$z = f(x, y) = 22 + 5x + 2y$$

***Ex₂*:** Find the Eq. for the plane through *A (0, 0, 1)*, *B (2, 0, 0)* and *C (0, 3, 0)*.



Sol: $\vec{V} = (x - x_0)i + (y - y_0)j + (z - z_0)k$

$$\overrightarrow{AB} = (2-0)i + (0-0)j + (0-1)k$$

$$\overrightarrow{AB} = 2i - k$$

$$\overrightarrow{AC} = 3j - k$$

$$\vec{N} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$\vec{N} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3i + 2j + 6k$$

Now we have a vector normal ($\vec{N}$) to the plane and point A (0, 0, 1) we can find the plane equation.

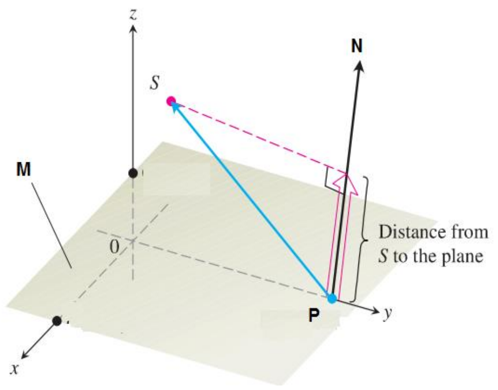
$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$

$$3x + 2y + 6z = 6$$

***<u>Note</u>*:** to find a plane eq. we must have a normal vector and a point within the plane.

6-**The Distance from the Point to a Plane**: (للاطلاع)

If P is a point on a plane with normal $\vec{N}$, then the distance from any point S to the plane is the length of the vector projection on to $\vec{N}$. Then the distance from S to the plane is:



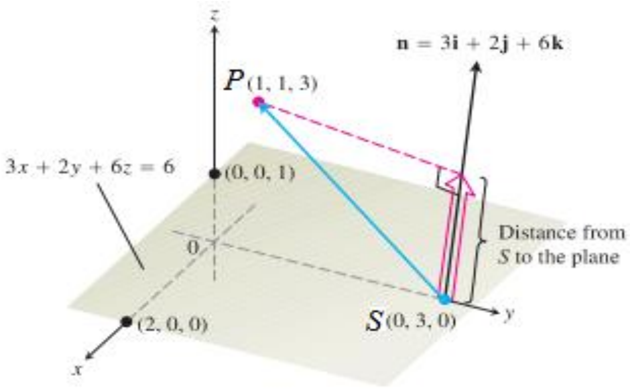
$$\boxed{d = \left| \overrightarrow{SP} \bullet \frac{\vec{N}}{|\vec{N}|} \right|}$$

Where $\vec{N} = Ai + Bj + Ck$ is normal to the plane.

***Ex₁*: Find the distance from S (1, 1, 3) to the plane 3x + 2y + 6z = 6.

Sol: $\vec{N} = 3i + 2j + 6k$

S point may be a point of intersection between the plane and the y-axis. Then x=z=0 then the point S is (0,3,0)



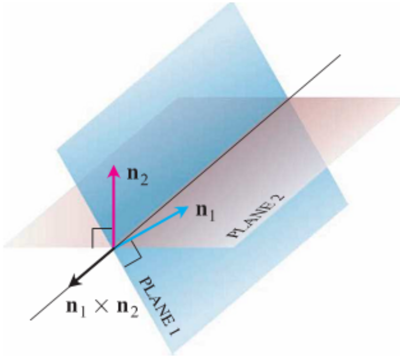
$$\overrightarrow{SP} = (1-0)i + (1-3)j + (3-0)k = i - 2j + 3k$$

$$d = \overrightarrow{SP}.\frac{\vec{N}}{|\vec{N}|} = i - 2j + 3k.\frac{3i + 2j + 6k}{\sqrt{9+4+36}}$$

$$d = (i - 2j + 3k).(\frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k)$$

$$d = \frac{3}{7} - \frac{4}{7} + \frac{18}{7} = \frac{17}{7} \quad \textit{length unit}$$

Ex: Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.



We find a vector parallel to the line and a point on the line.

The line of intersection of two planes is perpendicular to both planes' normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

$\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line.

To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$

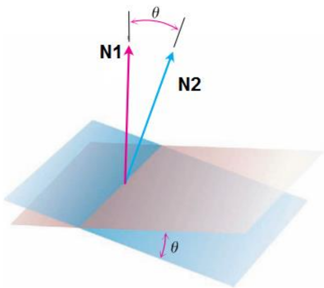
H.W:

1- Find the distance from the point P (2, -3, 4) to the plane $x + 2y + 2z = 13$.

2- If $\vec{A} = 3i + 2j,\quad \vec{B} = 5j + k$ find $\vec{A}.\vec{B}, |\vec{A}|, |\vec{B}|$ and the vector projection of $\vec{B}$ onto $\vec{A}, \left(proj_A^B\right)$.

7- Angles Between Planes

Two planes that are not parallel will intersect in a line. The angle between two intersecting planes is defined to be the angle between their normal vectors.



If the equations of planes are:

$$Ax + By + Cz = D$$
$$A'x + B'y + C'z = D'$$

Then the corresponding normal vectors are:

$$\vec{N}_1 = Ai + Bj + Ck$$
$$\vec{N}_2 = A'i + B'j + C'k$$

$$\vec{N}_1 . \vec{N}_2 = |\vec{N}_1||\vec{N}_2| \cos\theta$$

$$\boxed{\theta = \cos^{-1} \frac{\vec{N}_1 . \vec{N}_2}{|\vec{N}_1||\vec{N}_2|}} \qquad \text{.........(5)}$$

Ex₁: Find the angle between the planes $3x-6y-2z=0$ and $2x+y-2z=5$.

<u>Sol</u>:

$$\left.\begin{aligned}\vec{N}_1 &= 3i-6j-2k \\ \vec{N}_2 &= 2i+j-2k\end{aligned}\right\}$$ The vectors are normal to the planes.

$$\theta=\cos^{-1}\frac{\vec{N}_1.\vec{N}_2}{\left|\vec{N}_1\right|\left|\vec{N}_2\right|}$$

$$\left|\vec{N}_1\right|=\sqrt{3^2+6^2+2^4}=\sqrt{49}=7$$
$$\left|\vec{N}_2\right|=\sqrt{2^2+1^2+2^2}=\sqrt{9}=3$$

$$\theta=\cos^{-1}\frac{(3i-6j-2k).(2i+j-2k)}{7*3}$$
$$\theta=\cos^{-1}\frac{6-6+4}{21}=79^0$$

Ex₂: Find a vector parallel to the line of intersection of the plane *3x-6y - 2z =15* and *2x + y - 2z = 5*.

<u>Sol</u>:

The vector parallel to the line of intersection is the vector results from the cross product between the two normal vectors $\vec{N}_1,\vec{N}_2$.

$$\left.\begin{aligned}\vec{N}_1 &= 3i-6j-2k \\ \vec{N}_2 &= 2i+j-2k\end{aligned}\right\}$$ the vectors normal to the planes.

$$\vec{V}=\vec{N}_1\times\vec{N}_2=\begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix}=14i+2j+15k$$
$$\vec{V}=14i+2j+15k$$

<u>***H.W***</u>**₁**: find vector parallel to the line of intersection between two planes.

1) $x+y+z=1$ and $x+y=2$.

2) $x-2y+4z=2$ and $x+y-2z=5$

<u>***H.W***</u>**₂**: Find the angle between the planes $x+y=1$ and $2x+y-2z=2$

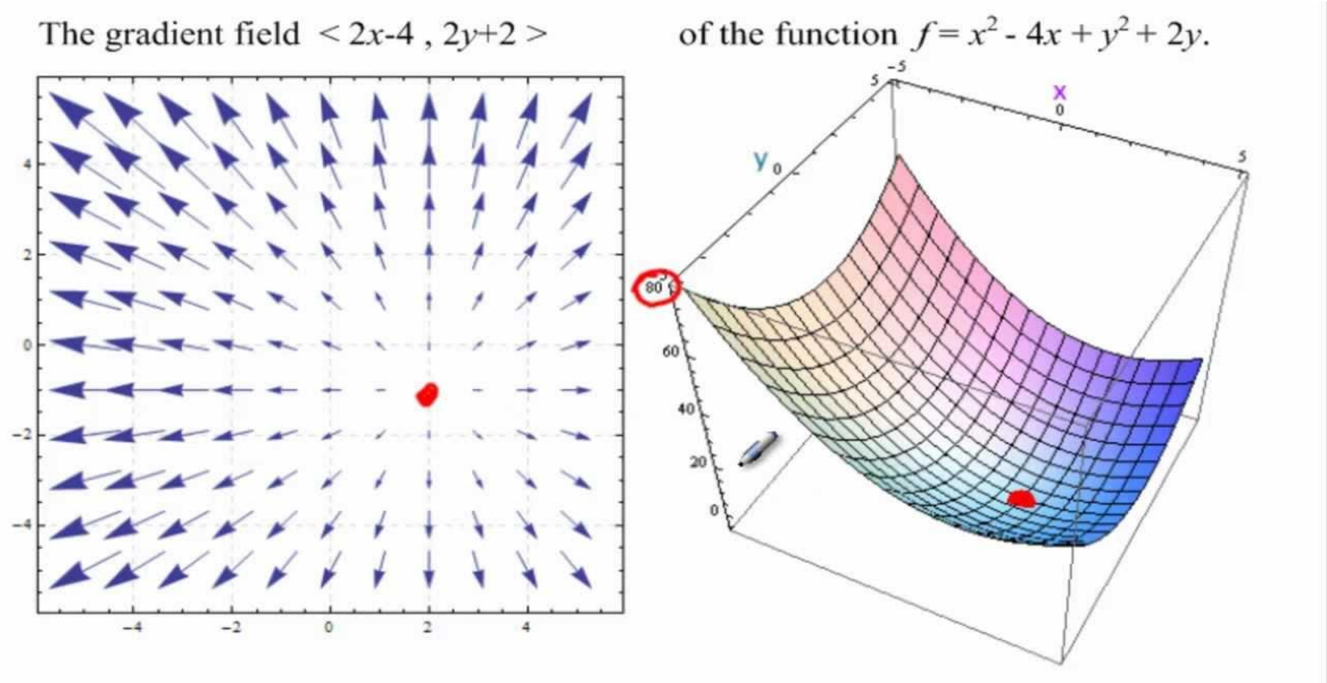
8- Gradient, Divergence and Curl of Vectors:

In three-dimensional Cartesian coordinate system (x, y, z) the Del operator (∇) is defined in term of partial derivative and given by:

$$\nabla \equiv \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

A- The gradient: In x-y plane to find the slope of the curve we use the derivative. Whereas in space we use the partial derivative to find the gradient (slope) the plane. Let we have a function $f(x, y, z)$ then :

$$grad\ f = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$



Ex1: find the gradient of $f(x, y, z) = 2y^3 + 4xz + 3x$.

Sol:

$$\frac{\partial f}{\partial x} = 4z + 3$$

$$\frac{\partial f}{\partial y} = 6y^2$$

$$\frac{\partial f}{\partial z} = 4x$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = (4z + 3)i + (6y^2) j + (4x)k$$

B- Divergence of a vector ($\vec{V}$): It is a scalar function can be found by:

Let we have a vector $\vec{V} = v_1 i + v_2 j + v_3 k$ then

$$div.\vec{V} = \nabla \bullet \vec{V} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \bullet (v_1 i + v_2 j + v_3 k)$$

$$\boxed{\nabla \bullet \vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}}$$

Ex1: Find the divergence of the vector $\vec{V} = (3xz)i + (2xy)j - (yz^2)k$

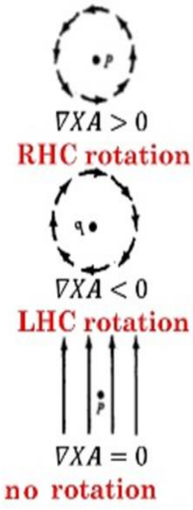
Sol:

$$\nabla \bullet \vec{V} = \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(-yz^2)$$

$$\nabla \bullet \vec{V} = 3z + 2x - 2yz$$

C- Curl of Vector field:

The curl of the vector $\vec{V}$ is a vector function that can be founded by the cross product between del operator and $\vec{V}$.



Let we have a vector $\vec{V} = v_1 i + v_2 j + v_3 k$ then:

$$Curl\ \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Ex1: Find the curl of the $\vec{V} = (3xz)i + (2xy)j$

Sol:

$$Curl\ \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xz & 2xy & 0 \end{vmatrix}$$

$$\nabla \times \vec{V} = \left(0 - \frac{\partial}{\partial z}(2xy)\right)i - \left(0 - \frac{\partial}{\partial z}(3xz)\right)j + \left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(3xz)\right)k$$

$$\nabla \times \vec{V} = 0i + (3x)j + (2y)k = (3x)j + (2y)k$$

Ex1: Find $\nabla \bullet \vec{V}\ and\ \nabla \times \vec{V}$, if $\vec{V} = -2yi + 2xj$.

Sol:

$$\nabla \bullet \vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(2x) + 0 = 0$$

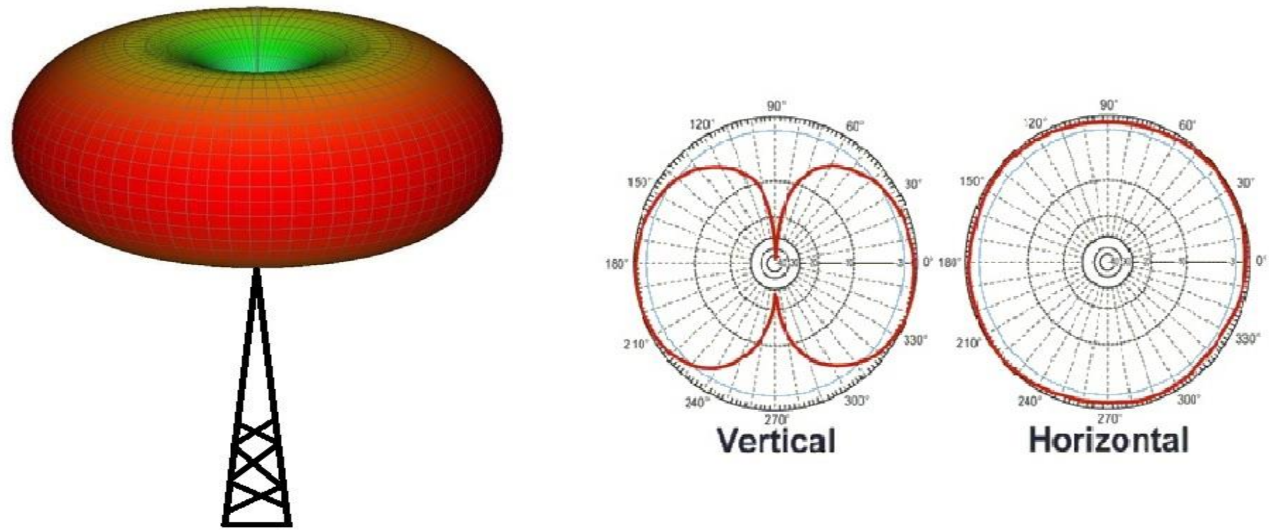
$$Curl\ \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & 2x & 0 \end{vmatrix}$$

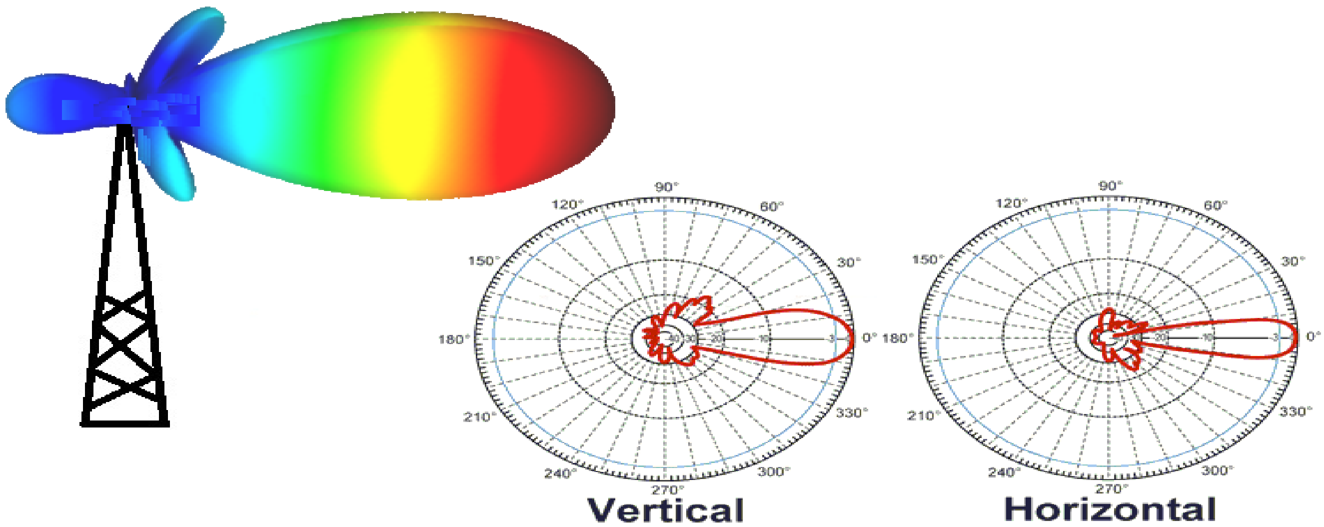
$$\nabla \times \vec{V} = \left(0 - \frac{\partial}{\partial z}(2x)\right)i - \left(0 - \frac{\partial}{\partial z}(-2y)\right)j + \left(\frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(-2y)\right)k$$

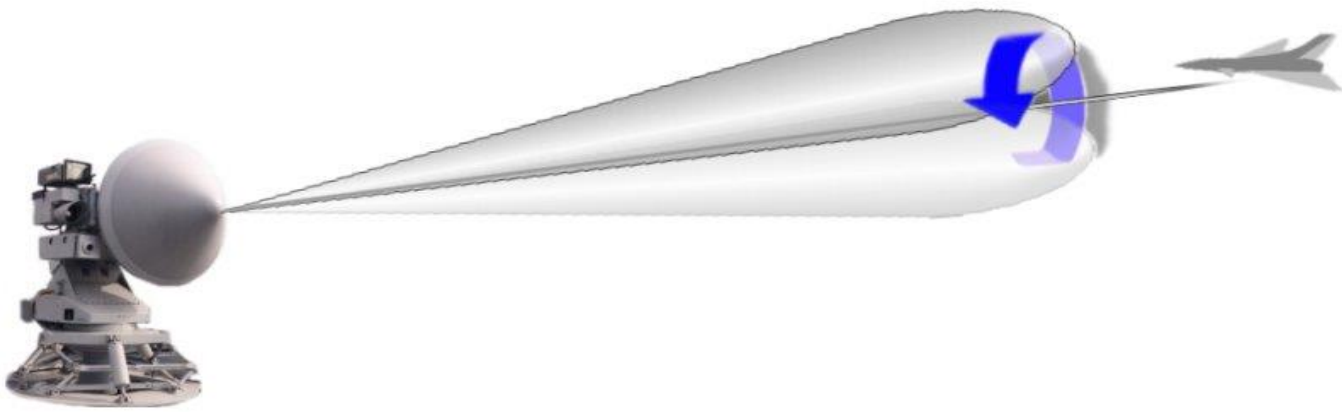
$$\nabla \times \vec{V} = 0i - 0j + 4k = 4k$$

<u>The purpose of polar coordinates:</u> (Sec. 10.5)

It is useful in the application that deals with the radiation pattern and tracking objects. Another benefit of polar coordinates is to simplify some of the complex integrals. It is also used in analysis of electrical circuits.





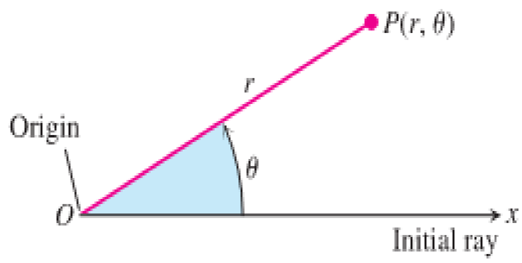


Definition of Polar Coordinates:

To define polar coordinate for the plane, we start with the origin "O" and initial ray (x-axis). Then each point in polar coordinate can be written in form of $P(r,\theta)$.

Where r: represents the direct distance from "O" to "P", the value of r can be positive or negative.

θ: angle bounded between initial ray and op ray. The value of θ is +ve when it measured counter clockwise and –ve when it measured clockwise.



Notice: The point in Cartesian coordinates has only one pair, whereas it has infinitely many pairs in polar coordinates as shown in example below.

Ex: Find all the polar coordinate pairs of the point $P\left(2,\dfrac{\pi}{6}\right)$.

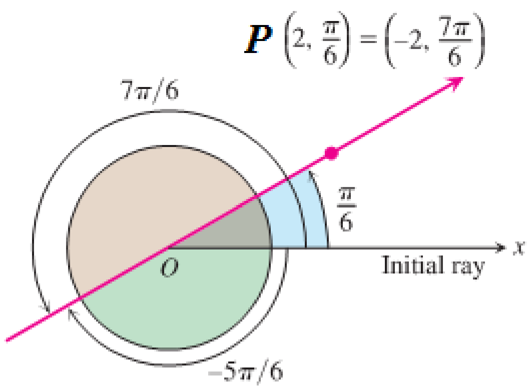
There will be two sets of points of P the first set when:

$r=+2$

Then the values of θ will be:

$$\theta=\frac{\pi}{6},\quad \frac{\pi}{6}\pm 2\pi,\quad \frac{\pi}{6}\pm 4\pi,\ldots\ldots\ldots\ldots$$

$$P\left(2,\frac{\pi}{6}\pm 2n\pi\right)\quad,\quad \text{where n}=0,\pm1,\pm2,\pm3\ldots\ldots$$



when:

$r=-2$

Then the values of θ will be:

$$\theta=\frac{7\pi}{6},\quad \frac{7\pi}{6}\pm 2\pi,\quad \frac{7\pi}{6}\pm 4\pi,\ldots\ldots\ldots\ldots$$

$$P\left(-2,\frac{7\pi}{6}\pm 2n\pi\right)\quad,\quad \text{where n}=0,\pm1,\pm2,\pm3\ldots\ldots$$

H.W: Find all the polar coordinate pairs of the point $P\left(-3,-\dfrac{\pi}{4}\right)$.

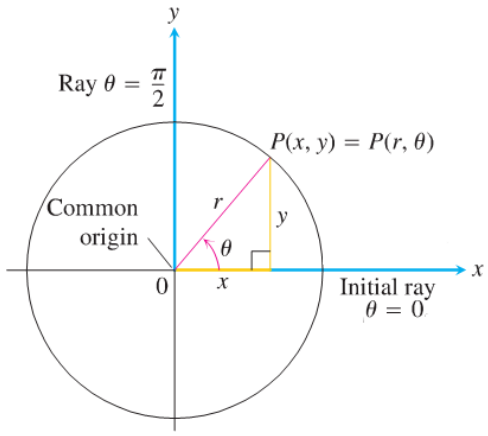
Transformation between polar and Cartesian coordinates:

To transform between polar and Cartesian coordinates, we use below equations:

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$



Ex: Replace the following polar equations by equivalent Cartesian equations:

1- $r\cos(\theta) = 2 \longrightarrow x = 2$

2- $r^2\cos(\theta)\sin(\theta) = 4 \longrightarrow xy = 4$

3- $r^2\cos^2(\theta) - r^2\sin^2(\theta) = 1 \longrightarrow x^2 - y^2 = 1$

4- $r = 1 + 2r\cos(\theta) \longrightarrow$

$$\sqrt{x^2+y^2} = 1 + 2x$$
$$x^2 + y^2 = (1+2x)^2$$
$$y^2 - 3x^2 - 4x - 1 = 0$$

5- $r = 1 - \cos(\theta) \longrightarrow$

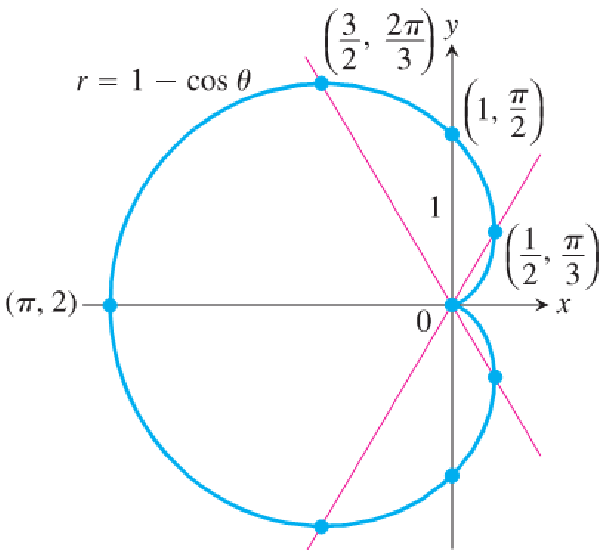
$$\sqrt{x^2+y^2} = 1 - \frac{x}{\sqrt{x^2+y^2}}$$
$$\sqrt{x^2+y^2} = \frac{\sqrt{x^2+y^2} - x}{\sqrt{x^2+y^2}}$$
$$x^2 + y^2 = \sqrt{x^2+y^2} - x$$
$$(x^2 + y^2 + x)^2 = x^2 + y^2$$
$$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$$

Graphing in Polar Coordinates:

To plot polar function we let the values of (θ) then substitute the assumed values in the function to find the values of (r).

Ex1: Graph the $r = 1 - \cos(\theta)$:

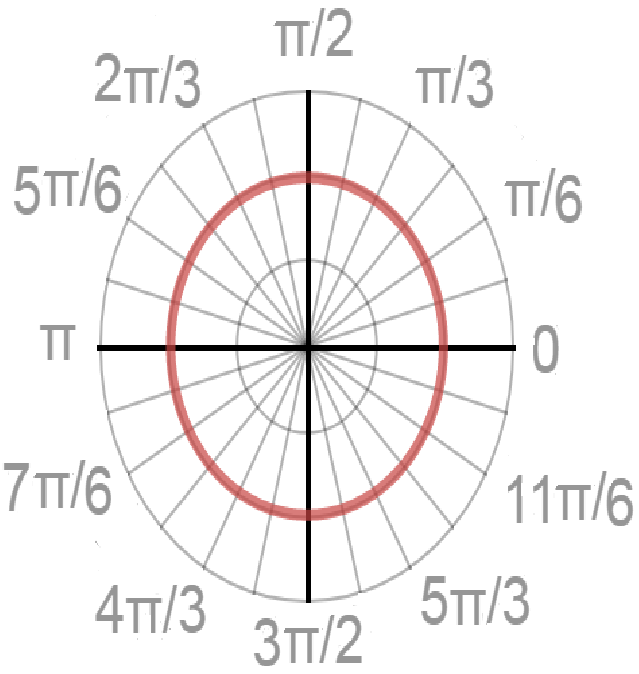
θ (radian)	θ (degree)	$r = 1 - \cos(\theta)$
0	0	(1-1) = 0
$\pi/3$	60	$(1-\frac{1}{2})= \frac{1}{2}$
$\pi/2$	90	(1-0) = 1
$2\pi/3$	120	$(1-(-\frac{1}{2})) = \frac{3}{2}$
π	180	(1-(-1)) = 2
$4\pi/3$	240	$\frac{3}{2}$
$3\pi/2$	270	1
$5\pi/3$	300	$\frac{1}{2}$
2π	360	0



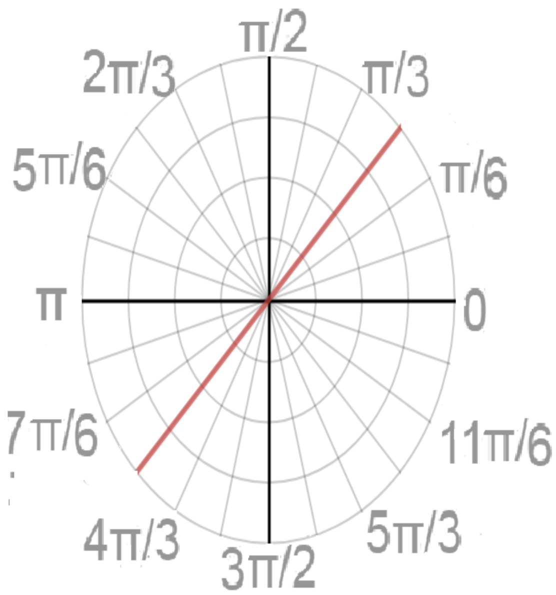
1- Circle and line Equations:

A-

Equation	Graph
$r = a$	Circle of radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray



r = a



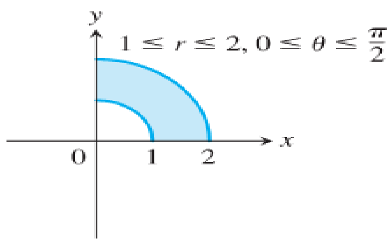
$\theta = \theta_0$

Ex: Graph the sets of points whose polar coordinates satisfy the following conditions:

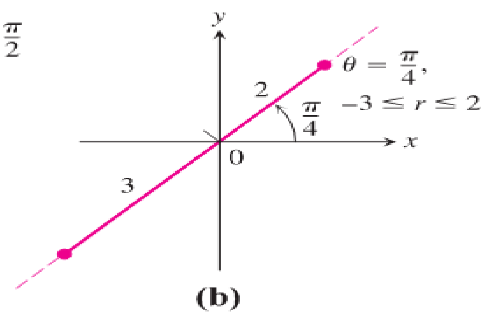
(a) $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \frac{\pi}{4}$

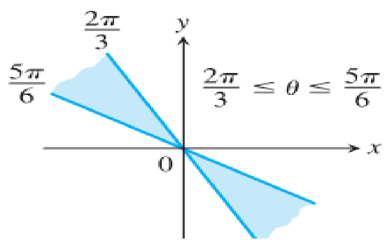
(c) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$ (no restriction on r)



(a)

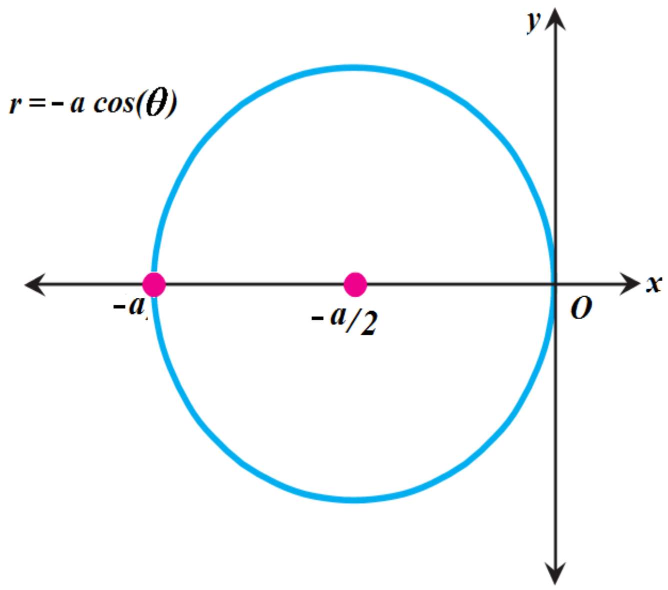


(b)

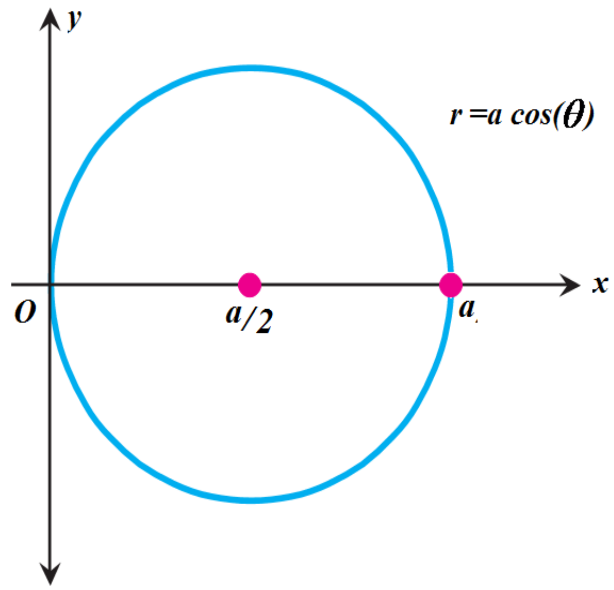


(c)

B- If the equation is $r = a\ \cos(\theta)$, then the graph is a circle on the *x*-axis with diameter of *a* as shown in figure below.

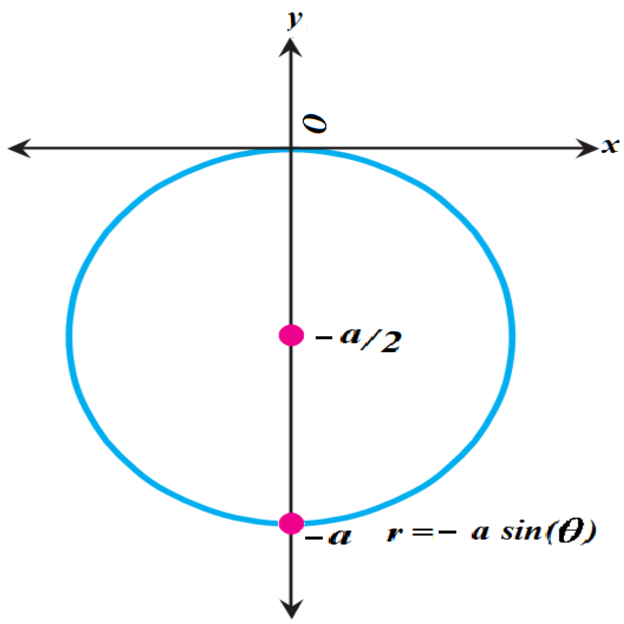


(a) when the value of a is negative

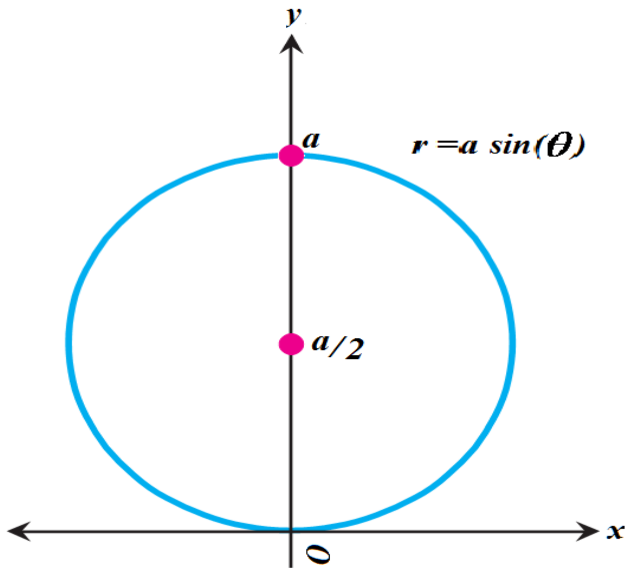


(b) when the value of a is positive

C- If the equation is $r = a \ \sin(\theta)$, then the graph is a circle on the *y*-axis with diameter of *a* as shown in figure below.



(a) when the value of a is negative



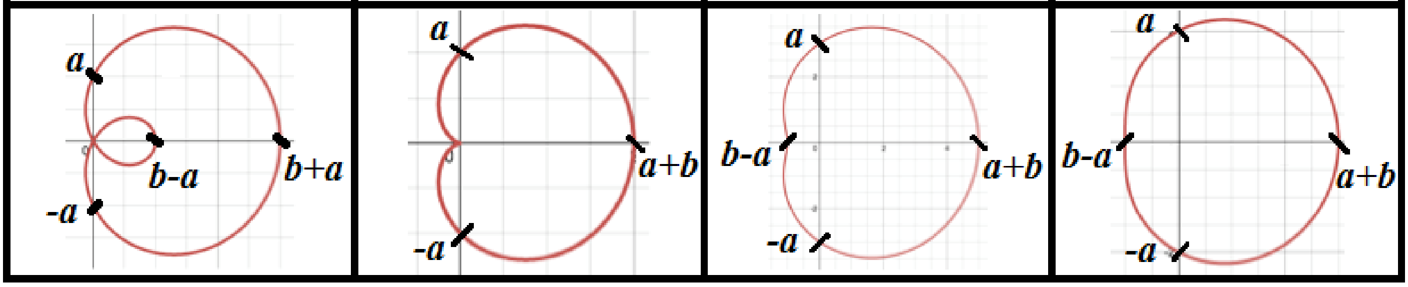
(b) when the value of a is positive

2- **Limacon Equations**:

It is also called the limaçon of Pascal. The word "limaçon" comes from the Latin *limax*, meaning "snail" or "helix". The equation of limacon is:

$$\boxed{\begin{array}{l} r = a \pm b\cos(\theta) \\ r = a \pm b\sin(\theta) \end{array}}$$

Inner loop Limacon	Heart/Cardioid	*Dimple Limacon*	*Convex Limacon*
$\frac{a}{b} < 1$	$\frac{a}{b} = 1$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} \geq 2$



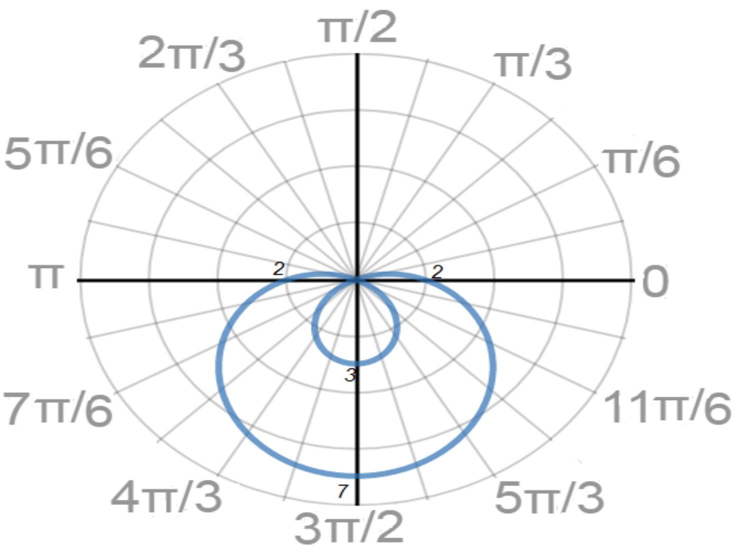
Notice:

$$* \cos(-\theta) = \cos(\theta)$$
$$* \sin(-\theta) = -\sin(\theta)$$

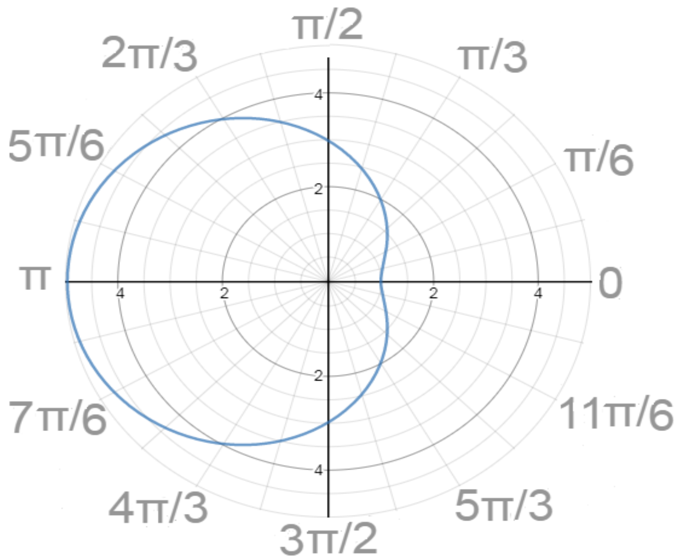
Ex2: Graph the

1) $r=2-5\sin(\theta)$ **2)** $r=3-2\cos(\theta)$

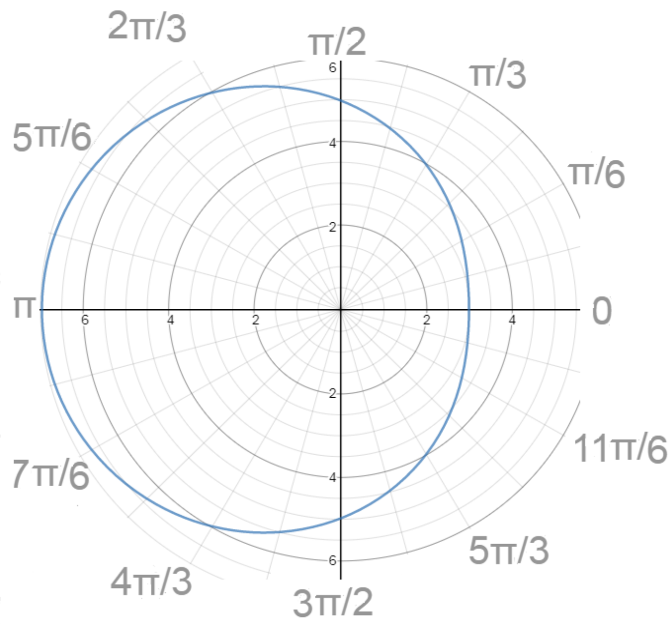
3) $r=5-2\cos(\theta)$ **4)** $r=2+2\sin(\theta)$



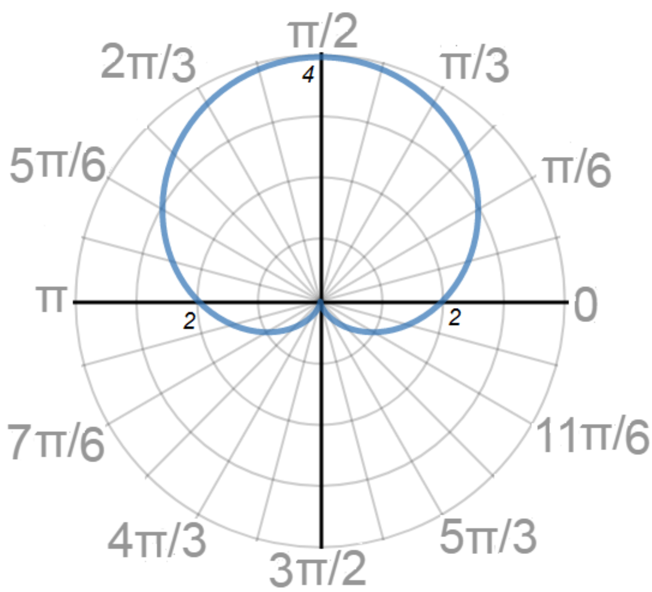
$r\ =\ 2\ -\ 5\ \sin(\ \theta\ )$



$r\ =\ 3\ -\ 2\ \cos(\theta\ )$



$r\ =\ 5\ -\ 2\ \cos(\ \theta\ )$



$r\ =\ 2\ +\ 2\ \sin(\ \theta\ )$

3-Rose Equations:

The equations of Rose are:

$$\boxed{\begin{aligned} r &= a\cos(n\theta) \\ r &= a\sin(n\theta) \end{aligned}}$$

To plot the rose equation:

1- Find the number of leaves which is equal to:

$No.\ \text{of leaves} = n$ …….When n is odd.

$No.\ \text{of leaves} = 2n$ ……. When n is even.

2- Find the angle of the 1st leaf by:

$r = a\cos(n\theta)$	$r = a\sin(n\theta)$
$\cos(n\theta) = 1$ $n\theta = 0$ $\theta = 0$ position of the 1st leaf at $\theta = 0$	$\sin(n\theta) = 1$ $n\theta = \frac{\pi}{2}$ $\theta = \frac{\pi}{2n}$ position of the 1st leaf at $\theta = \frac{\pi}{2n}$

3- Find the spacing angle between the leaves by:

$$SP = \frac{360}{No.\ of\ leaves}$$

4- Specify the positions of the other leaves with respect to the 1st leaf depending on the value of SP.

Ex:

1)

$r = 2\cos(2\theta)$

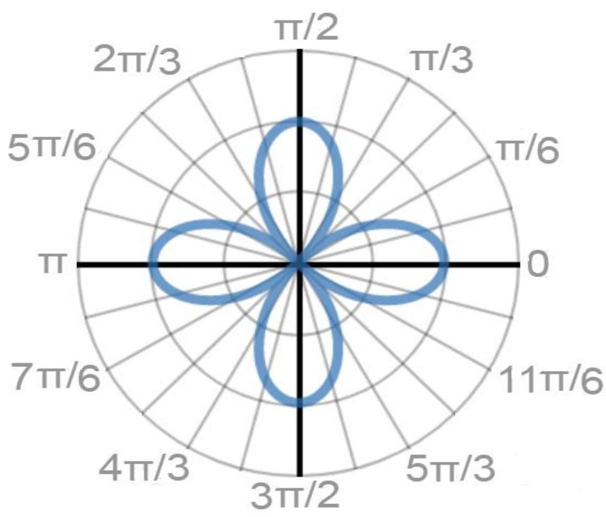
$No.\ \text{of leaves} = 2n = 4 \rightarrow (n : even)$

$n\theta = 0$

$\theta = 0$

position of the 1st leaf at $\theta = 0$

$$SP = \frac{360}{No.\ of\ leaves} = \frac{360}{4} = 90^0$$



2)

$r = 2\sin(2\theta)$

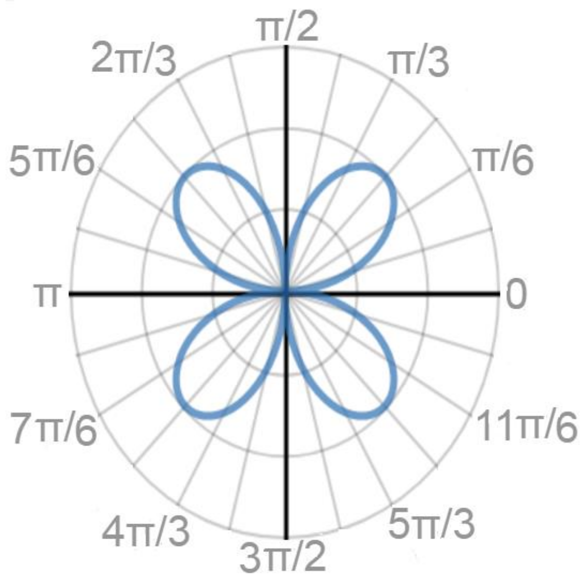
$No.\text{ of leaves} = n = 4 \rightarrow \ (n : even)$

$n\theta = \frac{\pi}{2}$

$2\theta = \frac{\pi}{2} \quad \rightarrow \quad \theta = \frac{\pi}{4}$

$\text{position of the 1st leaf at } \theta = \frac{\pi}{4}$

$SP = \frac{360}{No.\ of\ leaves} = \frac{360}{4} = 90^0$



3)

$r = 2\cos(3\theta)$

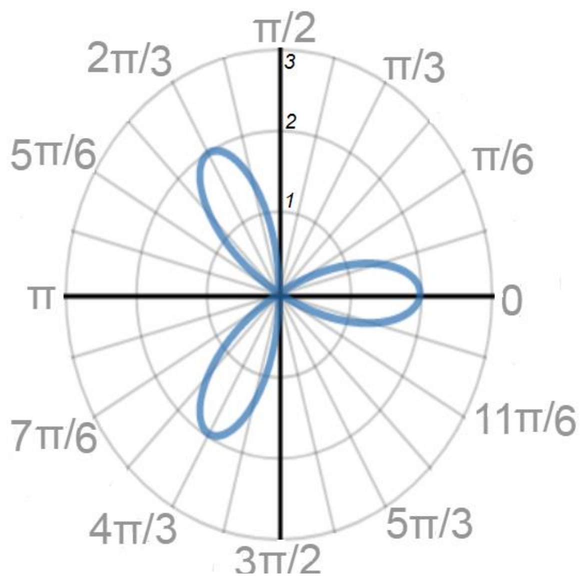
$No.\text{ of leaves} = n = 3 \rightarrow \ (n : odd)$

$n\theta = 0$

$\theta = 0$

$\text{position of the 1st leaf at } \theta = 0$

$SP = \frac{360}{No.\ of\ leaves} = \frac{360}{3} = 120^0$



4)

$r = 2\sin(3\theta)$

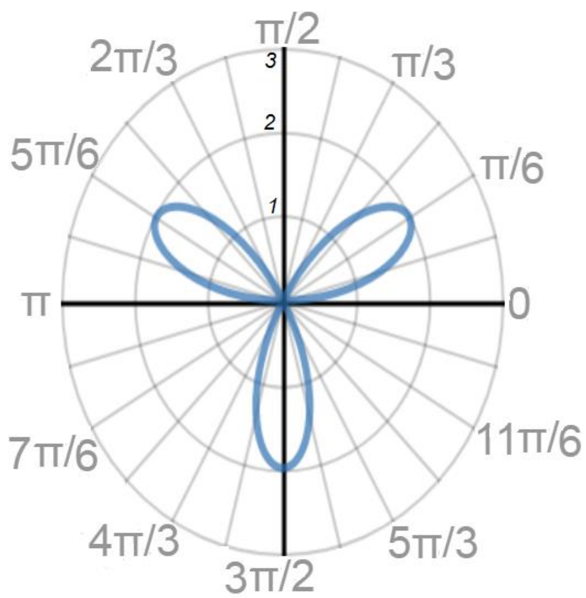
$No.\text{ of leaves} = n = 3 \rightarrow \ (n : odd)$

$n\theta = \frac{\pi}{2}$

$3\theta = \frac{\pi}{2} \quad \rightarrow \quad \theta = \frac{\pi}{6}$

$\text{position of the 1st leaf at } \theta = \frac{\pi}{6}$

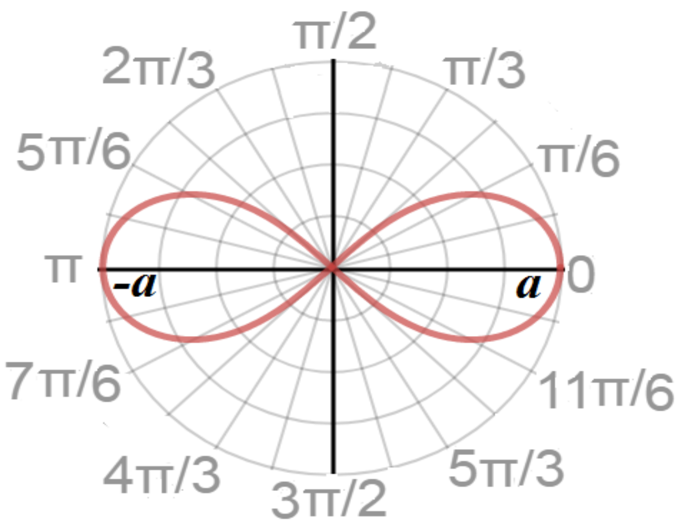
$SP = \frac{360}{No.\ of\ leaves} = \frac{360}{3} = 120^0$

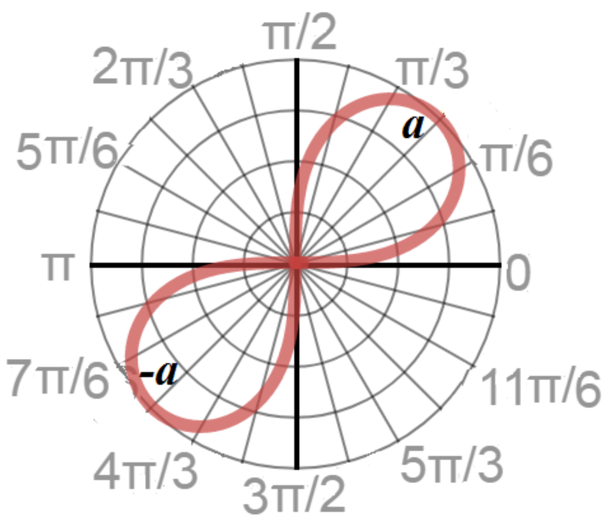


4-Lemniscate Equation:

The equations of lemniscates are:

$$\boxed{\begin{aligned} r^2 &= a^2\cos(2\theta) \\ r^2 &= a^2\sin(2\theta) \end{aligned}}$$





$$r^2 = a^2\cos(2\theta)$$

$$r^2 = a^2\sin(2\theta)$$

H.W1: name and sketch the following equations:

1) $r = 1 - \sin(\theta)$

2) $r = -1 + \sin(\theta)$

3) $r = 2 - 2\cos(\theta)$

4) $r = \sin(\theta) - 2$

5) $r = -\sin(3\theta)$

6) $r = \sin(5\theta)$

7) $r = -\cos(5\theta)$

8) $r^2 = \sin(\theta)$

9) $r^2 = \cos(\theta)$

10) $r = -2 + \sin(\theta)$

H.w2:

1. Which polar coordinate pairs label the same point?

a. $(3, 0)$ **b.** $(-3, 0)$ **c.** $(2, 2\pi/3)$

d. $(2, 7\pi/3)$ **e.** $(-3, \pi)$ **f.** $(2, \pi/3)$

g. $(-3, 2\pi)$ **h.** $(-2, -\pi/3)$

2. Which polar coordinate pairs label the same point?

a. $(-2, \pi/3)$ **b.** $(2, -\pi/3)$ **c.** (r, θ)

d. $(r, \theta + \pi)$ **e.** $(-r, \theta)$ **f.** $(2, -2\pi/3)$

g. $(-r, \theta + \pi)$ **h.** $(-2, 2\pi/3)$

3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

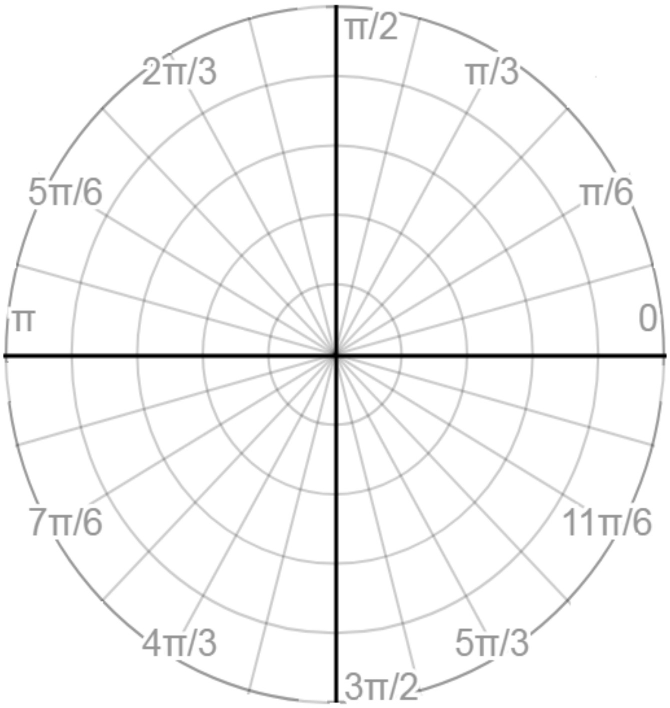
a. $(2, \pi/2)$ **b.** $(2, 0)$

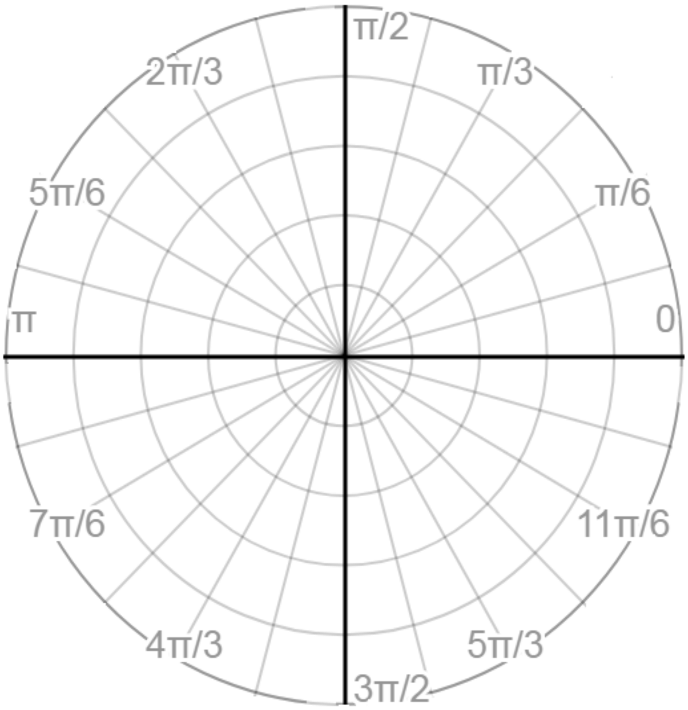
c. $(-2, \pi/2)$ **d.** $(-2, 0)$

4. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a. $(3, \pi/4)$ **b.** $(-3, \pi/4)$

c. $(3, -\pi/4)$ **d.** $(-3, -\pi/4)$


π/2
2π/3
π/3
5π/6
π/6
π
0
7π/6
11π/6
4π/3
5π/3
3π/2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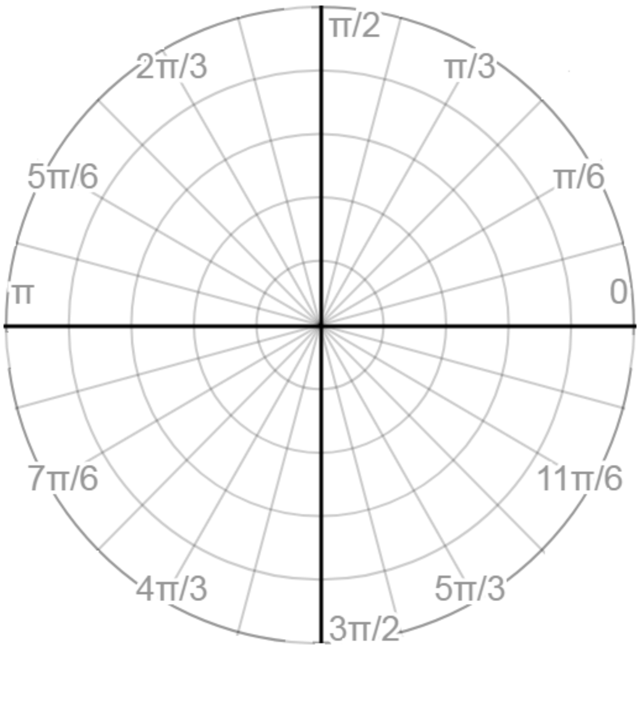
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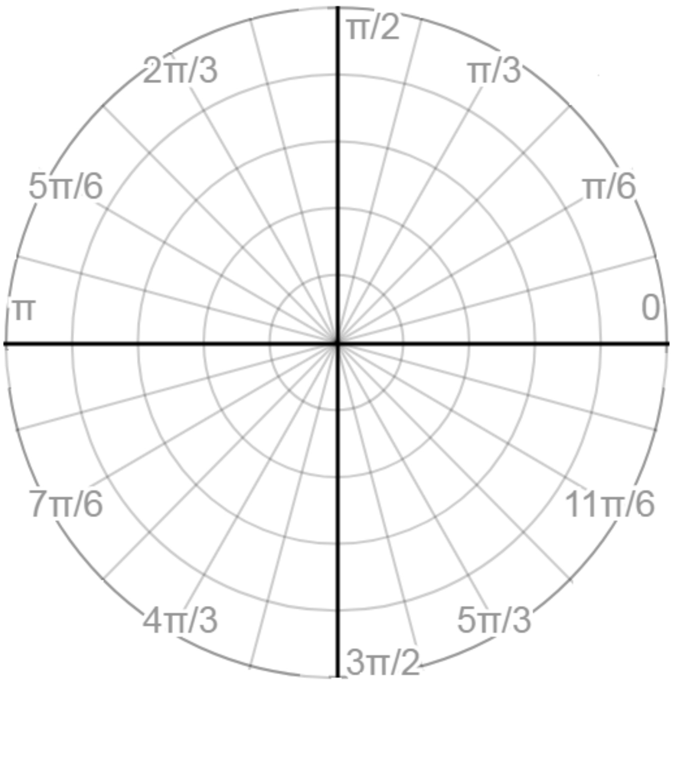
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