جامعة نينوى
كلية هندسة الالكترونيات
قسم هندسة النظم والسيطرة

Nineveh University

College Of Electronic
Systems and Control Engineering Department

ENGINEERING ANALYSIS II
الثاني الصف

Prepared by: Mr. Abdullah Ibrahim

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بسم الله الرحمن الرحيم

University of Mosul

College Electronics Engineering

Systems &Control Engineering Department

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أعداد:أ.عبدا لله إبراهيم عبدا لله

كلية هندسة الالكترونيات

Class: Second — Theory:2 Hrs/w

Subject: Engineering Analysis II — Tutorial:1 Hrs/w

syllabus

Session No.	Article	Hrs: 45	Page No.	Semester No.
	(الجزء الثاني)	Hrs		
1	Basic Probability and Statistics	20	1	2
2	Numerical Analysis	13	37	
3	Complex Variable Theory	12	57	

Textbook :

1. **Probability and Statistics for Engineers ,** 7th edition ,by Ronal A. Johnson, Miller & Freund’s Prentice Hall, 2005 .

2. **Advanced Engineering Mathematics:** By Kreyszig 10th edition, 2010

Basic Probability And Statistics

1. Statistics:

Statistics is the area of science that deals with collection, organization, analysis, and interpretation of data. It also deals with methods and techniques that can be used to draw conclusions about the characteristics of a large number of data points commonly called a **population**.

2.Probability: which measures the likelihood that an event will occur, is an important part of statistics. It is the basis of inferential statistics, where decisions are made under conditions of uncertainty

Definitions:

Random outcome : an outcome that cannot be predicted with certainty

An **experiment** is a process whose output is not known with certainty.

Sample Space (*S*) is the set of all possible outcomes of an experiment .

The outcomes are called **sample points** in ***S*.**

A **random variable** (X,Y,Z,--) is a function that assigns a real number to each point in ***S***.

An **Event** is the subset of the sample space.

Examples

EXPERIMENT	SAMPLE SPACE
Toss one coin	H, T
Roll a die	1, 2, 3, 4, 5, 6
Answer a true-false question	True, False
Toss two coins	HH, HT, TH, TT

1.Classical Probability

- The sample space is a collection of equally likely outcomes

$$P(A) = \frac{\text{outcomes in event } A}{\text{outcomes in the sample space}}$$

2.Empirical Probability (relative frequency)

- The outcomes of a random experiment are observed over repeated trials

$$P(A) = \frac{\text{times event } A \text{ occurs}}{\text{repetitions of the experiment}}$$

Note: The Law of Large Numbers states that relative frequency gets closer and closer to the true probability as the sample size increases

1. The probability of any event must be between 0 and 1. That is, $0 \le P(A) \le 1$ for any event *A*.

2. The sum of the probabilities for all simple events in a sample space must be 1.

3. The complement of event *A* consists of all outcomes in the sample space that do not make up event *A*, therefore

$$P(A^C) = 1 - P(A)$$

Example1:

For a card drawn from an ordinary deck, find the probability of getting
(a) queen (b) a 6 of clubs (c) a 3 or a diamond.

red	♥ hearts	A	2	3	4	5	6	7	8	9	10	J	Q	K
	♦ diamonds	A	2	3	4	5	6	7	8	9	10	J	Q	K
black	♠ spades	A	2	3	4	5	6	7	8	9	10	J	Q	K
	♣ clubs	A	2	3	4	5	6	7	8	9	10	J	Q	K

Solution:
(a) Since there are 4 queens and 52 cards, *P(queen) = 4/52 = 1/13.*
(b) Since there is only one 6 of clubs, then *P(6 of clubs) = 1/52.*
(c) There are four 3s and 13 diamonds, but the 3 of diamonds is counted twice in the listing. Hence there are only 16 possibilities of drawing a 3 or a diamond, thus *P(3 or diamond) = 16/52 = 4/13.*

Example 2:

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had AB blood. Set up a **frequency distribution**.

Type	Frequency
A	22
B	5
AB	2
O	21
	50=n

Example 3:

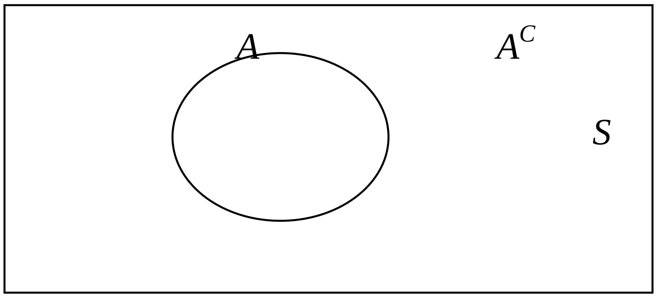
Find the following probabilities for the previous example.

A person has type O blood.
Solution: $P(\text{O}) = f/n = 21/50$.

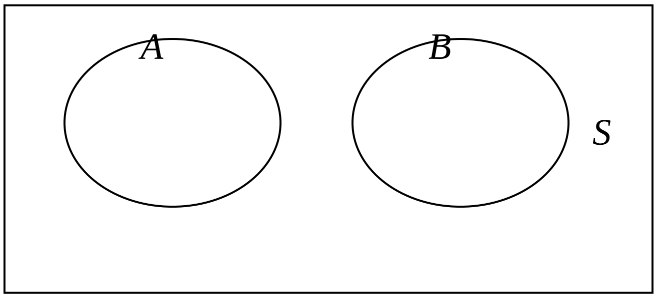
A person has type A or type B blood.
Solution: $P(\text{A or B}) = 22/50 + 5/50 = 27/50$.

3. A **<u>Venn diagram</u>** is useful for displaying relationships among events in a sample space.

The Venn diagram to show ***A*** and $\boldsymbol{A^C}$ might look like this:

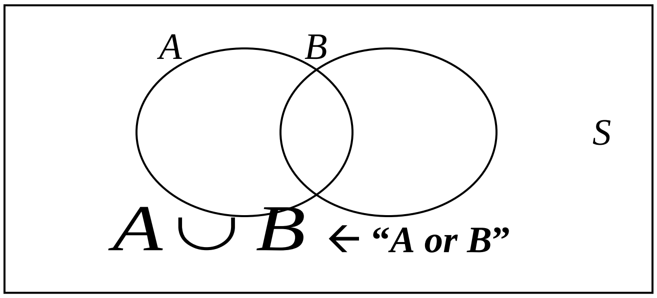


4. Two events are **<u>mutually exclusive</u>** (or <u>disjoint</u>) if they contain no common outcomes.
The Venn diagram to show two disjoint events *A* and *B* might look like this:

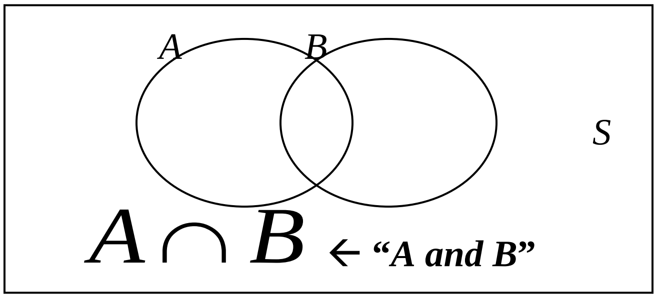


5. The **<u>union</u>** of two events *A* and *B* consists of all outcomes in the sample space that are in *A* **or** *B*, or both.

The Venn diagram to show the union of two events *A* and *B* might look like this:



6. The **<u>intersection</u>** of two events *A* and *B* consists of all outcomes in the sample space that are in both *A* **and** *B*.
The Venn diagram to show the intersection of two events *A* and *B* might look like this:



Example 4 .
An experiment consists of rolling a single die.

S = {1,2,3,4,5,6}

Define two events, A, B as follows.

A = {1,2,3,4}
B = {2,4,6}

Then

Union of A and B: AUB = {1,2,3,4,6}
Intersection of A and B: $A \cap B$ = {2,4}
Complement of A: A' = {5,6}

7.**<u>General Addition Rule</u>:** The probability of the union of *any* two events *A* and *B* is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If *A* and *B* are mutually exclusive, then $P(A \cap B) = 0$ and

$$P(A \cup B) = P(A) + P(B)$$

8.Two events are **<u>independent</u>** if the occurrence of one event does not change the probability of the other event.

Example 5.
An experiment consists of rolling a single die.
S = {1,2,3,4,5,6}

Define two events, A and B, as follows.
A: "roll an even number" A = {2,4,6}
B: "roll a five" B = {5}

Events A and B are mutually exclusive events; they have no elements in common.

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{6}$$

$$P(AUB) = P(A \text{ or } B) = P(A) + P(B) = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

Example 6:
In a hospital unit there are eight nurses and five physicians. Seven nurses and three physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

STAFF	FEMALES	MALES	TOTAL
NURSES	7	1	8
PHYSICIANS	3	2	5
TOTAL	10	3	13

Solution:
P(nurse or male)= P(nurse) + P(male) – P(male nurse) = 8/13 + 3/13 – 1/13
= 10/13

9.The **conditional probability** of event *A* given that event *B* has already occurred is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Note that the conditional probability of B given A is defined as

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

10. **General Multiplication Rule:** The probability of the intersection of *any* two events *A* and *B* is

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

or

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

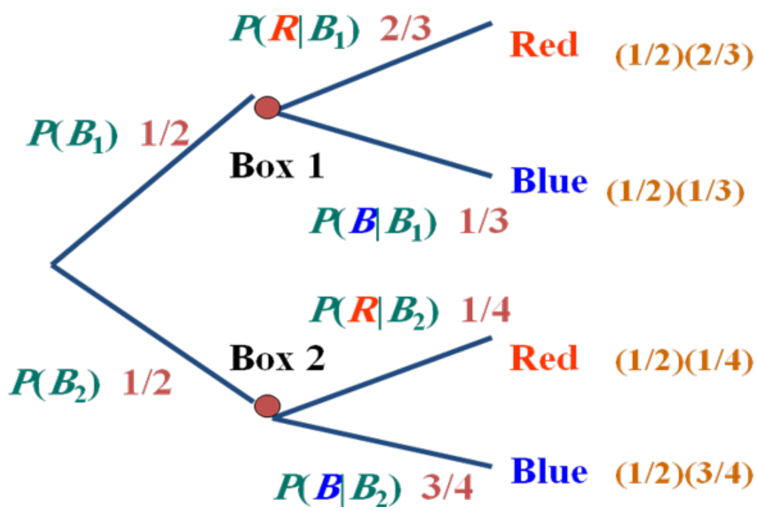
11. If two events *A* and *B* are independent, then

$$P(A) = P(A \mid B)$$
$$P(B) = P(B \mid A)$$
$$P(A \cap B) = P(A) \cdot P(B)$$

if any one is true, then all 3 are true
Furthermore, if any one of the above equations is true, then we can conclude that events *A* and *B* are independent.

Example 7:

Box 1 contains two red balls and one blue ball. Box 2 contains three blue balls and one red ball. A coin is tossed. If it falls heads up, box 1 is selected and a ball is drawn. If it falls tails up, box 2 is selected and a ball is drawn. Find the probability of selecting a red ball.



Solution: $P(\text{red}) = (1/2)(2/3) + (1/2)(1/4) = 2/6 + 1/8 = 8/24 + 3/24$
$= 11/24.$

Example 8:

Two fair dice are rolled . What is the conditional probability that the sum of the two face is 6 given that the two dice are showing different faces?

Let: A-event the two dice are showing different faces .

B-event the sum of the two face is 6 .

$$P(B/A)=\frac{p(B\cap A)}{p(A)}$$

	Die 2					
Die 1	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$$B = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

P(B)= $\frac{5}{36}$

BA= $\{(1,5),(2,4),(4,2),(5,1)\}$

P(BA)= $\frac{4}{36}$

P(A)= $\frac{30}{36}$, P(B/A)= $\frac{4/36}{30/36} = \frac{2}{15}$

Bayes Theorem:

From the multiplication rule, we know that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

From $\boxed{P(B|A)}$ **:** $P(A \cap B) = P(A)P(B|A)$

This can be substituted into $P(A|B)$ to give you: $\boxed{P(A|B) = \frac{P(A)P(B|A)}{P(B)}}$

This is known as Bayes Theorem, and is a very important result in probability, as it tells us how to "**turn conditional probabilities around**" that is, it tells us how to work out P(A/B) from P(B/A), and this is often very useful.

Also, $P(B)$ may be rewritten as:

$P(B) = P(A \cap B) + P(A' \cap B)$ (law of total probability)

This can be proven by the use of a Venn diagram, and rewriting $P(B)$**,**

we can also get: $P(B) = P(A)P(B|A) + P(A')P(B|A')$

This can then be substituted again into $P(A|B)$ to give you:

$$\boxed{P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A')P(B|A')}}$$ **(theorem of total probability)**

Bayes Theorem for partitions

Another important thing to notice that the use of the theorem of total probability in order to expand the bottom line of Bayes Theorem. In fact, this is done so often that Bayes Theorem is often stated in this form.
Suppose that we have a partition (A_1,…….,A_n) of a sample space S. Suppose
further that we have an event B, with P(B) > 0. Then, for each A_j, the probability of A_j given B is

$$\boxed{P(A_j \mid B) = \frac{P(A_j)P(B \mid A_j)}{\sum_{i=1}^{n} p(B / A_i)\, p(A_i)}}$$

Example 9: مرض بشع

A clinic offers you a free test for a very rare hideous disease. The test they offer is very reliable. If you have the disease it has a 98% chance of giving a positive result, and if you don't have the disease, it has only a 1% chance of giving a positive result. You decide to take the test, and find that you test positive — what is the probability that you have the disease?

Let P be the event "test positive" and D be the event "you have the disease".
We know that
P(P/D) = 0.98 and that $P(P/D^c) = 0.01$
We want to know P(D/P), so we use Bayes' Theorem.

$$P(D \mid P) = \frac{P(D)P(P \mid D)}{P(P)}$$

$$= \frac{P(D)P(P \mid D)}{P(D)P(P \mid D) + P(D^c)P(P \mid D^c)} = \frac{0.98 * p(D)}{0.98 * p(D) + 0.01 * (1 - p(D))}$$

So we see that the probability you have the disease given the test result depends on the probability that you had the disease in the first place. This is a rare disease, affecting only one in ten thousand people, so that
P(D) = 0.0001. Substituting this in gives

$$= \frac{0.98 * 0.0001}{0.98 * 0.0001 + 0.01 * (1 - 0.0001)} \approx 0.1$$

So, your probability of having the disease has increased from 1 in 10,000 to 1 in 100, but still isn't that much to get worried about! Note the crucial difference between P(P/D) and P(D/P).

Counting Rules: How many ways can it happen?

When outcomes are **equally likely** to occur (like when tossing a coin or rolling a die), you can use counting rules to find out how many outcomes are possible and then use that number to find probabilities.

1. **Multiplication rule** – when outcomes are selected from more than one set or group, multiply the number of outcomes for each set.

Example1: How many different meals can be made by pairing up 3 main courses and 4 side dishes?

Answer: 3 x 4 = 12 meals

(What if you have 2 dessert options? Then you can make 2 x 3 x 4 = 24 meals!)

In general if a task can be performed in n1 ways, a second task in n2 ways and a third task in n3 ways ……….., then the total number of distinct ways of performing all tasks together is $n1 \times n2 \times n3$……………

Example 2: How many 4 or 5 digit telephone numbers are possible, assuming the first is not zero?

ans : $9 \times 10 \times 10 \times 10 + 9 \times 10 \times 10 \times 10 \times 10 = 99\ 000$

Example 3: Toss two coin **:** m*n=2*2=4

Example 4: Throw two dice : m*n=6*6=36

2. **Permutation rule** – when outcomes are selected from only one set and the **order** that they are selected ***does*** **matter**, the total number of ways ***r*** outcomes can be chosen from ***n*** outcomes is

$$P_{n,r} = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

(for example $4! = 4\cdot 3\cdot 2\cdot 1 = 24$)

The ! is called a "factorial."

❖ **The number of permutations of n objects using all of them is n!**

Example 1: In how many ways can 5 people line up in a queue?

ans 5! = 120

Example2: Suppose that a **class president, vice-president**, **secretary**, and **treasure**r are to be randomly selected out of a group of **12** students nominated and that the order in which they are picked determines which office they will hold. **What is the probability of getting a specific set of class officers?**

$$P_{12,4} = \frac{12!}{(12-4)!} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$$
$$= 12\cdot 11\cdot 10\cdot 9 = 11880$$

Since there are 11,880 possible sets of class officers and they are all

equally likely, **the probability** of getting a specific set of class officers is

$$\frac{1}{11880}$$

Example3: How many three -letter codes are there using letters A,B,C and D if no letter can be repeated?

Solution :

Note ,the order does matter

$$ {}_4P_3 = \frac{4!}{(4-3)!} = 24 $$

Example 4: A club has 12 members. How many ways could a president, vice-president and treasurer be appointed.

$$ {}^{12}P_3 = \frac{12!}{(12-3)!} = 1\,320 $$

❖ Permutations with repeated elements

If a bag contains some objects in which m_1 are of type 1, m_2 are of type 2, ... m_k are of type k. The number of permutation is:

$$ \frac{(m_1 + m_2 + ... + m_k)!}{m_1! m_2! .... m_k!} $$

Example 5: How many ways can you permute the letters: **B A N A N A** ?

Of the **6** letters, there are **3 A's**, **2 N's**, and **1 B**.
The **2 N's** could be rearranged in **2! = 2** different ways.
The **3 A's** could be rearranged in **3! = 6** different ways.
So we need to divide 6! by both 6 and 2.
The number of ways to rearrange the letters in **BANANA** is $\frac{6!}{3!\,2!\,1!} = 60$

3.**<u>Combination rule</u>** – when outcomes are selected from only one set and the **order** that they are selected **does *not* matter**, the total number of ways ***r*** outcomes can be chosen from ***n*** outcomes is

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

<u>Example1</u>: Suppose there are 8 students in a group and that 5 of them must be selected to form a basketball team. How many *different* teams could be formed? What is the probability of ending up with one specific team?

Answer: Use the combination rule with n = 8 and r = 5 as shown below.

$$C_{8,5} = \binom{8}{5} = \frac{8!}{5!\cdot(3!)} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1\cdot(3\cdot2\cdot1)} = 8\cdot7 = 56$$

56 teams are possible and they are all equally is players are picked randomly, so the probability of ending up with one specific team is $\frac{1}{56}$

Suppose YOU were one of the 8 people to be selected for the team.

What is the probability that you would be selected to be on the team?

Since you must be on the team, we only need to select the other 4 players from the remaining 7, giving a total of $C_{7,4} = 35$ teams.

So, **the probability of YOU being on the team is** $\frac{35}{56} = 0.625$

Example2:How many committees of three can be selected from four people?

$$_4C_3 = \frac{4!}{(4-3)!\,3!} = 4$$

Example 3: From a club of 12 members, how many ways are there of selecting a committee of three?

$$^{12}C_3 = \frac{12!}{(12-3)!3!}$$

$$= 220$$

Example 4: A group consists of 8 boys and 5 girls:

(a) How many ways can you select 2 boys and 2 girls?

(b) How many ways can you select a committee of 4 containing at least 2 boys?

ans

(a) $^8C_2 \times {^5C_2} = 28 \times 10 = 280$

(b) At least 2 boys = 2 boys and 2 girls + 3 boys and 1 girl + 4boys

$$= {^8C_2} \times {^5C_2} + {^8C_3} \times {^5C_1} + {^8C_4} \times {^5C_0}$$

$$= 280 + 280 + 70$$

$$= 630$$

Probability Calculations Using Combinations / Permutations

Example 1: 4 chocolates are chosen at random from a box containing 6 with hard centers , and 8 with soft centers.
(a) Calculate the probability that 3 of the chocolates have soft centers.
(b) Calculate the probability that at least 3 of the chocolates have soft centers.

(a) Total number of ways of selecting 4 chocolates = ${}^{14}C_4 = 1\,001$

Number of ways of selecting 3 soft centers (and 1 hard) = ${}^{8}C_3 \times {}^{6}C_1$
$= 336$

$$P(\text{3 soft}) = \frac{336}{1001} = 0.3357$$

(b) Numbers of ways of selecting at least 3 soft = ${}^{8}C_3 \times {}^{6}C_1 + {}^{8}C_4 \times {}^{6}C_0$
$= 336 + 70$
$= 406$

$$P(\text{at least 3 soft}) = \frac{406}{1001} \quad \text{or } 0.4056$$

Example 2: A **4** digit security number is made using the digits 0, 1,……..9. If a number is made up at random, what is the probability that it contains the same digit repeated 3 times in a row.

(a) Total number of security codes = $\mathbf{10^4}$ = 10000.

(b) Total number of ways of getting **3** of the same in a row:

$$10 \times (1 \times 1 \times 1 \times 9 + 9 \times 1 \times 1 \times 1) = 180$$

(c) $$P(\text{3 in a row}) = \frac{180}{10000} = 0{\cdot}018$$

Random Variables and Probability Distributions:

In statistics we deal with random variables- variables whose observed value is determined by chance. Random variables usually fall into one of two categories: **discrete or continuous.**

A random variable (r.v.) is a function that associates a real number with each element in the sample space. Random variables will be denoted by uppercase letters and their observed numerical values by lowercase letters.

1.Discrete Random Variable. A random variable is discrete if it can assume at most a finite or a count ably infinite number of possible values.

Example 1. Two balls are drawn in succession without replacement from an urn containing 4 red and 3 black balls. The possible outcomes and values y of the random variable Y where Y is the number of red balls are:

Sample Space	y
RR	2
RB	1
BR	1
BB	0

2.Continuous Random Variable. A random variable is continuous if it can assume any value in some interval or intervals of real numbers and the probability that it assumes any specific value is 0.

3.Discrete Probability Distributions

Definition. The set of ordered pairs $(x, f(x))$ is a probability function, probability mass function or probability distribution of the discrete random variable X if,

(1) $f(x) \geq 0$

(2) $\sum_x f(x) = 1$

(3) $P(X = x) = f(x)$.

Example 1. A committee of size 5 is to be selected at random from 3 chemists and 5 mathematicians. Find the probability distribution (p.d.) for the number of chemists on the committee.

Let X be the number of chemists on the committee. Then $x: 0,1,2,3$.

$$P(X=0)=f(0)=\frac{\binom{3}{0}\binom{5}{5}}{\binom{8}{5}}=\frac{1}{56};\qquad P(X=1)=f(1)=\frac{\binom{3}{1}\binom{5}{4}}{\binom{8}{5}}=\frac{15}{56}$$

$$P(X=2)=f(2)=\frac{\binom{3}{2}\binom{5}{3}}{\binom{8}{5}}=\frac{30}{56};\qquad P(X=3)=f(3)=\frac{\binom{3}{3}\binom{5}{2}}{\binom{8}{5}}=\frac{10}{56}$$

Therefore the probability distribution of X is

x	0	1	2	3
$f(x)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$

4.Cumulative Distribution. The cumulative distribution $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x)=P(X\le x)=\sum_{t\le x} f(t),\quad for\ -\infty\le x\le\infty.$$

Example 2. Find the cumulative distribution of the number of chemists on the committee in example 1.

Using $F(x)$, show that $f(3)=\frac{10}{56}$.

$F(0)=f(0)=\frac{1}{56}$;

$F(1)=f(0)+f(1)=\frac{16}{56}$

$F(2)=f(0)+f(1)+f(2)=\frac{46}{56}$;

$F(3)=f(0)+f(1)+f(2)+f(3)=1$.

Hence,

$$F(x)=\begin{cases}0, & \text{if } x<0\\ \frac{1}{56} & \text{if } 0\le x<1\\ \frac{16}{56} & \text{if } 1\le x<2\\ \frac{46}{56} & \text{if } 2\le x<3\\ 1 & \text{if } x\ge 3\end{cases}$$

Now, $f(3)=F(3)-F(2)=1-\frac{46}{56}=\frac{10}{56}$.

5.Mean or Expected Value

The **mean** or **expected value** of a random variable *x* is the average value that we should expect for *x* over many trials of the experiment.

Notation: The mean or expected value of a random variable x will be represented by

$$\mu(x) \text{ or } E(x)$$

We can calculate the mean theoretically by using the formula:

$$E(x) = \mu(x) = \sum xP(x)$$

Examples:

let T be the random variable that represents the number of tails obtained when a coin is flipped three times. Then T has 4 possible values: 0, 1, 2, and 3. The probability distribution for T is given in the following table:

T	0	1	2	3
$P(T)$	1/8	3/8	3/8	1/8

$$E(T) = \sum T \times P(T) = \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) = 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{1}{8} = \frac{2\,3}{2}$$

6.Variance and Standard Deviation

Often, we are also interested in how much the values of a random variable differ from trial to trial. To measure this, we can define the variance and standard deviation for a random variable.

For a random variable *x*, the **variance of *x***, denoted by $\sigma^2(x)$ can be calculated by the formula:

$$\sigma^2(x) = \sum (x-\mu)^2 P(x)$$

The **standard deviation of *x***, denoted by $\sigma(x)$ is just the square root of $\sigma^2(x)$.

$$\sigma(x) = \sqrt{\sum (x-\mu)^2 P(x)}$$

As before, standard deviation estimates the average difference between a value of *x* and the average.

Example:

The probability distribution of the random variable Y, is given as

y	-3	-2	-1	0	1	2	3
P(y)	0.63	0.2	0.09	0.04	0.02	0.01	0.01

Find the mean and standared deviation of distribution.

m= $\sum_{i=1}^{N} xipi$ =-3(0.63)-2(0.2)-1(0.09)+0(0.04)+0.02+2(0.01)+3(0.01)

= -2.31

$\delta^2 = E(x^2) - m^2$

s.d= $\sqrt{\delta^2}$

$E(x^2) =$ (-3^2)0.63+(-2^2)0.2+(-1^2)0.09+0.02+(2^2)0.01+(3^2)0.01=6.71

$\delta^2 =$ 6.71-5.336=1.376 , s.d=1.172

Homework:

The probability distribution of the random variable X, is given as:

X	2	2.5	3	3.5	4	4.5
P(x)	0.07	0.36	0.21	0.19	0.1	0.07

1.Calculate
i) p(x $\leq$ 3.8).
ii)P(x>3.8).

2-Find the mean and standard deviation of distribution

7.Continuous Probability Distributions

Definition. (Probability Density Function) The function $f(x)$ is a probability density function for the continuous random variable X, defined over the set of real numbers R, if

(1) $f(x) \ge 0, \quad \text{for all } x \in R$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a < X < b) = \int_a^b f(x)\,dx$.

Note that for a continuous random variable X **,**

$$P(X = a) = \int_a^a f(x)\,dx = 0.$$

8.Cumulative Distribution. The cumulative distribution $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt \quad \text{for } -\infty < x < \infty.$$

Example 1. Suppose that a random variable X has a probability density function given by

$$f(x) = \begin{cases} kx(1-x) & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

(a) Find the value of k that makes this a probability function.

$$\int_0^1 kx(1-x)\,dx = 1 \Rightarrow k = 6$$

(b) Find $P(0.4 \le X \le 1)$

$$\int_{0.4}^{1} 6x(1-x)\,dx = 1 - 6\left[\frac{(0.4)^2}{2} - \frac{(0.4)^3}{3}\right] = 0.332$$

(c) Find $F(x) = P(X \le x)$ and sketch the graph of this function.

$$F(x) = \begin{cases} 0, & if \quad x \le 0 \\ 3x^2 - 2x^3, & if \quad 0 < x < 1 \\ 1, & if \quad x \ge 1 \end{cases}$$

9.Some Useful Probability Distributions

The observations generated by different statistical experiments have the same general type of behaviour. Random variables associated with these experiments can be described by essentially the same probability distribution and therefore can be represented by a single formula. The followings are the probability distributions that will be covered in this chapter:

- Binomial Distribution
- Poisson Distribution and Poisson Process
- Normal Distributions

9.1 The Binomial Distribution

Perhaps the most commonly used discrete probability distribution is the binomial distribution. An experiment which follows a binomial distribution will satisfy the following requirements (think of repeatedly flipping a coin as you read these):

1. The experiment consists of *n* identical trials, where *n* is fixed in advance.
2. Each trial has two possible outcomes, *S* or *F*, which we denote ``success'' and ``failure'' and code as 1 and 0, respectively.
3. The trials are independent, so the outcome of one trial has no effect on the outcome of another.
4. The probability of success, p is constant from one trial to another.

The random variable *X* of a binomial distribution counts the number of successes in *n* trials. The probability that *X* is a certain value *x* is given by the formula

$$P(X = x) = b(x, n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $0 \le p \le 1$ and $x = 0, 1, 2, ..., n$. Recall that the quantity $\binom{n}{x}$, ``*n* choose *x*,'' above is

$$\binom{n}{x} = \frac{n!}{x!(n-x)!},$$

We could use the formulas previously given to compute the mean and variance of *X*. However, for the binomial distribution these will always be equal to

$$E(X) = \boldsymbol{\mu} = np \quad \text{and} \quad Var(X) = \boldsymbol{\sigma}^2 = npq$$

Note: A particularly important example of the use of the binomial distribution is when sampling with replacement (this implies that p is constant).

Example 1. Suppose we have 10 balls in a bowl, 3 of the balls are red and 7 of them are blue. Define success S as drawing a red ball. If we sample *with replacement*, $P(S)$=0.3 for every trial. Let's say n=20, then $X \sim b(x, 20, 0.3)$ and we can figure out any probability we want. For example,

$$\begin{aligned} P(X=5) &= \binom{20}{5} 0.3^5 (1-0.3)^{20-5} \\ &= 15504(0.3^5)(0.7^{15}) = 0.1789. \end{aligned}$$

The mean and variance are

$$E(X) = 20(0.3) = 6, \qquad \mathrm{Var}(X) = 20(0.3)(0.7) = 4.2.$$

Example 2. The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

(a) at least 10 survive?
(b) from 3 to 7 survive?
(c) exactly 5 survive?

Solution. Probability of success= $p = 0.4$, and the probability of failure= $q = 0.6$.

$n = 15$ and X : no. of surviving patients

(a) $P(X \geq 10) = 1 - P(X \leq 9) = 1 - B(9; 15, 0.4) =$ 1-0.9662=0.0338

(b) $P(3 < X < 7) = P(4 \leq X \leq 6) = b(4; 15, 0.4) + b(5; 15, 0.4) + b(6; 15, 0.4) = 0.509$

(c) $P(X = 5) = b(5; 15, 0.4) = \binom{15}{5}(0.4)^5(0.6)^{15-5} = 0.186$

Example 3. A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 2 of the next 9 vehicles are from out of the state?

Solution. Probability of success= $p = 0.25$, and the probability of failure= $q = 1 - 0.25 = 0.75$.

$n = 9$ and X : no. of vehicles passing through the checkpoint

$$P(X < 2) = P(X \leq 1) = b(0; 9, 0.25) + b(1; 9, 0.25)$$

$$= \binom{9}{0}(0.25)^0(0.75)^9 + \binom{9}{1}(0.25)^1(0.75)^8 = 0.3$$

Example 4. Assuming that 6 in 10 automobile accidents are due mainly to speed violation,

(a) find the probability that among 8 automobile accidents 6 will be due mainly to a speed violation.
(b) Find the mean and variance of the number of automobile accidents for 8 automobile accidents.

Solution. Probability of success= $p = 6/10$, and the probability of failure

$q = 1 - 0.6 = 0.4$

$n = 8$ and X : no. of automobile accidents

(a) $P(X = 6) = b(6;8,6/10) = \binom{8}{6}\left(\frac{6}{10}\right)^6\left(1 - \frac{6}{10}\right)^{8-6} = 0.2090$

(b) The mean of the number of automobile accidents is

$\mu = E(X) = np = 8 * \frac{6}{10} = 4.8$.

The variance of the no. of auto. accidents is

$\sigma^2 = Var(X) = npq = 8 * \frac{6}{10} * \frac{4}{10} = 1.92$.

9.2 The Poisson Distribution

The Poisson distribution is most commonly used to model the number of random occurrences of some phenomenon in a specified unit of space or time. For example,

- The number of phone calls received by a telephone operator in a 10-minute period.
- The number of flaws in a bolt of fabric.
- The number of typos per page made by a secretary.

For a Poisson random variable, the probability that X is some value x is given by the formula

$$P(X = x) = f(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \qquad x = 0,1,2,\ldots$$

where λ is the average number of occurrences per unit time or region denoted by t. For the Poisson distribution,

$$E(X) = \lambda t \quad \text{and} \quad Var(X) = \lambda t.$$

Example 1. The number of false fire alarms in a suburb of Houston averages 2.1 per day. Assuming that a Poisson distribution is appropriate, the probability that 4 false alarms will occur on a given day is given by

$$P(X = 4) = \frac{2.1^4 e^{-2.1}}{4!} = 0.0992.$$

Example 2. During a laboratory experiment the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Example 3. A secretary makes 2 errors per page, on average. What is the probability that on the next page he or she will make

(a) 4 or more errors?
(b) no errors?

Example 4. The number of customers arriving per hour at a certain automobile service facility is assumed to follow a Poisson distribution with $\lambda = 7$.

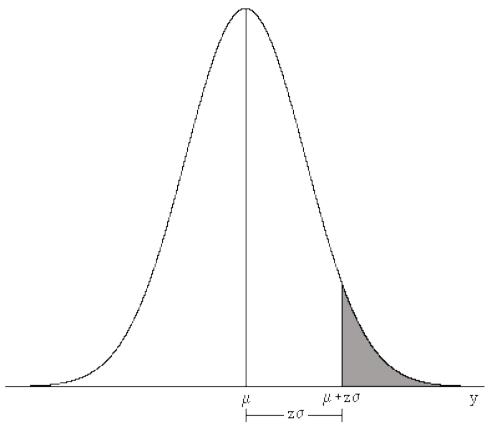
(a) Compute the probability that more than 10 customers will arrive in a 3-hour period.
(b) What is the mean number of arrivals during a 4-hour period?

Example 5. A restaurant chef prepares tossed salad containing, on average, 5 vegetables. Find the probability that the salad contains more than 5 vegetables

(a) on a given day
(b) on 3 of the next 4 days
(c) for the first time in April on April 5.

9.3 The Normal Distribution

The most important continuous probability distribution in the entire field of statistics is the normal distribution. Normal distributions are a family of distributions that have the same general shape. They are symmetric with scores more concentrated in the middle than in the tails. Normal distributions are sometimes described as bell shaped which are shown below. Notice that they differ in how spread out they are. The area under each curve is the same. The height of a normal distribution can be specified mathematically in terms of two parameters: the mean (μ) and the standard deviation (σ).



Definition. The density function of the normal random variable X, with mean μ and variance σ^2, is

$$N(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi\sigma}}e^{-(1/2)[(x-\mu)/\sigma]^2},\qquad -\infty<x<\infty$$

where $\pi = 3.14159....$ and $e = 2.71828$.

Standard normal distribution

The standard normal distribution is a normal distribution with a mean of 0 and a standard deviation of 1. Normal distributions can be transformed to standard normal distributions by the formula:

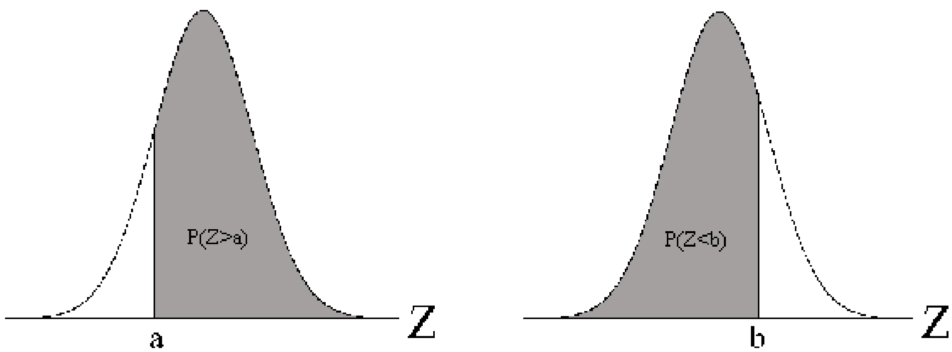
$$z=\frac{X-\mu}{\sigma}$$

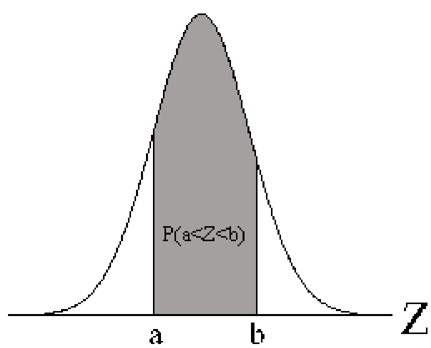
where X is a score from the original normal distribution, μ is the mean of the original normal distribution, and σ is the standard deviation of original normal distribution. The standard normal distribution is sometimes called the z distribution. A z score always reflects the number of standard deviations above or below the mean a particular score is. For instance, if a person scored a 70 on a test with a mean of 50 and a standard deviation of 10, then they scored 2 standard deviations above the mean. Converting the test scores to z scores, an X of 70 would be:

$$z=\frac{70-50}{10}=2$$

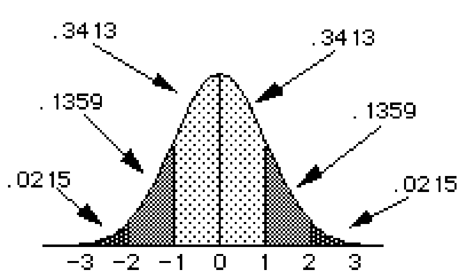
So, a z score of 2 means the original score was 2 standard deviations above the mean. Note that the z distribution will only be a normal distribution if the original distribution (X) is normal.

Note : The following figures give us the areas to the right of some $z-$value, to the left of some $z-$value and between two $z-$values.





Areas under portions of the standard normal distribution are shown to the right. About .68 (.34 + .34) of the distribution is between -1 and 1 while about .96 of the distribution is between -2 and 2.



Example 1. Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $z = 1.84$

(b) between $z = -1.97$ and $z = 0.86$

Solution. (a) $P(Z > 1.84) = 1 - P(Z \le 1.84) \overset{by\ table\ A3}{=} 1 - 0.9671 = 0.0329$.

(b)

$$P(-1.97 < Z < 0.86) = P(Z < 0.86) - P(Z < -1.97) \overset{by\ table\ A3}{=} 0.8051 - 0.0244 = 0.7807$$

Example 2. Given a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

Solution.

$$P(45 < X < 62) = P\left(\frac{45-50}{10} < Z < \frac{62-50}{10}\right) = P(-0.5 < Z < 1.2) = table(1.2) - table(-0.5)$$

$$= 0.8849\text{-}0.3088 = 0.5764$$

Example 3. Given a standard normal distribution, find the value of k such that

(a) $P(Z < k) = 0.0427$

(b) $P(Z > k) = 0.2946$

(c) $P(-0.93 < Z < k) = 0.7235$

Solution. (a) $P(Z < k) = 0.0427 \Rightarrow table(k) = 0.0427 \Rightarrow k = -1.72$

(b)

$$P(Z > k) = 0.2946 \Rightarrow P(Z > k) = 1 - P(Z \le k) = 0.2946$$

$$\Rightarrow P(Z \le k) = 1 - 0.2946 = 0.7054$$

$$\Rightarrow\ table(k) = 0.7054 \Rightarrow k = 0.54$$

(c) $P(-0.93 < Z < k) = 0.7235 \Rightarrow table(k) - table(-0.93) = 0.7235$

$$table(k) = 0.7235 + 0.1762 = 0.8997$$

$$\Rightarrow k = 1.28$$

10.Uniform Random Variables

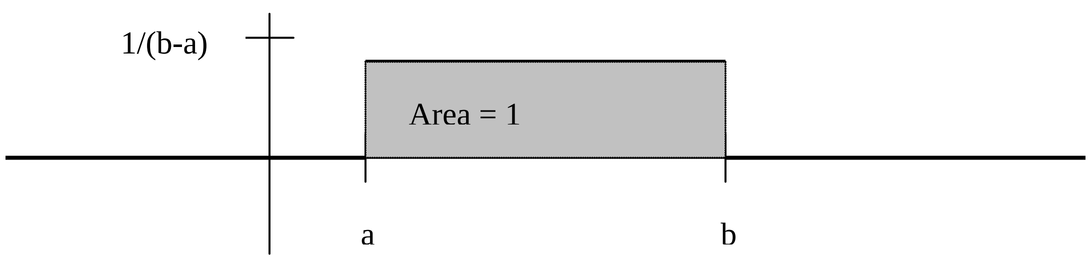
Probability Density Function Pdf:

We want to define a random variable X that is "equally likely" to take on any value in some finite interval (a,b). Formally this is nonsensical since the probability of a continuous random variable assuming a particular value is always 0. A better way of formalizing our intuition is that the probability of X falling in a subinterval of (a,b) should depend only on the length of the subinterval, not on its location within (a,b).

The random variable X that satisfies this condition is the uniform random variable. We write X~uniform(a,b). It has the probability density function.

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a < x < b \\ 0, & \text{otherwise} \end{cases}.$$

The graph of the pdf satisfies our intuition about "equal likelihood" of all intervals of a given length within (a,b). It also clearly has total area 1 under the pdf curve.

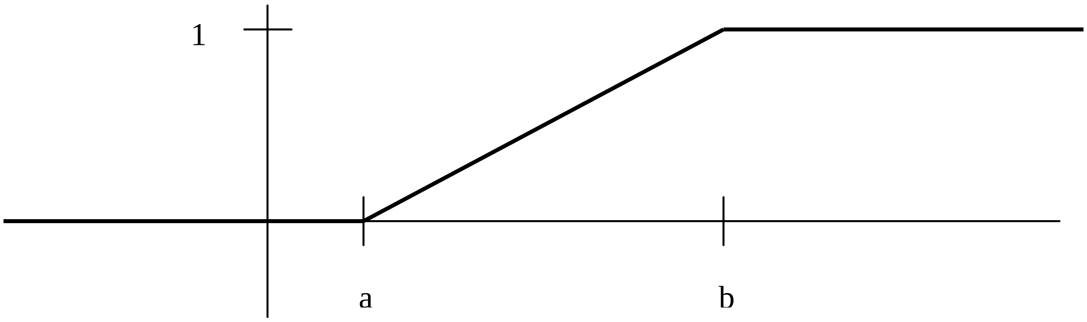


Cumulative Distribution Function Cdf:

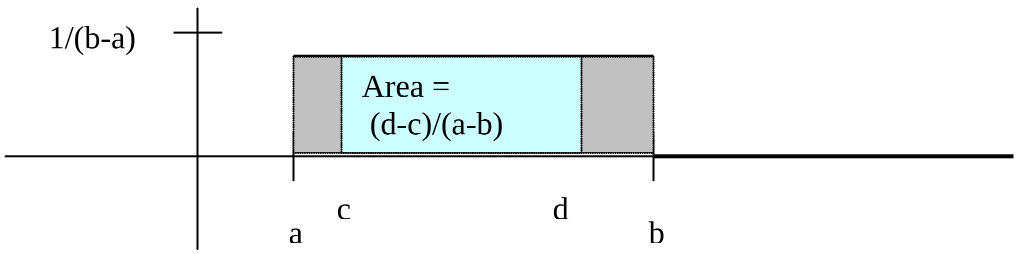
The cumulative distribution function F given below is easy to compute either by integration of the pdf or by finding the area of rectangles. Note that it has all the usual properties of a cdf: 0 to the left, 1 to the right, increasing and continuous inbetween.

$$F(x) = \begin{cases} 0, & \text{if } x \le a \\ \frac{x-a}{b-a}, & \text{if } a < x < b \\ 1, & \text{if } x \ge b \end{cases}$$

Here we see F and its properties graphically.



If we want to find the probability P(c<X<d) where a<c<d<b, then we can integrate formally, but it is easier to note that the probability is simply the ratio of the length of (c,d) to the length of (a,b).



Expected Value

Intuitively we anticipate E(X)=(a+b)/2, the midpoint of the interval. This turns out to be correct.

If X~uniform(a,b) we calculate

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx =$$

$$\int_a^b \frac{1}{b-a} xdx = \frac{1}{2(b-a)} x^2 \Big|_a^b = \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)} = \frac{b+a}{2}$$

Variance

Let X~uniform(a,b). We can find the variance of X using the shortcut formula $Var(X) = E(X^2) - \mu^2$. We proceed as follows.

$$E(X^2) = \int_a^b \frac{1}{b-a} x^2 dx = \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$E(X^2) - \mu^2 = \frac{b^2 + ab + a^2}{3} - \frac{(a+b)^2}{4}$$

Finally
$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3b^2 + 6ab + 3a^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$

Example 1. Imagine yourself blindfolded and having to cut a ribbon of length 1 yard suspended between two posts. Define a random variable to be the length of one of the two pieces of ribbon after you cut it. This is an example of a continuous random variable. Assuming that you have equal chance of making the cut at any point on the ribbon, the probability of cutting a piece of length say *d* is proportional to *d*, and since the total length is one, the probability is exactly *d*. This is an example of a *uniform* random variable. Its probability density function is $f(x) = 1$, for all $x \in [0,1]$.

If we want to find the probability that the length of the cut piece is less than 0.5 of a yard, we compute $P(X < 0.5) = \int_0^{0.5} f(x)\,dx = 0.5$. Similarly, the probability that its length is between 0.25 and 0.5 is:

$$P(0.25 < X < 0.5) = \int_{0.25}^{0.5} f(x)\,dx = 0.25$$

In general, a uniform random variable on an interval $[a,b]$ has a probability density function given by $f(x) = \dfrac{1}{b-a}$, for all $x \in [a,b]$.

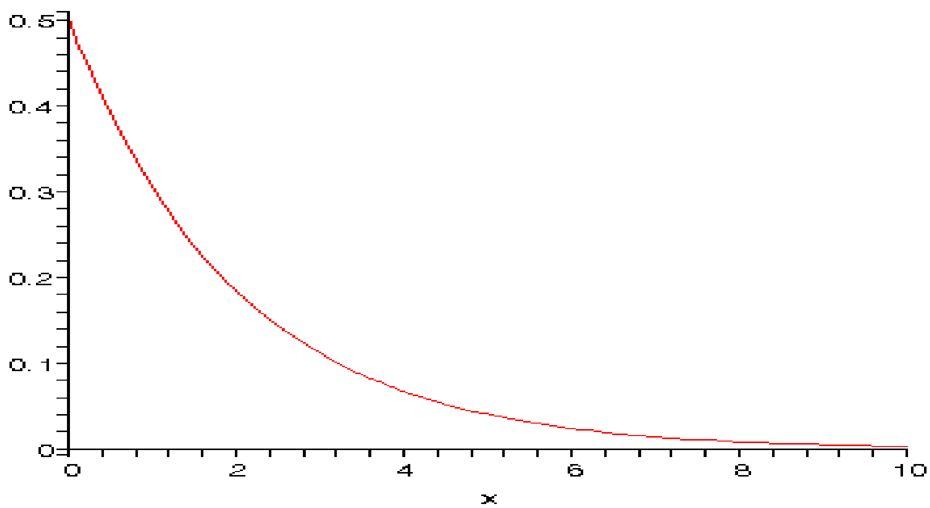
11.Exponential Random Variables

Probability Density Function Pdf:

Let λ be a positive real number. We write X~exponential(λ) and say that X is an exponential random variable with parameter λ if the pdf of X is

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & \text{if } x\geq 0\\ 0, & \text{otherwise}\end{cases}.$$

A look at the graph of the pdf is informative. Here is a graph produced by Maple for λ=0.5. Note that it has the same shape as every exponential graph with negative exponent (exponential decay). The tail shrinks to 0 quickly enough to make the area under the curve equal 1. Later we will see that the expected value of an exponential random variable is 1/λ (in this case 2). That is the balance point of the lamina with shape defined by the pdf.

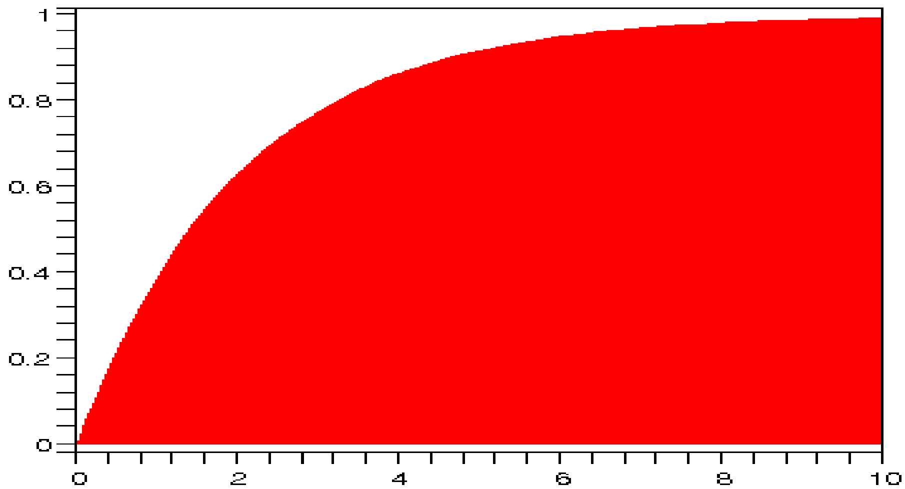


A simple integration shows that the total area under f is 1:

$$\int_{-\infty}^{\infty}\lambda e^{-\lambda x}dx=\lim_{t\to\infty}\int_0^t\lambda e^{-\lambda x}dx=\lim_{t\to\infty}-e^{-\lambda x}\Big|_0^t$$
$$=\lim_{t\to\infty}-e^{-\lambda t}-\left(-e^{-\lambda 0}\right)=-0+1=1$$

Cumulative Distribution Function Cdf:

Essentially the same computation as above shows that $F(x)=1-e^{-\lambda x}$. Here is the graph of the cdf for X~exponential(0.5).



As the coming examples will show, this formula greatly facilitates finding exponential probabilities.

Expected Value

So, if X~exponential(λ), then

$$E(X)=\int_0^\infty x\lambda e^{-\lambda x}dx=\lim_{t\to\infty}\int_0^t x\lambda e^{-\lambda x}dx.\text{ Let } u=x \text{ and } dv=\lambda e^{-\lambda x}dx.$$

Then $du=dx$ and $v=-e^{-\lambda x}$. So $E(x)=\lim_{t\to\infty}\left(uv\big|_0^t-\int_0^t vdu\right)$

$$=\lim_{t\to\infty}\left(xe^{-\lambda x}\big|_0^t-\int_0^t -e^{-\lambda x}dx\right)=\lim_{t\to\infty}\left(te^{-\lambda t}-0-\left[\frac{1}{\lambda}e^{-\lambda x}\Big|_0^t\right]\right)$$

$$=\lim_{t\to\infty}\left(te^{-\lambda t}-\left[\frac{1}{\lambda}e^{-\lambda t}-\frac{1}{\lambda}e^{-\lambda 0}\right]\right)=\lim_{t\to\infty}\left(te^{-\lambda t}-\frac{1}{\lambda}e^{-\lambda t}+\frac{1}{\lambda}\right)$$

$$=0-0+\frac{1}{\lambda}=\frac{1}{\lambda}$$

Variance: By a similar computation $Var(X)=\dfrac{1}{\lambda^2}$.

ex1-Suppose the wait time X for service at the post office has an exponential distribution with mean 3 minutes. If you enter the post office immediately behind another customer, what is the probability you wait over 5 minutes?

Since E(X)=1/λ=3 minutes, then λ=1/3, so X~exponential(1/3). We want

$$P(X > 5) = 1 - P(X \le 5) = 1 - F(5)$$
$$= 1 - \left(1 - e^{-\frac{1}{3}\cdot 5}\right) = e^{-\frac{5}{3}} \approx 0.189$$

ex2-Under the same conditions, what is the probability of waiting between 2 and 4 minutes? Here we calculate

$$P(2 \le X \le 4) = F(4) - F(2) = \left(1 - e^{-\frac{4}{3}}\right) - \left(1 - e^{-\frac{2}{3}}\right)$$
$$= e^{-\frac{2}{3}} - e^{-\frac{4}{3}} \approx 0.250$$

The trick in the previous example of calculating $P(a \le X \le b) = F(b) - F(a)$ is quite common. It is the reason the cdf is so useful in computing probabilities of continuous random variables.

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Numerical Analysis

1.Roots of Single Equations

1.1 Fixed-Point Iteration Method:

The method requires one initial guess only.

►**Procedure:**

Consider the equation f(x) = 0

1- Re-write the equation as x = g(x)

2- Assume an initial guess for the root = x_0 and calculate the first estimate of the root x_1 from: $x_1=g(x_0)$

3- Repeat step 2 several times until convergence is achieved, i.e.

$$x_{i+1}=g(x_i) \text{ until } \epsilon_a < \epsilon_s$$

Convergence Condition: |g(x)| < 1 in the region of interest.

Example1: Use the fixed-point iteration method to estimate the root of $f(x)=e^{-x}-x$ with an accuracy of $\epsilon_t= 5\%$. (The exact root is 0.56714329).

Solution:

The above iterative formula becomes

$$x_{i+1} = e^{-x_i}$$

Performing the iterations, we get:

Iter#	x_{i+1} (i=0,1,2,…………) **Let** $x_0=0$
1	$e^{-0} = 1$
2	$e^{-1} = 0.367879$
3	$e^{-0.367879} = 0.692201$
4	$e^{-0.692201} = 0.500473$
5	$e^{-0.500473} = 0.606244$
.	.
10	$e^{-0.571143} = 0.564879$

Example2: Use the fixed-point iteration method to estimate one of the roots of $f(x)=x^2-2x-3$ with an accuracy of $\epsilon_t = 5\%$. (The exact roots are: x=-1 and x=3).

Solution:

Alternative (1): Re-write the equation in the form: $x=\sqrt{2x+3}$, so that $g(x)=\sqrt{2x+3}$. Let $x_0=0$. and use the above iterative formula to generate the following results:

Iter #	x
1	1.73205
2	2.54246
.	.
.	.
6	2.99413

*Note that the method here, **converges** to the root **x=3**.*

Alternative (2): Re-write the equation in the form: $x=\dfrac{3}{x-2}$, so that

$$g(x)=\frac{3}{x-2}$$

For $x_0=0$, we get:

Iter #	x
1	-1.50000
2	-0.85714
.	.
.	.
5	-1.00549

*So that the method **converges** to the root **x = -1.***

Alternative (3): Re-write the equation in the form: $x = x^2 - x - 3$, so that $g(x) = x^2 - x - 3$. Starting with $x_0 = 0$, we get:

Iter #	x
1	-3.0000
2	9.00000
3	69.0000
4	4689.00

It is obvious that the method does **<u>not converge</u>** for the above choice of g(x).

Let us repeat the procedure with a different initial guess, say x_0=2.9 which is very close to one of the root. We get:

Iter #	x
1	2.51000
2	0.79010
.	….
.	….
5	90.6150
6	8117.47

It does **not converge**! Even if you try x_0 = -1.1 which is very close to the other root, you will find out that the method **diverges again**!! The reason will be clarified later.

Alternative (4): Re-write the equation in the form: $x = \frac{x^2 - 3}{2}$, so that $g(x) = \frac{x^2 - 3}{2}$. Starting with $x_0 = 0$, we get:

Iter #	x
1	-1.50000
2	-0.37500
.	….
14	-0.66217
15	-1.28076

It is obvious that the method **<u>diverges</u>** for this choice of g(x). The same thing will happen, even if we start with x_0 very close to one of the roots!!

1.2The Newton-Raphson Metod :

► **The method is based on the first order Taylor expansion, i.e.:**

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

When we hit the root, $f(x_{i+1}) = 0$, then:

$$x_{i+1} \approx x_i - \frac{f(x_i)}{f'(x_i)}$$

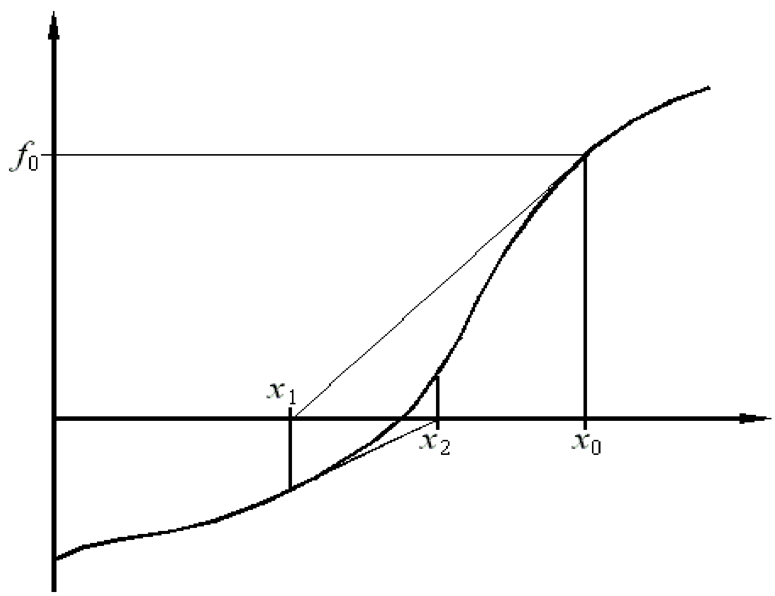
►**Procedure:**

4- Assume an initial guess for the root = x_0 and calculate the first estimate of the root x_1 from: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

5- Repeat step 1 several times until convergence is achieved, i.e.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \text{ until } \epsilon_a < \epsilon_s$$

The procedure is illustrated by the figure shown below.



Example1: Use Newton-Raphson method to estimate the root of $f(x)=e^{-x}-x$. Show all details of the iterations. Hint: the root is located between 0 and 1.

Iter	x_i	$f(x_i)$	$f'(x_i)=-e^{-x}-1$	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
1	0.	1.	-2.	0.5
2	0.5	0.1065	-1.6065	0.5663
3	0.5663	0.0013	-1.5676	0.5671
4	0.5671	0.0000	-1.5676	0.5671

Example2: Repeat Example1 starting with x0 = 5

Iter	x_i	$f(x_i)$	$f'(x_i)=-e^{-x}-1$	$x_{i+1}=x_n-\frac{f(x_i)}{f'(x_i)}$
1	5.	-4.9933	-1.0067	0.04016
2	0.04016	0.92048	-1.9606	0.5096
3	0.5096	0.0911	-1.6007	0.5665
4	0.5665	0.0010	-1.5675	0.5671

Example3:

Use the Newton- Raphson method ,with 1.5 as starting point ,to find solution of $f(x)= x-2\sin x$

$f(x)=x-2\sin x$, $x_0=1.5$, $f'(x)=1-2\cos x$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n-2\sin x_n}{1-2\cos x_n}$$

$$=\frac{2(\sin x_n-x_n\cos x_n)}{1-2\cos x_n}$$

x1	**2.0765582**
x2	1.9105066
x3	1.8956220
x4	1.89549427
x5	1.895494267033
x6	1.895494267033

1.3 Secant Method:

The method is used instead of Newton-Raphson method when the derivative of the function is difficult to obtain or if the slope of the function = 0 near the root. The formula of the method is obtained by replacing the exact derivative in Newton-Raphson's formula by $\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}$ so that

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{f(x_i)-f(x_{i-1})} f(x_i)$$

►Procedure:

6- Assume two initial guess x_0 and x_1 to calculate the first estimate of the root x_2 from: $x_2 = x_1 - \frac{x_1 - x_0}{f(x_1)-f(x_0)} f(x_1)$

7- Repeat step 1 several times until convergence is achieved, i.e.

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{f(x_i)-f(x_{i-1})} f(x_i) \quad \text{until } \epsilon_a < \epsilon_s$$

►Example3: Repeat Examples 1 using the secant method with the two initial guesses $x_0 = 0.$ and $x_1 = 1.$

Solution:

Iter	X_{i-1}	X_i	X_{i+1}
1	1.	2.	0.6127
2	2.	0.6127	0.563838
3	0.48714	0.563838	0.56717

►Example4: Repeat Examples 3 using the secant method with the two initial guesses $x_0 = 2.$ and $x_1 = 3.$

Solution:

Iter	X_{i-1}	X_i	X_{i+1}
1	2.	3.	0.2823
2	3.	0.2823	0.6570
.	.	.	.
5	0.5719	0.5671	0.5671

2.Numerical Solution of Ordinary Differential Equations (ODE)

An equation that consists of derivatives is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations. So, it is important for engineers and scientists to know how to set up differential equations and solve them.

Differential equations are of two types

1) ordinary differential equation (ODE)
2) partial differential equations (PDE).

An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable. Examples of ordinary differential equation include

1) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$, $\frac{dy}{dx}(0)=2, y(0)=4$,

2) $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+5\frac{dy}{dx}+y=\sin x$, $\frac{d^2y}{dx^2}(0)=12$, $\frac{dy}{dx}(0)=2$, $y(0)=4$

Note: In this first part, we will see how to solve ODE of the form

$$\frac{dy}{dx}=f(x,y), y(0)=y_0$$

In another section, we will discuss how to solve **higher order ordinary differential equations** or coupled (simultaneous) differential equations.

But first, How to write a first order differential equation in the above form?

Example 1

$$\frac{dy}{dx}+2y=1.3e^{-x}, y(0)=5$$

is rewritten as

$$\frac{dy}{dx}=1.3e^{-x}-2y, y(0)=5$$

In this case

$$f(x,y)=1.3e^{-x}-2y$$

Example 2

$$e^y\frac{dy}{dx}+x^2y^2=2Sin(3x), y(0)=5$$

is rewritten as

$$\frac{dy}{dx}=\frac{2Sin(3x)-x^2y^2}{e^y}, y(0)=5$$ In this case $$f(x,y)=\frac{2Sin(3x)-x^2y^2}{e^y}$$

2.1. Euler's Method

We will use Euler's method to solve an ODE under the form:

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

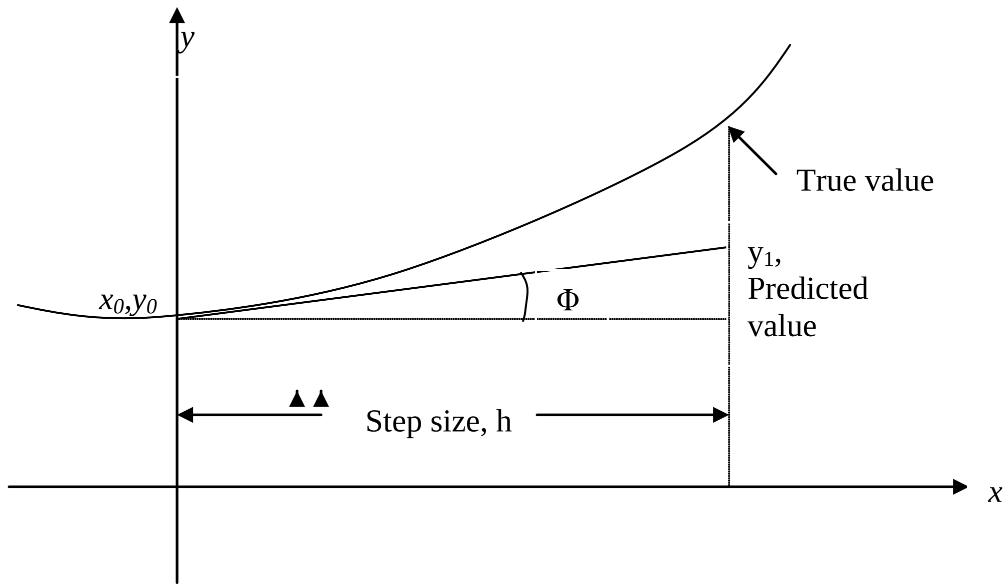


Figure 1. Graphical interpretation of the first step of Euler's method.

So the slope at $x=x_0$ as shown in the figure above

$$\text{Slope} = \frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$$

Thus

$$y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0)$$

If we consider $x_1 - x_0$ as a step size h, we get

$$y_1 = y_0 + f(x_0, y_0)\,h.$$

We are able now to use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, which is the predicted value at x_2,

$$y_2 = y_1 + f(x_1, y_1)\,h$$

$x_2 = x_1 + h$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$\boxed{y_{i+1} = y_i + f\left(x_i, y_i\right) h}$$

This formula is known as the Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

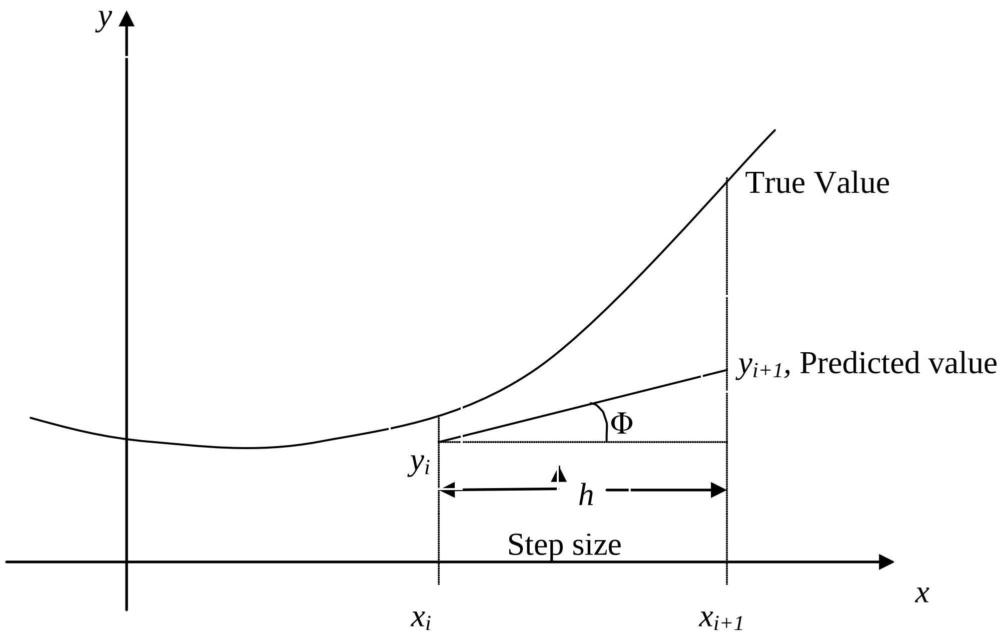


Figure 2. General graphical interpretation of Euler's method.

<u>Example1:</u>

Solve the differential equation ; $\frac{dy}{dx} = 2x^2 + 2y$, $y(0) = 1$, by Euler Method ,also find y on $0 \le x \le 0.3$ using $h = 0.1$

Euler Method; $\quad y_{i+1} = y_i + hf(y_i, x_i)$

$$y(0.1) = y(0) + (0.1)(2(0)^2 + 2(1))$$
$$= 1.2$$

i	x_i	$y(x_i)$ by *Euler*
0	***0***	***1***
1	***0.1***	***1.2***
2	***0.2***	***1.442***
3	***0.3***	***1.7384***

Example2:

Use Euler's method with h=0.14 to obtain a numerical solution of

$$y' = 1 + (y - x)^2$$

subject to y(1)=1.59 giving approximate values of y for $1 \le \chi \le 2.4$

y(1)=1.59 , h=0.14 , $1 \le x \le 2.4$

$$y(n+1) = y_n + h[1 + (y_n - x_n)^2]$$

$$y_1 = y_0 + 0.14\,[1 + (y_0 - x_0)^2]$$

$$= 1.59 + 0.14[1 + (1.50 - 1)^2] = 1.7787$$

n	x_n	y_n
0	1	1.59
1	1.14	1.7787
2	1.28	1.9758
3	1.42	2.1836
4	1.56	2.4052
5	1.7	2.6453
6	1.84	2.9104
7	1.98	3.2108
8	2.12	3.5629
9	2.26	3.9944
10	2.4	4.5554

3.Numerical Integration:

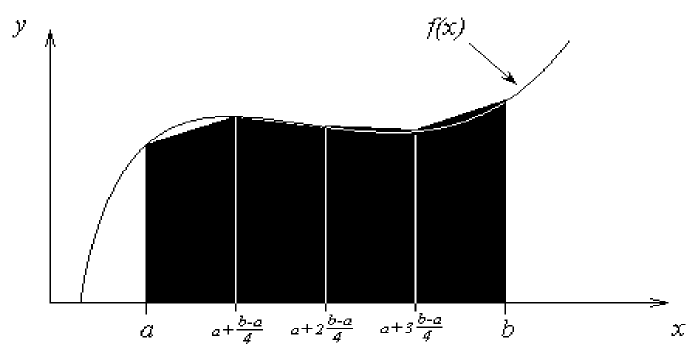
Realistically, to solve most engineering problems we require a numerical method, as the problem cannot be solved using an analytical technique. What do we mean by an analytical technique? Analytical techniques are based on algebraic methods. All the integration examples we have seen so far have been solved using analytical techniques such as integration by parts, substitution, partial fraction, etc. with these techniques you obtain an **exact** answer.

What does a numerical method involve?

Numerical methods are based on arithmetic operations. This normally involves calculations and generally with numerical methods your final result is an approximation. Here we will evaluate integrals with limits using numerical methods such as the **Trapezium rule** and **Simpson's rule.**

3.1 Trapezium Rule

What is the area represented by $\int_a^b ydx$, where $y = f(x)$?



The area under the curve $y = f(x)$ and the axis between *a* and *b*. We can approximate this area, $\int_a^b ydx$, by blocks of a trapezia.

We add all the blocks together, such that:

$$\text{Total area} = \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + \ldots\ldots + \frac{1}{2}h(y_{n-1} + y_n),$$

$$= \frac{1}{2}h\left[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \ldots\ldots + (y_{n-1} + y_n)\right]$$

$$= \frac{1}{2}h\left[y_0 + 2y_1 + 2y_2 + \ldots\ldots + 2y_{n-1} + y_n\right].$$

Remember the area under the curve is represented by $y = f(x)$ and so we have the formula known as the **trapezium rule:**

$$\int_a^b y\,dx \approx \frac{1}{2}h\left[y_0 + 2y_1 + 2y_2 + \ldots\ldots + 2y_{n-1} + y_n\right].$$

Note: the rule contains an approximation sign $\approx$ as we are finding an approximate solution and not an exact solution.

Remember that y_i represents the height of the given function at the point x_i, thus y_i is called the **ordinate**. Hence the trapezium rule can also be written as

$$\text{Area} \approx \frac{\text{width of block}}{2}\left[(\text{first ordinate}) + (\text{last ordinate}) + 2(\text{other ordinates})\right].$$

Example1: Evaluate $\int_0^1 x^2dx$ using the trapezium rule with 4 intervals.

Solution: Assume we are calculating $\int_a^b f(x)dx$ using the trapezium rule with *n* intervals.

Calculate uniform width: $h = \left(\frac{b-a}{n}\right) = \left(\frac{1-0}{4}\right) = \frac{1}{4} = 0.25$

Calculate function value $f(x)$ (use width $h = 0.25$ for 4 intervals i.e. $n = 0,1,2,3,4$ so going from *a* to *b* (*0* to *1*) we have x=*0, 0.25, 0.5, 0.75* and *1*):

n	x	y_n	$y=x^2$		
0	0	y_0	0		
1	0.25	y_1	0.0625		
2	0.5	y_2	0.25	$2(y_1+y_2+y_3)$	1.75
3	0.75	y_3	0.5625	$y_0+2(y_1+y_2+y_3)+y_4$	2.75
4	1	y_4	1	$1/8[y_0+2(y_1+y_2+y_3)+y_4]$	0.344

Calculate area using the Trapezium Rule:

$$\text{Area} \approx \frac{\text{width of block}}{2}\left[(\text{first ordinate}) + (\text{last ordinate}) + 2(\text{other ordinates})\right]$$

Therefore

$$\int_0^1 x^2 dx \approx \frac{0.25}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right] \approx \frac{0.25}{2}[0 + 1.75 + 1] \approx \frac{1}{8}[2.75] \approx 0.344$$

Example2: Evaluate $\int_0^1 e^{-x^2} dx$ using the trapezium rule with 4 intervals.

Solution: Calculate uniform width: $h = \left(\frac{b-a}{n}\right) = \left(\frac{1-0}{4}\right) = \frac{1}{4} = 0.25$

Calculate function value $f(x)$ (use width $h = 0.25$ for 4 intervals i.e. $n = 0,1,2,3,4$):

n	x	y_n	$z=-x^2$	e^z		
0	0	y_0	0	1		
1	0.25	y_1	-0.063	0.9394		
2	0.5	y_2	-0.25	0.7788	$2(y_1+y_2+y_3)$	4.576
3	0.75	y_3	-0.563	0.5698	$y_0+2(y_1+y_2+y_3)+y_4$	5.944
4	1	y_4	-1	0.3679	$1/8[y_0+2(y_1+y_2+y_3)+y_4]$	0.743

Calculate area using the Trapezium Rule:

$$\text{Area} \approx \frac{\text{width of block}}{2}\left[(\text{first ordinate}) + (\text{last ordinate}) + 2(\text{other ordinates})\right]$$

Therefore

$$\int_0^1 e^{-x^2}dx \approx \frac{0.25}{2}[y_0 + 2(y_1 + y_2 + y_3) + y_4] \approx \frac{0.25}{2}[1 + 4.576 + 0.368] \approx \frac{1}{8}[5.944] \approx 0.743$$

Example3: Evaluate $\int_0^2 (x^3 + 1)dx$ using the trapezium rule with 8 intervals.

Solution: Calculate uniform width: $h = \left(\frac{b-a}{n}\right) = \left(\frac{2-0}{8}\right) = \frac{2}{8} = 0.25$

Calculate function value $f(x)$ (use width $h = 0.25$ for 4 intervals i.e. $n = 0,1,2,3,4$):

n	x	y_n	x^3	x^3+1		
0	0	y_0	0	1		
1	0.25	y_1	0.0156	1.0156		
2	0.5	y_2	0.125	1.125	$2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)$	38.5
3	0.75	y_3	0.4219	1.4219	$y_0+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)+y_8$	48.5
4	1	y_4	1	2	1/8 $(y_0+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)+y_8)$	6.063
5	1.25	y_5	1.9531	2.9531		
6	1.5	y_6	3.375	4.375		
7	1.75	y_7	5.3594	6.3594		
8	2	y_8	8	9		

Calculate area using the Trapezium Rule:

$$\text{Area} \approx \frac{\text{width of block}}{2}[(\text{first ordinate}) + (\text{last ordinate}) + 2(\text{other ordinates})]$$

Therefore

$$\int_0^1 e^{-x^2}dx \approx \frac{0.25}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$$

$$\approx \frac{0.25}{2}[1 + 38.5 + 9] \approx \frac{1}{8}[48.5] \approx 6.063$$

TRY:

Evaluate $\int_0^{2\pi} \sin\frac{1}{2}x dx$ using the trapezium rule with 4 intervals

Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}dx$ using the trapezium rule with 8 intervals

3.2 Simpson's Rule

Here we state another numerical method used to find a value of an integral with limits, known as **Simpson's Rule**.

The general form of the equation of the second -degree parabola connecting the three point is

$$y = ax^2 + bx + c. \qquad (1)$$

The integration of this equation from -*h* to *h* gives the area contained in the two strips shown under the parabola (Figure 1).

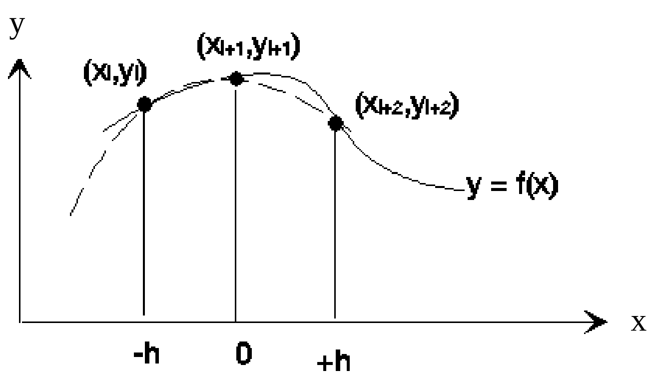


Figure 1. Approximation of area under curve by a second-degree parabola

Hence

$$A_{\text{2 strips}} = \int_{-h}^{h} f(x)dx = \int_{-h}^{h} (ax^2 + bx + c)dx = \{\frac{ax^3}{3} + \frac{bx2}{2} + cx\}\Big|_{-h}^{h} \qquad (2)$$

Substituting the limits into Equation yields

$$A_{\text{2 strips}} = \frac{2}{3} a h^3 + 2ch = \frac{h}{3}(2ah^2 + 6c) \qquad (3)$$

The constant a and c can be determined from the fact that points ($-h$, y_i), (0, y_{i+1}), and (h, y_{i+2}) must all satisfy Equation (1). The substitution of these three sets of coordinates into Equation (1) yields

$y_i = a(-h)^2 + b(-h) + c$

$y_{i+1} = c$

$y_{i+2} = a(h)^2 + b(h) + c.$

Solving these equations and the substitution of *a and c* into Equation (3) yields.

$$A_{2\ strips} = h/3(y_i + 4y_{i+1} + y_i+2) \qquad (4)$$

If the area under a curve is divided into N uniform strips (N even) the application of Equation (4) is resulting in Simpson's one-third rule

$$\int_a^b y dx \approx \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + ...) + 2(y_2 + y_4 + y_6 + ...) + y_n\right]. \qquad (5)$$

Here h is the uniform width, y_0, y_1, y_2, ……., y_n are the ordinates and n is the number of blocks, which must be **even**.

Hence Simpson's Rule can be written as

$$\boxed{\text{Area} \approx \frac{h}{3}\left[(\text{first ordinate}) + 4(\text{sum of odd ordinates}) + 2(\text{sum of even ordinates}) + (\text{last ordinate})\right]}$$

Note that to use this formula, the number of blocks *n* must be even.

Example1: Evaluate $\int_0^1 x^2 dx$ using Simpson's rule with 4 intervals.

Solution:

Calculate uniform width: $h = \left(\frac{b-a}{n}\right) = \left(\frac{1-0}{4}\right) = \frac{1}{4} = 0.25$

Calculate function value $f(x)$ (use width $h = 0.25$ for 4 intervals i.e. $n = 0,1,2,3,4$):

n	x	y_n	$y=x^2$		
0	0	y_0	0	(y_1+y_3)	0.625
1	0.25	y_1	0.0625	$2y_2$	0.5
2	0.5	y_2	0.25	$4(y_1+y_3)$	2.5
3	0.75	y_3	0.5625	$y_0+4(y_1+y_3)+2y_2+y_4$	4
4	1	y_4	1	$[1/4][1/3][y_0+4(y_1+y_3)+2y_2+y_4]$	0.3333

Calculate area using Simpson's Rule:

$$\int_a^b y dx \approx \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ...) + 2(y_2 + y_4 + y_6 + ...) + y_n]$$

Therefore

$$\int_{-1}^{1} x^3 dx \approx \frac{0.25}{3}[y_0 + 4(y_1 + y_3) + 2(y_2) + y_4] \approx \frac{0.25}{3}[0 + 4(0.063 + 0.563) + 2(0.25) + 1]$$

$$\approx \frac{0.25}{3}[0 + 4(0.625) + 2(0.25) + 1] \approx \frac{0.25}{3}[0 + 2.5 + 0.5 + 1] \approx \frac{0.25}{3}[4] \approx 0.333$$

Example2: Evaluate $\int_0^{\pi/2}\sqrt{\cos(x)}dx$ using Simpson's rule with 6 equal intervals.

Solution:

Calculate uniform width: $h=\left(\frac{b-a}{n}\right)=\left(\frac{\frac{\pi}{2}-0}{6}\right)=\frac{\frac{\pi}{2}}{6}=\frac{\pi}{12}$

Calculate function value $f(x)$ (use width $h=\frac{\pi}{12}$ for 6 intervals i.e. $n=0,1,2,3,4,5,6$):

n	x	y_n	y=cosx	y=√cosx		
0	0	y_0	1.000	1.000	PI	3.142
1	0.262	y_1	0.966	0.983	$(y_1+y_3+y_5)$	2.333
2	0.523	y_2	0.866	0.931	(y_2+y_4)	1.638
3	0.785	y_3	0.707	0.841	$4(y_1+y_3+y_5)$	9.333
4	1.047	y_4	0.500	0.707	$2(y_2+y_4)$	3.276
5	1.308	y_5	0.259	0.509	$y_0+4(y_1+y_3+y_5)+2(y_2+y_4)+y_6$	13.638
6	1.57	y_6	0.001	0.028	(PI/36) $[y_0+4(y_1+y_3+y_5)+2(y_2+y_4)+y_6]$	1.190

Calculate area using Simpson's Rule:

$$\int_a^b ydx \approx \frac{h}{3}\left[y_0+4(y_1+y_3+y_5+.......)+2(y_2+y_4+y_6+.......)+y_n\right]$$

Therefore

$$\int_0^{\pi/2}\sqrt{\cos(x)}dx \approx \frac{\frac{\pi}{12}}{3}\left[y_0+4(y_1+y_3+y_5)+2(y_2+y_4)+y_6\right]$$

$$\approx \frac{\pi}{36}\left[1+4(2.333)+2(1.638)+0.028\right]\approx\frac{\pi}{36}\left[13.638\right]\approx 1.190$$

<u>Example3</u>: Evaluate $\int_0^1 3x^3 e^{-x^2}\,dx$ using Simpson's rule with 8 intervals.

Solution: Calculate uniform width: $h = \left(\frac{b-a}{n}\right) = \left(\frac{1-0}{8}\right) = \frac{1}{8}$

Calculate function value $f(x)$ (use width $h = \frac{1}{8}$ for 8 intervals i.e. $n = 0,1,2,3,4,5,6,7,8$):

n	x	y_n	x^3	$z=-x^2$	e^z	$3x^3e^z$		
0	0	y_0	0.000	0.000	1.000	0.000		
1	0.125	y_1	0.002	-0.016	0.984	0.006	$(y_1+y_3+y_5+y_7)$	1.573
2	0.25	y_2	0.016	-0.063	0.939	0.044	$(y_2+y_4+y_6)$	1.057
3	0.375	y_3	0.053	-0.141	0.869	0.137	$4(y_1+y_3+y_5+y_7)$	6.294
4	0.5	y_4	0.125	-0.250	0.779	0.292	$2(y_2+y_4+y_6)$	2.114
5	0.625	y_5	0.244	-0.391	0.677	0.496	$y_0+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)+y_8$	9.512
6	0.75	y_6	0.422	-0.563	0.570	0.721	$(1/24)\,[y_0+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)+y_8]$	0.396
7	0.875	y_7	0.670	-0.766	0.465	0.935		
8	1	y_8	1.000	-1.000	0.368	1.104		

Calculate area using Simpson's Rule:

$$\int_a^b y\,dx \approx \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \ldots\ldots) + 2(y_2 + y_4 + y_6 + \ldots\ldots) + y_n]$$

Therefore

$$\int_0^1 3x^3 e^{-x^2}\,dx \approx \frac{\frac{1}{8}}{3}[y_0 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) + y_8]$$

$$\approx \frac{1}{24}[0 + 4(1.573) + 2(1.057) + 1.104] \approx \frac{1}{24}[9.512] \approx 0.396$$

Example4: Evaluate $\int_{-1}^{1} x^3 dx$ using Simpson's rule with 4 intervals

Solution:

Calculate uniform width: $h = \left(\frac{b-a}{n}\right) = \left(\frac{1--1}{4}\right) = \frac{1+1}{4} = \frac{2}{4} = 0.5$

Calculate function value $f(x)$ (use width $h = 0.5$ for 4 intervals i.e. $n = 0,1,2,3,4$):

n	x	y_n	$y=x^3$		
0	-1	y_0	-1	(y_1+y_3)	0
1	-0.5	y_1	-0.125	$2y_2$	0
2	0	y_2	0	$4(y_1+y_3)$	0
3	0.5	y_3	0.125	$y_0+4(y_1+y_3)+2y_2+y_4$	0
4	1	y_4	1	$(1/2)(1/3)[y_0+4(y_1+y_3)+2y_2+y_4]$	0

$$\int_a^b ydx \approx \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ...) + 2(y_2 + y_4 + y_6 + ...) + y_n]$$

$$\int_{-1}^{1} x^3 dx \approx \frac{0.5}{3}[y_0 + 4(y_1 + y_3) + 2(y_2) + y_4] \approx \frac{0.5}{3}[-1 + 4(-0.125 + 0.125) + 2(0) + 1]$$

$$\approx \frac{0.5}{3}[-1 + 4(0) + 2(0) + 1] \approx \frac{0.5}{3}[-1+1] \approx \frac{0.5}{3}[0] \approx 0$$

TRY:

Evaluate $\int_0^4 x^{\frac{1}{2}} dx$ using Simpson's rule with 8 intervals.

Evaluate $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ using Simpson's rule with 6 intervals

Complex Integration

- In this chapter we define and study **complex integration**.
- Integration in the complex plane is important for two reasons:
  - In application there occur *real integrals* that can be evaluated by *complex integration*, whereas the usual methods of real integral fail.
  - Derivatives of analytic functions can be established by complex integration.
- In this chapter the basic concept of complex integral is introduced, and the **Cauchy's integral theorem** that is of most importance in complex analysis is discussed.
- The **Cauchy's integral formula** is addressed in Sec..3, and **derivatives of an analytic function** are discussed in Sec..4.

1 Line Integral in the Complex Plane

- As in real calculus we distinguish between **definite integrals** and **indefinite integrals**.
- An **indefinite integral** is a function whose derivative equals a given analytic function in a region.
- *Complex definite integrals* are called (complex) **line integrals** and are written as

$$\int_C f(z)dz$$

where the **integrand** $f(z)$ is integrated over a given curve C in the complex plane, called the **path of integration**.

- We may express such a curve by a *parametric representation*

$$z(t) = x(t) + iy(t) \qquad (a \le t \le b),$$

which is useful for integration of *non-analytic functions*.

Definition of the Complex Line Integral

- Let C be a **smooth curve** with *continuous* and *nonzero derivative* $\dot{z} = dz/dt$, where

$$z(t) = x(t) + iy(t) \qquad (a \le t \le b).$$

Also let $f(z)$ be a continuous function.

- We now subdivide the interval $a \le t \le b$ by points

$$t_0(=a),\ t_1, \cdots,\ t_{n-1},\ t_n(=b).$$

To this subdivision there corresponds a subdivision of C by points

$$z_0,\ z_1, \cdots,\ z_{n-1},\ z_n(=Z)$$

where $z_j = z(t_j)$.

- On each portion of subdivision of C we choose an arbitrary point, say, a point ζ_1 between z_0 and z_1, a point ζ_2 between z_1 and z_2, etc.

- Then we may form the sum

$$S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m \quad \text{where} \quad \Delta z_m = z_m - z_{m-1}.$$

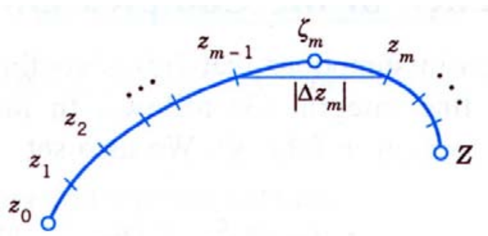


Fig. 1 . Complex line integral

- The limit of the finite sum obtained as $n \to \infty$ is called the **line integral** (or simply the integral) of $f(z)$ over the curve C and written by

$$\lim_{n\to\infty} S_n = \int_C f(z)dz$$

or by

$$\oint_C f(z)dz$$

if C is a closed path.

General Assumption. All paths of integration for complex line integrals are assumed to be **piecewise smooth**, that is, they consist of finitely many smooth curves.

Three Basic Properties Directly Implied by the Definition

1. **Linearity**. Integration is a linear operation.

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz.$$

2. **Sense reversal property**.

$$\int_{z_0}^{Z} f(z)dz = -\int_{Z}^{z_0} f(z)dz.$$

3. **Partitioning of path.**

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$

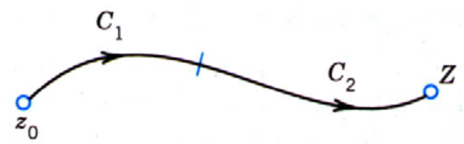


Fig. 2 Partitioning of path [formula (6)]

Existence of the Complex Line Integral

THEOREM. If $f(z)$ is *continuous* and C is at least *piecewise smooth*, the complex line integral of $f(z)$ exists such that

$$\int_C f(z)dz = \int_C udx - \int_C vdy + i\left[\int_C udy + \int_C vdx\right].$$

First Method: Indefinite Integration and Substitution of Limits

- This method is simpler than the next one, but is less general. It is restricted to *analytic functions*.

THEOREM 1 (Indefinite integration of analytic functions)

Let $f(z)$ be *analytic in a simply connected domain* D. Then there exists an *indefinite integral* of $f(z)$ in the domain D, that is, an *analytic function* $F(z)$ such that $F'(z) = f(z)$, and for all paths in D joining two points z_0 and z_1 we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0).$$

Note. We can write z_0 and z_1 instead of C, since we get the same value for all paths from z_0 to z_1.

EXAMPLE 1. $f(z) = z^2$.

$$\int_0^{1+i} z^2 dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i.$$

EXAMPLE 2. $f(z) = \cos z$.

$$\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z\Big|_{-\pi i}^{\pi i} = 2i \sinh \pi = 23.097i.$$

EXAMPLE 3. $f(z) = e^{z/2}$.

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2} dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0.$$

EXAMPLE 4. $f(z) = \dfrac{1}{z}$. D is the complex plane without zero and the real negative axis.

$$\int_{-i}^{i} \frac{dz}{z} = \mathrm{Ln}\,i - \mathrm{Ln}(-i) = \frac{i\pi}{2} - \left(-\frac{i\pi}{2}\right) = i\pi.$$

Second Method: Use of a Parametric Representation of the Path

- This method is not restricted to *analytic functions* but applies to any *continuous* complex function.

THEOREM 2. (Integration by the use of the path)

Let C be *piecewise smooth*, represented by $z = z(t)$, where $a \le t \le b$. Also let $f(z)$ be continuous on C. Then

$$\int_C f(z)dz = \int_a^b f\left[z(t)\right]\dot{z}(t)dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

PROOF. Let $z = x + iy$, then $\dot{z} = \dot{x} + i\dot{y}$. We also have $dx = \dot{x}dt$ and $dy = \dot{y}dt$. Thus

$$\begin{aligned}
\int_a^b f\left[z(t)\right]\dot{z}(t)dt &= \int_a^b (u+iv)(\dot{x}+i\dot{y})dt \\
&= \int_a^b (u+iv)(dx+idy) \\
&= \int_C \left[udx - vdy + i(udy + vdx)\right] \\
&= \int_C \left(udx - vdy\right) + i\int_C \left(udy + vdx\right) \\
&= \int_C f(z)dz.
\end{aligned}$$

Steps in applying Theorem 2

(A) Represent the path C in the form of $z(t)$ $(a \le t \le b)$.

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute $z(t)$ into $f\left[z(t)\right]$.

(D) Integrate $f\left[z(t)\right]\dot{z}(t)$ over t from a to b.

EXAMPLE 5. A basic result: Integral of $1/z$ around the unit circle

Show that

$$\oint_C \frac{dz}{z} = 2\pi i$$

where C is the unit circle, counter-clockwise.

Solution. We may represent the unit circle C by

$$z(t)=\cos t+i\sin t=e^{it} \qquad (0\le t\le 2\pi).$$

Since

$$f\left[z(t)\right]=\frac{1}{z(t)}=e^{-it} \quad \text{and} \quad \dot{z}(t)=ie^{it},$$

we have

$$\oint_C \frac{dz}{z}=\int_0^{2\pi} f\left[z(t)\right]\dot{z}(t)dt=\int_0^{2\pi} e^{-it}ie^{it}dt=2\pi i.$$

Caution! Theorem 1 cannot be applied to this problem, since the function $\frac{1}{z}$ is not analytic at $z=0$.

EXAMPLE 6. Integral of integer powers

Let $f(z)=(z-z_0)^m$ where m is an integer and z_0 a constant. Integrate counter-clockwise around the circle C of radius ρ with center at z_0.

Solution. We may represent C in the form

$$z(t)=z_0+\rho e^{it} \qquad (0\le t\le 2\pi).$$

Then we have

$$f(z)=(z-z_0)^m=\rho^m e^{imt}, \quad \dot{z}(t)=i\rho e^{it}.$$

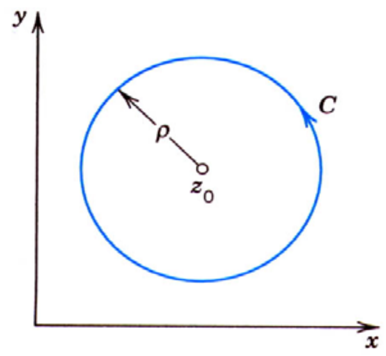


Fig. 3 Path in Example 6

Thus

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} f\left[z(t)\right]\dot{z}(t)dt = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it} dt$$
$$= i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt.$$

If $m \neq -1$, then

$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt = i\rho^{m+1}\frac{1}{i(m+1)}e^{i(m+1)t}\Big|_0^{2\pi}$$
$$= \frac{\rho^{m+1}}{m+1}\left[e^{i(m+1)2\pi}-1\right] = 0.$$

If $m = -1$, then

$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt = 2\pi i.$$

Hence

$$\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m=-1), \\ 0 & (m \neq -1 \text{ and integer}). \end{cases}$$

Dependence on path

- ☐ If we integrate a nonanalytic function $f(z)$ from a point z_0 to a point z_1 along *different paths*, the integral will in general have different values.

EXAMPLE 7. Integral of a nonanalytic function. Dependence on path

Integrate $f(z) = \operatorname{Re} z = x$ from 0 to $1+2i$ (a) along C^*, (b) along C_1 and C_2. Non-analytic, need to check.

Solution. **(a)** The path C^* in the figure can be represented by

$$z(t) = t + 2it \quad (0 \le t \le 1) \implies f\left[z(t)\right] = t, \quad \dot{z}(t) = 1+2i.$$

Thus

$$\int_{C^*} \operatorname{Re} z dz = \int_0^1 f[z(t)]\dot{z}(t)dt = \int_0^1 t(1+2i)dt = \frac{1}{2}(1+2i) = \frac{1}{2}+i.$$

(b) From the figure, we have

$$C_1: \ z(t)=t, \qquad \dot{z}(t)=1, \ f[z(t)]=x(t)=t \quad (0 \le t \le 1),$$
$$C_2: \ z(t)=1+it, \ \dot{z}(t)=i, \ f[z(t)]=x(t)=1 \quad (0 \le t \le 2\ )$$

Thus

$$\int_C \operatorname{Re} z dz = \int_{C_1} \operatorname{Re} z dz + \int_{C_2} \operatorname{Re} z dz = \int_0^1 t dt + \int_0^2 1 \cdot i dt = \frac{1}{2} + 2i.$$

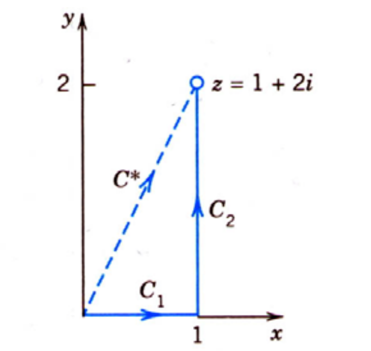


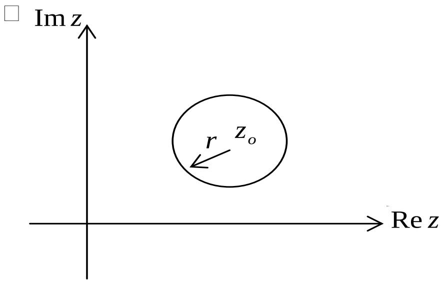
Fig. 4 Paths in Example 7

Curves and Regions in the Complex Plane

If x and y are real variables, then $z = x + iy$ is complex variable. Curves and regions in the complex plane may be represented by equations and inequalities involving z.

Example

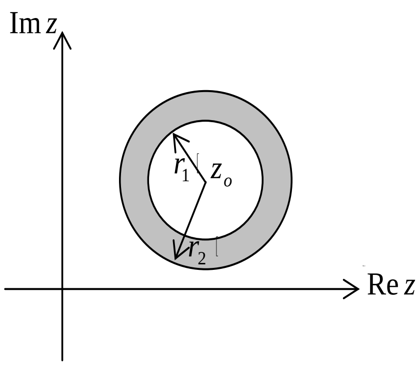
If $z_0 = x_0 + iy_0$ is a fixed complex number, $|z - z_0|$ is the distance between z and z_O, and $|z - z_0| = r$ is the equation of a circle centered at z_0 with radius r.



$|z - z_0| < r$ describes any point inside the circle and specifies an open circular disc.

$|z - z_0| \le r$ includes points on the circle and is a closed circular disc.

$|z - z_0| > r$ is the region exterior to the circle. Similarly, $r_1 < |z - z_0| < r_2$ describes an open annulus



2 Cauchy's Integral Theorem

- ☐ As discussed in Sec. 1, a line integral of a complex function $f(z)$ depends not merely on the endpoints of the path, but also on the choice of the path itself.
- ☐ However, if $f(z)$ is *analytic in a domain* D and D is *simply connected*, then the integral will not depend on path.
- ☐ This result follows from the famous **Cauchy's integral theorem,** the most important theorem in this chapter.
- ☐ To study Cauchy's integral theorem we need the following two concepts.
  - **A simple closed path** is a closed path that does not *intersect* or *touch itself*. For example, a circle is simple, but an 8-shaped curve is not.

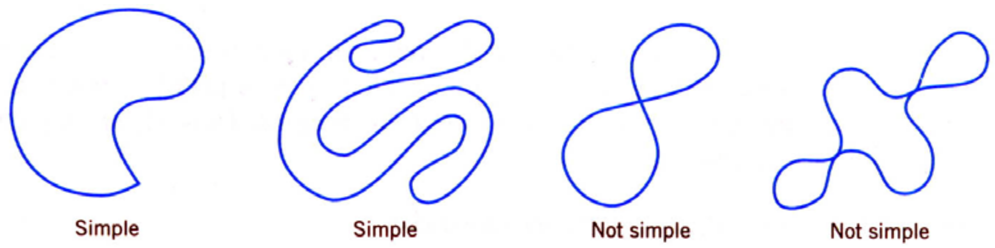


Fig. 5 Closed paths

  - **A simply connected domain** D is a domain such that every *simple closed path* in D encloses only points of D.

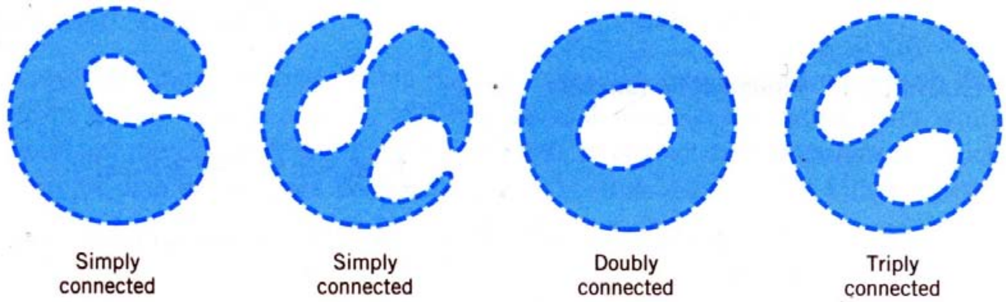


Fig. 6 Simply and multiply connected domains

  - **A doubly connected domain** is a domain that can be made simply connected by using a single barrier line.
  - **A triply connected domain** is a domain that can be made simply connected by using two barrier lines.

- ☐ A *simple closed path* is sometimes called **a contour** and an integral over such a path **a contour integral**.

THEOREM 1. Cauchy’s integral theorem

If $f(z)$ is *analytic in a simply connected domain* D, then for every *simple closed path* C in D,

$$\oint_C f(z)dz = 0.$$

PROOF. We have

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (udy + vdx).$$

Since $f(z)$ is analytic in D, $f'(z)$ exists in D. With assumption that $f'(z)$ is *continuous*, u and v have *continuous partial derivatives* in D. Hence from *Green’s theorem*, we have

$$\oint_C (udx - vdy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

$$\oint_C (udy + vdx) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy$$

where R is the region bounded by C. Since $f(z)$ is analytic, the *Cauchy-Riemann equations* $u_x = v_y$, $u_y = -v_x$ hold. Finally

$$\oint_C f(z)dz = 0.$$

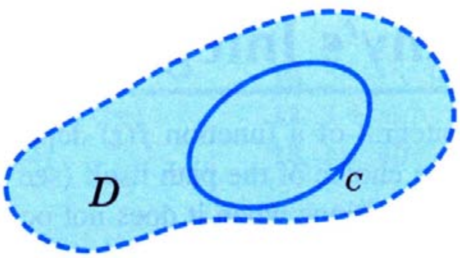


Fig. 7 Cauchy’s integral theorem

EXAMPLE 1. No singularities (Entire function)

$$\oint_C e^z dz = 0, \quad \oint_C \cos z dz = 0, \quad \oint_C z^n dz = 0 \ (n = 0,\ 1,\ 2, \cdots)$$

for any simply closed path.

EXAMPLE 2. Singularities outside contour

$$\oint_C \sec z dz = 0, \qquad \oint_C \frac{dz}{z^2+4} = 0$$

where C is the unit circle.

$\sec z = 1/\cos z$ is not analytic at $z = \pm\pi/2, \ \pm 3\pi/2, \cdots,$ but all these points lie outside C.

$1/(z^2+4)$ is not analytic at $z = \pm 2i,$ but these points also lie outside C.

EXAMPLE 3. Non-analytic function

$$\oint_C z^* dz = \int_0^{2\pi} e^{-it} ie^{it} dt = 2\pi i \neq 0$$

where C is the unit circle.

EXAMPLE 4. Analyticity sufficient, not necessary

$$\oint_C \frac{dz}{z^2} = 0$$

where C is the unit circle. (See Example 6 in Sec. 14.1)

==> The condition that $f(z)$ be analytic in D is *sufficient* rather than *necessary* for $\oint_C f(z)dz = 0.$

EXAMPLE 5. Simple connectedness essential

$$\oint_C \frac{1}{z} dz = 2\pi i \neq 0$$

for counter-clockwise integration around the unit circle.

C lies in the annulus $\frac{1}{2} < |z| < \frac{3}{2}$ where $\frac{1}{z}$ is analytic, but this domain is *not simply connected,* so that Cauchy's theorem cannot be applied.

THEOREM 2 (Independence of path)

If $f(z)$ is *analytic in a simply connected domain* D, then the integral of $f(z)$ is independent of path in D.

PROOF. Let z_1 and z_2 be any points in D. Integrate $f(z)$ as shown in Fig. 332, over a simple closed path. Then from Cauchy's integral theorem, we have

$$\int_{C_1} f dz + \int_{C_2^*} f dz = 0, \quad \text{thus} \quad \int_{C_1} f dz = -\int_{C_2^*} f dz.$$

Also

$$\int_{C_2^*} f dz = -\int_{C_2} f dz.$$

Finally

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

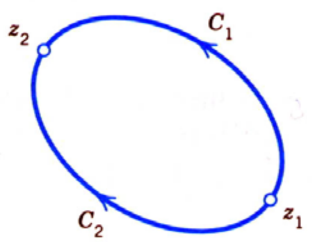


Fig. 8 Formula (2)

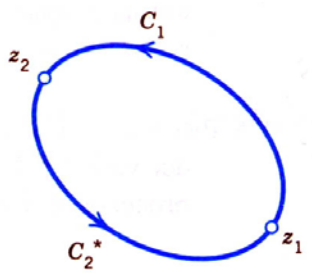


Fig. 9 Formula (2′)

Principle of Deformation of Path

As long as deforming path contains only points at which $f(z)$ is analytic, the integral retains the same value.

EXAMPLE 6. A basic result: Integral of integer powers

From Example 6 of Sec. 14.1 and the principle of deformation of path it follows that

$$\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

for **any counter-clockwise integration around any simple closed path containing** z_0 **in its interior.**

THEOREM 3. (Existence of an indefinite integral)

If $f(z)$ is *analytic in a simply connected domain* D, then there exists an *indefinite integral* $F(z)$ of $f(z)$ in D, which is analytic in D, and for all paths in D joining any two points z_0 and z_1 in D, the integral of D can be evaluated by

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

Cauchy's Theorem for Multiply Connected Domains

If a function $f(z)$ is analytic in any domain D^* that contains *a doubly connected domain* D and its boundary curves, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

where both integrals are taken counter-clockwise (or both clockwise).

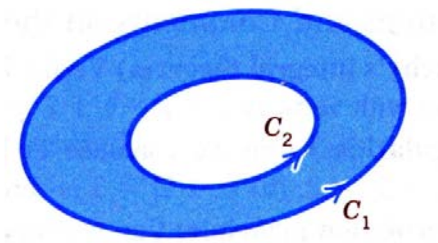


Fig. 10 Paths in (5)

PROOF. By two cuts $\tilde{C}_1$ and $\tilde{C}_2$ we cut D into two simply connected domains D_1 and D_2, where $f(z)$ is analytic.

By Cauchy's theorem the integral over the entire boundary of D_1 is zero.

The integral over the entire boundary of D_2 is also zero, and thus their sum is zero.

The integrals over the cuts $\tilde{C}_1$ and $\tilde{C}_2$ cancel each other. Hence

$$\text{Sum} = \oint_{C_1} f(z)dz + \oint_{C_2'} f(z)dz = \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0$$

Finally

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz.$$

For a triply connected domain, we have

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$

where C_2 and C_3 are inside C_1 and all the paths have the same direction.

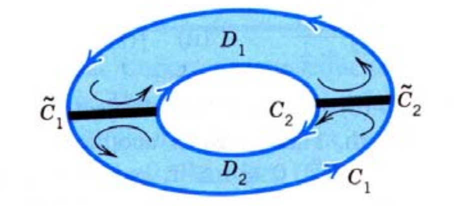


Fig. 11 Doubly connected domain

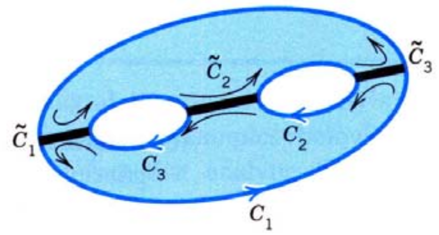


Fig. 12 Triply connected domain

3 Cauchy's Integral Formula

- ☐ The most important consequence of *Cauchy's integral theorem* is **Cauchy's integral formula**, which is useful for evaluating integrals.

THEOREM 1. (Cauchy's integral formula)

Let $f(z)$ be *analytic in a simply connected domain* D. Then for any point z_0 in D and *any simple closed path* C in D that encloses z_0, we have

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i \cdot f(z_0)$$

with the integration being taken counter-clockwise.

Alternatively

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

EXAMPLE 1. Cauchy's integral formula

$$\oint_C \frac{e^z}{z-2}\,dz = 2\pi i \cdot e^z\Big|_{z=2} = 2\pi i \cdot e^2 \approx 46.4268i$$

for any contour enclosing $z_0 = 2.$

EXAMPLE 2. Cauchy's integral formula

$$\oint_C \frac{z^3-6}{2z-i}\,dz = \oint_C \frac{\frac{1}{2}z^3-3}{z-\frac{1}{2}i}\,dz = 2\pi i\left(\tfrac{1}{2}z^3-3\right)\Big|_{z=i/2}$$
$$= \frac{\pi}{8} - 6\pi i$$

for any contour enclosing $z = \frac{1}{2}i.$

EXAMPLE 3. Integration around different contours

Integrate

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)}$$

counter-clockwise around each of the four circles.

Solution. Consider $g(z)$ is not analytic at $z = \pm 1.$

(a) The circle $|z-1| = 1$ encloses the point $z_0 = 1.$ Hence

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\frac{1}{z-1} \quad ==> \quad f(z) = \frac{z^2+1}{z+1}.$$

Thus

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_C \frac{f(z)}{z-1}\,dz = 2\pi i \cdot f(1)$$
$$= 2\pi i \cdot \frac{z^2+1}{z+1}\Big|_{z=1} = 2\pi i.$$

(b) gives the same as **(a)** by the *principle of deformation of path*.

(c) We have

$$g(z) = \frac{z^2+1}{z-1}\frac{1}{z+1} \quad ==> \quad f(z) = \frac{z^2+1}{z-1}.$$

Thus

$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{f(z)}{z+1}dz = 2\pi i \cdot f(-1)$$

$$= 2\pi i \cdot \left.\frac{z^2+1}{z-1}\right|_{z=-1} = -2\pi i.$$

(d) gives 0. Why?

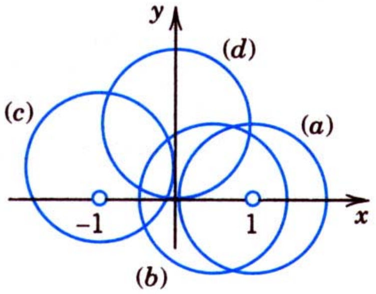


Fig. 12 Example 3

EXAMPLE 4. Use of partial fractions

Integrate $g(z) = (z^2-1)^{-1}\tan z$ around the circle $C: \quad |z| = 3/2$ (counter-clockwise).

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points are outside the contour.

$(z^2-1)^{-1}$ is not analytic at $z = \pm 1$. Using partial fraction, we have

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

From this we obtain

$$\oint_C \frac{\tan z}{z^2-1}dz = \frac{1}{2}\left[\oint_C \frac{\tan z}{z-1}dz - \oint_C \frac{\tan z}{z+1}dz\right]$$

$$= \frac{2\pi i}{2}\left[\tan 1 - \tan(-1)\right] = 2\pi i \cdot \tan 1 \approx 9.785i.$$

4 Derivatives of Analytic Functions

- In this section we use **Cauchy's integral formula** to show that *complex analytic functions have derivatives of all orders*.
- Indeed, if a *real function* is once differentiable, nothing follows about the *existence of second or higher derivatives*.

THEOREM 1. (Derivatives of an analytic function)

If $f(z)$ is *analytic in a domain* D, then it has *derivatives of all orders in* D, which are also analytic functions. The values of these derivatives at a point z_0 are given by the formulas

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz,$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz,$$

and in general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

C is any *simple closed path* in D and we integrate *counter-clockwise* around C.

EXAMPLE 1. Evaluation of line integrals

$$\oint_C \frac{\cos z}{(z-\pi i)^2}\,dz = 2\pi i \cdot (\cos z)'\Big|_{z=\pi i} = -2\pi i \cdot \sin \pi i$$

$$= 2\pi \sinh \pi$$

for any contour enclosing the point $z_0 = \pi i$ (counter-clockwise).

EXAMPLE 2.

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = \frac{2\pi i}{2!}(z^4 - 3z^2 + 6)''\Big|_{z=-i}$$

$$= \pi i\left(12z^2 - 6\right)\Big|_{z=-i} = -18\pi i$$

for any contour enclosing the point $-i$ (counter-clockwise).

EXAMPLE 3.

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}\,dz = 2\pi i\left(\frac{e^z}{z^2+4}\right)'\Bigg|_{z=1}$$

$$= 2\pi i\,\frac{e^z(z^2+4)-e^z\,2z}{(z^2+4)^2}\Bigg|_{z=1} \approx 2.050i$$

for any contour for which 1 lies inside and $\pm 2i$ lie outside (counter-clockwise).

5 Residues

A point z_0 is called a singular point of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0.

Definition 1. A singular point z_0 is said to be *isolated* if, in addition, there is a deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic.

Example 1. The function

$$\frac{z+1}{z^3(z^2+1)}$$

has the three isolated singular points $z = 0$ and $z = \pm i$.

5.1 Cauchy's Residue Theorem

Theorem (Cauchy). *Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points $z_k\ (k = 1, 2, \ldots, n)$ inside C, then*

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

$$\int_{C_k} f(z)dz = 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad (k = 1, 2, \ldots, n)$$

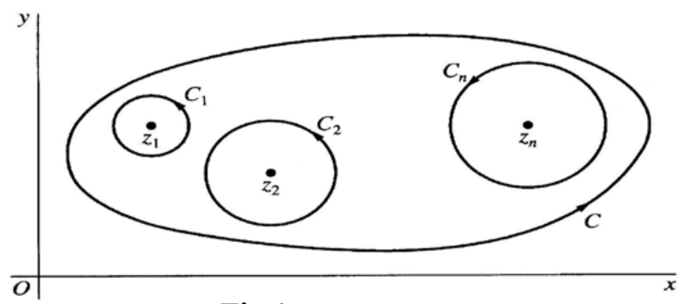


Fig.1

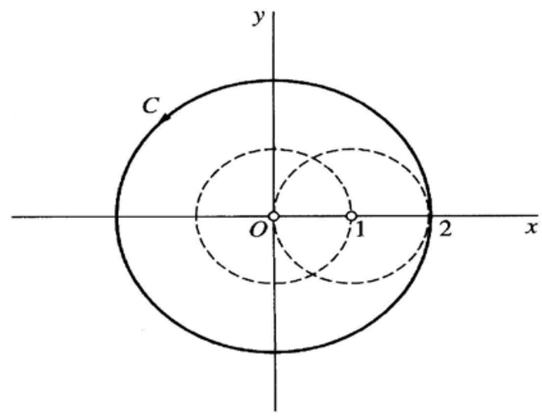
1.**Residues at Simple pole:**

$$res f(z)_{z=zo} = \lim_{z=z_0}[(z - z_0) f(z)]$$

2.**Residues at multiple pole:**

$$res f(z)_{z=zo} = \frac{1}{(m-1)!} \lim_{z=z_0}\left[\frac{d^{m-1}}{dz^{m-1}}(z - z_0)^m f(z)\right]$$

Example 1: Evaluate $\int_C \frac{5z-2}{z(z-1)}dz$ around the circle $|z|=2$, described counterclockwise



$$\int_{C_0} f(z)dz = 2\pi i(c_{-1}+b_{-1})$$

$$c_{-1} = \lim_{z\to z_0}\left[(z-z_0)f(z)\right] \qquad .$$

$$\int_C \frac{5z-2}{z(z-1)}dz = 2\pi i(c_{-1}+b_{-1}) = 10\pi i$$

Example2: Evaluate $\int_C \frac{50z}{(z+4)(z-1)^2}dz$ around the counterclockwise

$$\int_{C_0} f(z)dz = 2\pi i(c_{-1}+b_{-1})$$

$$res f(z)_{z=-4} = \lim_{z=-4}\left[\frac{50z}{(z-1)^2}\right] = -8$$

$$res f(z)_{z=1} = \frac{1}{(2-1)!}\lim_{z=1}\left[\frac{d^{2-1}}{dz^{2-1}}(z-1)^2 f(z)\right] = 8$$

$$\int_{C_0} f(z)dz = 2\pi i(8-8) = 0$$

$$\frac{1}{2}\int_C \frac{z^3}{z-j/2}dz = \frac{1}{2}\{2\pi j\times(-j/8)\} = \pi/8$$

Example 3

Integrate the function $\int_C \frac{dz}{z^2-1}$ around the circle $|z-1|=1$.

Solution

The integrand has singular points at $z=\pm 1$. We have $\int_C \frac{dz}{(z-1)(z+1)}$

Only the singular point $z = +1$ lies within the closed contour of integration, so we let $z_0 = +1$ and $f(z) = 1/(z+1)$:

$$\int_C \frac{dz}{z^2 - 1} = \int_C \frac{f(z)}{z-1} dz$$

Now $f(z_0) = 1/(1+1) = 1/2$. Thus

$$\int_C \frac{f(z)dz}{z-1} dz = 2\pi j \times (1/2) = \pi j$$

Example 4

Integrate the function $\int_C \frac{z^3}{(z+1)^3} dz$ where C is any closed contour enclosing the point $z = -1$.

Solution

We can write this in the form

$$\int_C \frac{f(z)}{(z-z_0)^3} dz$$

where

$$f(z) = z^3, \quad z_0 = -1$$

The function $f(z)$ is analytic on and inside the contour C so we can apply the formula

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi j}{n!} \frac{d^n f}{dz^n}\bigg|_{z_0}$$

Setting $n = 2$ puts the formula in the form we require:

$$\int_C \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi j}{2!} \frac{d^2 f}{dz^2}\bigg|_{z_0}$$

Now

$$\frac{d^2}{dz^2}(z^3) = 6z \rightarrow \frac{d^2 f}{dz^2}\bigg|_{z_0=-1} = -6$$

so that $\int_C \frac{f(z)}{(z-z_0)^3} dz = -6\pi j$

Example 5 : **Evaluate** $\oint_C \frac{1}{(z+1)^2(z+3)}dz$ **C:** $|z-2|$**=4**

Multiple pole at z=-1

I=2πj*Res$_{z=-1}$(f(z))

$$\operatorname{Re}s_{z=1}(f(z)) = \frac{1}{(2-1)!}\lim_{z\to-1}[\frac{d^{2-1}}{dz^{2-1}}(z+1)^2\frac{1}{(z+1)^2(z+3)}]$$

$$=\lim_{z\to-1}[\frac{d}{dz}\frac{1}{(z+3)}]=\frac{-1}{4} \quad =\frac{-1}{2}\pi j$$

Problem 1

(i) Under what conditions is Cauchy's Integral Formula valid?

(ii) Evaluate the integral $\int_C \frac{\cos z}{2z}dz$ where C is $|z|=1$ [ANSWER: πj]

(iii) Evaluate the integral $\int_C \frac{\tan z}{z-j}dz$ counterclockwise around the triangle with vertices $z=+1$, $z=2j$ and $z=-1$. Draw the triangle in the complex plane and show clearly the positions of the singular point(s) of the integrand. [ANSWER: $-2\pi\tanh(1)=-4.785$]

(iv) Evaluate the integral $\int_C \frac{2z+1}{z^2+z}dz$ where C is

(a) $|z|=\frac{1}{4}$, (b) $\left|z-\frac{1}{2}\right|=\frac{1}{4}$, (c) $|z|=2$ [ANSWER: (a) $2\pi j$, (b) 0, (c) $4\pi j$]

(v) Evaluate the integral $\int_C \frac{e^z}{z^2+1}dz$ where C is $|z|=2$ [ANSWER: $5.287j$]

Problem 2

[Formulas for derivatives of an analytic function]

Show that the three integrals $\int_C \frac{z^4}{(z-3j)^2}dz$, $\int_C \frac{\cos z}{z^2}dz$, and $\int_C \frac{e^{z^3}}{z^3}dz$, where C is the unit circle, are all zero