



Nineveh University
College Of Electronic Engineering
Systems and Control Engineering Department

Digital control

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الصف الثالث

Course Syllabus

Digital Control

Required Textbook

- 1-Digital Control Engineering Analysis and Design, by M. Sami Fadali, Antonio Visioli: Second Edition, 2013
- 2-Digital Control System analysis and design, by Philips, Nagle L.Fourth Edition ,2015

Course Topics:

CHAPTER 1 Introduction to Digital Control

- 1.1 Digital control system
- 1.2 Why digital control?
- 1.3 Examples of digital control systems

CHAPTER 2 Discrete-Time Systems

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- 2.3 Properties of the z-transform
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CHAPTER 3 Modeling of Digital Control Systems

- 3.1 Structure Of Control Systems
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CHAPTER 4 Stability of Digital Control Systems

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Introduction to Digital Control

1.1 Digital Control System

A digital control system model can be viewed from different perspectives including control algorithm, computer program, conversion between analog and digital domains, system performance etc. One of the most important aspects is the sampling process level.

In a digital control system, the control algorithm is implemented in a digital computer. The error signal is discretized and fed to the computer by using an A/D (analog to digital) converter. The controller output is again a discrete signal which is applied to the plant after using a D/A (digital to analog) converter. General block diagram of a digital control system is shown in Figure 1.

$e(t)$ is sampled at intervals of T . In the context of control and communication, sampling is a process by which a continuous time signal is converted into a sequence of numbers at discrete time intervals. It is a fundamental property of digital control systems because of the discrete nature of operation of digital computer. Figure 2 shows the structure and operation of a finite pulse width sampler, where (a) represents the basic block diagram and (b) illustrates the function of the same. T is the sampling period and p is the sample duration.

Automatic control is the science that develops techniques to steer, guide, control dynamic systems. These systems are built by humans and must perform a specific task. Examples of such dynamic systems are found in biology, physics, robotics, finance, etc.

Digital Control means that the control laws are implemented in a digital device, such as a **microcontroller** or a **microprocessor**. Such devices are light, fast and economical.

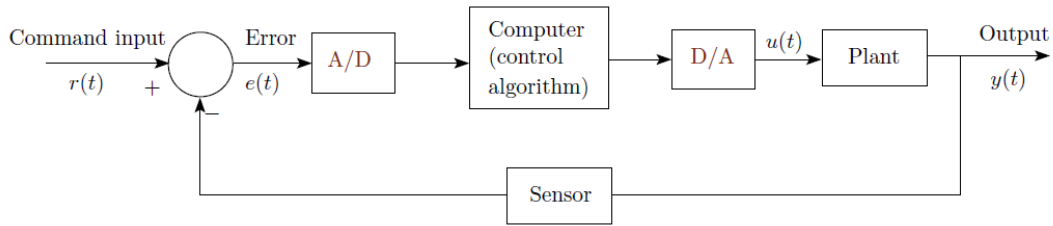


Figure 1 General block diagram of a digital control system

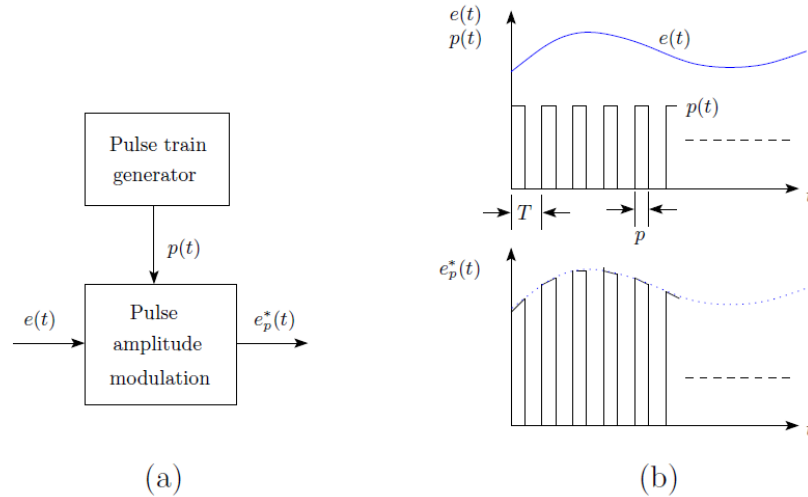


Figure 2 Basic structure and operation of a finite pulse width sampler

1.2 Why digital control?

Digital control offers distinct advantages over analog control that explain its popularity. Here are some of its many advantages:

Accuracy. Digital signals are represented in terms of zeros and ones with typically 12 bits or more to represent a single number. This involves a very small error as compared to analog signals, where noise and power supply drift are always present.

Implementation errors. Digital processing of control signals involves addition and multiplication by stored numerical values. The errors that result from digital representation and arithmetic are negligible. By contrast, the processing of analog signals is performed using components such as resistors and capacitors with actual values that vary significantly from the nominal design values.

Flexibility. An analog controller is difficult to modify or redesign once implemented in hardware. A digital controller is implemented in firmware or software and its modification is possible without a complete replacement of the original controller. Furthermore, the structure of the

digital controller need not follow one of the simple forms that are typically used in analog control. More complex controller structures involve a few extra arithmetic operations and are easily realizable.

Speed. The speed of computer hardware has increased exponentially since the 1980s. This increase in processing speed has made it possible to sample and process control signals at very high speeds. Because the interval between samples, the sampling period, can be made very small, digital controllers achieve performance that is essentially the same as that based on continuous monitoring of the controlled variable.

Cost. Although the prices of most goods and services have steadily increased, the cost of digital circuitry continues to decrease. Advances in very large-scale integration (VLSI) technology have made it possible to manufacture better, faster, and more reliable integrated circuits and to offer them to the consumer at a lower price. This has made the use of digital controllers more economical even for small, low-cost applications.

1.3 Examples of digital control systems

In this section, we briefly discuss examples of control systems where digital implementation is now the norm.

1.3.1 Closed-loop drug delivery system

Several chronic diseases require the regulation of the patient's blood levels of a specific drug or hormone. For example, some diseases involve the failure of the body's natural closed-loop control of blood levels of nutrients. Most prominent among these is the disease diabetes, where the production of the hormone insulin that controls blood glucose levels is impaired. To design a closed-loop drug delivery system, a sensor is utilized to measure the levels of the regulated drug or nutrient in the blood. This measurement is converted to digital form and fed to the control computer, which drives a pump that injects the drug into the patient's blood. A block diagram of the drug delivery system is shown in Figure 3.

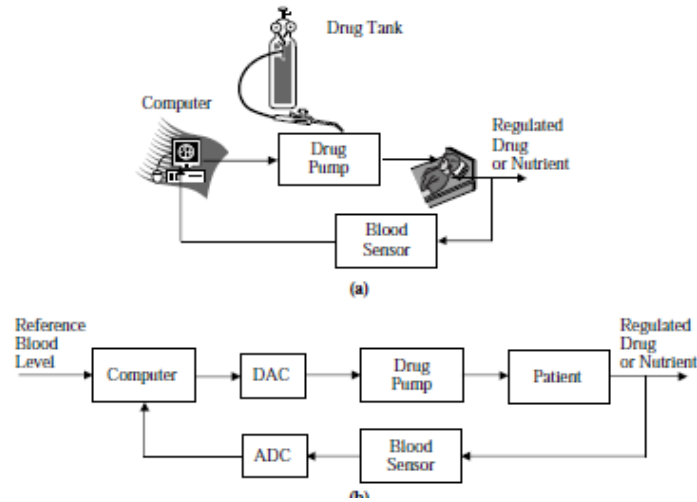


Figure.3 Block diagram of a drug delivery system.

تتطلب العديد من الأمراض المزمنة تنظيم مستويات دم المريض للدواء أو هرمون محددة. على سبيل المثال، بعض الأمراض تنطوي على فشل حلقة مغلقة سيطرة الجسم الطبيعية من مستويات الدم من المواد المغذية. ومن أبرز هؤلاء هو مرض داء السكري، حيث انخفضت قيمته إنتاج هرمون الأنسولين الذي يتحكم في مستويات السكر في الدم. تصميم نظام توصيل الدواء حلقة مغلقة، ويستخدم جهاز استشعار لقياس مستويات المخدرات الخاضعة للتنظيم أو المغذيات في الدم. يتم تحويل هذا القياس إلى شكل رقمي وتغذية لمراقبة الكمبيوتر، الذي يحرك مضخة أن يحقن الدواء في دم المريض. ويرد رسم بياني للنظام تسليم المخدرات في الشكل 3.

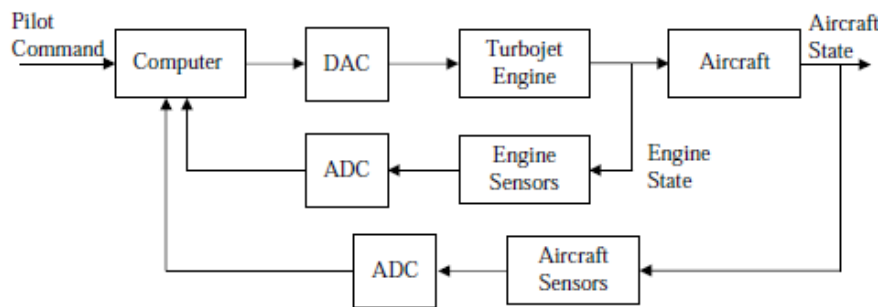
1.3.2 Computer control of an aircraft turbojet engine

To achieve the high performance required for today's aircraft, turbojet engines employ sophisticated computer control strategies. A simplified block diagram for turbojet computer control is shown in Figure.4. The control requires feedback of the engine state (speed, temperature, and pressure), measurements of the aircraft state (speed and direction), and pilot command.

لتحقيق الأداء العالي المطلوب للطائرات اليوم، محركات إرتكاس توظف استراتيجيات متطورة السيطرة على جهاز الكمبيوتر. ويرد رسم بياني مبسط للسيطرة على جهاز الكمبيوتر نفاث في الشكل 3. تتطلب مراقبة ردود الفعل من حالة المحرك (سرعة ودرجة الحرارة والضغط)، وقياسات حالة الطائرات (سرعة واتجاه)، والقيادة التجريبية.



(a)



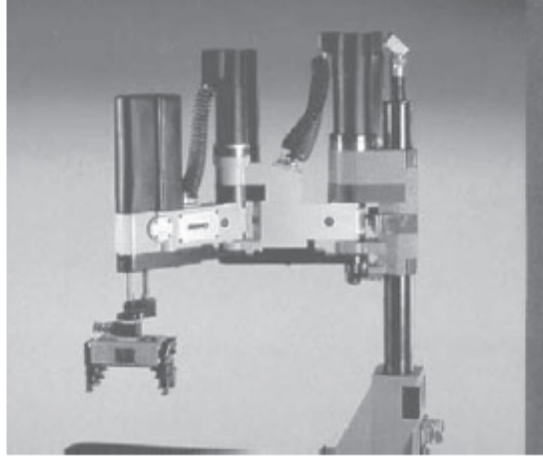
(b)

Figure .4-Turbojet engine control system. (a) F-22 military fighter aircraft. (b) Block diagram of an engine control system.

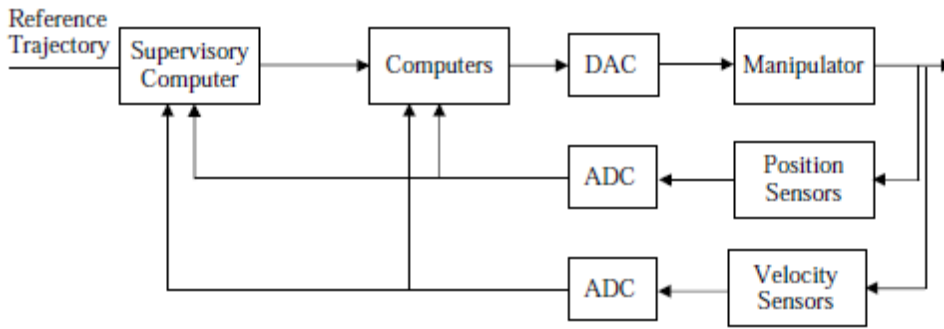
1.3.3 Control of a robotic manipulator

Robotic manipulators are capable of performing repetitive tasks at speeds and accuracies that far exceed those of human operators. They are now widely used in manufacturing processes such as spot welding and painting. To perform their tasks accurately and reliably, manipulator hand (or end-effector) positions and velocities are controlled digitally .

Each motion or degree of freedom (D.O.F.) of the manipulator is positioned using a separate position control system. All the motions are coordinated by a supervisory computer to achieve the desired speed and positioning of the end-effector. The computer also provides an interface between the robot and the operator that allows programming the lower-level controllers and directing their actions. A simple robotic manipulator is shown in Figure 1.5a, and a block diagram of its digital control system is shown in Figure 1.54b.



(a)



(b)

Figure 5 Robotic manipulator control system. (a) 3-D.O.F. robotic manipulator. (b) Block diagram of a manipulator control system.

المتلاعبين الروبوتية قادرة على أداء المهام المتكررة بسرعة ودقتها التي تتجاوز بكثير تلك التي من مشغلي الإنسان. تستخدم الآن على نطاق واسع في عمليات التصنيع مثل لحام البقعة والرسم. لأداء مهامهم بدقة وبشكل موثوق، يتم التحكم مناوور اليد (أو نهاية المستجيب) مواقع وسرعات رقمياً. يتم وضع كل حركة أو درجة حرية (.D.O.F.) من مناوور باستخدام نظام مستقل السيطرة على الموقف. يتم تنسيق جميع الاقتراحات التي كتبها كمبيوتر الإشرافي لتحقيق السرعة المطلوبة وتحديد المواقع من نهاية المستجيب. كما يوفر الكمبيوتر واجهة بين الروبوت والمشغل الذي يسمح برمجة وحدات التحكم المستوى الأدنى وتوجيه أعمالهم. ويرد مناوور الروبوتية بسيط في الشكل a 1.5، ويظهر رسم بياني من نظام التحكم الرقمي في الشكل b 1.5

Z-transform

The simple substitution

$$z = e^{Ts}$$

Converts the Laplace transform to the z transform. Making this substitution into the Laplace transform of the sampled signal

$$\begin{aligned} F^*(s) = Z [f^*(t)] &= F(z) = f(0) + f(T)e^{-Ts} + f(2T)e^{-2Ts} + \dots \\ &= f(0) + f(T)z^{-1} + f(2T)z^{-2} + \dots \\ &= \sum_{n=0}^{\infty} f(nT) z^{-n} \end{aligned} \quad (1)$$

where $F(z)$ designates the z transform of $f^*(t)$. Because only values of the signal at the sampling instants are considered, the z transform of $f(t)$ is the same as that of $f^*(t)$.

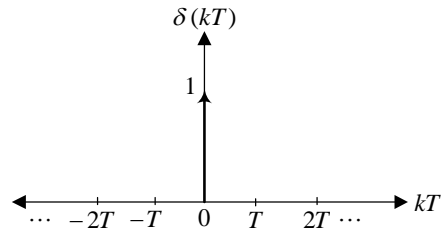
Z Transform by Definition:

In the following analysis, the z transform is derived using Eq.(1), where $f(nT)$ is the function for which the z transform will be obtained.

Impulse function:

The discrete unit impulse function is defined as

$$\delta(nT) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$



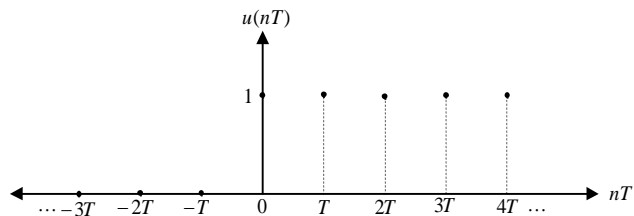
The z transform of the delta function $\delta(nT)$ can be given as

$$Z \{ \delta(nT) \} = \sum_{n=0}^{\infty} u_1(nT) z^{-n} = z^0 = 1$$

Discrete unit step function:

The discrete step function is defined as

$$u(nT) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



The z transform of the unit step

$$Z \{ u(nT) \} = U(z) = \sum_{n=0}^{\infty} u(nT) z^{-n} = 1 + z^{-1} + z^{-2} + \dots$$

Multiplying both sides of this last equation by z results in

$$z U(z) = z + 1 + z^{-1} + z^{-2} + z^{-3} + \dots = z + \sum_{n=0}^{\infty} z^{-n} = z + U(z)$$

$$U(z) (z - 1) = z \quad \text{or} \quad U(z) = \frac{z}{z - 1}$$

Discrete ramp function:

The discrete ramp function is defined as

$$x(nT) = \begin{cases} nT & n \geq 0 \\ 0 & n < 0 \end{cases}$$

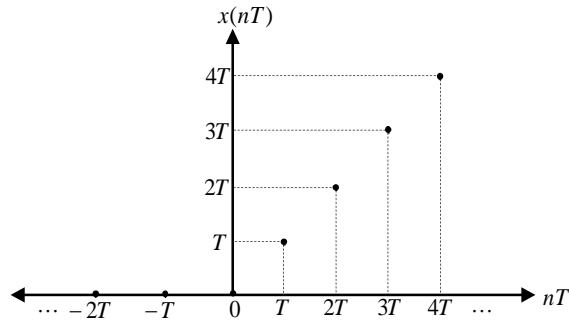
then

$$X(z) = \sum_{n=0}^{\infty} nT z^{-n} = T \sum_{n=0}^{\infty} n z^{-n}$$

since $n z^{-n} = -z \frac{d}{dz} (z^{-n})$, then

$$X(z) = -Tz \frac{d}{dz} \left(\sum_{n=0}^{\infty} z^{-n} \right)$$

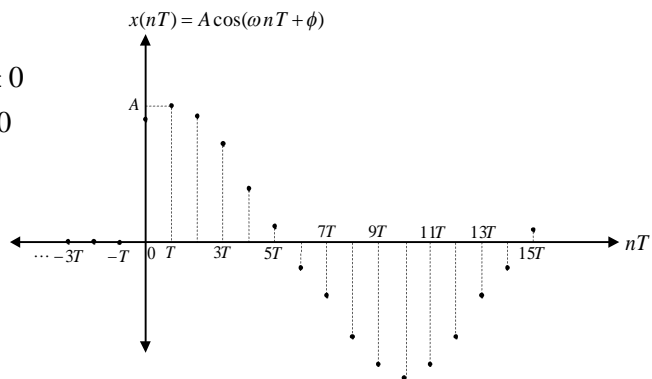
since $\sum_{n=0}^{\infty} z^{-n} = \frac{z}{(z-1)}$, then



Discrete cosine function:

Let

$$x(nT) = \begin{cases} A \cos(\omega nT + \phi) & n \geq 0 \\ 0 & n < 0 \end{cases}$$



The first step is the choice of the alternative representation of cosine function using Euler identity:

Then,

$$X(z) = \frac{A}{2} \sum_{n=0}^{\infty} e^{j\phi} e^{j\omega kT} z^{-n} + \frac{A}{2} \sum_{n=0}^{\infty} e^{-j\phi} e^{-j\omega kT} z^{-n}$$

$$= \frac{A}{2} \frac{z e^{j\phi}}{z - e^{-j\omega T}} + \frac{A}{2} \frac{z e^{-j\phi}}{z - e^{j\omega T}}$$

Discrete exponential decay function:

Let

$$x(nT) = \begin{cases} e^{-anT} & n \geq 0 \\ 0 & n < 0 \end{cases}$$

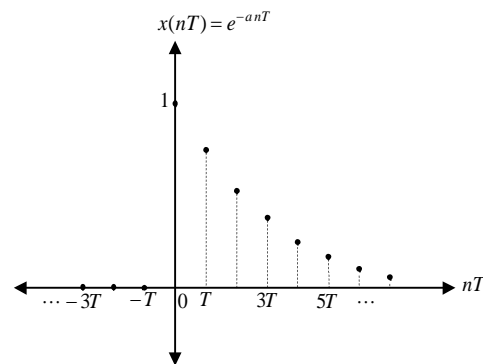
Then,

$$X(z) = \sum_{n=0}^{\infty} e^{-anT} z^{-n}$$

$$= \sum_{n=0}^{\infty} (e^{aT} z)^{-n}$$

$$= \frac{(e^{aT} z)}{(e^{aT} z) - 1}$$

$$= \frac{z}{z - e^{-aT}}, \quad |z| > e^{-aT}$$



In Table (1) is given a partial listing of Laplace transforms and corresponding z transforms for commonly encountered functions.

Z Transform Using Partial Fraction:

When the Laplace transform of a function is known, the corresponding z transform may be obtained by the partial fraction

Ex: Determine the z transform for the function whose Laplace transform is

$$F(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

From Table (1), the z transform corresponding to $1/s$ is $z/z-1$ and that corresponding to $1/s+1$ is $z/z - e^{-T}$. Thus,

$$F(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

Table (1) z transforms

Time function	Laplace Transform	Discrete Time function	Z transform
$\delta(t)$	1	$\delta(nT)$	1
$u(t)$	$\frac{1}{s}$	$u(nT)$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	nT	$\frac{zT}{(z-1)^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{(nT)^2}{2}$	$\frac{z(z+1)T^2}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	e^{-anT}	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$nT e^{-anT}$	$\frac{zT e^{-aT}}{(z-e^{-aT})^2}$
$a^{t/T}$	$\frac{1}{s - (1/T) \ln(a)}$	a^n	$\frac{z}{z-a} \quad (a > 0)$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\sin(\omega nT)$	$\frac{z \sin(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\cos(\omega nT)$	$\frac{z^2 - z \cos(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$

Ex: Determine the z transform of $\cos(\omega t)$.

It is known that the Laplace transform is $s/(s^2 + \omega^2)$. Performing a partial-fraction expansion gives

$$\mathcal{L}\{\cos(\omega t)\} \frac{s}{s^2 + \omega^2} = \frac{1/2}{s + j\omega} + \frac{1/2}{s - j\omega}$$

The corresponding z transform is

Z Transform Using Residue Method:

This is a powerful technique for obtaining z transforms. The z transform of $f^*(t)$ may be expressed in the form

$$F(z) = Z[f^*(t)] = \sum \text{residues of } F(s) \frac{z}{z - e^{sT}} \text{ at poles of } F(s)$$

When the denominator of $F(s)$ contains a linear factor of the form $s - r$ such that $F(s)$ has a first-order pole at $s = r$, the corresponding residue R is

$$R = \lim_{s \rightarrow r} (s - r) \left[F(s) \frac{z}{z - e^{sT}} \right]$$

When $F(s)$ contains a repeated pole of order q , the residue is

As is illustrated by the following examples, the determination of residues is similar to evaluating the constants in a partial-fraction expansion.

Ex: Determine the z transform of a unit step function.

For $F(s) = 1/s$, there is but one pole at $s=0$. The corresponding residue is

$$R = \lim_{s \rightarrow 0} s \left[\frac{1}{s} \frac{z}{z - e^{sT}} \right] = \frac{z}{z - 1}$$

Ex: Determine the z transform of e^{-at} .

For this function, $F(s) = 1/(s + a)$, which has but one pole at $s=-a$. Thus,

$$R = \lim_{s \rightarrow -a} (s + a) \left[\frac{1}{(s + a)} \frac{z}{z - e^{sT}} \right] = \frac{z}{z - e^{-aT}}$$

Ex: Determine the z transform of for the function whose Laplace transform is $F(s) = \frac{1}{s(s+1)}$

The poles of $F(s)$ occur at $s=0$ and $s=-1$. The residue due to the pole at $s=0$ is

$$R_1 = \lim_{s \rightarrow 0} s \left[\frac{1}{s(s+1)} \frac{z}{z - e^{sT}} \right] = \frac{z}{z - 1}$$

The residue due to the pole at $s=-1$ is

Adding these two residues results in

$$R = \sum_{i=1}^2 R_i = R_1 + R_2 = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}$$

Ex: Determine the z transform of $\cos(\omega t)$.

The Laplace transform is

$$F(s) = \frac{s}{s^2 + \omega^2} = \frac{s}{(s-j\omega)(s+j\omega)}$$

The poles are at $s = j\omega$ and $s = -j\omega$. Thus,

Adding these verify the previous result

$$R = \sum_{i=1}^2 R_i = R_1 + R_2 = \frac{z^2 - z \cos(\omega T)}{z^2 - 2z \cos(\omega T) + 1}$$

Ex: Determine the z transform corresponding to the function $f(t) = t$.

The Laplace transform is

$$F(s) = \frac{1}{s^2}$$

This has a second-order pole at $s=0$. Thus, the residue becomes

$$R = \frac{1}{(2-1)!} \lim_{s \rightarrow 0} \frac{d^{2-1}}{ds^{2-1}} \left[(s-r)^2 F(s) \frac{z}{z-e^{sT}} \right]$$

or

Theorems

→ Initial value theorem:

Suppose $f(nT)$ has z transform $F(z)$ and $\lim_{z \rightarrow \infty} F(z)$ exist, then the initial value $f(0)$ of $f(nT)$ is given by

Proof: Note that

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} = f(0) + f(T) \frac{1}{z} + f(2T) \frac{1}{z^2} + \dots$$

letting $z \rightarrow \infty$, the theorem is verified.

→ Final value theorem :

Suppose $f(nT)$ has z transform $F(z)$. Then,

Proof: Consider the following sums S_n and S_{n-1}

Dividing the second series by z and then subtracting the second from the first gives

$$\left(1 - \frac{1}{z}\right) f(0) + \dots + \left(1 - \frac{1}{z}\right) \frac{f[(n-1)T]}{z^{n-1}} + \frac{f(nT)}{z^n}$$

Taking the limit as z approaches 1 gives

When n is very large, $S_{n-1} \approx S_n \approx F(z)$. Thus, the final-value theorem given by Eq.(1) is verified.

Ex: For a discrete data system with transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z+1}{z^2 - 1.4z + 0.48}$$

and a unit step input for which the z transform is

$$U(z) = \frac{z}{z-1}$$

find the final value of the response sequence $y(nT)$. The response in the z domain is

$$Y(z) = \frac{z(z+1)}{(z^2 - 1.4z + 0.48)(z-1)}$$

and employing the final value theorem

$$y(\infty) = \lim_{z \rightarrow 1} \left[\frac{z-1}{z} Y(z) \right]$$

→ Shifting property:

Left shifting: When $f(nT)$ is delayed k sampling instants, the function $f(nT - kT)$ shown in Fig.(1-b) results. The value of $f(nT - kT)$ when $n = k$ is $f(0)$, the value when $n = k + 1$ is $f(T)$, etc. The z transform of $f(nT - kT)$ is

$$\begin{aligned} Z[f(nT - kT)] &= \sum_{n=0}^{\infty} f(nT - kT) z^{-n} \\ &= \underbrace{f(-kT)}_{n=0} + \underbrace{f(1-kT)}_{n=1} + \dots + \underbrace{f(0)}_{n=k} z^{-k} + \underbrace{f(1)}_{n=k+1} z^{-(k+1)} + \underbrace{f(2)}_{n=k+2} z^{-(k+2)} + \dots \\ &= f(0)z^{-k} + f(1)z^{-(k+1)} + f(2)z^{-(k+2)} + \dots \\ &= z^{-k} (f(0) + f(1)z^{-1} + f(2)z^{-2} + \dots) \\ &= z^{-k} F(z) \end{aligned}$$

∴

Right shifting: When the function $f(nT)$ of Fig.(1-a) is shifted one sampling period to the left, the function $f(nT + T)$ shown in Fig.(1-c) results. The value of $f(nT + T)$ when $n = 0$ is $f(T)$, the value when $n = 1$ is $f(2T)$, etc. The z transform of $f(nT + T)$ is

$$Z[f(nT + T)] = \sum_{n=0}^{\infty} f(nT + T) z^{-n} = f(T) + f(2T)z^{-1} + f(3T)z^{-2} + \dots$$

Multiplying through both sides by z^{-1} and adding $f(0)$ to both sides gives

$$z^{-1} Z[f(nT + T)] + f(0) = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \dots = F(z)$$

Thus,

$$Z[f(nT + T)] = z F(z) - z f(0)$$

Similarly, it follows that

$$Z[f(nT + 2T)] = z^2 F(z) - z^2 f(0) - z f(1)$$

In general,

➔ Multiplication by (nT):

The z transform of $nT f(nT)$ is

Proof: To verify this theorem, note that

$$\begin{aligned} -T z \frac{d}{dz} F(z) &= -T z \frac{d}{dz} [f(0) + f(T) z^{-1} + f(2T) z^{-2} + \dots] \\ &= T [f(T) z^{-1} + 2f(2T) z^{-2} + 3f(3T) z^{-3} + \dots] \\ &= T [f(T) z^{-1} + 2f(2T) z^{-2} + 3f(3T) z^{-3} + \dots] \\ &= \sum_{n=0}^{\infty} nT f(nT) z^{-nT} \\ &= Z[nT f(nT)] \end{aligned}$$

➔ Multiplication by a^n :

The z transform of $a^n f(nT)$ is

$$Z[a^n f(nT)] = F\left(\frac{z}{a}\right)$$

Proof: This theorem is readily proved by placing z by z/a in F(z)

$$F(z/a) = f(0) + \frac{a f(T)}{z} + \frac{a f(2T)}{z^2} + \dots$$

$$F(z/a) = f(0) + f(T) \left(\frac{z}{a}\right)^{-1} + f(2T) \left(\frac{z}{a}\right)^{-2} + \dots = \sum_{n=0}^{\infty} f(nT) \left(\frac{z}{a}\right)^{-n}$$

The right side is the z transform of $a^n f(nT)$.

Ex: The z transform of a unit step function,

$$f(nT) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

is given by $F(z) = \frac{z}{z-1}$, then

$$Z[a^n] = F(z/a) = \frac{(z/a)}{(z/a)-1} = \frac{z}{z-a}$$

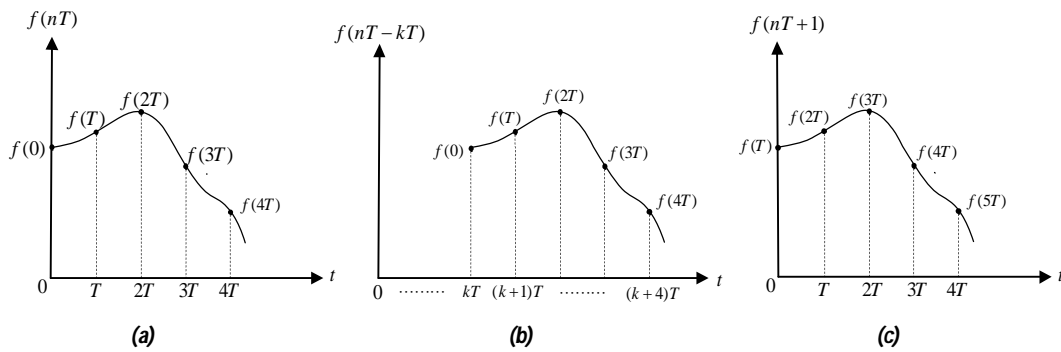


Figure (1) Translation of a discrete function $f(nT)$

A listing of z transform theorems and properties is given in Table(2)

Table (2) Properties of the z transforms

$f(nT)$	$Z [f(nT)]$
$a f(nT)$	$a F(z)$
$f_1(nT) + f_2(nT)$	$F_1(z) + F_2(z)$
$f(kT - nT)$	$z^{-n} F(z)$
$f(kT + T)$	$z F(z) - z f(0)$
$f(kT + 2T)$	$z^2 F(z) - z^2 f(0) - z f(T)$
$f(kT + nT)$	$z^n F(z) - z^n f(0) - z^{n-1} f(T) - \dots - z f[(n-1)T]$
$nT f(nT)$	$-zT \frac{d}{dz}(F(z))$
$e^{-a(nT)} f(nT)$	$F(z e^{aT})$
$a^{(nT)} f(nT)$	$F(z/a)$
$\frac{\partial}{\partial a} f(nT, a)$	$\frac{\partial}{\partial z} F(z/a)$

Inverse z transform

Inspection of Table (1) shows that z transform tend to be more complicated than corresponding Laplace transforms. Fortunately, there are some relatively simple techniques for obtaining inverse z transforms.

Partial-Fraction Method:

In this method, obtaining $x(nT)$ is based on the partial fraction expansion of $X(z)/z$ and the identification of each of the terms by the use of a table of z transforms. Note that the reason we expansion we $X(z)/z$ into partial fractions is that the functions of z appearing in tables of z transforms usually have the factor z in their numerators.

Consider $X(z)$ given by

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \dots + b_{m-1} z + b_m}{a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n} \quad (m \leq n)$$

- Factor the denominator polynomial of $X(z)$ and find the poles.
- Expand $X(z)/z$ into partial fractions so that each of the terms is easily recognizable in a table of z transforms.

Ex: Find $x(nT)$ if $X(z)$ is given by

$$X(z) = \frac{10z}{(z-1)(z-2)}$$

we first expand $X(z)/z$ into partial fractions as follows

Then we obtain

$$X(z) = \frac{-10z}{z-1} + \frac{10z}{z-2}$$

From table (1), one can obtain

$$Z^{-1} \left[\frac{z}{z-1} \right] = 1, \quad Z^{-1} \left[\frac{z}{z-2} \right] = 2^n$$

Hence

$$x(nT) = 10(-1 + 2^n) \quad n = 0, 1, 2, \dots$$

Ex: Find $f(nT)$ if $F(z)$ is given by

$$F(z) = \frac{(1 - e^{-T})z}{(z-1)(z - e^{-T})}$$

Performing a partial fraction expansion of $F(z)/z$ gives

$$\frac{F(z)}{z} = \left[\frac{(1 - e^{-T})}{(z-1)(z - e^{-T})} \right] = \left[\frac{K_1}{(z-1)} + \frac{K_2}{(z - e^{-T})} \right]$$

or

$$F(z) = \left[\frac{z}{(z-1)} - \frac{z}{(z - e^{-T})} \right]$$

From Table (1), the corresponding discrete time function

$$f(nT) = 1 - e^{-nT} \quad n = 0, 1, 2, \dots$$

→ Residue Method:

The third method of finding the inverse z transform is to use the inversion integral. Note that

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \dots$$

By multiplying both sides of this last equation by z^{n-1} , we obtain

$$F(z)z^{n-1} = f(0)z^{n-1} + f(T)z^{n-2} + f(2T)z^{n-3} + \dots \quad (*)$$

If $s = \sigma + j\omega$ is substituted in this last equation, we obtain $z = e^{(\sigma + j\omega)T}$, or

$$|z| = e^{\sigma T}, \quad z \neq \omega T$$

If the poles of $\mathbf{L}[x]$ lie to the left of the line $s = \sigma_1$ in the s plane, the poles of $\mathbf{Z}[x]$ will lie inside the circle with its center at the origin and radius equal to $e^{\sigma_1 T}$ in the z plane.

Suppose we integrate both sides of Eq.(*) along this circle in the counterclockwise direction:

$$\oint F(z)z^{n-1} dz = \oint f(0)z^{n-1} dz + \oint f(T)z^{n-2} dz + \dots + \oint f(T)z^{-1} dz + \dots$$

Applying Cauchy's theorem, we see that all terms on the right-hand side of this last equation are zero except one term

$$\oint f(T)z^{-1} dz$$

Hence

$$\oint F(z)z^{n-1} dz = \oint f(T)z^{-1} dz$$

from which we obtain the inversion integral for the z transform

$$x(nT) = \frac{1}{2\pi j} \oint X(z) z^{k-1} dz$$

which is equivalent to stating that

$$x(nT) = \sum [\text{residues of } X(z) z^{n-1} \text{ at poles of } X(z)]$$

In particular, the residue due to a first order pole at $z = r$ is

$$R = \lim_{z \rightarrow r} (z - r) [F(z) z^{n-1}]$$

Similarly, the residue due to a repeated pole of order q is

$$R = \frac{1}{(q-1)!} \lim_{z \rightarrow r} \frac{d^{q-1}}{dz^{q-1}} [(z-r)^q F(z) z^{n-1}]$$

Ex: Using residue method, find $f(nT)$ if $F(z)$ is given by

$$F(z) = \frac{(1 - e^{-T})z}{(z-1)(z - e^{-T})}$$

Application of the residue method to determine the inverse of the above equation

Adding these residues gives

$$f(nT) = 1 - e^{-nT} \quad n = 0, 1, 2, 3, \dots$$

Ex: Determine the inverse z transform for the function

$$F(z) = \frac{Tz}{(z-1)^2}$$

This function has a second-order pole at $z = 1$; thus

For $f(nT) = nT$, the corresponding time function is $f(t) = t$.

1-Structure Of Control Systems

The first approach for introducing a digital computer or a microprocessor into a control loop is indicated in Figure 1. The cascade: ADC -computer-DAC should behave in the same way as an analog controller (PID type), which implies the use of a high sampling frequency but the algorithm implemented on the computer is very simple (we just do not make use of the potentialities of the digital computer!).

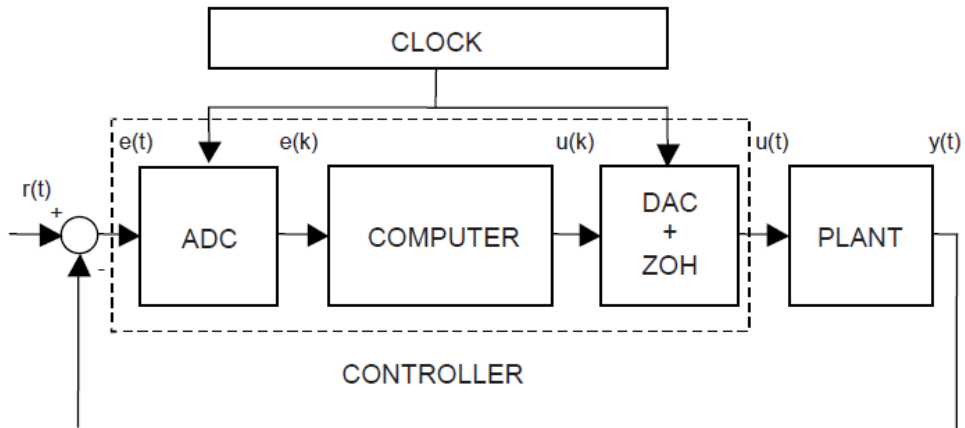


Figure 1. Digital realization of an « analog » type controller

A second and much more interesting approach for the introduction of a digital computer or microprocessor in a control loop is illustrated in Fig. 2

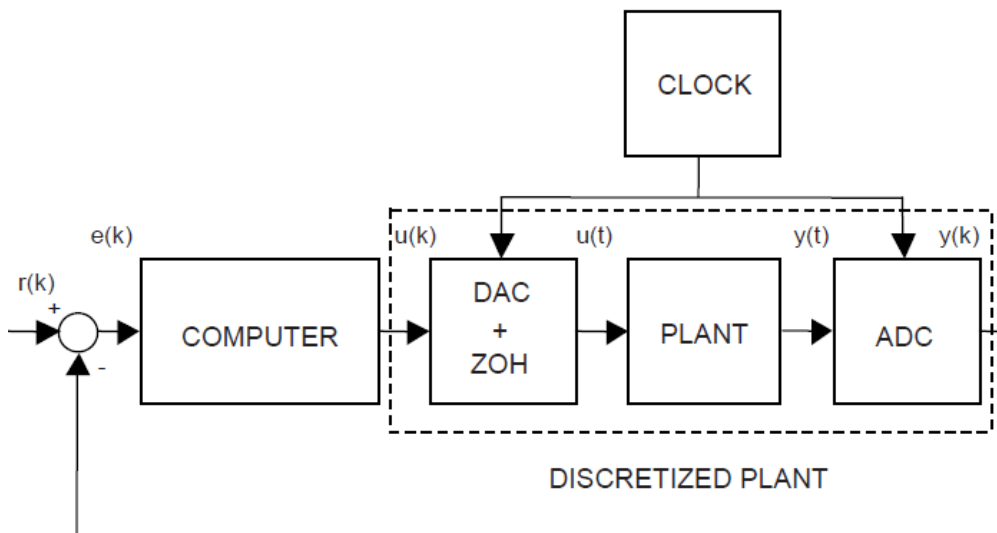


Figure 2. Digital control system

This discretized system is characterized by a “discrete-time model”, which describes the relation between the sequence of numbers $\{u(k)\}$ and the sequence of numbers $\{y(k)\}$. This model is related to the continuous-time

model of the plant. This approach offers several advantages. Among these advantages here we recall the following:

1. The sampling frequency is chosen in accordance with the “bandwidth” of the continuous-time system (it will be much lower than for the first approach).
2. Possibility of a direct design of the control algorithms tailored to the discretized plant models.
3. Efficient use of the computer since the increase of the sampling period permits the computation power to be used in order to implement algorithms which are more preferment but more complex than a PID controller, and which require a longer computation time.

2-ADC, the DAC and the ZOH

As indicated in Figure 3:

The **analog-to-digital converter** (ADC) implements two functions:

1. Analog signal sampling: this operation consists in the replacement of the continuous signal with a sequence of values equally spaced in the time domain (the temporal distance between two values is the sampling period), as these values correspond to the continuous signal amplitude at sampling instants.

2. Quantization: this is the operation by means of which the amplitude of a signal is represented with a discrete set of different values (quantized values of the signal), generally coded with a binary sequence.

The **digital-analog converter** (DAC) converts at the sampling instants a discrete signal, digitally coded, in a continuous signal. The zero-order hold (ZOH) keeps constant this continuous signal between two sampling instants (sampling period), in order to provide a continuous-time signal.

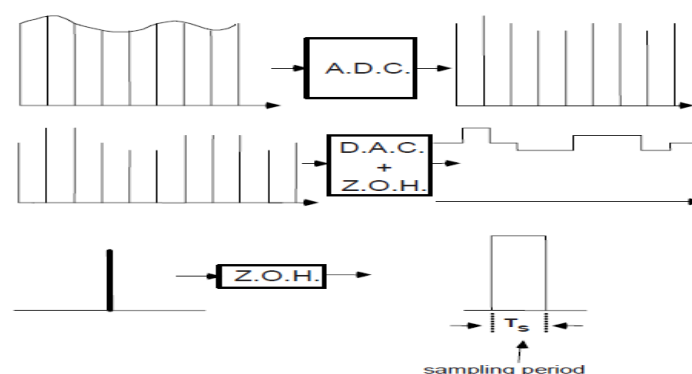
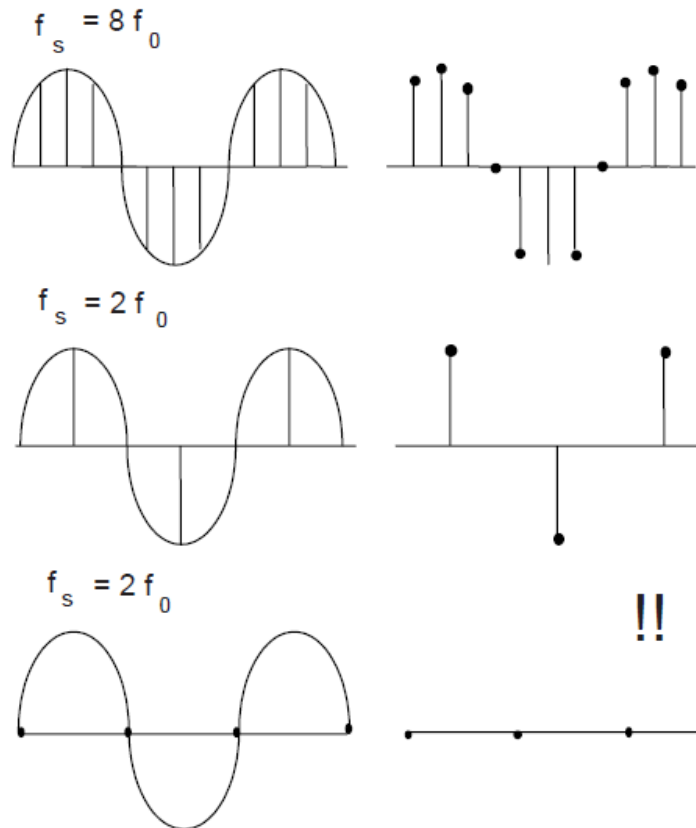


Figure 3. Operation of the ADC, DAC and the zero-order hold (ZOH)

3- Discretization and Overview of Sampled-data Systems:

It can be noted that, for a sampling frequency $f_s = 8f_0$, the continuous nature of the analog signal is unaltered in the sampled signal.



For the sampling frequency $f_s = 2f_0$, if the sampling is carried out at instants $2\pi f_0 t$ other than multiples of π , a periodic sampled signal is still obtained. However if the sampling is carried out at the instants where $2\pi f_0 t = n\pi$, the corresponding sampled sequence is identically zero.

In order to reconstruct a continuous signal from the sampled sequence, the sampling frequency must verify the condition (Nyquist's theorem):

$$f_s > 2 f_{\max} \text{ -----1}$$

Figure 5 illustrate the spectrum of the sampled signal. The upper part represents the case that the sampling frequency (f_s) is greater than twice the maximum frequency of the continuous signal f_{\max} . The spectrum of the sampled signal, if $f_s < 2f_{\max}$, is represented in the lower part of Figure .5.

The phenomenon of overlapping (aliasing) can be observed. This corresponds to the appearance of distortions. The frequency $(1/2)f_s$, which defines the maximum frequency (f_{\max}) admitted for a sampling with no distortions, is known as “Nyquist frequency” (or Shannon frequency).

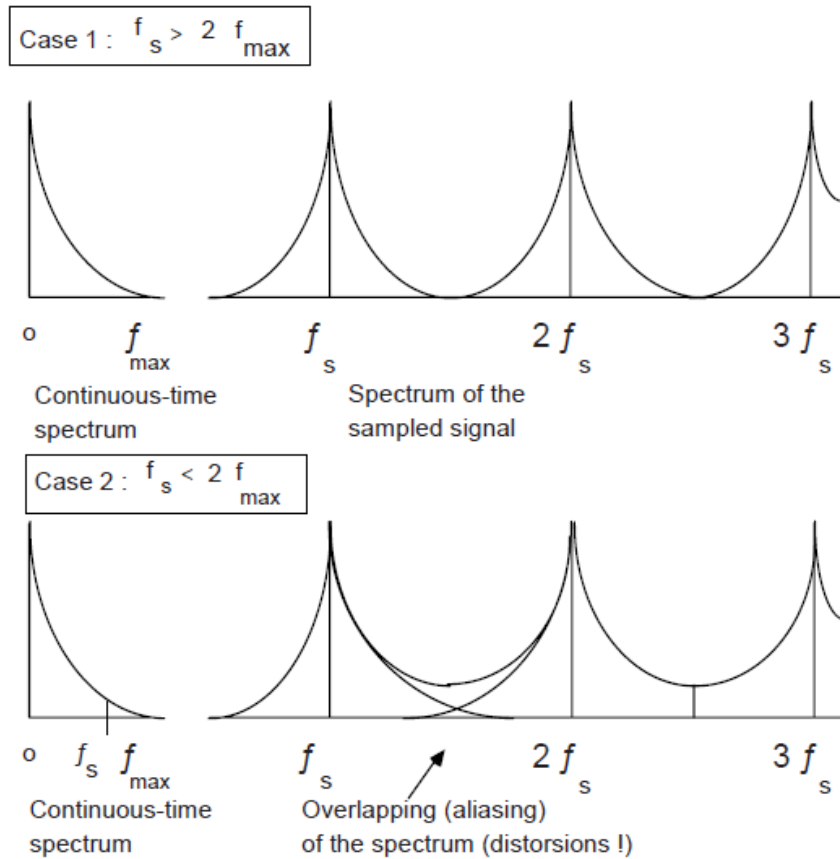


Figure5. Spectrum of a sampled signal

For a given sampling frequency, in order to avoid the folding (aliasing) of the spectrum and thus of the distortions, the analog signals must be filtered prior to sampling to ensure that:

$$f_s > 2 f_{max}$$

The filters used are known as “anti-aliasing filters”. A good anti-aliasing filter must have a minimum of two cascaded second-order cells ($f_{max} \ll (1/2) f_s$). An example of an anti-aliasing filter of this type is given in Figure 2.6. These filters must introduce a large attenuation at frequencies higher than $(1/2) f_s$, but their bandwidth must be higher than the required bandwidth of the closed loop system (generally higher than open loop system bandwidth).

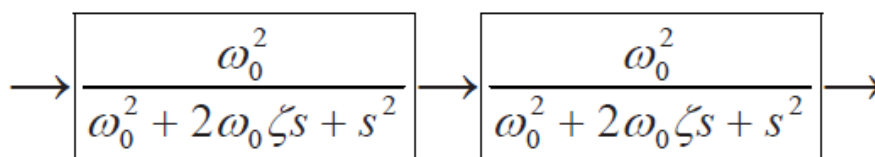
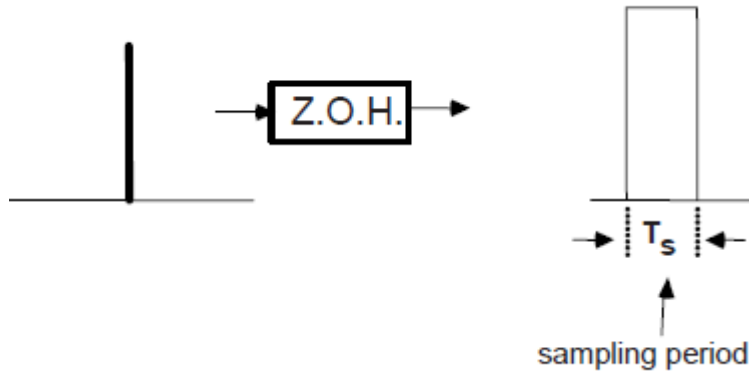


Figure 6. Anti-aliasing filter

1- The ZOH Equivalent of the plant transfer function: The ZOH transfer function



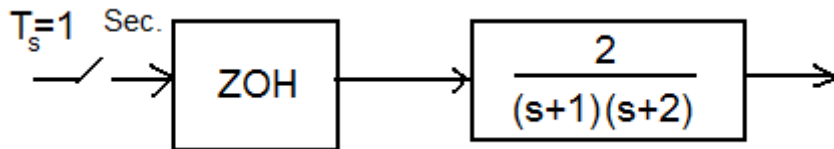
$g_{ZOH}(t) = u(t) - u(t - T_s)$ taking the Laplace transform;

$$G_{ZOH}(s) = \frac{1}{s} - \frac{e^{-sT_s}}{s} = \frac{1 - e^{-sT_s}}{s}$$

$$\therefore G_{ZOH}(z) = Z\left[\frac{1 - e^{-sT_s}}{s} G(s)\right] = (1 - z^{-1})Z\left[\frac{G(s)}{s}\right]$$

Example-1:

Find the ZOH equivalent for the following:



$$G(z) = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\} \text{ ----- (1)}$$

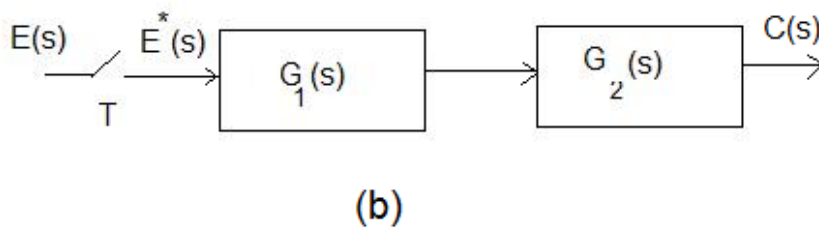
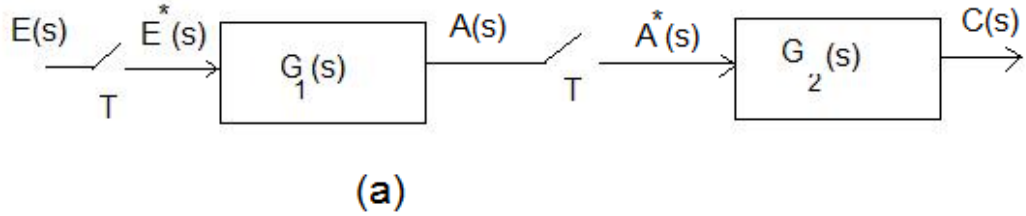
$$G(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2}$$

$$\text{So } Z\left\{\frac{G(s)}{s}\right\} = \frac{1}{1 - z^{-1}} + \frac{2}{1 - e^{-T_s} z^{-1}} + \frac{1}{1 - e^{-2T_s} z^{-1}}$$

$$\therefore G(z) = (1 - z^{-1}) \left[\frac{1}{1 - z^{-1}} + \frac{2}{1 - e^{-T_s} z^{-1}} + \frac{1}{1 - e^{-2T_s} z^{-1}} \right]$$

2- Pulse transfer function in Cascade:

Consider the two different cases below:



For **case (a)**

$$C(s) = G_2(s)A^*(s)$$

Thus after sampling $C(z) = G_2(z)A(z)$

Also $A(s) = G_1(s)E^*(s)$

So after sampling $A(z) = G_1(z)E(z)$

Therefore, $C(z) = G_1(z)G_2(z)E(z)$

i.e the total transfer function is the product of the pulse transfer functions.

For **case (b):**

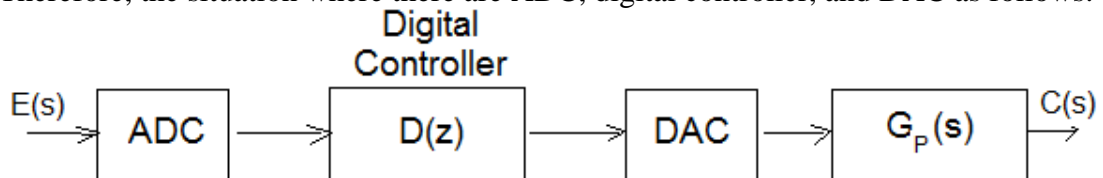
$$C(s) = G_1(s)G_2(s)E^*(s)$$

Thus after sampling $C(z) = G_1G_2(z)E(z)$

Where $G_1G_2(z) = Z\{G_1(s)G_2(s)\}$

Remember that $G_1G_2(z) \neq G_1(z)G_2(z)$

Therefore, the situation where there are ADC, digital controller, and DAC as follows:



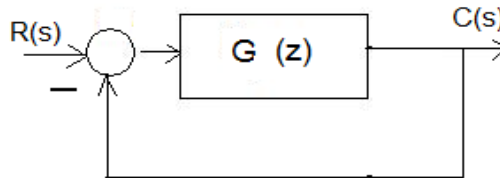
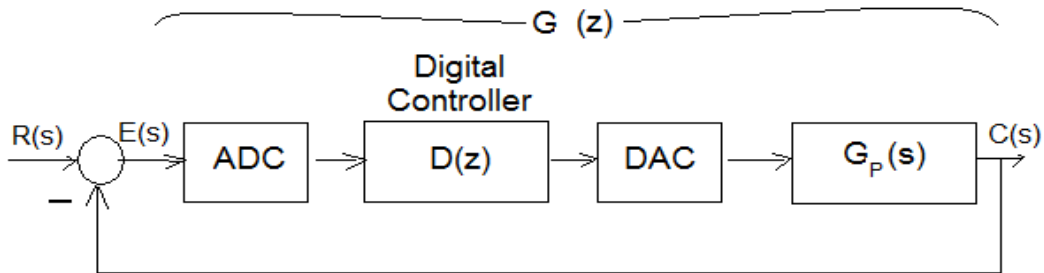
In this case

$$C(z) = G_{ZOH}(z)D(z)E(z)$$

3- Pulse transfer function in closed loop:

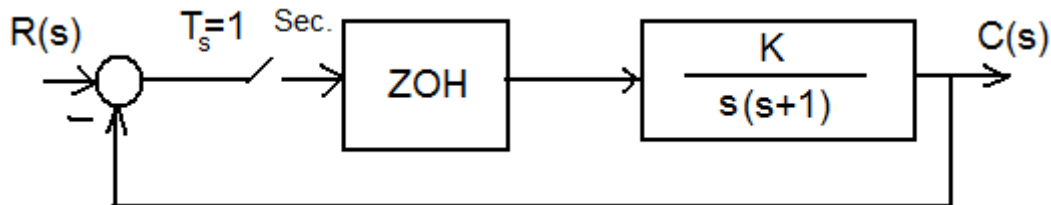
These two discrete-time systems are equivalent. They have the following closed-loop transfer function as:

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$



Example-2:

Find the closed-loop transfer function of the following system:



$$G(z) = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z}Z\left\{\frac{K}{s^2(s+1)}\right\} \quad \text{for } T_s = 1$$

$$G(z) = \frac{z-1}{z}Z\left\{\frac{K}{s^2(s+1)}\right\} = \frac{z-1}{z}Z\left\{\frac{K}{s^2} - \frac{K}{s} + \frac{K}{(s+1)}\right\}$$

$$G(z) = \frac{z-1}{z}\left\{\frac{T_s K z}{(z-1)^2} - \frac{K z}{(z-1)} + \frac{K z}{z - e^{-T_s}}\right\}$$

$$G(z) = \frac{K(0.368z + 0.264)}{z^2 - 1.368z + 0.368}$$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{K(0.368z + 0.264)}{z^2 + (0.368K - 1.368)z + (0.368 + 0.264K)}$$

Home Work:

Find the sequences $c(t)$ if $R(z)$ is a unit step (Assume $K=1$). Plot $c(t)$.

Because

then

$$c(0)=0, \quad c(T)=0.158, \quad c(2T)=0.349, \quad \text{and} \quad c(3T)=0.522$$

A plot of the response $c(nT)$ at the sampling instants is shown in Fig.(2). The long division method becomes quite cumbersome for computing $c(nT)$ for larger values of n . A more convenient procedure results from expressing the solution in the form of a difference equation.

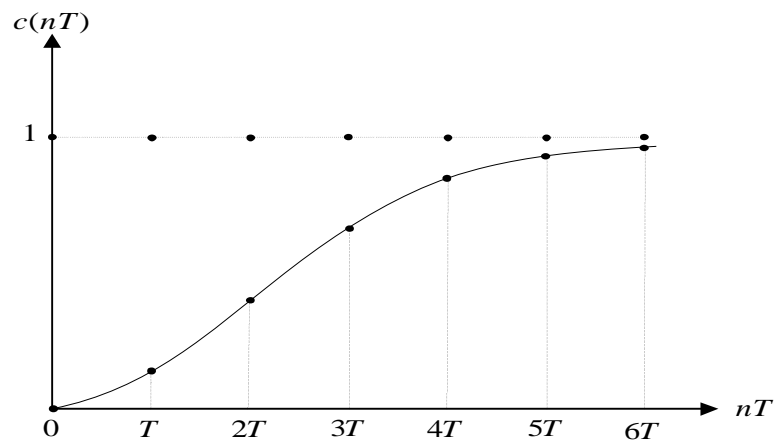


Figure (2) Sampled data system

→ Difference Equations:

To determine the inverse z transform by this method, one can write the equation for $C(z)$ in the form

$$C(z) = \frac{0.158z}{[z^2 - 1.21z + 0.368]} R(z)$$

Thus

$$C(z) - 1.21z^{-1}C(z) + 0.368z^{-2}C(z) = 0.158z^{-1}R(z)$$

Application of right shifting property

$$Z[f(nT - kT)] = z^{-k}F(z)$$

Then the preceding expression yields directly the difference equation

$$c(nT) = 1.21 c(nT - T) - 0.368 c(nT - 2T) + 0.158 r(nT - T)$$

This difference equation gives the value $c(nT)$ at the n th sampling instants in terms of values at the preceding sampling instants. Application of this result to obtain the values at the sampling instants gives

$$\begin{aligned}c(0) &= 0, \\c(T) &= 0.158 \quad r(0) = 0.158 \\c(2T) &= 1.21c(T) + 0.158 \quad r(T) = 0.349 \\c(3T) &= 1.21 c(2T) - 0.368 c(T) + 0.158 \quad r(2T) = 0.522\end{aligned}$$

Such recurrence relationships lend themselves very well to solution by a digital computer.

→ Partial-fraction expansion:

The response $c(nT)$ at the sampling instants may be also be obtained by performing a partial fraction expansion and then inverting. Thus

$$C(z) = z \left[\frac{0.158z}{(z-1)(z-0.61)^2} \right] = z \left[\frac{A}{(z-1)} + \frac{B_1}{(z-0.61)^2} + \frac{B_2}{(z-0.61)} \right]$$

The partial-fraction expansion constants are $A = 1$, $B_1 = -0.24$, and $B_2 = -1.0$.

Thus, $C(z)$ becomes

$$C(z) = \frac{z}{(z-1)} - 0.39 \frac{0.61z}{(z-0.61)^2} - \frac{z}{(z-0.61)}$$

By noting that

$$Z^{-1} \left[\frac{z}{z-1} \right] = 1, \quad Z^{-1} \left[\frac{z}{z-a} \right] = a^{nT}, \quad \text{and} \quad Z^{-1} \left[\frac{az}{(z-a)} \right] = nT a^{nT}$$

The inverse is found to be

With this method, the value $c(nT)$ at any sampling instants may be calculated directly without the need to compute the value at all the preceding instants.

Mapping of s-plane to z-plane

It is possible to map from the s plane to the z plane using the relationship

$$z = e^{sT}$$

Now

$$s = a \pm jb$$

Therefore,

$$z = e^{sT} = e^{(a \pm jb)T} = e^{aT} e^{\pm jbT} = r e^{j\theta} \quad (1)$$

where $r = |z| = e^{aT}$ and $\theta = bT$.

Since $\omega_s = 2\pi f_s = \frac{2\pi}{T}$, then $\theta = bT = \left(\frac{2\pi}{T}\right)T = 2\pi$, where ω_s is the switching frequency in rad/sec and f_s is the switching frequency in Hz. Equation (1) results in a polar diagram in the z plane as shown in Fig.(1).

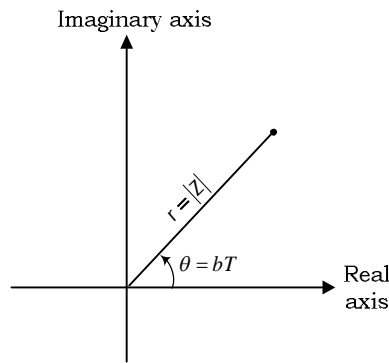


Figure (1) Mapping from the s to the z plane

Two horizontal lines of constant b are shown in the s-plane of Fig.(2.a). The corresponding paths in the z-plane are radial straight lines.

$$z = e^{sT} = e^{(a \pm jb)T} = e^{aT} e^{\pm jbT}$$

The angle of inclination of these radial lines is $\theta = \pm bT$.

Two vertical lines of constant a (i.e. constant settling time) are shown in Fig.(2.b). The corresponding paths in the z-plane are circles of radius e^{aT} . For negative values of a the circles are inside the unit circle of the z-plane. For positive values of a the circles lie outside the unit circle of the z-plane. Thus, one can conclude that the left-hand side (stable) of the s plane corresponds to a region within a circle of unity radius (the unit circle) in the z plane.

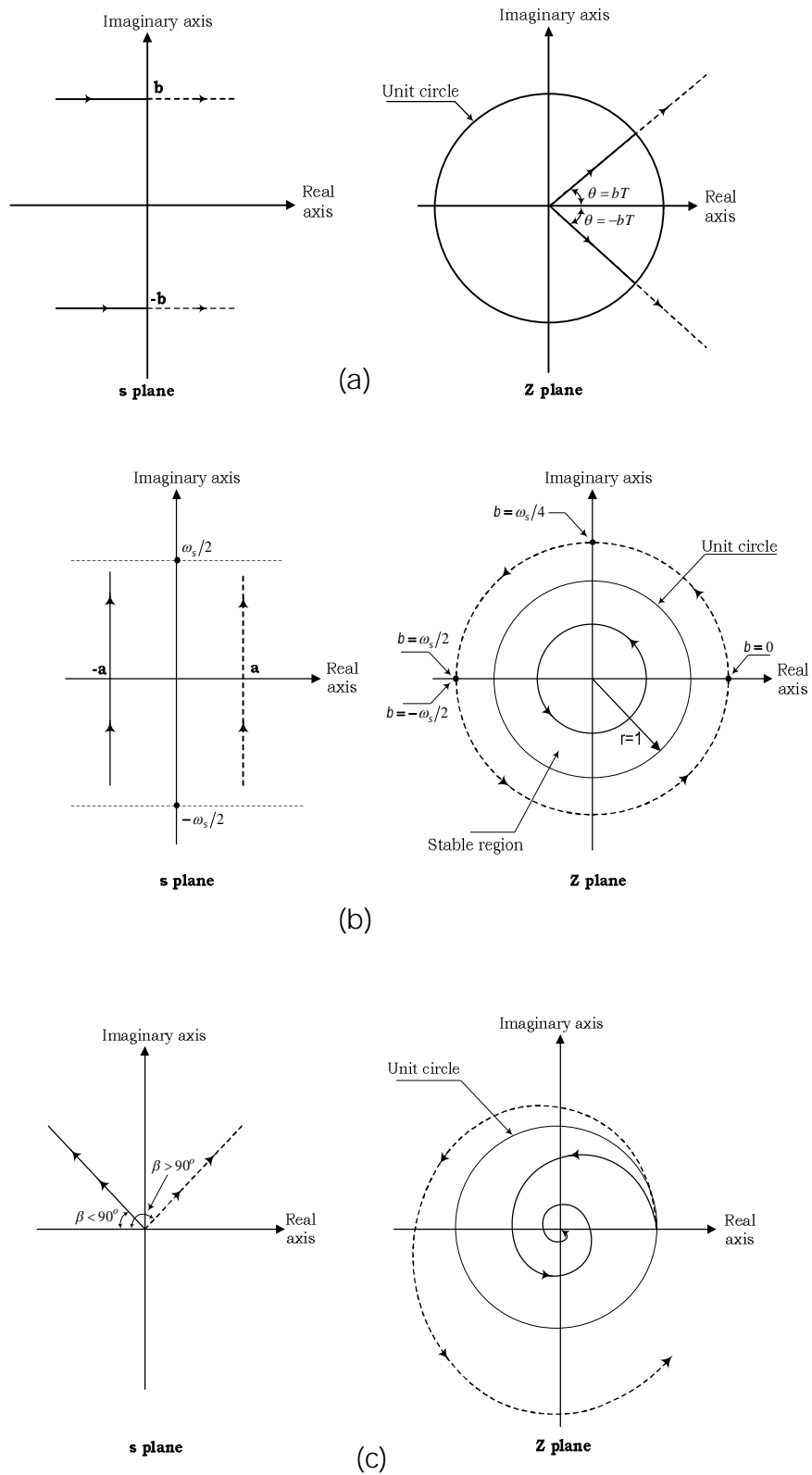


Figure (2) Corresponding paths in the *s* plane and *z* plane

Radial lines of constant damping ratio $\zeta = \cos \beta$ are shown in Fig.(1.c).

In polar coordinates, $s = a \pm jb = -\zeta \omega_n \pm j \omega_n \sqrt{1-\zeta^2}$. Thus,

$$z = e^{-\zeta \omega_n T} e^{\pm j \omega_n T \sqrt{1-\zeta^2}}$$

The corresponding paths in the z-plane are logarithmic spirals. For $\beta < 90^\circ$ the spirals decay within the unit circle, and for $\beta > 90^\circ$ the spirals grow outside the unit circle.

Consider now how a given point, $z = r e^{j\theta}$, in the z plane maps back into the s plane. For

$$z = r e^{j\theta} = e^{sT} = e^{(a \pm jb)T}$$

Equating real and imaginary parts shows that

$$\ln(r) = aT$$

$$\theta = \pm bT$$

This verifies the fact that a circle of constant r in the z plane is a vertical line of constant a in the s plane. Similarly, a ray at angle θ in the z plane is a horizontal line of constant b in the s plane.

Ex: Find the corresponding locations of points in the s-plane into z plane

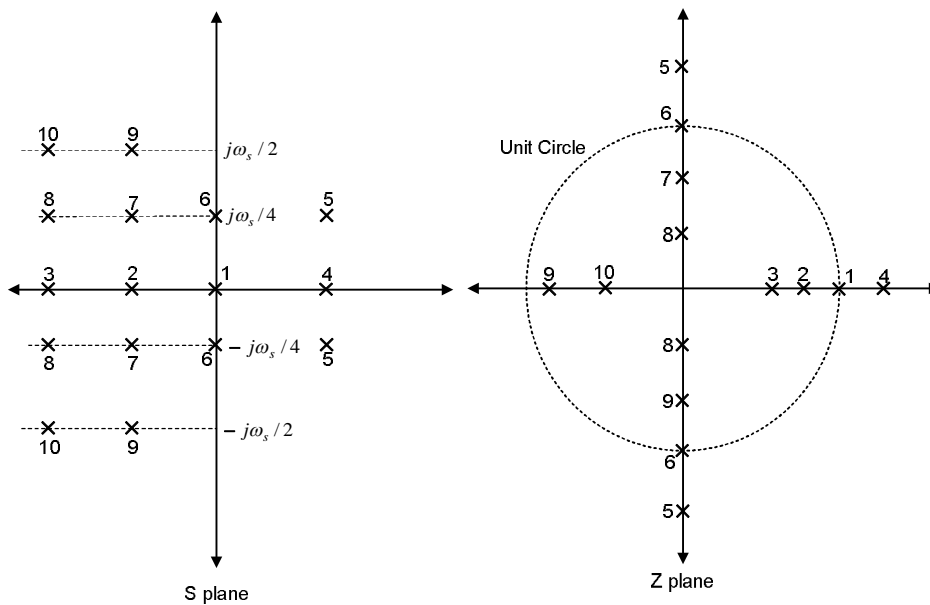


Figure (3) Corresponding pole locations between the s plane and the z plane

Since $\Rightarrow r = e^{aT}$ and

Points 3, 2, 1 and 4:

$$\begin{aligned} b_1, b_2, b_3 \text{ and } b_4 = 0 &\Rightarrow \theta = bT \Rightarrow \theta_{1,2,3,4} = 0 \\ a_1 = 0 &\Rightarrow r_1 = e^{0T} = 1, \\ &\text{and} \Rightarrow r_2(e^{a_2T}) > r_3(e^{a_3T}), \\ a_4 > 0 &\Rightarrow \end{aligned}$$

Upper points 10 and 9:

$$\begin{aligned} b_9 \text{ and } b_{10} = \frac{\omega_s}{2} = \frac{2\pi/T}{2} = \frac{\pi}{T} &\Rightarrow \theta_9 = b_9T = \pi \text{ and } \theta_{10} = b_{10}T = \pi \\ &\text{and } a_9 > a_{10} \Rightarrow \end{aligned}$$

Upper points 10 and 9:

$$\begin{aligned} b_9 \text{ and } b_{10} = -\frac{\omega_s}{2} = -\frac{2\pi/T}{2} = -\frac{\pi}{T} &\Rightarrow \theta_9 = b_9T = -\pi \text{ and } \theta_{10} = b_{10}T = -\pi \\ &\text{and } a_{10} < a_9 \Rightarrow \end{aligned}$$

Therefore, the lower and upper points 9 and 10 coincide on each other. The same argument may be performed with the other points. This results in the corresponding points at z plane.

Ex: The time-response characteristics of the z-plane pole locations are illustrated in Fig.(4). Since $z = e^{sT}$, the response characteristics are a function of both s and T.

The poles in the s-plane occur at $s = a \pm jb$. These poles result in a system transient-response term of the form $k_1 e^{at} \cos(\omega t \pm \phi)$. When sampling occurs, these s-plane poles result in z-plane poles at

$$z = e^{sT} \Big|_{s = a \pm jb} = e^{aT} e^{\pm j b T} = r e^{\pm j \theta}$$

The roots of the characteristic equation that appear at $z = r e^{\pm j \theta}$ result in a transient response term of the form

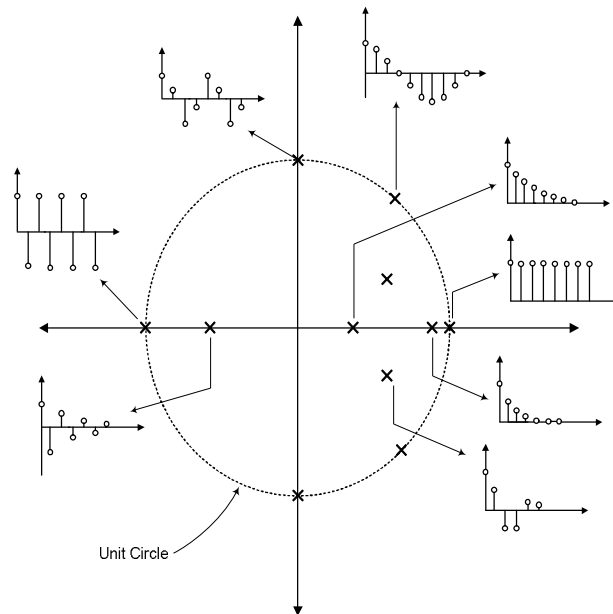


Figure (4) Transient response characteristics of the z plane pole locations.

Lines of Constant Damping Ratio ζ :

In the s-domain, the lines of constants damping ratio ζ are rays originating at the origin while the curves representing constant undamped natural frequency ω_n are quarter circles, as shown in Fig.(5).

Figure (6) shows the real and imaginary parts of the complex variables s expressed in terms of ζ and ω_n . That is

$$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

The equivalent point in the z-plane is found by applying the transformation $z = e^{sT}$ to obtain

$$z = e^{-\zeta\omega_n T} e^{j\omega_n T\sqrt{1-\zeta^2}} \tag{2}$$

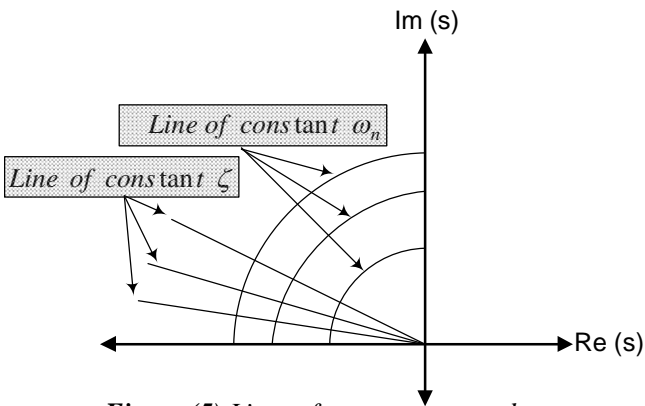


Figure (5) Lines of constant ω_n and curves of constant ζ

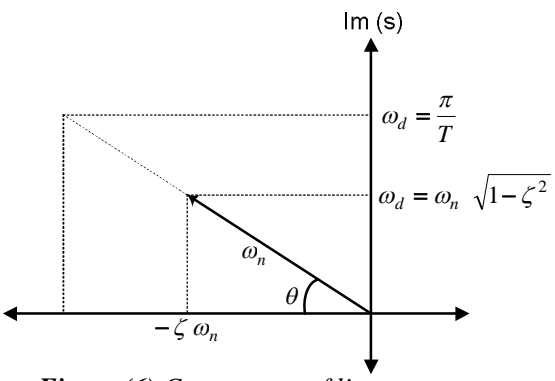


Figure (6) Components of line of constant ω_n

If in Eq.(2) we fix ζ and vary ω_n we will plot a log spiral curve, since the magnitude of z will vary exponentially with ω_n , while the phase varies linearly. As shown in Fig.(6), we only need consider the portion of the ray of constant damping ratio between the origin and the point where the ray intersects the edge of the primary strip.

For $\omega_n = 0$ a ray of constant damping ratio starts at the point

$$z = e^0 = 1$$

The other end of the array in the s plane touches the edge of the primary strip. At the point of intersection

$$\omega_n \sqrt{1 - \zeta^2} = \frac{\pi}{T}$$

or, equivalently,

$$\omega_n = \frac{\pi}{T\sqrt{1 - \zeta^2}}$$

Hence

Thus z is a vector of length

and angle 180° . Note that the larger ζ the shorter the length of the vector.

The log spiral curves connecting the end points of the curves for $\zeta = 0.1, 0.2, \dots, 0.9$ in increments of 0.1 are shown in Fig.(7).

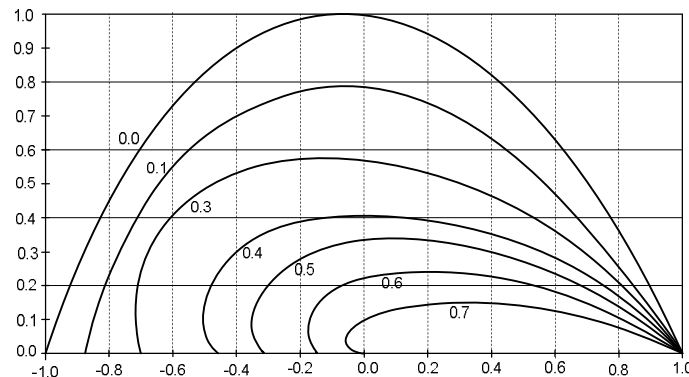


Figure (7) curves of constant ζ

Curves of constant natural frequency ω_n :

To find the curves of constant ω_n we again use the transformation $z = e^{sT}$, but this time we fix ω_n and vary ζ . It is customary to let

$$\omega_n = \frac{k\pi}{10T} \quad k = 0, 1, 2, \dots, 10$$

Then

(3)

Eq.(3) can be used to plot the curves of constant ω_n by holding ω_n constant and varying ζ between zero and one. When $\zeta = 0$, corresponding to $s = jk\pi/10$

$$z = e^0 e^{jk\pi\sqrt{1-0^2}/10}$$

In this case z , is a vector of length one and angle $k\pi/10$ rad. Thus, all the curves of constant ω_n originate on the unit circle at the angles

$$k \times \frac{180^\circ}{10} = k \times 18^\circ, \quad k = 1, 2, 3, \dots, 10.$$

At the other end of each these curves, $\zeta = 1$, and

$$z = e^{-k\pi T/10T} e^{jk\pi T\sqrt{1-1}/10T} = e^{-k\pi/10} \quad k = 1, 2, 3, \dots, 10.$$

These points lie on the positive real axis in the z plane. The smaller k , the larger $e^{-k\pi/10}$. The curves that connect these end points are shown in Fig.(8).

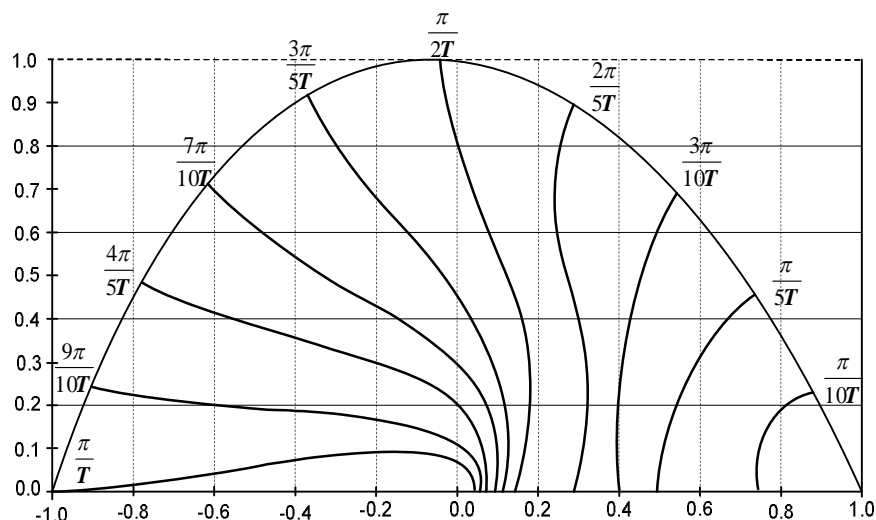


Figure (8) curves of constant ω_n

We see that the curves show increasing distortion as k increases. For $k=1$, the curve is very close to a quarter circle centered at $z=1$. The curves for $k=2$ and $k=3$ still have the general shape of a quarter circle, but for $k>3$ they do not.

Ex: Map the shaded area in Fig.(9) in the s -plane into corresponding poles in the z plane. In Fig.(8), the lines of constant ω_n are labeled

$$\frac{\pi}{T}, \frac{\pi}{5T}, \frac{3\pi}{10T}, \dots, \frac{\pi}{T}$$

indicating the value of ω_n that corresponds to each curve. As noted, the curves end at the angles

By combining the curves of constant ζ and constant ω_n we can locate points in the z plane with any desired combination of damping ratio and natural frequency.

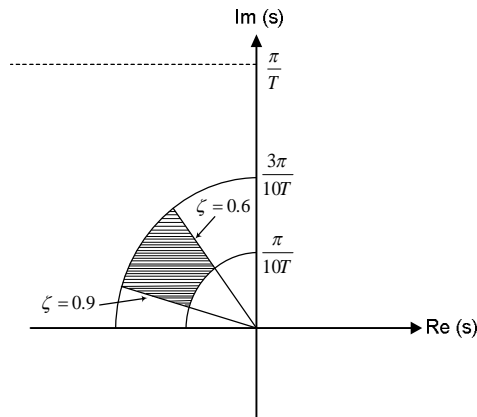


Figure (9) Desired pole locations in s -plane

In Fig.(9), poles locations in the s plane with damping ratios between 0.6 and 0.9 and natural frequencies between $\pi/10T$ and $3\pi/10T$ are in the shaded area. The corresponding poles in the z plane are shown in Fig.(10).

Notes:

- The curves of constant ζ do not depend on T

- The curves of constant ω_n depends on T, and then, on the sampling rate. For instance, for a sampling rate of 10 Hz, the poles in the shaded region will have natural frequencies between 0.5 and 1.5 Hz, or one tenth and three tenths of the maximum frequency that can be sampled without aliasing, namely, 5 Hz. On the other hand, if the sampling rate is 100 Hz poles in this same region will have natural frequencies between 5 and 15 Hz.

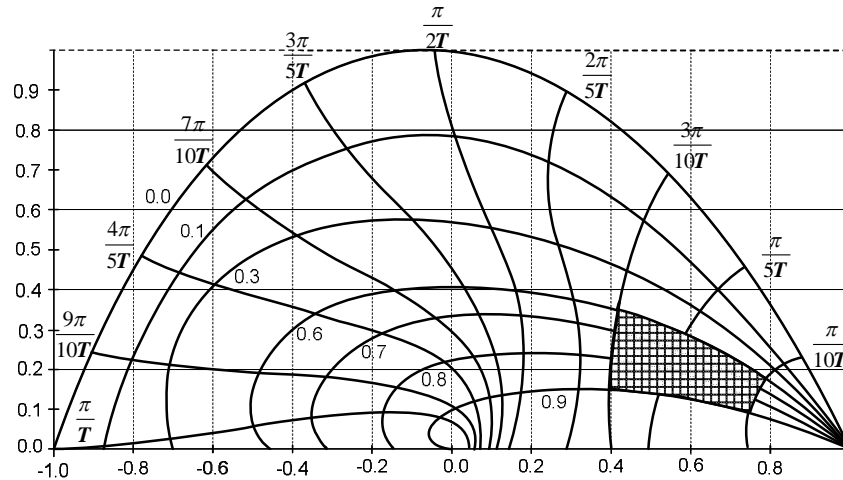


Figure (10) Desired pole locations in z-plane

The primary strip:

Suppose we map the primary strip of the s plane into the z plane. We begin by mapping the points of a vertical line

$$s = a + jb$$

where $a < 0$ is fixed. Under the mapping $z = e^{sT}$, a point on this line maps to

$$z = e^{(a+jb)T} = e^{aT} e^{jbT}$$

The term e^{aT} is a real number that can be thought of as a scaling factor for the unit phasor e^{jbT} . If $-\pi/T \leq b \leq \pi/T$, and a is fixed, with $a < 0$, then the mapping of this portion of the vertical line in the s plane to the z plane is a circle with radius $e^{-aT} < 1$ as shown in Fig.(11). If $a > 0$, the line segment maps to a circle with radius greater than one, as shown in the figure. It should be noted that

$$-\pi/T \leq b \leq \pi/T \Rightarrow -\pi \leq \theta \leq \pi$$

The area confined between $-\pi/T \leq b \leq \pi/T$ is called *the primary strip*. One can easily see from Fig.(11) that the width of the primary strip is $2\pi/T$. The other strips of the same width as that of primary strip are called the secondary strips.

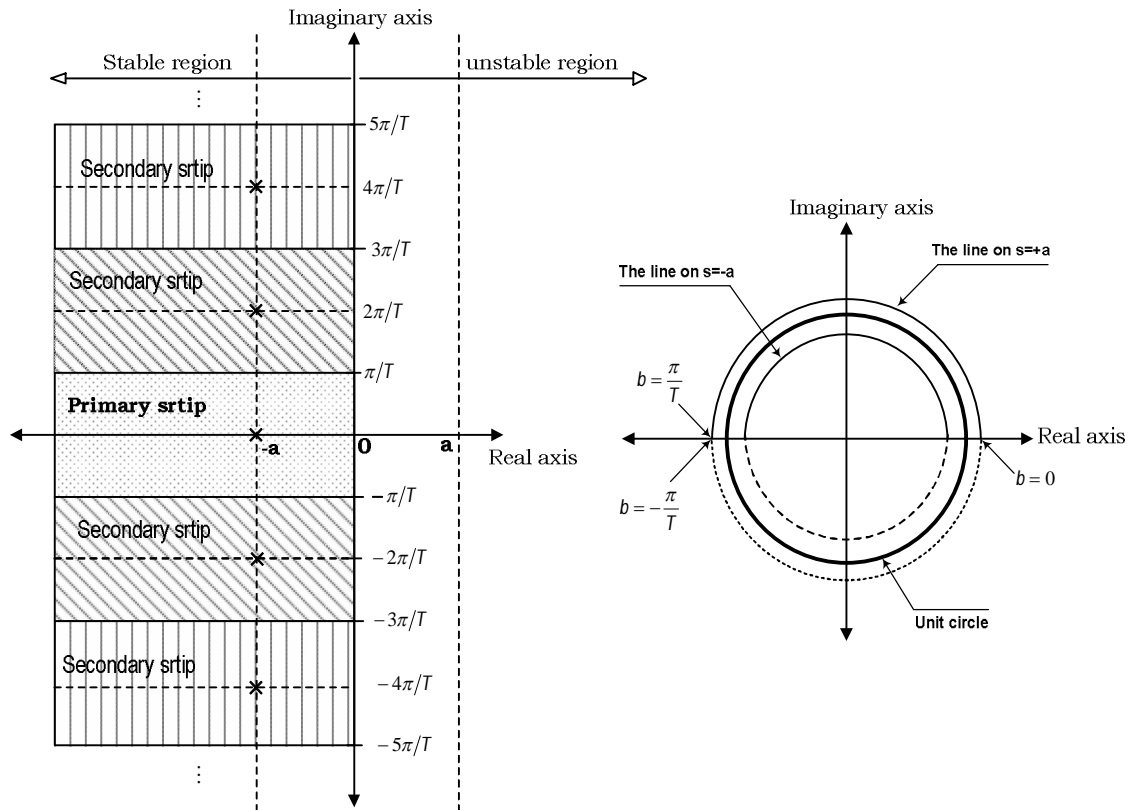


Figure (11) Mapping of the Primary strip into z-plane

Jury's Test

Jury's test is a stability test which has some advantages over the Routh's test for continuous-data system. In general, given the polynomial in z ,

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_2 z^2 + a_1 z + a_0 = 0 \quad (1)$$

where a_0, a_1, \dots, a_n are real coefficients. Assuming that a_n is positive, or that it can be made positive by changing the signs of all coefficients, the following table is made:

Row	z^0	z^1	z^2	z^{n-k}	z^{n-1}	z^n
1	a_0	a_1	a_3	a_{n-k}	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	a_k	a_1	a_0
3	b_0	b_1	b_2	b_{n-k}	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	b_k	b_0	
5	c_0	c_1	c_2		c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}		c_0		
\vdots	\vdots	\vdots	\vdots				
$2n-5$	p_0	p_1	p_2	p_3				
$2n-4$	p_3	p_2	p_1	p_0				
$2n-3$	q_0	q_1	q_2					

Note that the elements of the $(2k+2)$ th row ($k=0,1,2,\dots$) consists of the coefficients of the $(2k+1)$ th row are written in the reverse order. The elements in the table are defined as

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$

$$\dots q_0 = \begin{vmatrix} p_0 & p_3 \\ p_3 & p_0 \end{vmatrix}, \quad q_2 = \begin{vmatrix} p_0 & p_1 \\ p_3 & p_2 \end{vmatrix}$$

The necessary and sufficient conditions for the polynomial $F(z)$ to have no roots on and outside the unit circle in the z -plane are:

$$\left. \begin{array}{l}
 F(1) > 0 \\
 F(-1) = \begin{cases} > 0 & n \text{ even} \\ < 0 & n \text{ odd} \end{cases} \\
 \left. \begin{array}{l} |a_0| < |a_n| \\ |b_0| > |b_{n-1}| \\ |c_0| > |c_{n-2}| \\ |d_0| > |d_{n-3}| \\ \vdots \\ |q_0| > |q_2| \end{array} \right\} (n-1) \text{ constraints}
 \end{array} \right\} \quad (2)$$

For a second-order system, $n=2$, Jury's tabulation contains only one row. Therefore, the requirements listed in Eq.(2) are reduced to

$$F(1) > 0, \quad F(-1) < 0 \quad \text{and} \quad |a_0| < |a_n|$$

As in the Routh-Hurwitz criterion which is used for stability testing of linear continuous-data, occasionally the first element of a row or a complete row of the tabulation may be zero before the tabulation is scheduled to terminate. These cases are referred as singular cases. In Jury's tabulation a singular case is signified by either having the first and the last elements of a row be zero, or having a complete row of zeros.

The Singular Cases:

When some or all of the elements of a row in the Jury's tabulation are zero, the tabulation ends prematurely. This situation is referred to as the *singular case*. The singular case can be eliminated by expanding and contracting the unit circle infinitesimally, which is equivalent to moving the roots off the unit circle. The transformation for this purpose is

$$z = (1 + \varepsilon) z \quad (3)$$

where ε is a very small real number. When ε is a positive number in Eq.(3), the radius of the unit circle is expanded to $1 + \varepsilon$, and when ε is negative, the radius of the unit circle is reduced to $1 - \varepsilon$. This is equivalent

to moving the roots slightly. The difference between the number of roots found inside (or outside) the unit circle when the unit circle is expanded or contracted by ε is the number of roots on the circle.

The transformation in Eq.(3) is actually ver easy to apply, since

$$(1 \pm \varepsilon)^n \cong (1 \pm n\varepsilon) z^n \quad (4)$$

This means that the coefficient of z^n term is multiplied by $(1 \pm n\varepsilon)$.

Example1:

If the characteristic equation of a system is

$$F(z) = z^2 + z + 0.25 = 0$$

The first two conditions of Jury's test in Eq.(2) lead to

and

Since $n=2$ is even, these results satisfy the $F(1)>0$ and $F(-1)<0$ requirements for stability. Next, we tabulate the coefficients of $F(z)$ according to Jury's test; we have

Since $2n-3=1$, Jury's tabulation consists of only one row. The result is

and thus the system is stable, and all roots are inside the unit circle.

Example 2:

Consider the equation

$$F(z) = z^3 + 3.3z^2 + 3z + 0.8 = 0$$

which has roots at $z=-0.5$, -0.8 , and -2 .

From Jury's test, $F(1)=8.1$ and $F(-1)=0.1$. For odd n , since $F(-1)$ is not negative, $F(z)$ has at least one root outside the unit circle.

Example 3:

For the following characteristic equation

$$z^2 - z(1.48 - 0.025K) + (0.5026 + 0.0204K) = 0$$

Find the range of K for stability.

The first two conditions of Jury's test in Eq.(2) lead to

$$F(1) =$$

$$F(-1) =$$

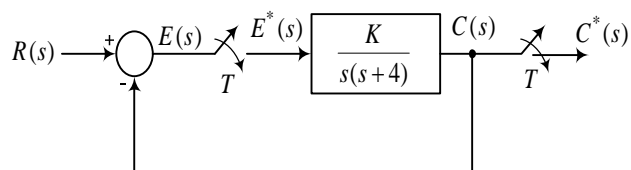
Since $n=2$ is even, these results satisfy the $F(1)>0$ and $F(-1)<0$ requirements for stability. Next, we tabulate the coefficients of $F(z)$ according to Jury's test; we have

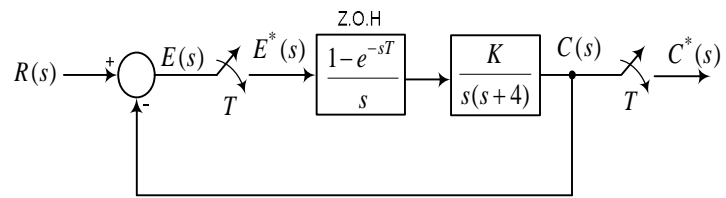
Since $2n-3=1$, Jury's tabulation consists of only one row. The result is

$$|a_0| =$$

Since $K > 0$, then the range of stability is _____ and for _____

HW: For the following block diagrams, use Routh-Hurwitz criterion and Jury's test to find the range of K for stable system. ($T=0.25$ sec)





Root Locus in the z-plane

As with the continuous systems, the root locus of a discrete system is a plot of the locus of the roots of the characteristic equation

$$1 + GH(z) = 0$$

in the z-plane as a function of the open-loop gain constant K. The closed-loop system will remain stable providing the loci remain within the unit circle.

Root Locus Construction Rules:

These are similar to those given in continuous systems.

- ❑ Starting points ($K = 0$). The root loci start at the open-loop poles.
- ❑ Termination points ($K = \infty$). The root loci terminate at the open-loop zeros when they exist, otherwise at ∞ .
- ❑ Number of distinct root loci (branches): This is equal to the order of the characteristic equations (or the number of poles of open loop transfer function).
- ❑ Symmetry of root loci: The root loci are symmetric about the real axis.
- ❑ Root locus locations on the real axis: A point on the real axis is part of the loci if the sum of the open-loop poles and zeros to the right of the point concerned is odd.
- ❑ Break away (in) points. The points at which a locus breaks away from (or break in) the real axis can be found by letting K as a function of z, taking the derivative of dK/dz and then setting the derivative equal to zero.
- ❑ Unit circle crossover: This can be obtained by determining the value of K for marginal stability using Jury test or Routh-Hurwitz criterion.

1. Root Locus without Zero Order Hold

Ex: Sketch the root locus for the diagram shown in Fig.(1)

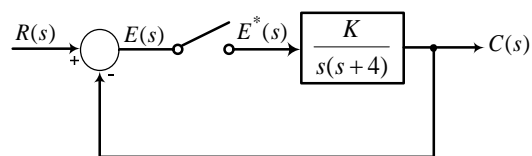


Figure (1) Sample-data system

The z-transform for the output $C(z)$ is

$$C(z) = \frac{G(z)}{1 + G(z)} R(z)$$

The z-transformed characteristic equation is

$$1 + G(z) = 0$$

The partial fraction expansion for G(s) is

$$G(s) = \frac{K}{4} \left(\frac{1}{s} - \frac{1}{s+4} \right)$$

The corresponding z transform is

$$G(z) = \frac{K}{4} \left(\frac{z}{z-1} - \frac{z}{z-e^{-4T}} \right) = \left(\frac{K}{4} \right) \frac{z(1-e^{-4T})}{(z-1)(z-e^{-4T})}$$

For T=0.25 sec.

❑ Open-loop poles and zeros:

Poles: $z = 1$ and $z = 0.368$

Zeros: $z = 0$

❑ Number of branches: Number of branches equals No. of poles=2.

❑ Root locus locations on the real axis: The root locus on the real axis lies between poles ($z = 1$ and $z = 0.368$) and to the left of zero ($z=0$).

❑ Break away and in points:

The characteristic equation is

$$1 + G(z) = 1 + K \frac{0.158 z}{(z-1)(z-0.368)} = 0$$

or

$$\text{Then } \frac{dK}{dz} = - \left(\frac{1}{0.158} \right) \frac{z(2z-1.368) - (z^2 - 1.368z + 0.368)}{(z)^2} = 0$$

or

To find the value of K at break away and in points, we use the magnitude condition:

The gain K at breakaway point:

$$K = \left| \frac{(z-1)(z-0.368)}{0.158 z} \right|_{z=0.606} = \left[\frac{|(z-1)||z-0.368|}{|0.158 z|} \right]_{z=0.606}$$

The gain K at break in point:

$$K = \left. \frac{(z-1)(z-0.368)}{0.158z} \right|_{z=-0.606}$$

□ Crossing points of z-plane imaginary axis:

In general $z = a + jb$, and when the root locus crosses the imaginary axis of the z-plane, then the real part becomes zero, or $z = jb$. Substitute this value in the characteristic equation one can obtain:

$$z^2 - 1.368z + 0.368 + 0.158Kz = 0$$

$$(jb)^2 - 1.368(jb) + 0.368 + 0.158K(jb) = 0$$

or

$$-b^2 - j1.368b + 0.368 + j0.158Kb = 0$$

$$\underbrace{(-b^2 + 0.368)}_{\text{Real}} + \underbrace{j(-1.368b + 0.158Kb)}_{\text{Imaginary}} = 0$$

Two equations will be obtained:

$$-b^2 + 0.368 = 0 \quad \text{and} \quad -1.368b + 0.158Kb = 0$$

From the first equation one can obtain the point of interception of root locus with the imaginary axis

$$-b^2 + 0.368 = 0 \Rightarrow b = \pm 0.606 \rightarrow z = \pm j0.606$$

Substitute the value of b at the second equation, the value of gain K at the imaginary axis becomes

$$-1.368b + 0.158Kb = 0 \Rightarrow -1.368 \times 0.606 + 0.158K \times 0.606 = 0 \rightarrow K = 8.658$$

Alternatively, one can use the magnitude condition to find the value of K at imaginary axis crossing points: (use either $z = j0.606$ or $z = -j0.606$)

$$K = \left. \frac{(z-1)(z-0.368)}{0.158z} \right|_{z=j0.606} = \left[\frac{|(z-1)||z-0.368|}{|0.158z|} \right]_{z=j0.606}$$

$$= \left[\frac{|(j0.606-1)||j0.606-0.368|}{|0.158j0.606|} \right]$$

- K for marginal stability: Using Routh-Hurwitz criterion (or Jury test), the value of K as the root locus crosses the unit circle into the unstable region is

$$K = 17.316$$

- Unit circle crossover: Inserting $K = 17.3$ into the characteristic equation

$$1 + G(z) = 1 + K \frac{0.158 z}{(z-1)(z-0.368)} = 0 \Rightarrow 1 + 17.316 \times \frac{0.158 z}{(z-1)(z-0.368)} = 0$$

$$\Rightarrow z^2 + 1.367z + 0.368 \Rightarrow \text{The roots are } z = \pm 1$$

- Angle of asymptotes

$$\lambda = \frac{(2n+1)180}{p-z} \quad n=0,1,2,3$$

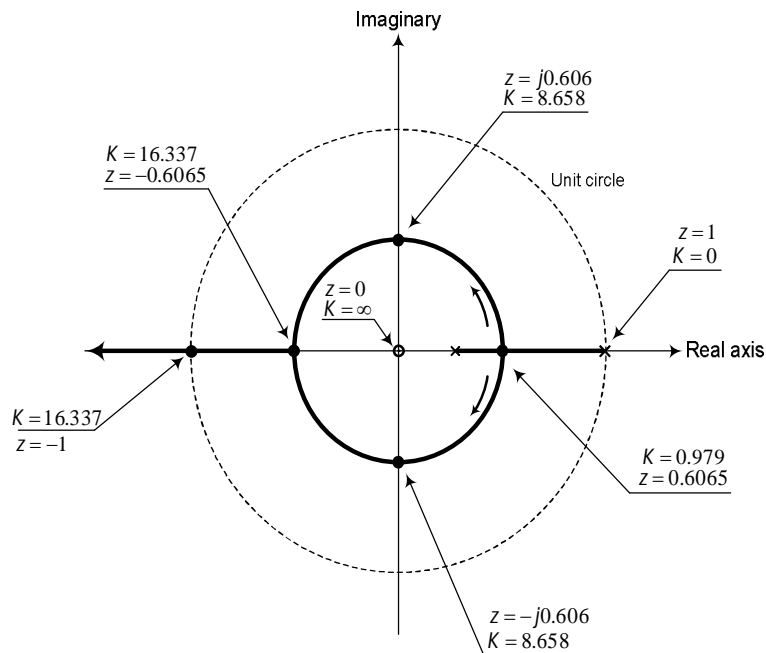
where p =number of poles and z is the number of zeros. Thus λ becomes

$$\lambda = 180$$

The real axis interception of the asymptotes is

$$\sigma_x = \frac{\sum_0^p z_p - \sum_0^z z_z}{p-z} = \frac{1+0.368-0}{2-1} = 1.368$$

The complete root-locus plot may now be constructed as shown in the following figure



Ex2: for the diagram shown in Fig.(2),

- ❑ Sketch the root locus for $T=1/4$ sec.
- ❑ Plot the response of the system to a unit step function and for $K=4$.

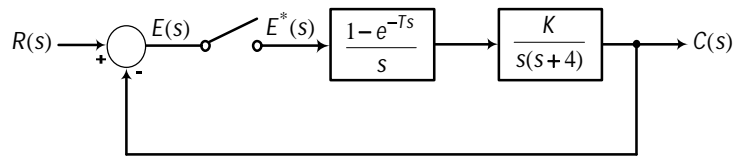


Figure (2) Sample-data system

For $T=1/4$, one can show that $G(z)$ has the form

$$G(z) = \frac{0.368K(z + 0.717)}{16(z - 1)(z - 0.368)}$$

The z-transform closed-loop transfer function

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$

The characteristic equation of the above transfer function is

$$1 + G(z) = 0$$

or

$$z^2 + (0.023K - 1.368)z + (0.368 + 0.01649K) = 0$$

- ❑ Open-loop poles and zeros:

Poles:

Zeros:

Number of branches: Number of branches equals No. of poles=2.

- ❑ Root locus locations on the real axis: The root locus on the real axis lies between poles ($z = 1$ and $z = 0.368$) and to the left of zero ($z = 0.717$).
- ❑ Break away and in points:

The characteristic equation is

$$1 + G(z) = 1 + K \frac{0.368(z + 0.717)}{16(z - 1)(z - 0.368)} = 0$$

or

Then

$$\frac{dK}{dz} = -(43.478) \frac{(z + 0.717)(2z - 1.368) - (z^2 - 1.368z + 0.368)}{(z + 0.717)^2} = 0$$

or

$$(2z + 0.066z - 0.980) - (z^2 - 1.368z + 0.368) = 0 \Rightarrow z^2 + 1.434z - 1.348 = 0$$

Then

To find the value of K at break away and in points, we use the magnitude condition:

The gain K at breakaway point:

$$\begin{aligned} K &= \left| (43.478) \frac{(z-1)(z-0.368)}{(z+0.717)} \right|_{z=0.647} = 43.478 \left[\frac{|(z-1)||z-0.368|}{|z+0.717|} \right]_{z=0.647} \\ &= 43.478 \left[\frac{|0.647-1| \times |0.647-0.368|}{|0.647+0.717|} \right] = 3.139 \end{aligned}$$

The gain K at break in point:

$$\begin{aligned} K &= 43.478 \left| \frac{(z-1)(z-0.368)}{(z+0.717)} \right|_{z=-2.081} = 43.478 \left[\frac{|(z-1)||z-0.368|}{|z+0.717|} \right]_{z=-2.081} \\ &= 43.478 \left[\frac{|-2.081-1| \times |-2.081-0.368|}{|-2.081+0.717|} \right] = 240.511 \end{aligned}$$

□ Crossing points of z-plane imaginary axis:

In general $z = a + jb$, and when the root locus crosses the imaginary axis of the z-plane, then the real part becomes zero, or $z = jb$. Substitute this value in the characteristic equation one can obtain:

$$(jb)^2 + (0.023K - 1.368)jb + (0.368 + 0.01649K) = 0$$

or

Two equations will be obtained:

$$(0.368 + 0.01649K - b^2) = 0 \quad \text{and} \quad b(0.023K - 1.368) = 0$$

From the second equation one can determine the value of gain at the point of root-locus interception with the imaginary axis

$$b(0.023K - 1.368) = 0 \Rightarrow K = 59.478$$

Substitute the value of K into the first equation, the value of z at the imaginary axis becomes

$$(0.368 + 0.01649 K - b^2) = 0 \Rightarrow$$

Then, $z = \pm j1.161$ at the imaginary axis of the z-plane.

- K for marginal stability: Using Routh-Hurwitz criterion (or Jury test), the value of K as the root locus crosses the unit circle into unstable region is

$$K = 38.3$$

- Unit circle crossover: Inserting $K = 38.3$ into the characteristic equation

$$z^2 + (0.023 \times 38.3 - 1.368)z + (0.368 + 0.01649 \times 38.3) = 0$$

or

$$z^2 - 0.4871z + 0.999 = 0 \Rightarrow z = 0.2435 \pm j0.9693 \text{ (crossing points)}$$

- Angle of asymptotes

$$\lambda = \frac{(2n+1)180}{p-z} \quad n=0,1,2,3$$

where p=number of poles and z is the number of zeros. Thus λ becomes

$$\lambda = 180$$

The real axis interception of the asymptotes is

The complete root-locus plot may now be constructed as shown in the figure below. Let it now be desired to determine the response of this system to a unit step function for the case in which $K=4$. It follows that

$$G(z) = \frac{0.368K(z+0.717)}{16(z-1)(z-0.368)} = \frac{0.092(z+0.171)}{z^2 - 1.368z + 0.367}$$

The z-transform closed-loop transfer function

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.092(z+0.171)}{(z^2 - 1.368z + 0.368) + 0.092(z+0.717)}$$

Thus,

$$C(z) - 1.276 z^{-1} C(z) + 0.434 z^{-2} C(z) = 0.092 z^{-1} R(z) + 0.066 z^{-2} R(z)$$

The corresponding recursive time difference equation is given by

The substitution of $c(nT)=r(nT)=0$ for $n<0$ and $r(nT)=1$ for $K \geq 0$ yields the following values for $c(nT)$ at the sampling instants:

$$c(0)=0$$

$$c(T) =$$

$$c(2T) = c($$

$$c(3T)=c$$

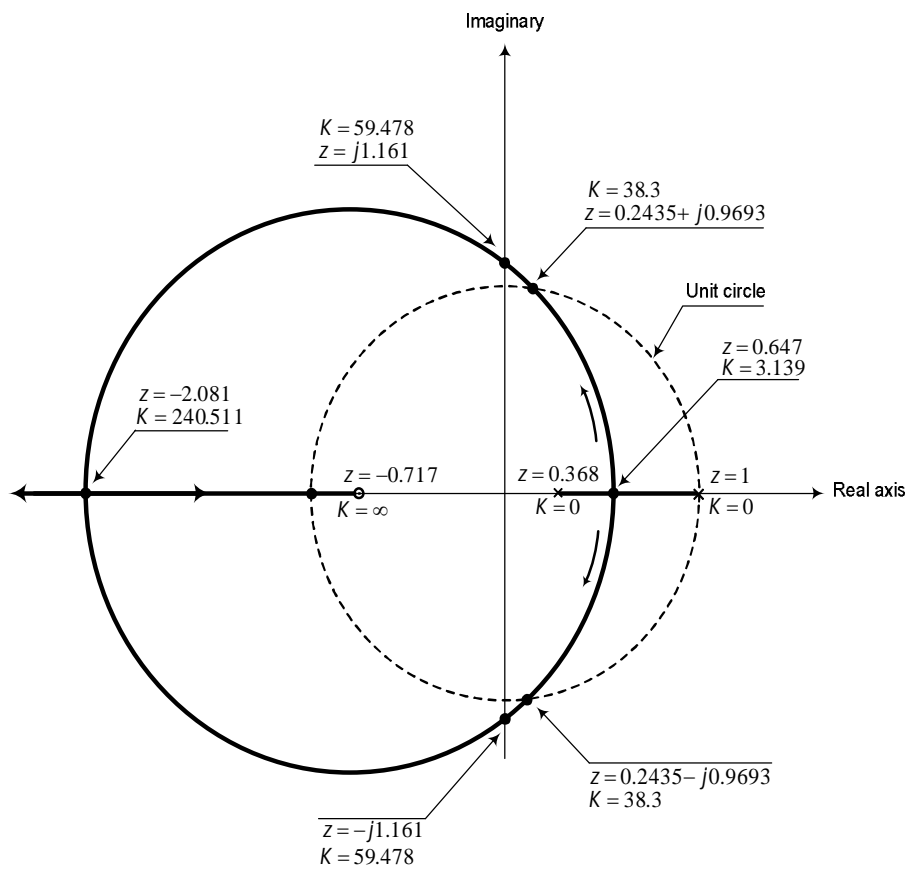


Figure (3) Root-locus plot for $(z - 1)(z - 0.368) + 0.023K(z + 0.717)$.

Steady State Error

An important characteristic of a control system is its ability to follow, or track, certain inputs with a minimum of error. The control system designer attempts to minimize the system error to certain anticipated inputs. In this section the effects of the system transfer characteristics on the steady-state system errors are considered.

Consider the system of Fig.(1). The signal $e(t)$ is defined as the error; that is,

$$e(t) = r(t) - b(t)$$

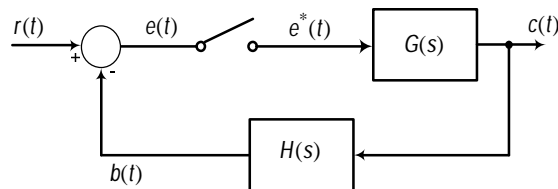


Figure (1) A digital control system

Since it is difficult to describe $e(t)$ in a digital control system, the sampled error $e^*(t)$ or the error at the sampling instants $e(kT)$ is usually used. Thus, the steady-state error at the sampling instants is defined as

$$E_{ss}^* = \lim_{k \rightarrow \infty} e(kT) = \lim_{t \rightarrow \infty} e^*(t)$$

Using the z-transform, *the final value theorem* leads to

$$(1)$$

For the system shown in Fig.(1), the z-transform of the error signal is written

$$E(z) = \frac{R(z)}{1 + GH(z)}$$

Substituting the last equation into Eq.(1), we have

$$(2)$$

This expression shows that the steady state error depends on the reference input $R(z)$, as well as the loop transfer function $GH(z)$. In the following, three basic types of input signals will be considered: step function, ramp function and parabolic function.

□ Steady State Error due to a Step Function input:

Let the reference input to the system of Fig.(1) be a step function of magnitude h . The z-transform of $r(t)$ is

$$R(z) = h \frac{z}{z-1}$$

Substituting the last equation into Eq.(2), we have

(3)

Let the step-error constant be defined as

$$K_p = \lim_{z \rightarrow 1} GH(z)$$

Then Eq.(3) becomes

$$E_{ss}^* = \frac{h}{1 + K_p}$$

Thus, for the steady-state error due to a step function input to be zero, the step-error constant K_p must be infinite. This implies that the transfer function $GH(z)$ must have at least one pole at $z=1$.

□ Steady State Error due to a Ramp Function input:

For a ramp function, $r(t) = ht$, the z-transform of $r(t)$ is

$$R(z) = \frac{hTz}{(z-1)^2}$$

Substitute the previous equation into Eq.(2), we have

(4)

Let the ramp-error constant be defined as

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} [(z-1) GH(z)] \quad (5)$$

then, Eq.(4) becomes

$$E_{ss}^* = \frac{h}{K_v} \quad (6)$$

The ramp-error constant K_v is meaningful only when the input to the system is a ramp function. Again, Eq.(6) is valid only if the function after the limit sign in Eq.(2) does not have any poles on or outside the unit circle $|z|=1$. This means that the closed-loop digital control system must at least be asymptotically stable.

Equation (6) shows that in order for E_{ss}^* due to a ramp function input be zero, K_v must equal infinity. From Eq.(5) we see that this is equivalent to the requirement of $(z-1)GH(z)$ having at least one pole at $z=1$, or $GH(z)$ having two poles at $z=1$.

□ Steady State Error due to a Parabolic Function input:

For a parabolic function, $r(t) = h \frac{t^2}{2}$, the z-transform of $r(t)$ is

$$R(z) = \frac{hT^2 z(z+1)}{2(z-1)^3}$$

From Eq.(2), the steady-state error at the sampling instants is written as

$$E_{ss}^* = \frac{T^2}{2} \lim_{z \rightarrow 1} \frac{h(z+1)}{(z-1)^2 [1+GH(z)]}$$

or

(7)

Now, let the parabolic-error constant be defined as

$$K_a = \frac{1}{T^2} \lim_{z \rightarrow 1} [(z-1)^2 GH(z)] \quad (8)$$

Then, Eq.(7) becomes

$$E_{ss}^* = \frac{h}{K_a} \quad (9)$$

In a similar manner we must point out that the parabolic-error constant is associated only with the parabolic function input, and should not be used with any of the other types of inputs.

Effects of Sampling on the Steady-State Error:

If the open-loop transfer function of Fig.(1) is of the following form:

$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s) \cdots (1+T_m s)}{s^j (1+T_1 s)(1+T_2 s) \cdots (1+T_n s)} \quad (10)$$

where the T's are nonzero real or complex constants, the type of the system is equal to j. The error constants for the continuous-data system are defined as

$$\text{Step-error constant: } K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

$$\text{Ramp-error constant: } K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

$$\text{Parabolic-error constant: } K_a = \lim_{s \rightarrow 0} s^2 G(s)H(s)$$

According to the above equations, one can easily conclude that, for instance, a type-0 system will have a constant steady-state error due to a step function input, and infinite error due to all higher-order inputs. A type-1 system (j=1) will have a zero steady-state error due to a step-function input, a constant error due to a ramp function input, and infinite error due to all

higher-order inputs. Table (1) summarizes the relationships between the system type, and the error constants for the continuous-data systems.

Table (1)

Type of System	K_p	K_v	K_a
0	K	0	0
1	∞	K	0
2	∞	∞	K

We will evaluate the error constants of digital control system shown in Fig.(1) for the cases of $j=0,1$ and 2 as follows:

□ Type 0 ($j=0$)

In this type $j=0$ and Eq.(10) becomes

$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s)\cdots(1+T_m s)}{(1+T_1 s)(1+T_2 s)\cdots(1+T_n s)}$$

where we assume that the open loop transfer function has more pole than zeros. The z transform of $G(s)H(s)$ is

(11)

Performing partial fraction expansion to the function inside the bracket in the last equation, we have

$$GH(z) = Z \left\{ \frac{K_1}{(1+T_1 s)} + \frac{K_2}{(1+T_1 s)} + \cdots + \frac{K_n}{(1+T_n s)} \right\}$$

$$= \{ \text{Terms with nonzero poles} \}$$

It is important to note that the terms due to the nonzero poles do not contain the term $(z-1)$ in the denominator. Thus, the step-error constant is

$$K_p = \lim_{z \rightarrow 1} GH(z) = \lim_{z \rightarrow 1} [\text{terms with nonzero poles}] = \text{constant}$$

Substituting Eq.(11) into the ramp-error constant of Eq.(8), we get

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) GH(z) = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) [\text{terms with nonzero poles}]$$

$$= \lim_{z \rightarrow 1} \left[\frac{(z-1)}{T} \right] \lim_{z \rightarrow 1} [\text{terms with nonzero poles}] = \lim_{z \rightarrow 1} \left[\frac{(z-1)}{T} \right] [\text{constant}] = 0$$

Similarly,

$$K_a = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 GH(z) = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 [\text{terms with nonzero poles}]$$

$$= \lim_{z \rightarrow 1} \left[\frac{(z-1)^2}{T^2} \right] \lim_{z \rightarrow 1} [\text{terms with nonzero poles}] = \lim_{z \rightarrow 1} \left[\frac{(z-1)^2}{T^2} \right] [\text{const tan } t] = 0$$

□ Type 1 (j=1)

In this type j=1 and Eq.(10) becomes

$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s) \cdots (1+T_m s)}{s(1+T_1 s)(1+T_2 s) \cdots (1+T_n s)}$$

The z transform of $G(s)H(s)$ is

(12)

Performing partial fraction expansion to the function inside the bracket in the last equation, we have

$$GH(z) = Z \left\{ \frac{K}{s} + \text{terms due to the nonzero poles} \right\}$$

$$GH(z) = \left\{ \frac{Kz}{z-1} + \text{terms due to the nonzero poles} \right\}$$

Thus, the step-error constant is

$$K_p = \lim_{z \rightarrow 1} GH(z) = \lim_{z \rightarrow 1} \left[\frac{Kz}{z-1} + \text{terms due to the nonzero poles} \right] = [\infty + \text{value}] = \infty$$

Substituting Eq.(12) into the ramp-error constant of Eq.(8), we get

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) GH(z) = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) \left[\frac{Kz}{z-1} + \text{terms due to the nonzero poles} \right]$$

$$= \lim_{z \rightarrow 1} \left[\frac{Kz}{T} + \frac{(z-1)}{T} (\text{terms due to the nonzero poles}) \right] = \frac{K}{T}$$

Similarly,

$$= \lim_{z \rightarrow 1} \left[\frac{Kz(z-1)}{T^2} + \frac{Kz(z-1)^2}{T^2} (\text{terms due to the nonzero poles}) \right] = 0$$

□ Type 2 (j=2)

For a type-1 system j=1, Eq.(11) becomes

$$GH(z) = Z \left[\frac{K(1+T_a s)(1+T_b s) \cdots (1+T_m s)}{s^2 (1+T_1 s)(1+T_2 s) \cdots (1+T_n s)} \right] \quad (13)$$

$$GH(z) = Z \left\{ \frac{K}{s^2} + \frac{K_1}{s} + \text{terms due to the nonzero poles} \right\}$$

Then, the step-error constant is

$$K_p = \lim_{z \rightarrow 1} GH(z) = \lim_{z \rightarrow 1} \left[\frac{KTz}{(z-1)^2} + \frac{K_1 z}{z-1} + \text{terms due to the nonzero poles} \right] = \infty$$

The ramp-error constant is

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) GH(z) = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) \left[\frac{KTz}{(z-1)^2} + \frac{K_1 z}{z-1} + \text{terms due to the nonzero poles} \right]$$

The parabolic-error constant

$$K_a = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 GH(z) = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 \left[\frac{KTz}{(z-1)^2} + \frac{K_1 z}{z-1} + \text{terms due to the nonzero poles} \right]$$

$$= \lim_{z \rightarrow 1} \left[\frac{Kz}{T} + \frac{K_1 z(z-1)}{T^2} + \frac{(z-1)^2}{T^2} (\text{terms due to the nonzero poles}) \right] = \left[\frac{K}{T} + 0 + 0 \right] = \frac{K}{T}$$

One can summarize the above in the following Table

Table (1)

Type of System	K_p	K_v	K_a
0	constant	0	0
1	∞	K/T	0
2	∞	∞	K/T

It would seem that K_v and K_a all depend on the sampling period T

Ex1: Calculate the steady-state errors for the system of Fig(1), in which the open-loop transfer function is given as

$$G(s) = \left(\frac{1 - e^{-Ts}}{s} \right) \left(\frac{K}{s(s+1)} \right)$$

Thus

$$G(z) = K Z \left[\frac{1 - e^{-Ts}}{s^2(s+1)} \right] = K \frac{(z-1)}{z} Z \left[\frac{1}{s^2(s+1)} \right]$$

Since $G(z)$ has one pole at $z=1$, the steady state error to unit step is zero, and to ramp input is $1/K$ provided that the system is stable.

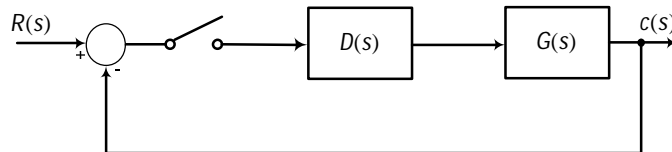
Ex2: Consider the system of Fig.(1), where $H(s) = 1$ and

$$G(z) = \frac{1 - e^{-T}}{z - e^{-T}}$$

Suppose that the design specification for this system requires that the steady state error to a unit ramp input be less than 0.01. Thus, it is necessary that the open-loop transfer function have a pole at $z=1$. Since the plant does not contain a pole at $z=1$, a digital compensator of the form

$$D(z) = \frac{K_i z}{z-1} + K_p$$

will be added to produce the resultant system shown in figure below.



The compensator, called a PI or proportional-plus-integral compensator, is of a form commonly used to reduce steady-state errors. Employing the expressions above for $D(z)$ and $G(z)$, we see that

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} \left[\frac{(K_i + K_p)z - K_p}{(z-1)} \right] \left[\frac{1 - e^{-T}}{z - e^{-T}} \right] = \frac{K_i}{T}$$

Thus K_i must equal $(100T)$ for the required steady-state error, provided that the system is stable. The latter point is needed an important consideration since the error analysis is meaningful unless the stability is guaranteed.

HW1: If a zero-order hold is included immediately after the sampler in the digital control system of Figure (1), then

- ❑ Follow the same above argument, show that the step, ramp and parabolic error constants are the same as given in Table (1) for continuous system.
- ❑ Do these error constants depend on the sampling period T ? Why?

HW2: For the simplified digital control system in the figure shown below, find the step, ramp and parabolic error constants. Express the results in terms of the system parameters.

