Nineveh University

College Of Electronic

Systems and Control Engineering Department

ENGINEERING ANALYSIS

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Purpose and Structure of the lecture

This lecture provides a comprehensive, thorough, and up-to-date treatment of engineering mathematics. It is intended to introduce students of engineering to those areas of applied mathematics that are most relevant for solving practical problems

The subject matter is arranged into four parts as follows:

A. Ordinary Differential Equations (ODEs) in Chapter 1
B. Laplace Transform in Chapter 2
C. Matrix Theory in Chapter 3
D. Multiple Integrals in Chapter 4

Textbook :

1. Calculus : By Finny and Thomas. Prentice Hall, 8th edition, 2002.

2. Calculus : By Weir, Hass and Thomas Prentice Hall , 12th edition 2010.

3. Advanced Engineering Mathematics: By Kreyszig 9th edition, 2006

4. Advanced Engineering Mathematics: By Kreyszig 10th edition, 2011

BASIC INTEGRATION RULES

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
$\frac{d}{dx}[C]=0$	$\int 0dx=C$
$\frac{d}{dx}[kx]=k$	$\int kdx=kx+C$
$\frac{d}{dx}[kf(x)]=kf'(x)$	$\int kf(x)dx=k\int f(x)dx+C$
$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$	$\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx+C$
$\frac{d}{dx}[x^n]=nx^{n-1}$	$\int x^n dx=\frac{x^{n+1}}{n+1}+C,\ n\neq -1$
$\frac{d}{dx}[\sin x]=\cos x$	$\int \cos xdx=\sin x+C$
$\frac{d}{dx}[\cos x]=-\sin x$	$\int \sin xdx=-\cos x+C$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 xdx=\tan x+C$
$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan xdx=\sec x+C$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int \csc^2 xdx=-\cot x+C$
$\frac{d}{dx}[\csc x]=-\csc x\cot x$	$\int \csc x\cot xdx=-\csc x+C$

Laplace Transforms of some Basic Functions

f(t)	L (f(t))	f(t)	L (f(t))
1. 1	$\frac{1}{s}$	9 $\cos^2 kt$	$\frac{s^2+2k^2}{s(s^2+4k^2)}$
2. t	$\frac{1}{s^2}$	10 e^{at}	$\frac{1}{s-a}$
3. t^n	$\frac{n!}{s^{n+1}}$	11 sinh k t	$\frac{k}{s^2-k^2}$
4. $t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$	12 cosh kt	$\frac{s}{s^2-k^2}$
5. $t^{1/2}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	13 $\sinh^2$ kt	$\frac{2k^2}{s(s^2-4k^2)}$
6. sin k t	$\frac{k}{s^2+k^2}$	14 $\cosh^2$ kt	$\frac{s^2-2k^2}{s(s^2-4k^2)}$
7. cos k t	$\frac{s}{s^2+k^2}$	15 $t\,e^{at}$	$\frac{1}{(s-a)^2}$
8. $\sin^2 kt$	$\frac{2k^2}{s(s^2+4k^2)}$	16 $t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$ **n a positive integer**
17. e^{at} sin kt	$\frac{k}{(s-a)^2+k^2}$	31. H (t-a)=$u_a(t)$	$\frac{e^{-as}}{s}, s>0$
18. e^{at} cos kt	$\frac{s-a}{(s-a)^2+k^2}$	32. δ (t)	**1**
19. e^{at} sinh kt	$\frac{k}{(s-a)^2-k^2}$	33. δ (t-to)	e^{-st_0}

20. e^{at} cosh kt	$\frac{s-a}{(s-a)^2-k^2}$	34. e^{at} f(t)	**F(s-a)**
21. t sin kt	$\frac{2ks}{{}^{y}(s^2+k^2)^2}$	35. f(t-a) H(t-a)	**$e^{-as}F(s)$**
22. t cos kt	$\frac{s^2-k^2}{(s^2+k^2)^2}$	36. $f^{(n)}(t)$	**s^n F(s)-s^{n-1}f(0) .....-$f^{(n-1)}$(0)**
23. sin kt + kt cos kt	$\frac{2ks^2}{(s^2+k^2)^2}$	37. $t^n f(t)$	**$(-1)^n \frac{d^n}{ds^n}$ F(s)**
24. sin kt-kt cos kt	$\frac{2k^3}{(s^2+k^2)^2}$	38. $\int_0^t f(u)\, g(t-u)du$	**F(s) G(s)**
25. t Sinh k t	$\frac{2ks}{(s^2-k^2)^2}$	39. $\frac{\sin at}{t}$	**arc tan** $\left(\frac{a}{s}\right)$
26. t cosh kt	$\frac{s^2+k^2}{(s^2-k^2)^2}$	40. $\frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
27. $\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$		
28. $\frac{ae^{at}-e^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$		
29. 1-cos kt	$\frac{k^2}{s(s^2+k^2)}$		
30. $\frac{e^{at}-e^{bt}}{t}$	$\ln\frac{s-a}{s-b}$		

CHAPTER 1

Differential Equations

A differential equation is an equation that involves one or more derivatives, or differentials. Differential equations are classified by:

1. ***Type:*** Ordinary or partial.
2. ***Order:*** The order of differential equation is the highest order derivative that occurs in the equation.
3. ***Degree:*** The exponent of the highest power of the highest order derivative.

A differential equation is an **ordinary D.Eqs**. if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variable, the D.Eqs. is a **partial D.Eqs**..

$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ is a partial D.Eqs..

Ex1:

$\frac{dy}{dx} = 5x + 3$ 1st order-1st degree

Ex2:

$\left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{d^2 y}{dx}\right)^5$ 3rd order-2nd degree

Ex3:

$4\frac{d^3 y}{dx^3} + \sin x \frac{d^2 y}{dx^2} + 5xy = 0$ 3rd order-1st degree

Exercise: Find the order and degree of these differential equations.

1. $\frac{dy}{dx} + \cos x = 0$ ans:1st order-1st degree
2. $3dx + 4y^2 dy = 0$ ans:1st order-1st degree
3. $\frac{d^2 y}{dx^2} + y = y^2$
4. $(y'')^2 + 2y' = x^2$
5. $y''' + 2(y'')^2 = xy$

Solution

The solution of the differential equation in the unknown function y and the independent variable x is a function y(x) that satisfies the differential equation.

Ordinary Differential Equations:

Ordinary Differential Equations are equations involve derivatives.

A. <u>First Order D.Eqs.</u>

1- Variable Separable.
2- Homogeneous.
3- Linear.
4- Exact.

1- Variable Separable:

A first order D.Eq. can be solved by integration if it is possible to collect all y terms with dy and all x terms with dx, that is, if it is possible to write the D.Eq. in the form

$$f(x)\ dx + g(y)\ dy = 0$$

then the general solution is:

$$\int f(x)dx + \int g(y)dy = c$$ where **c** is an arbitrary constant.

Ex.1:

Solve $\frac{dy}{dx} = e^{x+y}$

Sol.:

$$\frac{dy}{dx} = e^x \cdot e^y$$

$$\frac{dy}{e^y} = e^x dx$$

$$\int e^{-y} dy = \int e^x \cdot dx$$

$$-\int e^{-y} \cdot (-dy) = \int e^x dx \quad \Rightarrow \quad -e^{-y} = e^x + c$$

Ex.2:

Solve $(1+x)\frac{dy}{dx} = x(y^2+1)$

Sol.:

$$\int \frac{dy}{(y^2+1)} = \int \frac{x}{x+1} dx$$

$$\tan^{-1} y = \int dx - \int \frac{1}{x+1} dx$$

$$\tan^{-1} y = x - \ln|x+1| + c$$

Ex.3: Solve $\frac{dy}{dx}=(y-x)^2 \quad \cdots(1)$

Sol.: *put* $y-x=\mathrm{u}$, $\frac{\mathrm{dy}}{\mathrm{d}x}-1=\frac{du}{dx} \Rightarrow \frac{\mathrm{dy}}{\mathrm{d}x}=\frac{du}{dx}+1$ ….. (2)

$$\frac{du}{dx}+1=u^2 \Rightarrow \int\frac{\mathrm{du}}{\mathrm{u}^2-1}=\int dx$$

$$\therefore \int\left[\frac{1/2}{u-1}+\frac{-1/2}{u+1}\right]du=\int dx$$

$$\frac{1}{2}\left[\ln(\mathrm{u}-1)-\ln(\mathrm{u}+1)\right]=x+c$$

$$\frac{1}{2}\ln\frac{\mathrm{u}-1}{\mathrm{u}+1}=x+c$$

$$\frac{u-1}{u+1}=e^{2x+c}$$

Exercise: Separate the variables and solve**.**

1. $x(2y-3)dx+(x^2+1)dy=0$ ans: $(x^2+1)(2y-3)=c$
2. $dy=e^{x-y}\,dx$ ans: $e^y=e^x+c$
3. $\sin x\frac{dy}{dx}+\cosh 2y=0$ ans: $\sinh 2y-2\cos x=c$
4. $xe^y dy+\frac{x^2+1}{y}dx=0$ ans: $e^y(y-1)+\frac{x^2}{2}+\ln|x|=c$
5. $\sqrt{2xy}\frac{dy}{dx}=1$ ans: $\frac{\sqrt{2}}{3}y^{\frac{3}{2}}=x^{\frac{1}{2}}+c$

2- Homogeneous:

Some times a D.Eq. which variables can't be separated can be transformed by a change of variables into an equation which variables can be separated. This is the case with any equation that can be put into form:

$$\frac{dy}{dx} = f(\frac{y}{x}) \ldots (1)$$

Such an equation is called homogenous.

Put $\frac{y}{x} = u \quad \Rightarrow \quad y = ux, \ \frac{dy}{dx} = u + x \cdot \frac{du}{dx}$ and (1) becomes

$$x \cdot \frac{du}{dx} + u = f(u)$$

Ex.1:

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

Sol.:

$$\frac{dy}{dx} = \frac{1 + \frac{y^2}{x^2}}{\frac{y}{x}} \quad \Rightarrow \text{ homo. Put } \frac{y}{x} = u \quad \Rightarrow \quad \frac{dy}{dx} = x \cdot \frac{du}{dx} + u$$

$$x \cdot \frac{du}{dx} + u = \frac{1 + u^2}{u} \quad \Rightarrow \quad x \cdot \frac{du}{dx} = \frac{1 + u^2 - u^2}{u}$$

$$x \cdot \frac{du}{dx} = \frac{1}{u} \quad , \quad \int u \cdot du = \int \frac{dx}{x}$$

$$\frac{u^2}{2} = \ln x + c \quad \Rightarrow \quad \frac{y^2}{2x^2} = \ln x + c$$

Ex.2: Solve the homogenous D.Eq $xdy + 2ydx = 0$

Sol.: $xdy = 2ydx \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$ put $\frac{y}{x} = u \quad \Rightarrow \quad \frac{dy}{dx} = x \cdot \frac{du}{dx} + u$

$$x \cdot \frac{du}{dx} + u = 2u \qquad \ln|x| - \ln|u| = c \quad \Rightarrow \frac{x}{u} = c \Rightarrow \frac{x^2}{y} = c$$

Exercise: Show that the following differential equations are homogenous and solve.

1. $(x^2+y^2)dx+xy\,dy=0$ ans: $x^2(x^2+2y^2)=c$
2. $x^2dy+(y^2-xy)dx=0$ ans: $y = \frac{x}{\ln x - c}$
3. $(xe^{\frac{y}{x}} + y)dx - xdy = 0$ ans: $\ln|x| + e^{\frac{-y}{x}} = c$

3 - Linear

The equation of the form $\frac{dy}{dx}+p\cdot y=Q$ where P and Q are functions of only x or constant is called linear in y and $\frac{dy}{dx}$.

Find integrating factor $(I.f.)=e^{\int Pdx}$, then the general solution is

$$y\cdot(I.f.)=\int(I.f.)\ Q.dx$$

Ex.1: Solve $\frac{dy}{dx}-\frac{y}{x}=x\cdot e^{x}$

$P(x)=-\frac{1}{x},\quad Q(x)=x.e^{x}$

$(I.f.)=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=\frac{1}{x}$

Solution is

$$y\cdot\frac{1}{x}=\int\frac{1}{x}\cdot xe^{x}\cdot dx$$

$$\frac{y}{x}=e^{x}+c$$

Ex.2:

Solve $\frac{dy}{dx}+x.y=x$

$P=x,\ Q=x$

$(I.f.)=e^{\int xdx}=e^{\frac{x^2}{2}}$

Solution is

$$y\cdot e^{\frac{x^2}{2}}=\int e^{\frac{x^2}{2}}\cdot x\cdot dx$$

$$y\cdot e^{\frac{x^2}{2}}=e^{\frac{x^2}{2}}+c\Rightarrow y=1+ce^{\frac{-x^2}{2}}\ \text{is the solution}$$

Exercise:

1. $\frac{dy}{dx}+2y=e^{-x}$ ans: $y=e^{-x}+ce^{-2x}$
2. $x\frac{dy}{dx}+3y=\frac{\sin x}{x^2}$ ans: $x^3y=c-\cos x$
3. $xdy+ydx=ydy$ ans: $x=\frac{y}{2}+\frac{c}{y}$

4- Exact

The equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

General Solution is

$$c = \int M dx + \int (terms\ \ in\ \ N\ \ \text{do not contains}\ \ x)\text{dy}$$

Ex.1:

Show that the following D.Eq. are exact D.Eq.

a) $(3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 + 2x \quad , \quad \frac{\partial N}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The D.Eq. is exact.

b) $[x\cos(x+y) + \sin(x+y)]dx + (x\cos(x+y)dy = 0$

$$\frac{\partial M}{\partial y} = -x\sin(x+y) + \cos(x+y)$$

$$\frac{\partial N}{\partial x} = -x\sin(x+y) + \cos(x+y)$$

$\therefore$ the D.Eq. is exact.

Ex.2: Is the D.Eq. $\frac{dy}{dx} = -\frac{(x^2+y^2)}{2xy}$ exact or not?

Sol.

$$2xydy = -(x^2 + y^2)dx$$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \therefore \text{theD.Eq. is exact}$$

Ex.3:

Solve the exact D.Eqs. in Ex.1(a) above $(3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy = 0$

Sol.

$$c = \int (3x^2y + 2xy)dx + \int 2y\text{dy}$$

$$= 3\text{y} \cdot \frac{\text{x}^3}{3} + 2y \cdot \frac{\text{x}^2}{2} + 2 \cdot \frac{\text{y}^2}{3}$$

the solution is $\text{x}^3\text{y} + \text{x}^2\text{y} + \text{y}^2 = c$

Ex.4:

Solve $(x+y)dx+(x+y^2)dy=0$

Sol.

$$\frac{\partial M}{\partial y}=1 \quad , \quad \frac{\partial N}{\partial x}=1$$

$\therefore$ theD.Eq. is exact

$$c=\int Mdx+\int(terms\ \ in\ \ N\ \text{ do not contains } \text{x)dy}$$

$$=(x+y)dx+\int y^2\text{dy}$$

$$=\frac{\text{x}^2}{2}+xy+\frac{\text{y}^3}{3}$$

the solution is $\frac{\text{x}^2}{2}+\text{xy}+\frac{\text{y}^3}{3}=c$

Exercise:

1. $(2+ye^{xy})dx+(xe^{xy}-2y)dy=0$ ans: $c=2x+e^{xy}-y^2$
2. $(\tan x+\tan y)dy+(y\sec^2 x+\sec x\tan x)dx=0$ ans: $c=y\tan x-\ln\cos y+\sec x$
3. $(2xy+y^2)dx+(x^2+2xy-y)dy=0$ ans: $x^2y+y^2x-y^2/2=c$

Problems:

Solve the following differential equations:

1- $y\ln y dx+(1+x^2)dy=0$

2- $e^{x+2y}dy-e^{y-2x}dx=0$

3- $(2x+y)dx+(x-2y)dy=0$

4- $x\,dy=(y+x\cos^2(\frac{y}{x}))dx$

5- $x(\ln y-\ln x)dy=y(1+\ln y-\ln x)dx$

6- $x\,dy+(2y-x^2-1)dx=0$

7- $\cos y\,dx+(x\sin y-\cos^2 y)dy=0$

8- $(1+y^2)dx+(2xy+y^2+1)dy=0$

9- $(e^x+\ln y)dx+(\frac{x+y}{y})dy=0$

10- $x(1+e^y)dx+\frac{1}{2}(x^2+y^2)e^y dy=0$

B. Second Order Differential Equations:

The second order linear differential equations with constant coefficient has the general form is:

$$ay'' + by' + cy = F(x) \qquad \ldots(1),$$

where a, b and c are constants.

If $F(x) = 0$ then (1) is called homogenous.

If $F(x) \neq 0$ then (1) is called non homogenous.

Ex:

1) $y''-x^2y'+\sin x\, y=0$ is linear, 2nd order, homo.
2) $y''-(y')^2+ y=\sin x$ is non linear, 2nd order, non homo.
3) $y''+2yy'=\ln x$

a) Homogeneous.

b) Nonhomogeneous.

a. Undeterminant coefficients.

b. Variation of parameters.

a) The Second order linear homogenous D.Eq. with constant coefficient:

The general form is

$$ay'' + by' + cy = 0 \qquad \ldots(2)$$

where a, b and c are constants.

<u>The general solution</u>

Put $y'=Dy$ and $y''=D^2y$ in eq. (2) (D is an operator)

$\Rightarrow a D^2y+bDy+cy=0$

$\Rightarrow (aD^2 + bD + c)y = 0$ (using D-operator)

now substitute D by r and leave y then $ar^2 + br + c = 0$

is called **characteristic equation** of the differential equation and the solution of this equation (the roots r) give the solution of the differential equation where

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

There are two values of r :

1- ***real (equal and not equal).***
2- ***complex.***

Case 1: If $b^2 - 4ac \succ 0$ then r_1 and r_2 are distinct ($r_1 \neq r_2$) and real roots, and the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: If $b^2 - 4ac = 0$ then $r_1 = r_2 = \mathrm{r}$, and the general solution is:

$$y = (c_1 + c_2 x)e^{rx}$$

Case 3: If $b^2 - 4ac \prec 0$ then the roots are two complex conjugate roots $r = \alpha \pm i\beta$, $i = \sqrt{-1}$, and the general solution is:

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Ex.1: Solve $y'' - 2y' - 3y = 0$

Solution:

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$, $\mathrm{y} = 1$, $\mathrm{y}' = r$, $\mathrm{y}'' = r^2$

$(r+1)(r-3) = 0$

$r + 1 = 0 \quad \Rightarrow \quad r = -1$

$r - 3 = 0 \quad \Rightarrow \quad r = 3$

the general solution is

$y = c_1 e^{-x} + c_2 e^{3x}$

Ex.2: Solve $y'' - 6y' + 9y = 0$

Solution:

$y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 3$

$\therefore \quad y = (c_1 + c_2 x)e^{3x}$

Ex.3: Solve $y'' + y' + y = 0$

Solution:

$$y'' + y' + y = 0$$

$$r^2 + r + 1 = 0 \qquad \mathrm{a} = 1, \mathrm{b} = 1, \mathrm{c} = 1$$

$$r = \frac{-b \pm \sqrt{1 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} \qquad = \frac{-1 \pm \sqrt{3}i}{2}$$

$$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i \qquad \alpha = \frac{-1}{2} \quad , \quad \beta = \frac{\sqrt{3}}{2}$$

$$\therefore \quad y = e^{\frac{-1}{2}x} (c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x)$$

Exercise: solve

1. 4y''-12y'+5y=0 ans: $y=c_1e^{(1/2)x}+ c_2e^{(5/2)x}$
2. 3y''-14y'-5y=0 ans: $y=c_1e^{5x}+ c_2e^{(-1/3)x}$
3. 4y''+y=0 ans: $y=c_1\cos(x/2)+ c_2\sin(x/2)$
4. y''-8y'+16y=0 ans: $y=c_1e^{4x}+ c_2xe^{4x}$
5. y''+9y=0 ans: $y=c_1\cos 3x+ c_2\sin 3x$

b) The Second order linear non homogenous D.Eq. with constant coefficient:

The general form is: $ay'' + by' + cy = F(x)$ …(3)

where a, b and c are constants.

<u>The general solution</u>

If y_h is the solution of the homo. D.Eq. $ay'' + by' + cy = 0$, then the general solution of eq. (3) is:

$$y = y_h + y_p \qquad y_h \text{ (complementary function)}$$
$$y_p \text{ (porticular integral)}$$

i) y_h is y homo.
ii) y_p (use the table)

Methods of finding y_p:

There are two methods:

1) Undetermined coefficients:

In this method y_p depends on the roots r_1, and r_2 of characteristic equation and on the form of $F(x)$ in eq. (3) as follows:

$F(x)$	Choice of y_p	M.R.
kx^n nth degree polynomial	$k_n x^n + k_{n-1}x^{n-1} + k_{n-2}x^{n-2} + \cdots + k_0$	0
ke^{px}	ce^{px}	p
$k\sin\beta x$ or $k\cos\beta x$	$c_1\cos\beta x + c_2\sin\beta x$	$\mp i\beta$

Note: For repeated term (root), multiply by x.

Ex.1: Use the table to write y_p

1) $F(x) = 3x^2$, $k = 3$, $n = 2$

$$y_p = k_2 x^2 + k_1 x + k_0$$

2) $F(x) = \frac{-1}{2} e^{-3x}$, $k = \frac{-1}{2} \Rightarrow c$

$$y_p = ce^{-3x}$$

3) $F(x) = 2\cos 3x$, $k = 2$, $\beta = 3$

$$y_p = c_1 \cos 3x + c_2 \sin 3x$$

4) $F(x) = 3x^2 - 3x + 5 - 2e^{3x}$, $k = -3$, $c = -2$

$$y_p = k_2 x^2 + k_1 x + k_0 + ce^{3x}$$

5) $F(x) = 2\cos x - \frac{1}{2}\sin x$

$$y_p = c_1 \cos x + c_2 \sin x$$

6) $F(x) = \sin x - \cos 2x$

$$y_p = c_1 \cos x + c_2 \sin x + A\cos 2x + B\sin 2x$$

Ex.2: Solve $y'' - y' - 2y = 4x^2$ …. (1)

Solution:

y" –y'-2y=0

the char. Eq. r^2-r-2=0

$$(r+1)(r-2) = 0$$

$$r_1 = -1,\ r_2 = 2$$

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

f(x)=$4x^2$ is polynomial of second degree then

$$y_p = k_2 x^2 + k_1 x + k_0 \quad \text{... (2)}$$

$$\Rightarrow \quad y_p' = 2k_2 x + k_1 \ , \quad y_p'' = 2k_2$$

Substitution gives

$$2k_2-(2k_2x+k_1)-2(k_2x^2+k_1x+k_0)=4x^2$$

$coeff.of\ x^2: -2k_2=4 \quad \Rightarrow \quad k_2=-2$

$coeff.of\ x: -2k_2-2k_1=0 \quad \Rightarrow \ k_1=2$

$const: 2k_2-k_1-2k_0=0 \quad \Rightarrow \ k_0=-3$

$$y_p=-2x^2+2x-3$$

$$y_g=y_h+y_p=(c_1e^{-x}+c_2e^{2x})-2x^2+2x-3$$

Ex.3: $y''-y'-2y=e^{3x}$

Solution:

$y''-y'-2y=e^{3x} \qquad ....(1)$

$y''-y'-2y=0$

$r^2-r-2=0$

$(r-2)(r+1)=0 \Rightarrow \ r_1=2,\ r_2=-1$

$y_h=(c_1e^{2x}+c_2e^{-x})$, Put

$y_p=ce^{3x} \qquad ....(2)$

$y'_p=3ce^{3x} \quad , \quad y''_p=9ce^{3x}$

Substitute In (1)

$9ce^{3x}-3ce^{3x}-2ce^{3x}=e^{3x}$

$9c-3c-2c=1 \Rightarrow 4c=1 \quad \Rightarrow \quad c=\frac{1}{4}$

In (2)$\Rightarrow$ $y_p=\frac{1}{4}e^{3x}$

$$y_g=y_h+y_p=c_1e^{2x}+c_2e^{-x}+\frac{1}{4}e^{3x}$$

قاعدة التعديل Modification rule

1) اذا كان $F(x)=kx^n$ **وكان احد جذري المعادلة القياسية = 0 ← يضرب** y_p **السابق في** x.

2)

a - اذا كان $F(x)=ke^{px}$ **وكان احد جذري المعادلة القياسية = p ← يضرب** y_p **السابق في** x.

b - اذا كان $F(x)=ke^{px}$ **وكان جذري المعادلة القياسية = p ← يضرب** y_p **السابق في** x^2.

3) اذا كان $F(x)=\begin{cases} k\cos\beta x \\ k\sin\beta x \end{cases}$ **وكان** $\alpha=0$, $r=\mp i\beta$ **← يضرب** y_p **السابق في** x.

Ex.4: Solve y''+y= sinx

Solution:

y''+y=0

$r^2+1=0$, $r^2=-1 \Rightarrow r=\pm i$, $\alpha=0$, $\beta=1$

$y_h=c_1\cos x+c_2\sin x$

$y_p=x(c_3\cos x+c_4\sin x)$, $y'_p=x(-c_3\sin x+c_4\cos x)+(c_3\cos x+c_4\sin x)$

$y''_p=x(-c_3\cos x-c_4\sin x)+(-c_3\sin x+c_4\cos x)+(-c_3\sin x+c_4\cos x)$

Substitution gives

$-2c_3\sin x+2c_4\cos x=\sin x$

$-2c_3=1 \Rightarrow c_3=-1/2$, $2c_4=0 \Rightarrow c_4=0$ $\qquad y_g=c_1\cos x+c_2\sin x-\frac{x}{2}\cos x$

2- Variation of parameters.

Let $y_h=c_1u_1+c_2u_2$ be the homogenous solution of $ay''+by'+cy=F(x)$ and the particular solution has the form $y_p=u_1v_1+u_2v_2$ where v_1 and v_2 are unknown functions of x which must be determined, first solve the following linear equations for v'_1 and v'_2:

$v'_1u_1+v'_2u_2=0$

$v'_1u'_1+v'_2u'_2=F(x)$

which can be solved with respect to v'_1 and v'_2 by Grammar rule as follows

$$D=\begin{vmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{vmatrix},\quad D_1=\begin{vmatrix} 0 & u_2 \\ F(x) & u'_2 \end{vmatrix},\quad D_2=\begin{vmatrix} u_1 & 0 \\ u'_1 & F(x) \end{vmatrix}$$

and $v'_1=\dfrac{D_1}{D},\quad v'_2=\dfrac{D_2}{D}$

by integration of v'_1 and v'_2 with respect to x we can find v_1 and v_2.

Ex.1:

Solve $y''-y'-2y=e^{3x}$ ……. (1)

$y_h=c_1e^{-x}+c_2e^{2x}$, hence

$u_1=e^{-x} \quad\Rightarrow\quad u_1'=-e^{-x}$

$u_2=e^{2x} \quad\Rightarrow\quad u_2'=2e^{2x}$

$y_p=v_1u_1+v_2u_2$

$v_1'u_1+v_2'u_2=0$

$v_1'u_1'+v_2'u_2'=F(x)$

$v_1'(e^{-x})+v_2'(e^{2x})=0$

$v_1'(-e^{-x})+v_2'(2e^{2x})=e^{3x}$

Solving this system by Cramer rule gives

$$D=\begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix}=3e^{x},\quad D_1=\begin{vmatrix} 0 & e^{2x} \\ e^{3x} & 2e^{2x} \end{vmatrix}=-e^{5x},\quad D_2=\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{3x} \end{vmatrix}=e^{2x}$$

$$v'_1 = \frac{-e^{5x}}{3e^x} = \frac{-1}{3}e^{4x} \Rightarrow v_1 = \int \frac{-1}{3}e^{4x} = -\frac{1}{12}e^{4x},$$

$$v'_2 = \frac{e^{2x}}{3e^x} = \frac{1}{3}e^x \Rightarrow v_2 = \int \frac{1}{3}e^x = \frac{1}{3}e^x$$

$$\therefore y_p = -\frac{1}{2}e^{4x}e^{-x} + \frac{1}{3}e^x e^{2x} = \frac{1}{4}e^{3x}$$

the general solution is : $y = c_1e^{-x} + c_2e^{2x} + \frac{1}{4}e^{3x}$

Ex.2: solve

$$y''+y=\sec x$$

Solution:

$y''+y=0$

$r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r = \pm i \qquad \alpha=0, \beta=1$

$y_h=c_1\cos x + c_2\sin x, \quad u_1=\cos x, u_2=\sin x, f(x)=\sec x$

$y_p= v_1u_1+v_2u_2$

$= v_1\cos x +v_2\sin x$ then

$$v_1'(\cos x) + v_2'(\sin x) = 0$$
$$v_1'(-\sin x) + v_2'(\cos x) = \sec x$$

$$D = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1,$$

$$D_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \ \sec x = -\sin x \frac{1}{\cos x} = -\tan x,$$

$$D_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$v_1' = \frac{-\tan x}{1} = -\tan x \Rightarrow v_1 = \int \frac{-\sin x}{\cos x}dx = \ln|\cos x|$$

$$v'_2 = 1 \Rightarrow v_2 = \int dx = x$$

$y_p = \ln|\cos x| \cos x + x \sin x$

$y_g = c_1\cos x + c_2\sin x + \ln|\cos x| \cos x + x \sin x$

Exercise: Solve

1. $y''-2y'+y = e^x \ln x$ 2. $y''-2y'+y = \frac{e^x}{x^5}$ 3. $y''+4y=\sin^2 2x$

CHAPTER 2

Laplace Transform

Laplace transform has many applications in engineering such as in electrical circuit, mechanical vibration and control engineering. The main idea behind the Laplace Transform is that we can solve an equation containing differential equation and integral terms by transform the equation in *t* domain to one in *s* domain with the intention that the problem easier to solve. In mathematics, it is used for solving ordinary differential equation and integral equations. We begin our discussion with the definition of Laplace transform

Definition of Laplace Transform

Let $f(t)$ be a function defined over $[0,\infty)$. The integral

$$\int_0^\infty e^{-st} f(t)dt$$

is called the Laplace transform of $f(t)$ if the integral exist. We write

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)d$ t where $\mathcal{L}$ interpreted as an operator.

In general, the function to be transformed is denoted by a lowercase letter, while its Laplace transform will be denoted by the corresponding uppercase letter. For examples,

$$\mathcal{L}\{f(t)\} = F(s), \quad \mathcal{L}\{g(t)\} = G(s), \quad \text{and} \quad \mathcal{L}\{y(t)\} = Y(s).$$

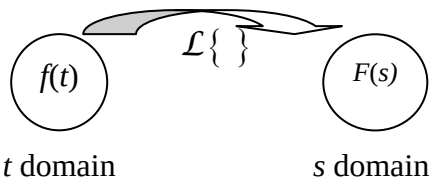


Figure 2.1: The Laplace transform operator

The Laplace transform is one-to-one function, that is for all ordinary functions, given $F(s)$ the corresponding function $f(t)$ is determined uniquely, just as $f(t)$ determines $F(s)$ uniquely.

I. Laplace Transform of Simple Functions

The Laplace transform of some important elementary functions are given in example below.

1. $\mathrm{L}[1] = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$

2. $\mathrm{L}[e^{at}] = \int_0^\infty e^{at}e^{-st}dt = \frac{e^{-(s-a)t}}{-(s-a)}\Bigg|_0^\infty = \frac{1}{s-a}$

3. $\mathrm{L}[e^{iat}] = \frac{1}{s-ia} \Rightarrow \mathrm{L}[\cos at + i\sin at] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

$\therefore \mathrm{L}[\cos at] = \frac{s}{s^2+a^2}$, and $\mathrm{L}[\sin at] = \frac{a}{s^2+a^2}$

4. $\mathrm{L}[\sinh at] = \mathrm{L}[\frac{e^{at}-e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} - \frac{1}{s+a}) = \frac{a}{s^2-a^2}$

$\mathrm{L}[\cosh at] = \mathrm{L}[\frac{e^{at}+e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} + \frac{1}{s+a}) = \frac{s}{s^2-a^2}$

5. Using the definition, determine the Laplace Transforms of $f(t) = t$

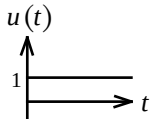
$$\mathcal{L}\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)d$$

$$\mathcal{L}\{f(t)\} = \int_{t=0}^{\infty} e^{-st}tdt$$

$$= \left[\frac{te^{-st}}{-s}\right]_{t=0}^{\infty} - \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{1}{s^2}$$

6. Unit step function

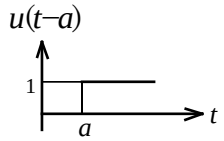
$$u(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$$



$$\mathrm{L}[u(t)]=\int_0^\infty u(t)e^{-st}dt$$

$$=\int_0^\infty e^{-st}dt=\frac{1}{s}$$

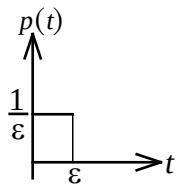
$$\Rightarrow u(t-a)=\begin{cases}1 & t>a\\ 0 & t<a\end{cases}$$



$$\mathrm{L}[u(t-a)]=\frac{e^{-as}}{s}$$

7. Unit impulse function (Dirac delta function)

(a) square wave function $p(t)=\begin{cases}\frac{1}{\varepsilon} & 0\le t\le \varepsilon\\ 0 & t>\varepsilon\end{cases}$



$$\mathrm{L}\ [p(t)]=\int_0^\infty p(t)e^{-st}dt=\int_0^\infty \frac{1}{\varepsilon}[u(t)-u(t-\varepsilon)]e^{-st}dt$$

$$=\frac{1}{\varepsilon}\left(\frac{1}{s}-\frac{e^{-\varepsilon s}}{s}\right)=\frac{1-e^{-\varepsilon s}}{s\varepsilon}$$

(b) **unit impulse function** $\delta(t)=\lim_{\varepsilon\to 0}p(t)$ **(also called singular function, Dirac delta function)**

$$\mathrm{L}\ [\delta(t)]=\mathrm{L}\ [\lim_{\varepsilon\to 0}p(t)]=\lim_{\varepsilon\to 0}\{\mathrm{L}\ [p(t)]\}=\lim_{\varepsilon\to 0}\frac{1-e^{-\varepsilon s}}{s\varepsilon}=\lim_{\varepsilon\to 0}\frac{se^{-\varepsilon s}}{s}=1$$

II. Standard Table of Laplace Transform

Table of Laplace Transform are shown below. These are generally used after analyzing some elementary function by using definition.

$f(t)=L^{-1}\{F(s)\}$	$F(s)=L\{f(t)\}$
a	$\frac{a}{s}$
$t^n,\ n=1,2,3,...$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

2. Properties of Laplace Transform

1. Linearity

$$\mathrm{L}\,[af(t)+bg(t)]=\int_0^\infty [af(t)+bg(t)]e^{-st}dt = a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt = aF(s)+bG(s)$$

Ex. 1

Find the Laplace transform of $\cos^2 t$.

$$\text{Solution}: \mathrm{L}\ [\cos^2 t] = \mathrm{L}\ [\frac{1+\cos 2t}{2}] = \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+2^2}\right) = \frac{s^2+2}{s(s^2+4)}$$

2.Shifting

The First Shifting Theorem: (frequency shifting property)

a) If $\mathcal{L}$ (f(t)) = F(s).

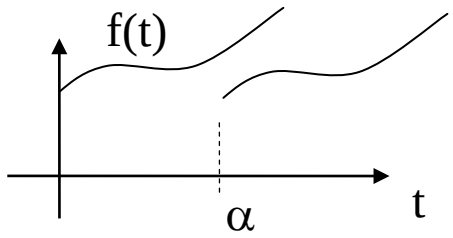
Then $\mathcal{L}$ $\{e^{at}f(t)\}$ = F (s-a)

$$F(s-a)=\int_0^\infty f(t)e^{-(s-a)t}dt = \int_0^\infty [e^{at}f(t)]e^{-st}dt = \mathrm{L}[e^{at}f(t)]$$

b) The second shifting theorem:(Delay)

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L}$ $\{f(t-a)u(t-a)\}$ = e^{-as} F(s)



$$\mathrm{L}[f(t-a)u(t-a)] = \int_0^\infty f(t-a)u(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt$$

Let $\tau = t-a$, then

$$\mathrm{L}[f(t-a)u(t-a)] = \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sa}F(s)$$

Ex. 2

What is the Laplace transform of the function: $f(t)=\begin{cases}0, & t<4\\ 2t^3, & t\ge 4\end{cases}$.

Solution: $f(t)=2t^3u(t-4)$

$$\mathrm{L}\,[f(t)] = \mathrm{L}\,\{2[(t-4)^3+12(t-4)^2+48(t-4)+64]\,u(t-4)\}$$

$$= 2e^{-4s}\left(\frac{3!}{s^4}+12\times\frac{2!}{s^3}+48\times\frac{1}{s^2}+\frac{64}{s}\right) = 4e^{-4s}\left(\frac{3}{s^4}+\frac{12}{s^3}+\frac{24}{s^2}+\frac{32}{s}\right)$$

3. Scaling

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L}$ (f(at)) = $\frac{1}{a}F(\frac{s}{a})$

$$\mathrm{L}[f(at)] = \int_0^\infty f(at)e^{-st}dt$$

Let $\tau = at$, then

$$\mathrm{L}[f(at)] = \int_0^\infty f(\tau)e^{-s\frac{\tau}{a}}d\frac{\tau}{a} = \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F(\frac{s}{a})$$

Ex. 3

Find the Laplace transform of cos2t.

Solution: $\because$ L $[\cos t] = \frac{s}{s^2+1}$

$$\therefore \mathrm{L}\ [\cos 2t] = \frac{1}{2}\frac{\frac{s}{2}}{(\frac{s}{2})^2+1} = \frac{s}{s^2+4}$$

4. Derivative

a)Time - differentiation property

(i) Transform of the First Derivative

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathrm{L}\ [f'(t)] = sF(s) - f(0)$

$$\mathrm{L}\ [f'(t)] = \int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}dt \qquad (2.1)$$

$$\mathrm{L}\ [f'(t)] = sF(s) - f(0)$$

where $u = e^{-st}$ $\quad v' = f'(t)$

$u' = -se^{-st}$ $\quad v = \int f'(t) = f(t)$

(ii)Transform of the Second Derivative

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

If $f(t), f'(t), f''(t), \cdots, f^{(n-1)}(t)$ are continuous, and $f^{(n)}(t)$ is piecewise continuous, and all of them are exponential order functions, then

$$\mathrm{L}\ [f^{(n)}(t)] = s^nF(s) - \sum_{i=1}^{n} s^{n-i}f^{(i-1)}(0)$$

b)frequency - differentiation property

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L}$ (t f(t)) $= -\dfrac{d}{ds}F(s)$

In general $\mathcal{L}(t^n f(t)) = -\dfrac{d^n}{ds^n}F(s)$

Ex. 4

Find the Laplace transform of $t\,e^t$.

Solution: $\mathcal{L}\ (e^t) = \dfrac{1}{s-1} \Rightarrow \mathcal{L}\ (te^t) = -\dfrac{d}{ds}\left(\dfrac{1}{s-1}\right) = \dfrac{1}{(s-1)^2}$

Ex. 5

$f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$, find L[$f'(t)$].

Solution: $f(t) = t^2[u(t) - u(t-1)]$

$$\mathcal{L}\ [f(t)] = \mathcal{L}\ [t^2u(t)] - \mathcal{L}\ [t^2u(t-1)] = \frac{2!}{s^3} - \mathcal{L}\ \{[(t-1)+1]^2 u(t-1)\}$$

$$= \frac{2}{s^3} - \mathcal{L}\ \{[(t-1)^2 + 2(t-1) + 1]u(t-1)\}$$

$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}\right)$$

$$\mathcal{L}\ [f'(t)] = sF(s) - f(0) - e^{-s}[f(1^+) - f(1^-)]$$

$$= \left[\frac{2}{s^2} - e^{-s}\left(\frac{2}{s^2} + \frac{2}{s} + 1\right)\right] - 0 - e^{-s}(0-1) = \frac{2}{s^2} - e^{-s}\left(\frac{2}{s^2} + \frac{2}{s}\right)$$

5. Integration

a)Time - integration property

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L}\int_{o}^{t} f(\tau)d\tau = \frac{F(s)}{s}$

$$\mathrm{L}[\int_0^t f(\tau)d\tau] = \int_0^\infty \int_0^t f(\tau)d\tau e^{-st}dt$$

$$= \frac{1}{-s}\left[e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty - \int_0^\infty f(t)e^{-st}dt\right] = \frac{1}{s}F(s)$$

$$\Rightarrow \mathrm{L}[\int_0^t\int_0^t\ldots\int_0^t f(t)dtdt\cdots dt] = \frac{1}{s^n}F(s)$$

b)frequency - integration property

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{s}^{\infty} F(s)\,ds$

$$\int_s^\infty F(s)ds = \int_s^\infty\int_0^\infty f(t)e^{-st}dtds = \int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$

$$= \int_0^\infty f(t)\frac{e^{-st}}{-t}\Big|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathrm{L}\ [\frac{f(t)}{t}]$$

$$\Rightarrow \int_s^\infty\int_s^\infty\cdots\int_s^\infty F(s)dsds\cdots ds = \mathrm{L}\ [\frac{1}{t^n}f(t)]$$

Ex. 6

Find (a) L $[\frac{1-e^{-t}}{t}]$ (b) L $[\frac{1-e^{-t}}{t^2}]$.

Solution: (a)L $[1-e^{-t}] = \frac{1}{s} - \frac{1}{s+1}$

$$\text{L}\ [\frac{1-e^{-t}}{t}] = \int_s^\infty (\frac{1}{s} - \frac{1}{s+1})ds = \ln s - \ln(s+1)\Big|_s^\infty = \ln\frac{s}{s+1}\Big|_s^\infty$$

$$= 0 - \ln\frac{s}{s+1} = \ln\frac{s+1}{s}$$

$$(b)\text{L}\ [\frac{1-e^{-t}}{t^2}] = \int_s^\infty \ln\frac{s+1}{s}ds = s\ln\frac{s+1}{s}\Big|_s^\infty - \int_s^\infty s(\frac{1}{s+1} - \frac{1}{s})ds$$

$$= s\ln\frac{s+1}{s}\Big|_s^\infty + \int_s^\infty \frac{1}{s+1}ds = \left[s\ln\frac{s+1}{s} + \ln(s+1)\right]_s^\infty$$

$$= \left[(s+1)\ln(s+1) - s\ln s\right]_s^\infty = s\ln s - (s+1)\ln(s+1)$$

Ex. 7

Find (a) $\int_0^\infty \frac{\sin kt\, e^{-st}}{t}dt$ (b) $\int_{-\infty}^\infty \frac{\sin x}{x}dx$.

Solution: $(a)\int_0^\infty \frac{\sin kte^{-st}}{t}dt = \text{L}\ [\frac{\sin kt}{t}]$

$$\because \text{L}\ [\sin kt] = \frac{k}{s^2+k^2}$$

$$\text{L}\ [\frac{\sin kt}{t}] = \int_s^\infty \frac{k}{s^2+k^2}ds = \frac{1}{k}\int_s^\infty \frac{1}{(\frac{s}{k})^2+1}ds$$

$$= \tan^{-1}\frac{s}{k}\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{k}$$

$$(b)\int_{-\infty}^\infty \frac{\sin x}{x}dx = 2\int_0^\infty \frac{\sin x}{x}dx$$

$$= 2\lim_{\substack{k\to 1\\ s\to 0}}\int_0^\infty \frac{\sin kte^{-st}}{t}dt$$

$$= 2\lim_{\substack{k\to 1\\ s\to 0}}(\frac{\pi}{2} - \tan^{-1}\frac{s}{k}) = \pi$$

6. Convolution theorem

If $\mathcal{L}\{ f(t) \} = F(s)$
and $\mathcal{L}\{ g(t) \} = G(s)$
then the convolution of $f(t)$ and $g(t)$ is denoted by $(f * g)(t)$, is defined by

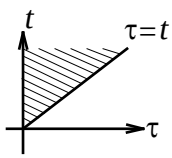
$$\boxed{(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau}$$

and the Laplace transform of the convolution of two functions is the product of the separate Laplace transforms:

$$\boxed{\mathcal{L}\{(f*g)(t)\} = F(s)G(s)}$$

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty \int_0^t f(\tau)g(t-\tau)d\tau e^{-st}dt$$

$$= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dtd\tau = \int_0^\infty f(\tau)\int_\tau^\infty g(t-\tau)e^{-st}dtd\tau$$



Let $u = t - \tau$, $du = dt$, then

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty f(\tau)\int_0^\infty g(u)e^{-s(u+\tau)}dud\tau$$

$$= \int_0^\infty f(\tau)e^{-s\tau}d\tau\int_0^\infty g(u)e^{-su}du = F(s)G(s)$$

Ex. 8

Find the Laplace transform of $\int_0^t e^{t-\tau}\sin 2\tau\, d\tau$.

Solution: $\because \mathcal{L}\ [e^t] = \dfrac{1}{s-1}$, $\mathcal{L}\ [\sin 2t] = \dfrac{2}{s^2+4}$

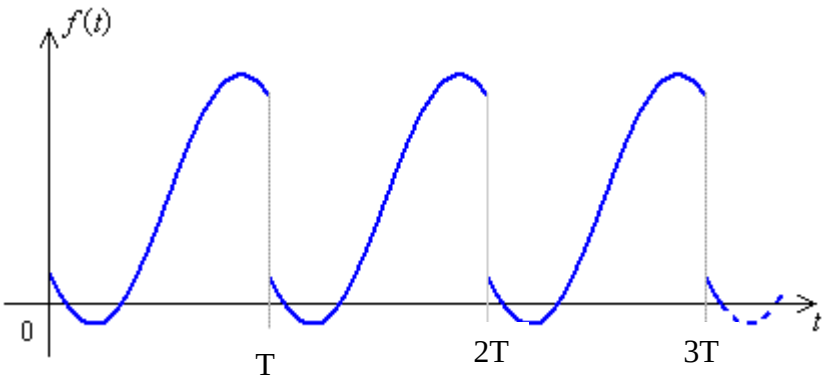
$$\therefore \mathcal{L}\left[\int_0^t e^{t-\tau}\sin 2t\, d\tau\right] = \mathcal{L}\ [e^t * \sin 2t] = \mathcal{L}\ [e^t]\cdot\mathcal{L}\ [\sin 2t]$$

$$= \frac{1}{s-1}\cdot\frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

7. Periodic Function: $f(t+T)=f(t)$

If the constant $T>0$ and $f(t+T) = f(t)$ for all $t>0$, then $f(t)$ is a **periodic function** of t, with period T.

Example of a periodic function (with one finite discontinuity in each period):



$$L\ [f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

and $\int_T^{2T} f(t)e^{-st}dt = \int_0^T f(u+T)e^{-s(u+T)}du = e^{-sT}\int_0^T f(u)e^{-su}du$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st}dt = e^{-2sT}\int_0^T f(u)e^{-su}du$$

$$\therefore L\ [f(t)] = (1+e^{-sT}+e^{-2sT}+\cdots\cdots)\int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

Ex. 9

Find the Laplace transform of $f(t) = \frac{k}{p}t,\ 0<t<p,\ f(t+p) = f(t)$.

Solution: $L\ [f(t)] = \frac{1}{1-e^{-ps}}\int_0^p \frac{k}{p}te^{-st}dt$

$$= \frac{1}{1-e^{-ps}}\frac{k}{p}[\frac{1}{-s}(te^{-st}\Big|_0^p - \int_0^p e^{-st}dt)]$$

$$= \frac{-k}{ps(1-e^{-ps})}(te^{-st}+\frac{1}{s}e^{-st})\Big|_0^p$$

$$= \frac{-k}{ps(1-e^{-ps})}(pe^{-sp}+\frac{e^{-sp}}{s}-\frac{1}{s})$$

8. Initial Value Theorem:

$\because$ L $[f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty} sF(s) - f(0) \Rightarrow 0 = \lim_{s\to\infty} sF(s) - f(0)$

we get initial value theorem $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Deduce general initial value theorem : $\lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{s\to\infty}\frac{F(s)}{G(s)}$

9. Final Value Theorem:

L $[f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to 0} sF(s) - f(0) \Rightarrow$

$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0) \Rightarrow$ final value theorem : $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

General final value theorem : $\lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{s\to 0}\frac{F(s)}{G(s)}$

Ex. 10

Find L $[\int_0^t \frac{\sin x}{x}dx]$.

Solution : Let $f(t) = \int_0^t \frac{\sin x}{x}dx \Rightarrow f'(t) = \frac{\sin t}{t},\ f(0) = 0$

$$\text{L}\ [tf'(t)] = \text{L}\ [\sin t] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}\text{L}\ [f'(t)] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = \frac{1}{s^2+1} \Rightarrow \frac{d}{ds}[sF(s)] = -\frac{1}{s^2+1}$$

$$sF(s) = -\tan^{-1}s + C$$

From the initial value theorem, we get

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$0 = -\frac{\pi}{2} + C \quad \therefore C = \frac{\pi}{2}$$

$$sF(s) = \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\frac{1}{s}$$

$$F(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

[Exercise] Find L $\left[\int_t^{\infty} \frac{e^{-x}}{x} dx\right]$.

Solution: Let $f(t) = \int_x^{\infty} \frac{e^{-x}}{x} dx \Rightarrow f'(t) = -\frac{e^{-t}}{t}, \lim_{t\to\infty} f(t) = 0$

$$L\ [tf'(t)] = L\ [-e^{-t}] = -\frac{1}{s+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = -\frac{1}{s+1}$$

$$\frac{d}{ds}[sF(s)] = \frac{1}{s+1}$$

$$sF(s) = \ln(s+1) + C$$

From the final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$0 = 0 + C \Rightarrow C = 0, \text{ and } F(s) = \frac{\ln(s+1)}{s}$$

[Note] $\int_0^t \frac{\sin x}{x} dx$, and $\int_t^{\infty} \frac{e^{-x}}{x} dx$ are called sine, and exponential integral function, respectively.

[Exercises]

Find the Laplace transform of the problems:

1. $e^{-\alpha t}(A\cos\beta t + B\sin\beta t)$ 2. $t^2\cos t$ 3. $u(t-\pi)\cos t$

4. $\int_t^{\infty} \frac{\cos x}{x} dx$ (cosine integral function) 5. Find the value of the integral $\int_0^{\infty} te^{-2t}\cos t dt$

6. Find the value of the integral $\int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} dt$

[Ans.] 1. $\frac{A(s+\alpha) + B\beta}{(s+\alpha)^2 + \beta^2}$ 2. $\frac{2s(s^2-3)}{(s^2+1)^3}$ 3. $\frac{-se^{-\pi s}}{s^2+1}$

4. $\frac{\ln(s^2+1)}{2s}$ 5. $\frac{3}{25}$ 6. $\ln 3$

I. Properties of Laplace Transform

Property	Original Function	Transformed Function
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$
Shifting	$f(t-a)\,u(t-a)$	$e^{-as}F(s)$
	$e^{at}f(t)$	$F(s-a)$
Scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots-f^{(n-1)}(0)$
	$(-t)^nf(t)$	$\frac{d^nF(s)}{ds^n}$
Integration	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
	$\frac{1}{t}f(t)$	$\int_s^\infty F(s)ds$
Convolution	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
Periodic Function	$f(t)=f(t+T)$	$\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$
Initial Value Theorem	$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$	
	$\lim_{t\to 0}\frac{f(t)}{g(t)}=\lim_{s\to\infty}\frac{F(s)}{G(s)}$	
Final Value Theorem	$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$	
	$\lim_{t\to\infty}\frac{f(t)}{g(t)}=\lim_{s\to 0}\frac{F(s)}{G(s)}$	

3.Inverse Laplace Transform

I. Inversion from Basic Properties

1. Linearity

Ex. 1.

$(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}]$ $(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}]$.

Solution: $(a)\mathcal{L}^{-1}[\frac{2s+1}{s^2+4}]=\mathcal{L}^{-1}[2\frac{s}{s^2+2^2}+\frac{1}{2}\frac{2}{s^2+2^2}]=2\cos 2t+\frac{1}{2}\sin 2t$

$(b)\mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}]=\mathcal{L}^{-1}[4\frac{s}{s^2-4^2}+\frac{4}{s^2-4^2}]=4\cosh 4t+\sinh 4t$

2. Shifting

Ex. 2.

$(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]$ $(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]$.

Solution: $(a)\ \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]=\mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]$

$\because \mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}]=e^{-t}\sin t$

and $\mathcal{L}\ [f(t-a)u(t-a)]=e^{-as}F(s)$

$\therefore \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]=e^{-(t-\pi)}\sin(t-\pi)u(t-\pi)=-e^{-(t-\pi)}\sin tu(t-\pi)$

$(b)\ \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]=\mathcal{L}^{-1}[\frac{2(s+\frac{3}{2})}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}]=2e^{-\frac{3}{2}t}\cosh\frac{t}{2}$

3. Scaling

Ex. 3.

$\mathcal{L}^{-1}[\frac{4s}{16s^2-4}]$.

Solution: $\mathcal{L}^{-1}[\frac{4s}{16s^2-4}]=\mathcal{L}^{-1}[\frac{4s}{(4s)^2-2^2}]=\frac{1}{4}\cosh 2\cdot\frac{1}{4}t=\frac{1}{4}\cosh\frac{t}{2}$

4. Derivative

Ex. 4.

$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]$ $(b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]$.

solution: $(a)\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}\Rightarrow\mathcal{L}[t\sin\omega t]=-\frac{d}{ds}(\frac{\omega}{s^2+\omega^2})=\frac{2\omega s}{(s^2+\omega^2)^2}$

Let $F(t)=t\sin\omega t\Rightarrow\mathcal{L}[F'(t)]=s\cdot\frac{2\omega s}{(s^2+\omega^2)^2}-F(0)$

$$\mathcal{L}[F'(t)]=2\omega\frac{s^2}{(s^2+\omega^2)^2}=2\omega[\frac{(s^2+\omega^2)-\omega^2}{(s^2+\omega^2)^2}]=2\omega[\frac{1}{s^2+\omega^2}-\frac{\omega^2}{(s^2+\omega^2)^2}]$$

$$=2\mathcal{L}[\sin\omega t]-\frac{2\omega^3}{(s^2+\omega^2)^2}$$

$$\frac{1}{(s^2+\omega^2)^2}=\frac{1}{2\omega^3}\cdot\mathcal{L}[2\sin\omega t-F'(t)]$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}\cdot[2\sin\omega t-F'(t)]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

(b) Let $\mathcal{L}[f(t)]=\ln\frac{s+a}{s+b}=\ln(s+a)-\ln(s+b)$

$$\mathcal{L}[tf(t)]=-\frac{d}{ds}[\ln(s+a)-\ln(s+b)]=\frac{1}{s+b}-\frac{1}{s+a}=\mathcal{L}[e^{-bt}-e^{-at}]$$

$$\therefore f(t)=\frac{e^{-bt}-e^{-at}}{t}$$

5. Integration

Ex. 5.

$(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})]$ $(b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]$.

Solution: $(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})]=\mathcal{L}^{-1}[\frac{1}{s(s+1)}-\frac{1}{s^2(s+1)}]=\int_0^t e^{-t}dt-\int_0^t\int_0^t e^{-t}dtdt$

$$=-(e^{-t}-1)+\int_0^t(e^{-t}-1)dt=-(e^{-t}-1)-(e^{-t}-1)-t=2-2e^{-t}-t$$

$$(b)\mathcal{L}[e^{-bt}-e^{-at}]=\frac{1}{s+b}-\frac{1}{s+a}$$

$$\mathcal{L}[\frac{e^{-bt}-e^{-at}}{t}]=\int_s^\infty(\frac{1}{s+b}-\frac{1}{s+a})ds=\ln\frac{s+b}{s+a}\Big|_s^\infty=\ln\frac{s+a}{s+b}$$

$$\therefore\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]=\frac{e^{-bt}-e^{-at}}{t}$$

6. Convolution

Ex. 6.

$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]$ $(b)\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}]$.

Solution: $(a)\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}\Rightarrow\mathcal{L}[\frac{1}{\omega}\sin\omega t]=\frac{1}{s^2+\omega^2}$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{\omega^2}\int_0^t\sin\omega\tau\sin\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega^2}\int_0^t\frac{1}{2}[\cos(\omega\tau-\omega t+\omega\tau)-\cos(\omega\tau+\omega t-\omega\tau)]d\tau$$

$$=\frac{1}{2\omega^2}\int_0^t[\cos(2\omega\tau-\omega t)-\cos\omega t]d\tau=\frac{1}{2\omega^2}\left[\frac{1}{2\omega}\sin(2\omega\tau-\omega t)-\tau\cos\omega t\right]_0^t$$

$$=\frac{1}{2\omega^2}\{[\frac{1}{2\omega}(\sin\omega t-\sin(-\omega t)]-t\cos\omega t\}=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

$(b)\mathcal{L}[\frac{1}{\omega}\sin\omega t]=\frac{1}{s^2+\omega^2}$ $\mathcal{L}[\cos\omega t]=\frac{s}{s^2+\omega^2}$

$$\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}]=\frac{1}{\omega}\int_0^t\sin\omega\tau\cos\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega}\int_0^t\frac{1}{2}[\sin(\omega\tau+\omega t-\omega\tau)+\sin(\omega\tau-\omega t+\omega\tau)]d\tau$$

$$=\frac{1}{2\omega}\int_0^t[\sin\omega t+\sin(2\omega\tau-\omega t)]d\tau=\frac{1}{2\omega}\left[\tau\sin\omega t+\frac{-1}{2\omega}\cos(2\omega\tau-\omega t)\right]_0^t$$

$$=\frac{1}{2\omega}\{t\sin\omega t-\frac{1}{2\omega}[\cos\omega t-\cos(-\omega t)]\}=\frac{t}{2\omega}\sin\omega t$$

[Exercises] 1. $\mathcal{L}^{-1}[s^{-3/2}]$ 2. $\mathcal{L}^{-1}[\frac{2n\pi T}{T^2s^2+(2n\pi)^2}]$ 3. $\mathcal{L}^{-1}[\frac{s-4}{s^2-8s-9}]$

4. $\mathcal{L}^{-1}[\frac{se^{-2\pi s/3}}{s^2+1}]$ 5. $\mathcal{L}^{-1}[\frac{1}{s}\tan^{-1}\frac{1}{s}]$ 6. $\mathcal{L}^{-1}[\ln\frac{s-1}{s+1}]$ 7. $\mathcal{L}^{-1}[\frac{s+1}{s^2+s+1}]$

8. $\mathcal{L}^{-1}[\ln(1+\frac{1}{s^2})]$ 9. $\mathcal{L}^{-1}[\frac{1}{s^2(s+1)^2}]$ 10. $\mathcal{L}^{-1}[\frac{\sqrt{s}+1}{s^3}]$

[Ans.] 1. $2\sqrt{t}/\Gamma(\frac{1}{2})$ or $2\sqrt{\frac{t}{\pi}}$ 2. $\sin\frac{2n\pi t}{\mathrm{T}}$ 3. $e^{4t}\cosh 5t$ 4. $\cos(t-\frac{2\pi}{3})u(t-\frac{2\pi}{3})$

5. $\int_0^t\frac{\sin u}{u}du$ 6. $\frac{2\sinh t}{t}$ 7. $e^{-\frac{t}{2}}(\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)$ 8. $\frac{2(1-\cos t)}{t}$

9. $e^{-t}(t+2)+t-2$ 10. $\frac{4}{3\sqrt{\pi}}t^{3/2}+\frac{1}{2}t^2$

II. Partial Fraction

If $F(s)=\dfrac{P(s)}{Q(s)}$, where $\deg[P(s)]<\deg[Q(s)]$

1. $Q(s)=0$ with unrepeated factors $(s-a_i)$

$$\frac{P(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots+\frac{A_n}{s-a_n}$$

$$A_k=\lim_{S\to a_k}[\frac{P(s)}{Q(s)}(s-a_k)]$$

Ex. 7.

$$L^{-1}[\frac{s+1}{s^3+s^2-6s}].$$

Solution: $\dfrac{s+1}{s^3+s^2-6s}=\dfrac{s+1}{s(s-2)(s+3)}=\dfrac{A_1}{s}+\dfrac{A_2}{s-2}+\dfrac{A_3}{s+3}$

$$A_1=\lim_{s\to 0}\frac{s+1}{(s-2)(s+3)}=-\frac{1}{6}$$

$$A_2=\lim_{s\to 2}\frac{s+1}{s(s+3)}=\frac{3}{10}$$

$$A_3=\lim_{s\to -3}\frac{s+1}{s(s-2)}=\frac{-2}{15}$$

$$L^{-1}[\frac{s+1}{s^3+s^2-6s}]=\frac{-\frac{1}{6}}{s}+\frac{\frac{3}{10}}{s-2}+\frac{\frac{-2}{15}}{s+3}=-\frac{1}{6}+\frac{3}{10}e^{2t}-\frac{2}{15}e^{-3t}$$

2. $Q(s)=0$ with repeated factors $(s-a_k)^m$

$$\frac{P(s)}{Q(s)}=\frac{C_m}{(s-a_k)^m}+\frac{C_{m-1}}{(s-a_k)^{m-1}}+\cdots\cdots+\frac{C_1}{s-a_k}$$

$$\frac{P(s)}{Q(s)}(s-a_k)^m=C_m+C_{m-1}(s-a_k)+C_{m-2}(s-a_k)^2+\cdots+C_1(s-a_k)^{m-1}$$

$$C_m=\lim_{s\to a_k}[\frac{P(s)}{Q(s)}(s-a_k)^m]$$

$$C_{m-1}=\lim_{s\to a_k}\{\frac{d}{ds}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}$$

$$C_{m-2}=\lim_{s\to a_k}\{\frac{d^2}{ds^2}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{2!}$$

……

$$C_1=\lim_{s\to a_k}\{\frac{d^{m-1}}{ds^{m-1}}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{(m-1)!}$$

$$L^{-1}[\frac{P(s)}{Q(s)}]=e^{a_k t}[C_m\frac{t^{m-1}}{(m-1)!}+C_{m-1}\frac{t_{m-2}}{(m-2)!}+\cdots+C_2t+C_1]$$

Ex. 8.

$$L^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}].$$

Solution: $\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}=\frac{C_2}{s^2}+\frac{C_1}{s}+\frac{A_1}{s-1}+\frac{A_2}{s-2}+\frac{A_3}{s-3}$

$$C_2=\lim_{s\to 0}\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}=\frac{-12}{-6}=2$$

$$C_1=\lim_{s\to 0}\frac{d}{ds}[\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}]$$

$$=\frac{4(-1)(-2)(-3)-(-12)[(-2)(-3)+(-1)(-3)+(-1)(-2)]}{[(-1)(-2)(-3)]^2}=\frac{-24+12\times 11}{6^2}=3$$

$$A_1=\lim_{s\to 1}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-2)(s-3)}=\frac{-1}{2}$$

$$A_2=\lim_{s\to 2}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-3)}=\frac{8}{-4}=-2$$

$$A_3=\lim_{s\to 3}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)}=\frac{9}{18}=\frac{1}{2}$$

$$L^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}]=2t+3-\frac{1}{2}e^t-2e^{2t}+\frac{1}{2}e^{3t}$$

3. $Q(s)=0$ with unrepeated factor $(s-\alpha)^2+\beta$, where $\beta>0$

$$\frac{P(s)}{Q(s)}=\frac{As+B}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]=As+B$$

$$\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\}=A(\alpha+i\beta)+B$$

$$R+iI=(A\alpha+\beta)+iA\beta$$

where R and I are the real and imaginary parts of $\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\}$, respectively

then, $\begin{cases} A\alpha+B=R \\ A\beta=I \end{cases}$, where we can get A and B, and

$$L^{-1}[\frac{P(s)}{Q(s)}]=L^{-1}[\frac{A(s-\alpha)+(A\alpha+B)}{(s-\alpha)^2+\beta^2}]=e^{\alpha t}\left(A\cos\beta t+\frac{A\alpha+B}{\beta}\sin\beta t\right)$$

Ex. 9.

$L^{-1}[\frac{s^2}{s^4+4}]$.

$$\text{Solution}:\frac{s^2}{s^4+4}=\frac{s^2}{(s^2)^2+2\cdot s^2\cdot 2+2^2-2\cdot s^2\cdot 2}=\frac{s^2}{(s^2+2)^2-(2s)^2}$$

$$=\frac{s^2}{(s^2+2s+2)(s^2-2s+2)}=\frac{A_1s+B_1}{(s+1)^2+1}+\frac{A_2s+B_2}{(s-1)^2+1}$$

$$\lim_{s\to-1+i}\frac{s^2}{(s-1)^2+1}=A_1(-1+i)+B_1\Rightarrow\frac{-2i}{4-4i}=(-A_1+B_1)+iA_1$$

$$\frac{8-8i}{32}=(-A_1+B_1)+iA_1\Rightarrow A_1=-\frac{1}{4},B_1=0$$

$$\lim_{s\to1+i}\frac{s^2}{(s+1)^2+1}=A_2(1+i)+B_2\Rightarrow\frac{2i}{4+4i}=(A_2+B_2)+iA_2$$

$$\frac{8+8i}{32}=(A_2+B_2)+iA_2\Rightarrow A_2=\frac{1}{4},B_2=0$$

$$L^{-1}[\frac{s^2}{s^4+4}]=L^{-1}[\frac{-\frac{1}{4}(s+1)+\frac{1}{4}}{(s+1)^2+1}+\frac{\frac{1}{4}(s-1)+\frac{1}{4}}{(s-1)^2+1}]$$

$$=\frac{e^{-t}}{4}(-\cos t+\sin t)+\frac{e^{t}}{4}(\cos t+\sin t)$$

4. Q(s) = 0 with repeated complex factor $[(s-\alpha)^2+\beta]^2$, where $\beta > 0$

$$\frac{P(s)}{Q(s)} = \frac{As+B}{[(s-\alpha)^2+\beta^2]^2} + \frac{Cs+D}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2 = As+B+(Cs+D)[(s-\alpha)^2+\beta^2]$$

$$\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\} = A(\alpha+i\beta)+B$$

$R_1 + iI_1 = (A\alpha+B)+iA\beta \Rightarrow \{\begin{matrix} A\alpha+B=R_1 \\ A\beta = I_1\end{matrix}$, where A and B can be obtained

$$\lim_{s\to\alpha+i\beta}\frac{d}{ds}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\} = A+[C(\alpha+i\beta)+D]\lim_{s\to\alpha+i\beta}\frac{d}{ds}[(s-\alpha)^2+\beta^2]$$

$$R_2 + iI_2 = A+[C(\alpha+i\beta)+D]2i\beta = (A-2C\beta^2)+i(2\alpha\beta C+2\beta D)$$

$\Rightarrow \{\begin{matrix} A-2C\beta^2 = R_2 \\ 2\alpha\beta C+2\beta D = I_2\end{matrix}$, where we get C and D, hence

$$L^{-1}[\frac{P(s)}{Q(s)}] = L^{-1}\{\frac{A(s-\alpha)+(A\alpha+B)}{[(s-\alpha)^2+\beta^2]^2}\} + L^{-1}[\frac{C(s-\alpha)+(C+D)}{(s-\alpha)^2+\beta^2}]$$

$$= e^{\alpha t}\{[\frac{At}{2\beta}\sin\beta t + (A\alpha+B)\frac{1}{2\beta^3}(\sin\beta t - \beta t\cos\beta t)] + [C\cos\beta t + (C\alpha+D)\frac{1}{\beta}\sin\beta t]\}$$

Ex. 10.

$$L^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}].$$

Solution: $\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2} = \frac{As+B}{[(s-1)^2+1]^2} + \frac{cs+D}{(s-1)^2+1}$

$$\lim_{s\to 1+i}(s^3-3s^2+6s+4) = A(1+i)+B$$

$$2i = (A+B)+iA \Rightarrow A=2, B=-2$$

$$\lim_{s\to 1+i}\frac{d}{ds}(s^3-3s^2+6s+4) = A+[c(1+i)+D]\lim_{s\to 1+i}\frac{d}{ds}[(s-1)^2+1]$$

$$0 = A+(c+ic+D)2i = (A-2c)+2i(c+D)$$

$$c=1, D=-1$$

$$L^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}] = L^{-1}\{\frac{2(s-1)}{[(s-1)^2+1]^2}\} + L^{-1}[\frac{s-1}{(s-1)^2+1}]$$

$$= e^t(2\cdot\frac{t}{2}\sin t + \cos t) = e^t(t\sin t + \cos t)$$

[Exercises] 1. $L^{-1}[\frac{s+1}{bs^2+7s+2}]$ 2. $L^{-1}[\frac{s-1}{(s+3)(s^2+2s+2)}]$

3. $L^{-1}[\frac{s}{(s^2-2s+2)(s^2+2s+2)}]$ 4. $L^{-1}[\frac{11s^3-47s^2+56s+4}{(s-2)^3(s+2)}]$

5. $L^{-1}[\frac{s^2}{s^4+4}]$ 6. $L^{-1}[\frac{1}{(s^2-1)^3}]$

[Ans] 1. $\frac{1}{2}e^{-\frac{t}{2}}-\frac{1}{3}e^{-\frac{2}{3}t}$ 2. $\frac{1}{5}e^{-t}(4\cos t-3\sin t)-\frac{4}{5}e^{-3t}$ 3. $\frac{1}{2}\sin t\sinh t$

4. $e^{2t}(2t^2-t+5)+6e^{-2t}$ 5. $\frac{1}{2}(\cosh t\sin t+\sinh t\cos t)$ 6. $\frac{1}{8}[(3+t^2)\sinh t-3t\cosh t]$

III. Differentiation with Respect to a Number

Ex. 11.

$L^{-1}[\frac{1}{(s^2+\omega^2)^2}]$.

Solution: $\frac{d}{d\omega}(\frac{1}{s^2+\omega^2})=\frac{-2\omega}{(s^2+\omega^2)^2}\Rightarrow L^{-1}[\frac{d}{d\omega}(\frac{1}{s^2+\omega^2})]=L^{-1}[\frac{-2\omega}{(s^2+\omega^2)^2}]$

$-2\omega L^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{d}{d\omega}L^{-1}[\frac{1}{s^2+\omega^2}]=\frac{d}{d\omega}(\frac{1}{\omega}\sin\omega t)=-\frac{1}{\omega^2}\sin\omega t+\frac{t}{\omega}\cos\omega t$

$L^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$

[Exercise] $L^{-1}[\frac{s}{(s^2+\omega^2)^2}]$ [Ans.] $\frac{t}{2\omega}\sin\omega t$

4.Applied to Solve Differential Equations

I. Ordinary Differential Equations with Constant Coefficients

Ex. 1.

$y''+y'+y=g(x),\ y(0)=1,\ y'(0)=0,$ where $g(x)=\begin{cases}1 & 0<x<3\\ 3 & x>3\end{cases}$.

Solution: $g(x)=u(x)+2u(x-3)$

$$[s^2Y-sy(0)-y'(0)]+[sY-y(0)]+Y=\frac{1}{s}+2\frac{e^{-3s}}{s}$$

$$(s^2+s+1)Y=s+1+\frac{1}{s}+2\frac{e^{-3s}}{s}$$

$$Y=\frac{s+1}{s^2+s+1}+\frac{1}{s(s^2+s+1)}+\frac{2e^{-3s}}{s(s^2+s+1)}$$

$$=\frac{s+1}{s^2+s+1}+(\frac{1}{s}-\frac{s+1}{s^2+s+1})+2e^{-3s}(\frac{1}{s}-\frac{s+1}{s^2+s+1})$$

$$\frac{s+1}{s^2+s+1}=\frac{(s+\frac{1}{2})+\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\Rightarrow L^{-1}[\frac{s+1}{s^2+s+1}]=e^{-\frac{x}{2}}(\cos\frac{\sqrt{3}}{2}x+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x)$$

$$y(x)=u(x)+2u(x-3)\{1-e^{-\frac{x-3}{2}}[\cos\frac{\sqrt{3}}{2}(x-3)+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(x-3)]\}$$

Ex. 2.

$y'''(t)-2y''(t)+5y'(t)=0,\ y(0)=0,\ y'(0)=1,\ y(\frac{\pi}{8})=1$.

Solution: $[s^3Y-s^2y(0)-sy'(0)-y''(0)]-2[s^2Y-sy(0)-y'(0)]+5[sY-y(0)]=0$

$$y''(0)=c$$

$$Y=\frac{s+c-2}{s(s^2-2s+5)}=\frac{A}{s}+\frac{Ps+Q}{(s-1)^2+2^2}$$

$$A=\lim_{s\to 0}\frac{s+c-2}{s^2-2s+5}=\frac{c-2}{5}$$

$$P(1+2i)+Q=\lim_{s\to 1+2i}\frac{s+c-2}{s}=\frac{-1+c+2i}{1+2i}=\frac{c+3}{5}+\frac{4-2c}{5}i$$

$$P=\frac{2-c}{5},\quad Q=\frac{2c+1}{5}$$

$$y(t)=\frac{c-2}{5}+e^t(\frac{2-c}{5}\cos 2t+\frac{c+3}{10}\sin 2t)$$

$$y(\frac{\pi}{8})=1\Rightarrow 1=\frac{c-2}{5}+e^{\frac{\pi}{8}}(\frac{2-c}{5}\frac{1}{\sqrt{2}}+\frac{c+3}{10}\frac{1}{\sqrt{2}})\Rightarrow c=7$$

$$\therefore\ y(t)=1+e^t(-\cos 2t+\sin 2t)$$

II. Ordinary Differential Equations with Variable Coefficients

Ex. 3.

$ty'' + (1-2t)y' - 2y = 0,\ \ y(0) = 1, y'(0) = 2.$

$$\text{Solution}: -\frac{d}{ds}[s^2Y - sy(0) - y'(0)] + \{[sY - y(0)] + 2\frac{d}{ds}[sY - y(0)]\} - 2Y = 0$$

$$(-s^2Y' - 2sY + 1) + [(sY - 1) + 2(sY' + Y)] - 2Y = 0$$

$$(-s^2 + 2s)Y' + (-2s + s + 2 - 2)Y = 0$$

$$-(s-2)Y' = Y \Rightarrow \frac{dY}{Y} = -\frac{ds}{s-2} \Rightarrow \ln Y = -\ln(s-2) + c_1$$

$$Y = \frac{c}{s-2} \Rightarrow y(t) = ce^{2t}$$

$$y(0) = 1, \therefore 1 = c,\ y(t) = e^{2t}$$

III. Simultaneous Ordinary Differential Equations

Ex. 4.

$$\begin{cases} \dfrac{dx}{dt} = 2x + y + 2e^{5t} \\ \dfrac{dy}{dt} = x + 2y + 3e^{2t} \end{cases}, \quad x(0) = y(0) = 0\,.$$

$$\text{Solution}: \begin{cases} sX - x(0) = 2X + Y + \dfrac{2}{s-5} \\ sY - y(0) = X + 2Y + \dfrac{3}{s-2} \end{cases} \Rightarrow \begin{cases} (s-2)X - Y = \dfrac{2}{s-5} \\ -X + (s-2)Y = \dfrac{3}{s-2} \end{cases}$$

$$X = \frac{(s-2)\dfrac{2}{s-5} + \dfrac{3}{s-2}}{(s-2)^2 - 1} = \frac{2s^2 - 5s - 7}{(s-1)(s-2)(s-3)(s-5)}$$

$$Y = \frac{\dfrac{2}{s-5} + (s-2)\dfrac{3}{s-2}}{(s-2)^2 - 1} = \frac{3s - 13}{(s-1)(s-3)(s-5)}$$

$$X = \frac{5/4}{s-1} + \frac{-3}{s-2} + \frac{1}{s-3} + \frac{3/4}{s-5} \Rightarrow x(t) = \frac{5}{4}e^t - 3e^{2t} + e^{3t} + \frac{3}{4}e^{5t}$$

$$Y = \frac{-5/4}{s-1} + \frac{1}{s-3} + \frac{1/4}{s-5} \Rightarrow y(t) = -\frac{5}{4}e^t + e^{3t} + \frac{1}{4}e^{5t}$$

[Exercises] Solve the following equations:

1. $y'' + 3y' + 2y = u(x-1),\ y(0) = 0,\ y'(0) = 1.$

2. $y'' + 2y' + 2y = e^{-t}\sin t,\ y(0) = 0,\ y'(0) = 1.$

3. $y''' + y' = x,\ y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.$

4. $y'' + 2y' + 2y = r(t),\ y(0) = 0,\ y'(0) = 0,$ where $r(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 1 \\ 1, & t > 1 \end{cases}.$

5. $f(t) = -1 + t - 2\int_0^t f(t-\alpha)\sin\alpha\, d\alpha.$

6. $y''(x) + y'(x) - 2y(x) = 0,\ y(0) = 1,\ \lim_{x\to\infty} y(x) = 0.$

7. $y''(t) + y(t) = 1,\ y(0) = 1,\ y(\frac{\pi}{2}) = 0.$

8. $ty''(t) - (4t-2)y'(t) - 4y(t) = 0,\ y(0) = y'(0) = 0.$

9. $t^2y'' - 2y = 2t,\ y(0) = 0,\ y(2) = 2.$

10. $\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + x + y = 1 \\ \dfrac{dy}{dt} - 2x - y = 0 \end{cases},\quad x(0) = 0,\ y(0) = 1.$

11. $\begin{cases} \dfrac{d^2y_1}{dt^2} = -y_1 - y_2 + 1 \\ \dfrac{dy_2}{dt} = y_1 + y_2 \end{cases},\quad y_1(0) = y_1'(0) = y_2(0) = 0.$

[Ans] 1. $[e^{-2(x-1)} + e^{-(x-1)}]u(x-1) + e^{-2x} + e^{-x}$

2. $e^{-t}(\frac{3}{2}\sin t - \frac{t}{2}\cos t)$ 3. $-1 + \frac{t^2}{2} + \cos t + \sin t$

4. $(-\frac{1}{2} + \frac{t}{2} + \frac{1}{2}e^{-t}\cos t) - [-\frac{1}{2} + \frac{1}{2}(t-1) + \frac{1}{2}e^{-(t-1)}\cos(t-1)]u(t-1)$

5. $-\cos\sqrt{3}t + \sin\sqrt{3}t$ 6. e^{-2x} 7. $1 - \sin t$ 8. 0 9. $t^2 - t$

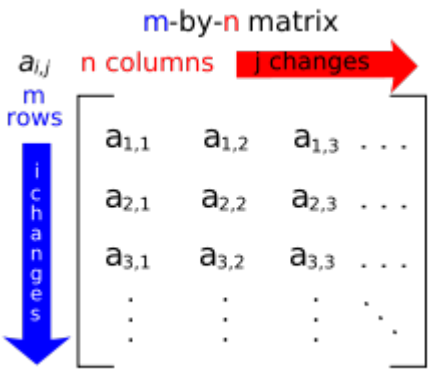
10. $\begin{cases} x(t) = e^{-t} - 1 \\ y(t) = 2 - e^{-t} \end{cases}$ 11. $\begin{cases} y_1(t) = -t + \dfrac{2}{\sqrt{3}}e^{\frac{t}{2}}\sin\dfrac{\sqrt{3}}{2}t \\ y_2(t) = t + 1 - e^{\frac{t}{2}}\cos\dfrac{\sqrt{3}}{2}t \end{cases}$

CHAPTER 3

MATRIX THEORY

In mathematics, a **matrix** is a rectangular table of *elements* (or *entries*), which may be numbers or, more generally, any abstract quantities that can be added and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformations and to record data that depend on multiple parameters. Matrices are described by the field of matrix theory. Matrices can be added, multiplied, and decomposed in various ways, which also makes them a key concept in the field of linear algebra.

In this article, the entries of a matrix are real or complex numbers unless otherwise noted.



Basic operations

Sum

Two or more matrices of identical dimensions *m* and *n* can be added. Given *m*-by-*n* matrices **A** and **B**, their **sum A+B** is the *m*-by-*n* matrix computed by adding corresponding elements:

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= (a_{i,j})_{1\le i\le m;\,1\le j\le n} + (b_{i,j})_{1\le i\le m;\,1\le j\le n} \\ &= (a_{i,j}+b_{i,j})_{1\le i\le m;1\le j\le n}.\end{aligned}$$

$$\begin{bmatrix}1 & 3 & 1\\ 1 & 0 & 0\\ 1 & 2 & 2\end{bmatrix} + \begin{bmatrix}0 & 0 & 5\\ 7 & 5 & 0\\ 2 & 1 & 1\end{bmatrix} = \begin{bmatrix}1+0 & 3+0 & 1+5\\ 1+7 & 0+5 & 0+0\\ 1+2 & 2+1 & 2+1\end{bmatrix} = \begin{bmatrix}1 & 3 & 6\\ 8 & 5 & 0\\ 3 & 3 & 3\end{bmatrix}.$$

Scalar multiplication

Given a matrix **A** and a number *c*, the scalar multiplication *c***A** is computed by multiplying every element of **A** by the scalar *c* (i.e. $(c\mathbf{A})_{i,j} = c \cdot a_{i,j}$). For example:

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot -3 \\ 2 \cdot 4 & 2 \cdot -2 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}.$$

Matrix multiplication

Multiplication of two matrices is well-defined only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If **A** is an *m*-by-*n* matrix and **B** is an *n*-by-*p* matrix, then their **matrix product AB** is the *m*-by-*p* matrix given by:

$$(\mathbf{AB})_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,n}b_{n,j}$$

for each pair (*i*,*j*). For example:

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1 \times 3 + 0 \times 2 + 2 \times 1) & (1 \times 1 + 0 \times 1 + 2 \times 0) \\ (-1 \times 3 + 3 \times 2 + 1 \times 1) & (-1 \times 1 + 3 \times 1 + 1 \times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}.$$

Matrix multiplication has the following properties:

- (**AB**)**C** = **A**(**BC**) for all *k*-by-*m* matrices **A**, *m*-by-*n* matrices **B** and *n*-by-*p* matrices **C** ("associatively").
- (**A**+**B**)**C** = **AC**+**BC** for all *m*-by-*n* matrices **A** and **B** and *n*-by-*k* matrices **C** ("right distributive").
- **C**(**A**+**B**) = **CA**+**CB** for all *m*-by-*n* matrices **A** and **B** and *k*-by-*m* matrices **C** ("left distributive").

Matrix multiplication is not commutative; that is, given matrices **A** and **B** and their product defined, then generally **AB** ≠ **BA**. It may also happen that **AB** is defined but **BA** is not defined.

Transposition

Transposing a matrix means converting and *m by n* matrix into an *n by m* matrix, by "flipping" the rows and columns.

$$x_{i,j} = a_{j,i}$$

It is denoted by a superscript T, eg:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

As an aside, there is an interesting relationship between transposition and multiplication:

$(A \times B)^T = B^T \times A^T$

$A = (A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(kA)^T = k(A)^T$

If you are interested, you can prove this for yourself fairly easily. Hint – look at the definition of matrix multiply, and try swapping the subscripts!

Equality

2 matrices are considered to be *equal* if they are of the same order, and if all their corresponding elements are equal.

Square Matrix

A square matrix is one where the number of rows and columns are equal, eg a *2 by 2* matrix, a *3 by 3* matrix etc.

Unit (Identity) Matrix

A unit matrix is a square matrix in which all the elements on the leading diagonal are 1, and all the other elements are 0, eg:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zero (Null) Matrix

A zero, or null, matrix is one where every element is zero, eg

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Diagonal Matrix

A diagonal matrix is a square matrix in which all the elements are zero except for the elements on the leading diagonal, eg:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Symmetric Matrix

A symmetric matrix is a square matrix where

$a_{i,j} = a_{j,i}$ i.e. ($A = A^T$)

for all elements, i.e., the matrix is symmetrical about the leading diagonal. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

Skew Symmetric Matrix

A skew symmetric matrix is a square matrix where

$a_{i,j} = -a_{j,i}$ i.e. ($A = -A^T$)

for all elements. I.e., the matrix is anti-symmetrical about the leading diagonal. This of course requires that elements along the **diagonal must be zero**. For example

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

Orthogonal Matrix

An orthogonal matrix is a square matrix which produces a identity matrix if it is multiplied by its own transpose, i.e.:

$A \times A^T = I$

A x $A^T = I = A^T$ x A

Or $\boldsymbol{A^{-1} = A^T}$

RANK

The RANK of a matrix is equal to the highest order non-zero determinant that can be formed from its sub-matrices

$$A = \begin{bmatrix} 4 & 5 & 2 & 14 \\ 3 & 9 & 6 & 21 \\ 8 & 10 & 7 & 28 \\ 1 & 2 & 9 & 5 \end{bmatrix}$$

det $A = 0$

$$\begin{vmatrix} 4 & 5 & 2 \\ 3 & 9 & 6 \\ 8 & 10 & 7 \end{vmatrix} = 63$$

Rank of $A = 3$

The rank of a matrix can also be measured by the maximum number of linearly independent columns of A

This also equals the maximum number of linearly independent rows

$$1\begin{bmatrix} 4 \\ 3 \\ 8 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 9 \\ 10 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \\ 7 \\ 9 \end{bmatrix} + (-1)\begin{bmatrix} 14 \\ 21 \\ 28 \\ 5 \end{bmatrix} = 0$$

$c_1\underline{a}_1 + c_2\underline{a}_2 + c_3\underline{a}_3 + c_4\underline{a}_4 = 0$

A FULL COLUMN RANK matrix has the same number of linearly independent columns (rows) equal to the number of columns.

A FULL ROW RANK matrix has the same number of linearly independent rows (columns) equal to the number of rows.

If *A* does not have full row and column rank it is **SINGULAR** : **det(A)=0**
i.e. rank(A)=n-1
If *A* does have full row and column rank it is **NON-SINGULAR** : **det(A)≠0**
i.e. rank(A)=n
rank (I_n) = n
rank (k*A*) = rank (*A*)
rank (A^T) = rank (*A*)
If *A* is (m x n) then rank (*A*) is $\leq$ min {m, n}
rank $AB \leq$ min{rank (*A*), rank (*B*)}

Inverse Matrix

The inverse of a square matrix A , sometimes called a reciprocal matrix, is a matrix A^{-1} such that

$$A A^{-1} = I \tag{1}$$

where I is the identity matrix. A square matrix A has an inverse iff the determinant $|A| \neq 0$. A matrix possessing an inverse is called nonsingular, or invertible.

The matrix inverse of a square matrix m may be taken in *Mathematic* using the function Inverse[*m*].

For a 2×2 matrix

$$A \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Is :

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For a 3 x 3 Matrix it's inverse would be :

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{13} & a_{12} \\ a_{33} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{11} \\ a_{32} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}.$$

Eigen values And Eigenvectors

Let A be a $n \times n$ matrix.

λ **is an** Eigen value **for** A **if there exists a vector** $X \neq 0$ **such that**

$$AX = \lambda X$$

If such a vector X **exists, it is said to be an** eigenvector **associated with the Eigen value** λ**.**

(I) <u>How to find the Eigen values and eigenvectors</u>

Suppose that λ is an Eigen value for A.

Then there exists a non-zero vector $X = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$ **such that** $AX = \lambda X$ **or, equivalently, such that** $[A - \lambda I_n]X = 0$**, where** I_n **is the** $n \times n$ **identity matrix.**

Since $[A - \lambda I_n]X = 0$ **has a non-trivial solution** X**, the matrix** $[A - \lambda I_n]$ **is not invertible, i.e.** $|A - \lambda I_n| = 0.$

Note that $|A - \lambda I_n|$ **is a polynomial in** λ**, so you get the Eigen values of** A **by finding the roots of** $|A - \lambda I_n|$.

Say λ_0 **is such an Eigen value. In order to find the eigenvectors associated with** λ_0**, you have to solve the system** $AX = \lambda_0 X$ **for** $x_1, \dots, x_n$**.**

(II) Example: two by two matrices

Let $A=\begin{pmatrix}-3 & 2\\ -2 & 1\end{pmatrix}$.

To find the Eigen values of A*, form the matrix* $A-\lambda I_2$*, find its determinant* $|A-\lambda I_2|$ *and solve the equation* $|A-\lambda I_2|=0$*:*

$$A-\lambda I_2=\begin{pmatrix}-3 & 2\\ -2 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-3-\lambda & 2\\ -2 & 1-\lambda\end{pmatrix}.$$

Now,

$$|A-\lambda I_2|=(-3-\lambda)(1-\lambda)-(2)(-2)=\lambda^2+2\lambda+1,$$

so that the Eigen values of A **are the roots of** $\lambda^2+2\lambda+1=(\lambda+1)^2$**, i.e.** A **has a repeated Eigen value:** $\lambda_0=-1$**.**

Now we have to solve the system $[A-\lambda I_n]X=0$**. Here** $\lambda_0=-1$**, so that,**

$$\begin{pmatrix}-2 & 2\\ -2 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

A vector $X=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ **is therefore an eigenvector associated with the Eigen value** $\lambda_0=-1$ **if and only if its coordinates satisfy**

$$\begin{cases}-2x_1 & +2x_2=0\\ -2x_1 & +2x_2=0\end{cases},$$

i.e. if and only if $x_1=x_2$.

Hence the eigenvectors associated with the Eigen value $\lambda_0=-1$ **are of the form** $X=\alpha\begin{pmatrix}1\\ 1\end{pmatrix}$**, where** α **is a real number. Normalized** $X=\dfrac{\alpha\begin{bmatrix}1\\ 1\end{bmatrix}}{\sqrt{\alpha^2+\alpha^2}}=\dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$

(III) <u>Example: three by three matrices</u>

IF $A=\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Find the Eigen values and the Eigen vector of the matrix A .

$$|A-\lambda I|=\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$$=(1-\lambda)\{(1-\lambda)(2-\lambda)-1\}-(1-\lambda)=\lambda^3-4\lambda^2+3\lambda$$

$$=\lambda(\lambda-1)(\lambda-3)$$

When $\lambda=0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0 \Leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\approx\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\approx\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_2=x_3, x_1=x_2=x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=k\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k\neq 0$$

For $k=1$, the eigenvectors is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

When $\lambda=1$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\Leftrightarrow \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\sim\begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_2=0, x_1=x_2-x_3=-x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=k\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, k\neq 0$$

When $\lambda = 3$

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_2 = -2x_3, x_1 = x_2 + x_3 = -x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, k \neq 0$$

Example 3

Show that $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is diagonalizable.

Proof:

For $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, we have Eigen values $\lambda_1 = 5$, $\lambda_2 = -1$. For $\lambda_1 = 5$ one of the corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda_2 = -1$, one of the corresponding eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Take $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$. As the eigenvectors are linearly independent,

$$P^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1}$$

exists and $P^{-1} = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix}$. Also $P^{-1}AP = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$.

It concludes that $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is diagonalizable.

Example 4

Diagonalize $A=\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, if possible..

Proof:

$$|A-\lambda I|=\begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$=-\lambda^3-3\lambda^2+4=-(\lambda-1)(\lambda+2)^2$$

When $\lambda=1$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0 \Leftrightarrow \begin{bmatrix} 3 & 3 & 0 \\ -3 & -6 & -3 \\ 0 & 3 & 3 \end{bmatrix}\sim\begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix}\sim\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x_2=-x_3, x_1=-x_2=x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, k\neq 0$$

When $\lambda=-2$

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

$$\Leftrightarrow \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\Rightarrow x_3=t, x_2=s, x_1=-s-t$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -s-t \\ s \\ t \end{pmatrix}=t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}+s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, t^2+s^2\neq 0$$

$A=\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ has three linearly independent eigenvectors, therefore

$A=\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ is diagonalizable and

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}=\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Cayley-Hamilton Theorem:

An $n \times n$ matrix A satisfies its characteristic equation.

Example 5:

Find the characteristic equation of $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$ and verify Cayley-Hamilton Theorem.

Hence find the inverse of the matrix.

Solution: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$ $\therefore$ Characteristic eqn. of A is

$\lambda^3 - \lambda^2[1+1-3] + \lambda[-9-9-1] + 2 \cancel{6} 0$

i.e $\lambda^3 + \lambda^2 - 19\lambda + 26 = 0$

By **Cayley-Hamilton theorem** $\therefore A^3 + A^2 - 19A + 26I = 0$.

<u>Verification:</u>

$$\therefore A^2 = A.A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 9 & 2 & -7 \\ 5 & 9 & -1 \\ -10 & -7 & 21 \end{pmatrix} 0$$

$$\therefore A^3 = A^2.A = \begin{pmatrix} 9 & 2 & -7 \\ 5 & 9 & -10 \\ -10 & -7 & 21 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{pmatrix} = \begin{pmatrix} -16 & -21 & 45 \\ -43 & -16 & 67 \\ 67 & 45 & -104 \end{pmatrix}$$

Substituting in the characteristic equation

$$\begin{pmatrix} -16 & -21 & 45 \\ -43 & -16 & 67 \\ 67 & 45 & -104 \end{pmatrix} + \begin{pmatrix} 9 & 2 & -7 \\ 5 & 9 & -10 \\ -10 & -7 & 21 \end{pmatrix} - \begin{pmatrix} 19 & -19 & 38 \\ -38 & 19 & 57 \\ 57 & 38 & -57 \end{pmatrix} + \begin{pmatrix} 26 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 26 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence verified.

Now to find the inverse of the matrix A, premultiply the characteristic equation by A^{-1}

$$\therefore A^2 + A - 19I + 26A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{26}\left(19I - A - A^2\right)$$

$$= \frac{1}{26}\left[\begin{pmatrix} 1 & \cancel{0}9 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{pmatrix} - \begin{pmatrix} 9 & 2 & -7 \\ 5 & 9 & -10 \\ -10 & -7 & 21 \end{pmatrix}\right] = \frac{1}{26}\begin{pmatrix} 9 & -5 & 5 \\ -3 & 9 & 7 \\ 7 & 5 & 1 \end{pmatrix}$$

Example 6: Given $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, use Cayley-Hamilton Theorem to find the inverse of A and also find A^4

Solution:

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

i.e., $(1-\lambda)\,[(1-\lambda)(1-\lambda)-1] + 3[-2-(1-\lambda)] = 0$

i.e., $(1-\lambda)^3 - (1-\lambda) - 6 - 3 + 3\lambda = 0$

i.e., $1 - 3\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda - 9 + 3\lambda = 0$

i.e., $-\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$

i.e., $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

To find A^{-1}, multiplying by A^{-1}, $A^2 - 3A - I + 9A^{-1} = 0$

$\therefore \quad A^{-1} = \frac{1}{9}[-A^2 + 3A + I]$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

To find A^4:

We have $A^3 - 3A^2 - A + 9I = 0$

i.e., $A^3 = 3A^2 + A - 9I$ (1)

Multiplying (1) by A, we get,

$A^4 = 3A^3 + A^2 - 9A$ $\therefore$

$= 3(3A^2 + A - 9I) + A^2 - 9A$ using (1)

$= 10A^2 - 6A - 27I$

$$= 10\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

Homework

1. Let $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$

a-Use the Eigen values method to compute $A^4 = P\begin{bmatrix} \lambda_1^4 & 0 \\ 0 & \lambda_2^4 \end{bmatrix}P^{-1}$, where P is the Eigen vectors of A.

b-Compute e^A and cosA

2. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 6 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$, Compute $\sin A$ *and* A^{-2}

3. If $B = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$, Show that $B^{-1} = \frac{1}{12}[B^2 - 8B + 19I]$

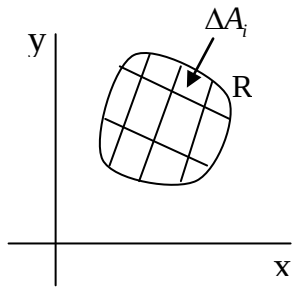
CHAPTER 4

Multiple integrals

The theory of multiple integral looks like the theory of infinite integral for one variable .If f(x,y) is a continuous function in a closed region **R,** if we divide the area to n number of areas ΔA_i, if we choose a point (x_i, y_i)in each sub-region $\mathbf{\Delta A}_i$ and form the sum $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$ thus , we have the following definition

<u>Definition of double integral</u>

Let ***f(x,y)*** be a function of two variables defined on a closed region ***R*** **.** Then the double integral of ***f*** over ***R*** is given by



$$\iint f(x,y)dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

When ***f(x, y)*** **=** 1 on ***R*** then $\lim_{n\to\infty}\sum_{i=1}^{n}\Delta A_i$ gives the area A

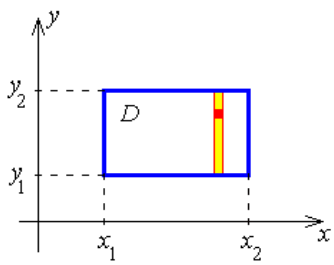
$$\boldsymbol{A} = \iint_R dA$$

when *z = f(x,y)* represents a surface the then the volume *V* of the solid above the region R and below the surface *z = f(x,y)* is given by*:*

$$\boldsymbol{V} = \iint_R f(x,y)\, dA$$

<u>First case:</u> the integration limits are constants

<u>Example 1</u>



Evaluate $\int_{-1}^{1}\int_{0}^{3}(x^2+y^2)\, dy\, dx$

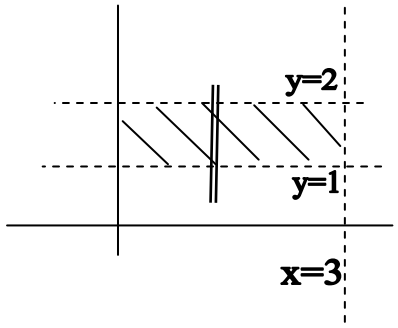
Solution

$$I = \int_{-1}^{1}\int_{0}^{3}(x^2+y^2)\, dy\, dx = \int_{-1}^{1}\left(\int_{0}^{3} x^2+y^2\ dy\right) dx = \int_{-1}^{1} x^2 y + \frac{y^3}{3}\Big|_0^3\, dx = \int_{-1}^{1} 3x^2+9\, dx$$

$$= 3\frac{x^3}{3}+9x\Big|_{-1}^{1} = 20$$

Example 2:

1) Evaluate $\int_0^3\int_1^2 (1+8xy)\,dydx$



sketch: since $dydx \Rightarrow$ vertical

$y=1$, $y=2$

$$\int_0^3\int_1^2 (1+8xy)\,dydx = \int_0^3 (y+8x\frac{y^2}{2})\Big|_1^2 dx$$

$$= \int_0^3 \{[2+4x(4)]-[1+4x(1)]\}\,dx$$

$$= \int_0^3 \{[2+16x]-[1+4x]\}\,dx$$

$$= \int_0^3 \{1+12x\}\,dx$$

$$= (x+12\frac{x^2}{2})\Big|_0^3$$

$$= (3+6(9))-(0) = (3+54) = \mathbf{57}$$

Second case: The integration limits are variables

1.If R is region of type one

Taking a vertical lamina means that we will integrate first with respect to y and in this case the integration limits will be a function of x , then integrate the result with respect to y which will be defined through a constant limits

$$I = \int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx$$

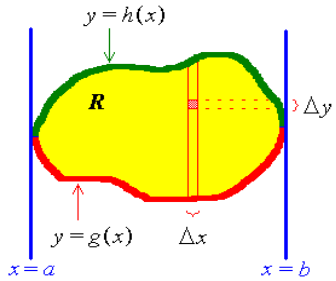


Fig. 1

2. If R is region of type Two

Taking a horizontal lamina means that we will integrate first with respect to x and in this case the integration limits will be a function of y , then integrate the result with respect to x which will be defined through a constant limits

$$I = \int_{y=c}^{y=d}\int_{x=p(y)}^{x=q(y)} f(x,y)\,dx\,dy$$

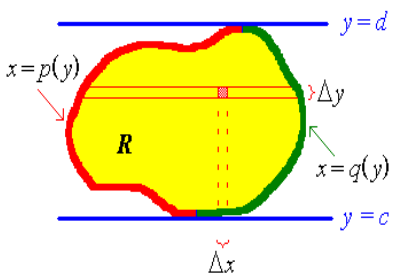


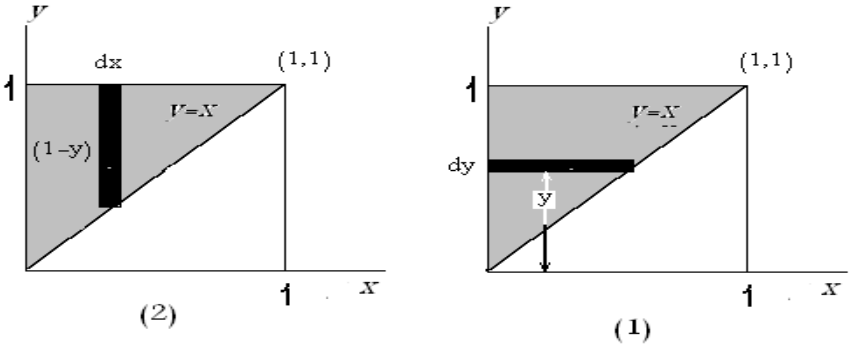
Fig. 2

Example1

Evaluate $\iint_R x^3 y^2 \, dA$ where R is the region bounded by y = x;

y = 1 and x=0

Solution



First Solution(figure.1)

$$\iint_R x^3 y^2 \, dA = \int_0^1 \int_{x=0}^{x=y} x^3 y^2 dxdy = \int_0^1 y^2 \frac{x^4}{4}\Big|_0^y dy = \frac{1}{4}\int_0^1 y^6 dy = \frac{y^7}{28}\Big|_0^1 = \frac{1}{28}$$

Second solution (figure2)

$$\iint_R x^3 y^2 \, dA =$$

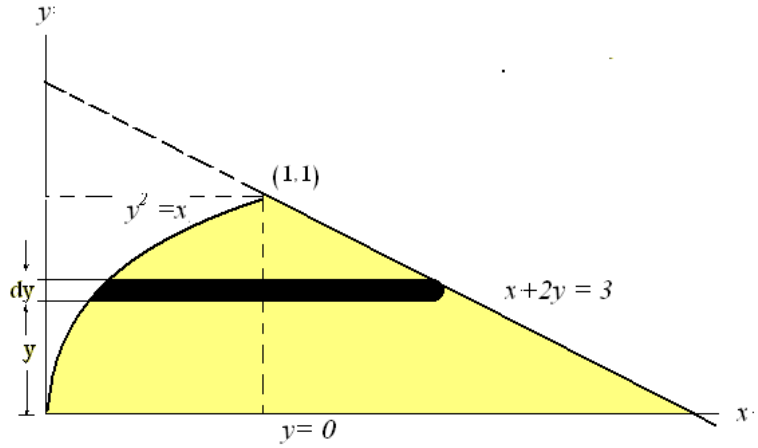
$$\int_0^1 \int_x^1 x^3 y^2 dydx = \int_0^1 x^3 \frac{y^3}{3}\Big|_x^1 dx = \frac{1}{3}\int_0^1 x^3 (1 - x^3)\, dx = \frac{1}{3}\left(\frac{x^4}{4} - \frac{x^7}{7}\right)\Big|_0^1 = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) = \frac{1}{28}$$

Example2

Evaluate $\iint_R x + y \, dxdy$ where R is the region bounded by y^2 =x;

x+2y = 3 and y= 0 in the first quadrant

Solution



$$\iint_R x + y \; dxdy =$$

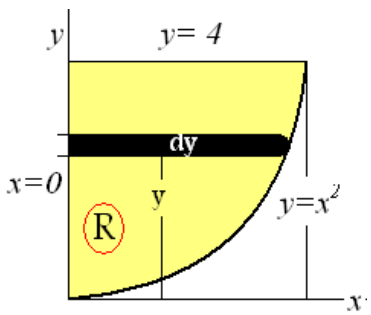
$$\int_0^1 \int_{y^2}^{3-2y} (x+y)dxdy = \int_0^1 \frac{x^2}{2} + yx \Big|_{y^2}^{3-2y} dy = \int_0^1 \left\{ \frac{(3-2y)^2 - (y^2)^2}{2} + y\left[(3-2y) - y^2\right] \right\} dy$$

$$= \int_0^1 (\frac{9}{2} - 3y - y^3 - y^4)dy = 2.55$$

<u>Example3</u>

Evaluate $\iint_R x\, e^{y^2} \; dA$ where R is the region bounded by $y=x^2$; x=0 and y= 4

Solution



$$\iint_R x\, e^{y^2} \; dA = \int_0^4 \int_0^{\sqrt{y}} x\, e^{y^2} dxdy = \int_0^4 \frac{x^2}{2}\Big|_0^{\sqrt{y}} e^{y^2} dy = \int_0^4 \frac{y}{2} e^{y^2} dy = \frac{1}{4} e^{y^2}\Big|_0^4 = \frac{1}{4}(e^{16} - 1)$$

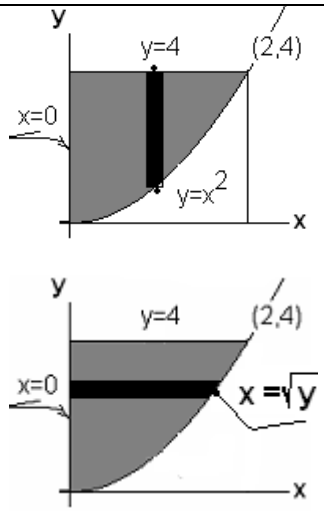
Reversing the order of Integration

A problem may become easier when the order of integration is reversed or changed. Which means some integrals may be impossible to be evaluated with respect to one of the variables but can be done with respect to the other one

Example1

Evaluate $\int_0^2\int_{x^2}^4 x\,e^{y^2}\,dA = \int_0^2\int_{x^2}^4 x\,e^{y^2}\,dydx = \int_0^4\int_0^{\sqrt{y}} x\,e^{y^2}\,dxdy$

$$= \int_0^4 \frac{x^2}{2}\Big|_0^{\sqrt{y}} e^{y^2}\,dy = \frac{1}{2}\int_0^4 ye^{y^2}\,dy = \frac{1}{4}\int_0^4 e^{y^2}\,dy^2 = \frac{1}{4}e^{y^2}\Big|_0^4 = \frac{e^{16}-1}{4}$$



Example2

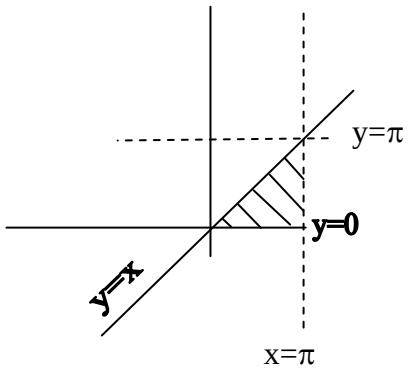
Evaluate $\int_0^\pi\int_y^\pi \frac{\sin x}{x}\,dx\,dy$

From left $x = y$

From right $x = \pi$

value of y , from $0 \Rightarrow x$

reverse the order



$$\Rightarrow \int_0^\pi\int_y^\pi \frac{\sin x}{x}\,dx\,dy = \int_0^\pi\int_0^x \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^\pi \frac{\sin x}{x}\cdot y\Big|_0^x dx = \int_0^\pi \frac{\sin x}{x}\cdot x\,dx$$

$$= \int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -(-1-1) = \mathbf{2}$$

Change to polar coordinates

The relation between Cartesian and polar coordinates is very famous and can be given by:

$x = r\cos\theta \qquad y = r\sin\theta$

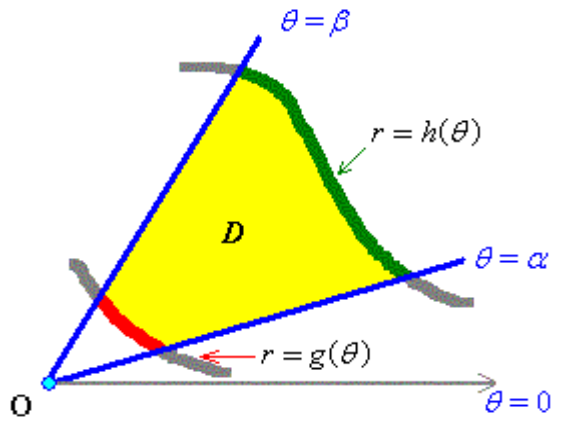
or may be it can be written as $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}(y/x)$

Suppose that D is the region shown in Figure , it is clear that if we try to take vertical or horizontal lamina we will get more than one region beside the integration limits will include roots which will makes the second integral very complicated

So changing to polar coordinates will transfer the segment area *dA (dxdy)* to another area in polar plane given by J dr dθ where J in the Jacobian and it is equal to r in the case of changing from Cartesian to Polar, so

$$\iint_D f(x,y)dA = \iint_D f(x,y)dxdy = \iint_D f(r\cos\theta, r\sin\theta)\, r\, drd\theta$$

In general, in plane polar coordinates,



$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example1 Evaluate $\int_1^2 \int_0^x \frac{1}{(x^2+y^2)^{3/2}}\, dy\, dx$

Solution

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\sec\theta} \frac{1}{r^3} r\,dr\,d\theta$$

$$= \int_0^{\pi/4} -\frac{1}{r} \Big|_{\sec\theta}^{2\sec\theta} d\theta$$

$$= \int_0^{\pi/4} (\cos\theta - \frac{1}{2}\cos\theta)\, d\theta = \frac{\sqrt{2}}{4}$$

Example 2

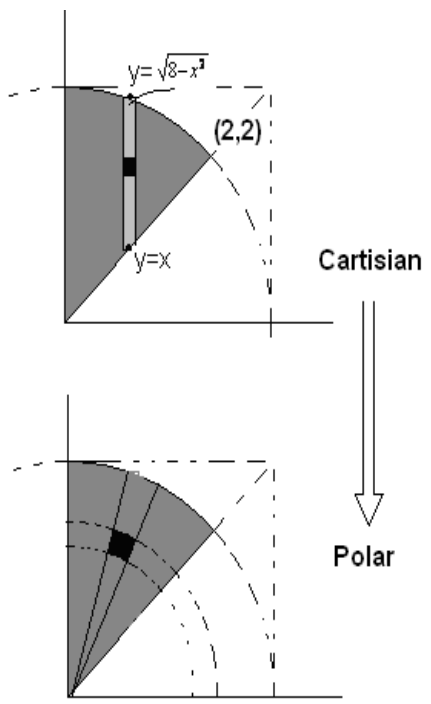
Evaluate $\int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2}\, dydx$

$$= \int_0^{2\sqrt{2}} \int_{\pi/4}^{\pi/2} \frac{1}{5+r^2}\, r\,d\theta\,dr$$

$$= \int_0^{2\sqrt{2}} \theta\Big|_{\pi/4}^{\pi/2} \frac{1}{5+r^2}\, r\,dr$$

$$= \frac{\pi}{4} \int_0^{2\sqrt{2}} \frac{1}{5+r^2}\, r\,dr$$

$$= \frac{\pi}{8} \ell n(5+r^2)\Big|_0^{2\sqrt{2}} = \frac{\pi}{8} \ell n\left(\frac{13}{5}\right)$$



Example 3

Evaluate $\int_0^\infty e^{-x^2}\,dx$

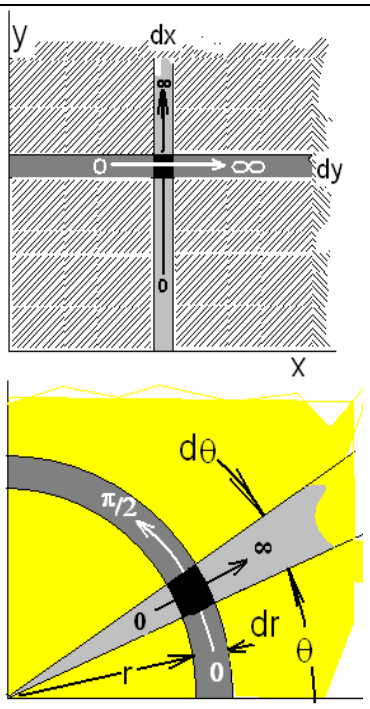
Answer

$$\int_0^\infty e^{-x^2}dx \times \int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$$

$$= \int_0^\infty\int_0^{\pi/2} e^{-r^2}\, r\,d\theta\,dr = \int_0^\infty e^{-r^2}\,\theta\Big|_0^{\pi/2}\, r\,dr$$

$$= \frac{\pi}{4}\int_0^\infty e^{-r^2}\;2r\,dr = \frac{\pi}{4}\left[-e^{-r^2}\right]_0^\infty = \frac{\pi}{4}$$

$$\Rightarrow \left(\int_0^\infty e^{-x^2}dx\right)^2 = \frac{\pi}{4}\quad,\quad \Rightarrow \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \quad \triangleleft$$



Example 4

Find the area enclosed by one loop of the curve $r = \cos 2\theta$.

Boundaries:

$$0 \le r \le \cos 2\theta\ ;\quad -\frac{\pi}{4} \le \theta \le +\frac{\pi}{4}$$

Area:

$$A = \iint_D 1\,dA = \int_{-\pi/4}^{+\pi/4}\int_0^{\cos 2\theta} 1\,r\,dr\,d\theta$$

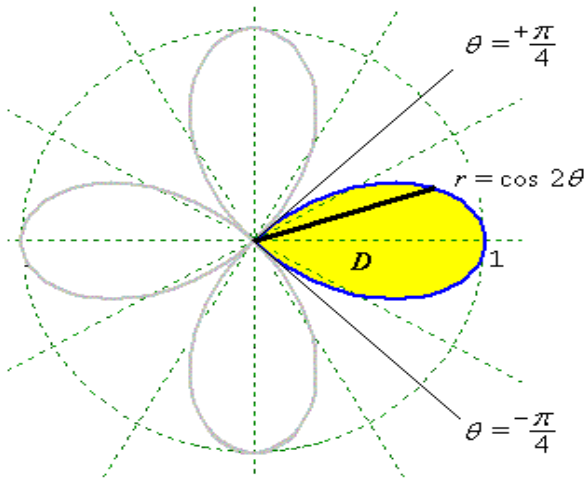
$$= \int_{-\pi/4}^{+\pi/4}\left[\frac{r^2}{2}\right]_0^{\cos 2\theta} d\theta$$

$$= \int_{-\pi/4}^{+\pi/4}\left(\frac{\cos^2 2\theta}{2} - 0\right)d\theta$$

$$= \int_{-\pi/4}^{+\pi/4}\frac{\cos 4\theta + 1}{4}\,d\theta$$

$$= \left[\frac{\sin 4\theta}{16} + \frac{\theta}{4}\right]_{-\pi/4}^{+\pi/4} = \left(0+\frac{\pi}{16}\right) - \left(0-\frac{\pi}{16}\right)$$

Therefore $A = \underline{\underline{\frac{\pi}{8}}}$

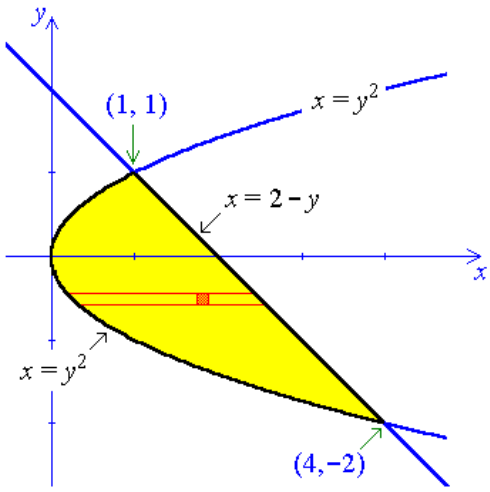


Example 5

Evaluate $I = \iint_R \left(6x + 2y^2\right) dA$

where R is the region enclosed by the parabola $x = y^2$ and the line $x + y = 2$.

The upper boundary changes form at $x = 1$.
The left boundary is the same throughout R.
The right boundary is the same throughout R.
Therefore choose horizontal strips.



$$I = \int_{-2}^{1} \int_{y^2}^{2-y} \left(6x + 2y^2\right) dx\, dy$$

$$I = \int_{-2}^{1} \left[3x^2 + 2xy^2\right]_{x=y^2}^{x=2-y} dy$$

$$= \int_{-2}^{1} \left(\left(3(2-y)^2 + 2(2-y)y^2\right) - \left(3y^4 + 2y^4\right)\right) dy$$

$$= \int_{-2}^{1} \left(\left(12 - 12y + 3y^2\right) + \left(4y^2 - 2y^3\right) - 5y^4\right) dy$$

$$= \int_{-2}^{1} \left(12 - 12y + 7y^2 - 2y^3 - 5y^4\right) dy$$

$$= \left[12y - 6y^2 + \frac{7}{3}y^3 - \frac{1}{2}y^4 - y^5\right]_{-2}^{+1}$$

$$= \left(12 - 6 + \frac{7}{3} - \frac{1}{2} - 1\right) - \left(-24 - 24 - \frac{56}{3} - 8 + 32\right)$$

Therefore

$$I = \underline{\underline{\frac{99}{2}}}$$

<u>Example 6</u>

Evaluate each of the following integrals.

1) $\iint_R y\sqrt{1+x^3}\,dA$ where R is the triangle with vertices (0, 0), (3, 0) and (3, 2)

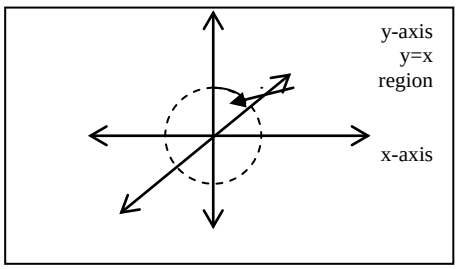
Solution:

$$\iint_{\mathbf{R}} \mathbf{y}\sqrt{1+\mathbf{x}^3}\,\mathbf{dA} = \int_0^3 \int_0^{2\mathbf{x}/3} \mathbf{y}\sqrt{1+\mathbf{x}^3}\,\mathbf{dydx} = \int_0^3 \frac{\mathbf{y}^2}{2}\Big|_0^{2\mathbf{x}/3} \left(1+\mathbf{x}^3\right)^{1/2}\mathbf{dx} = \frac{2}{9}\int_0^3 \mathbf{x}^2\left(1+\mathbf{x}^3\right)^{1/2}\mathbf{dx}$$

$$= \frac{2}{27}\cdot\frac{\left(1+\mathbf{x}^3\right)^{3/2}}{3/2}\Bigg|_0^3 = \frac{4}{81}\left[28\sqrt{28}-1\right] \approx 7.267262885$$

answer =7.26726288

2). $\iint_R \frac{1}{\left(x^2+y^2+1\right)^2}dA$ where R is the region in the first quadrant bounded by the circle $x^2+y^2=9$ $x=0,\ and\ \ y=x$



Solution:

$$\iint_R \frac{1}{\left(x^2+y^2+1\right)^2}dA = \int_{\pi/4}^{\pi/2}\int_0^3 \frac{1}{\left(r^2+1\right)^2}r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_0^3 \left(r^2+1\right)^{-2} r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_{\pi/4}^{\pi/2} \left(r^2+1\right)^{-1}\Big|_0^3 = -\frac{1}{2}\left(\frac{1}{10}-1\right)\int_{\pi/4}^{\pi/2} d\theta = \frac{9}{20}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$= \frac{9\pi}{80} \approx 0.3534291735$$

Triple Integrals

A triple integral is an integral taken over a volume of space.

Definition 1: The Definite Integral of a Function of Three Variables

For any function $f(x, y, z)$ defined in the rectangular box $Q = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$, and $\|P\|$ (the norm of the partition) defined as the maximum diagonal of any rectangular box region in the partition, the **triple integral** of f over Q is defined as:

$$\iiint_Q f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(u_i, v_i, w_i)\,\Delta V_i ,$$

provided the limit exists and is the same for all choices of the evaluation points $(u_i, v_i, w_i) \in Q_i$ for $i = 1, 2, \ldots, n$. In this case, we say f is **integrable** over Q.

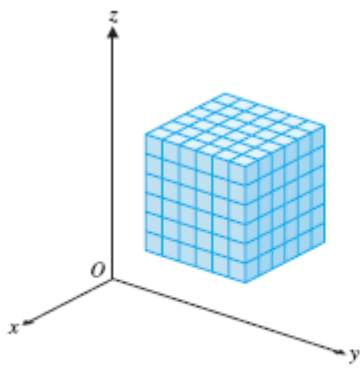


Fig. 1-a

Partition of the box Q

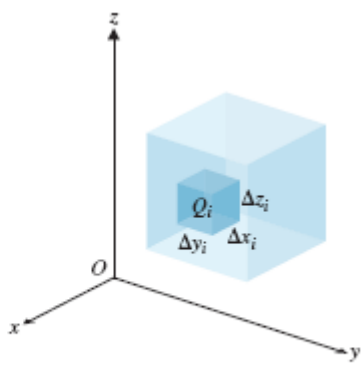


Fig. 1- b

Typical box Q_i

Fubini's Theorem: (Order of Integration is Interchangeable)

If a function $f(x, y)$ is integrable on the box $Q = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$, then we can write the triple integral of f over Q as:

$$\begin{aligned}
\iiint_Q f(x,y,z)\,dV &= \int_r^s\int_c^d\int_a^b f(x,y,z)\,dxdydz \\
&= \int_c^d\int_r^s\int_a^b f(x,y,z)\,dxdzdy \\
&= \int_a^b\int_r^s\int_c^d f(x,y,z)\,dydzdx \\
&= \int_r^s\int_a^b\int_c^d f(x,y,z)\,dydxdz \\
&= \int_c^d\int_a^b\int_r^s f(x,y,z)\,dzdxdy \\
&= \int_a^b\int_c^d\int_r^s f(x,y,z)\,dzdydx
\end{aligned}.$$

Definition 2: Triple Integral of a Function of Three Variables Over Any Bounded Volume

For any function $f(x, y, z)$ defined on the bounded solid Q, we define the triple integral of f over Q as:

$$\iiint_Q f(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(u_i, v_i, w_i)\,\Delta V_i\,,$$

provided the limit exists and is the same for all choices of the evaluation points $(u_i, v_i, w_i) \in Q_i$ for $i = 1, 2, \ldots, n$. In this case, we say f is **integrable** over Q.

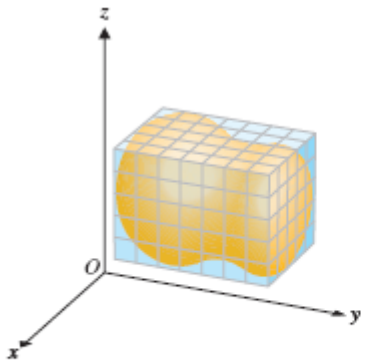


Fig. 2- a

Partition of a solid

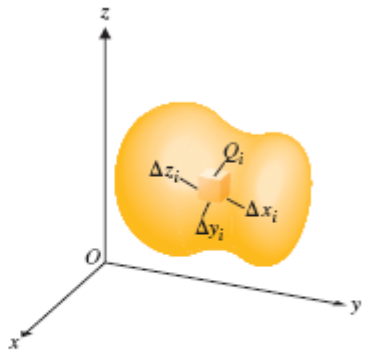


Fig. 2- b

Typical rectangle in inner partition of solid

Special Case of a Triple Integral over a Bounded Volume:

If Q has the form

$Q = \{(x, y, z) \mid (x, y) \in R$ (a bounded region in the xy plane) and $g_1(x, y) \le z \le g_2(x, y)\}$, then

$$\iiint_Q f(x,y,z)\,dV = \iint_R \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\,dA$$

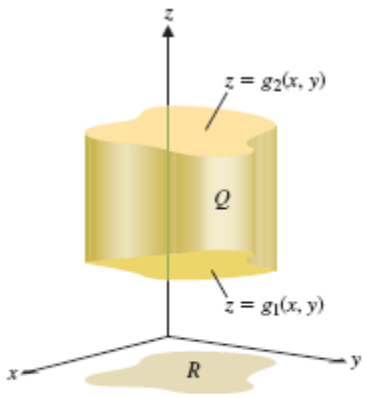


Fig. 3

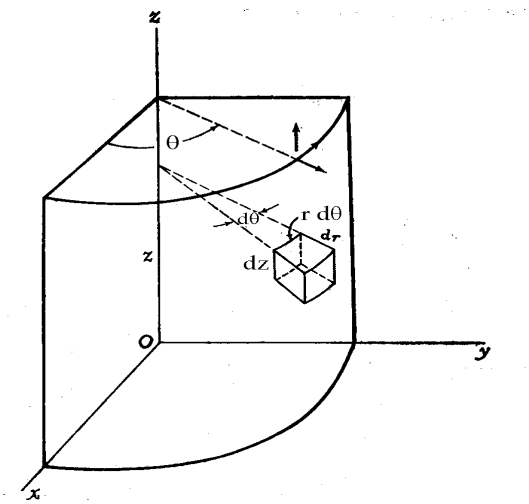
Solid with defined top and bottom surfaces

Note:
The most common choices for non-Cartesian coordinate systems in D^3 are:
1.Cylindrical Polar Coordinates:

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

For which the differential volume is

$$dv = rdrd\phi dz$$

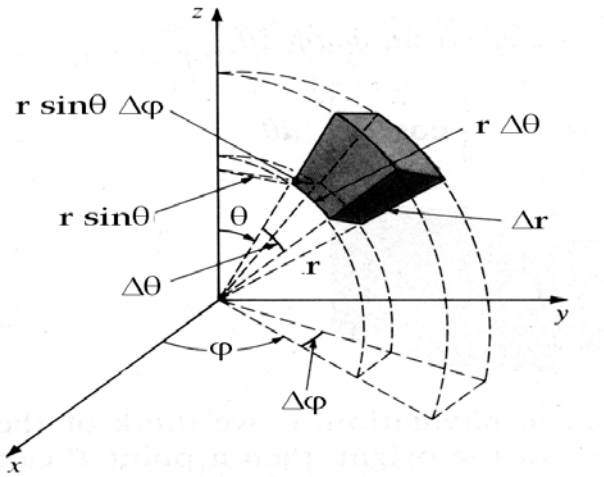


2.Spherical Polar Coordinates:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

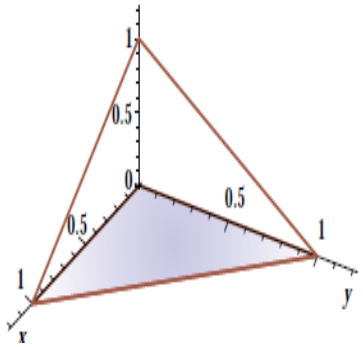
For which the differential volume is

$$dv = r^2\sin\theta drd\theta d\phi$$



Example 1:

Use a triple integral to find the volume of the tetrahedron T bounded by the plane $x + y + z = 1$, and the planes $x = 0$, $y = 0$, and $z = 0$.



$$V = \iiint_T f(x,y,z)\,dv$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dz$$

$$= \int_0^1 \int_0^{1-x} z\Big]_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$$

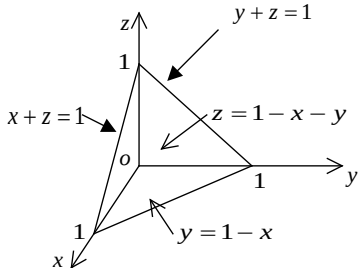
$$= \int_0^1 \left[y - xy - \frac{1}{2}y^2 \right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2 \right) dx$$

$$= \frac{1}{6}$$

Example 2:

Evaluate $\iiint_V \frac{1}{(x+y+2z+1)^3}\,dxdydz$ where V is the region enclosed by the planes $x=0, y=0,\ z=0$ and $x+y+z=1$.



Sol:

The projection of V on xy-plane is σ_{xy} which is bounded by $x=0, y=0, y=1-x$.

$$\iint_{\sigma_{xy}}\left[\int_0^{1-x-y}\frac{1}{(x+y+2z+1)^3}dz\right]dxdy=\int_0^1\left[\int_0^{1-x}\left[\int_0^{1-x-y}\frac{1}{(x+y+2z+1)^3}dz\right]dy\right]dx$$

$$=\int_0^1\left[\int_0^{1-x}\left[\frac{-1}{4(x+y+2z+1)^2}\right]_0^{1-x-y}dy\right]dx=\int_0^1\left[\int_0^{1-x}\left[\frac{1}{4(x+y+1)^2}-\frac{1}{4(3-x-y)^2}\right]dy\right]dx$$

$$=\int_0^1\left[\frac{-1}{4(x+y+1)}-\frac{1}{4(3-x-y)}\right]_0^{1-x}dx=\int_0^1\left[-\frac{1}{8}-\frac{1}{8}+\frac{1}{4(x+1)}+\frac{1}{4(3-x)}\right]dx$$

$$=\left[-\frac{x}{4}+\frac{1}{4}\log(x+1)-\frac{1}{4}\log(3-x)\right]_0^1=\frac{1}{4}(\log 3-1)$$

Example 3:

Use a triple integral to find the volume bounded by $x^2+4y^2=4,\ z=0,\ and\ z=1-y^2$

$$V=4\int_0^2\int_0^{\sqrt{1-\frac{x^2}{4}}}\int_0^{1-y^2}dzdydx=4.7124$$

Example 4:

Use the fact that mass is the product of volume and density to find the mass of the solid bounded by the coordinate planes and 3x+2y+z=6 with density x.

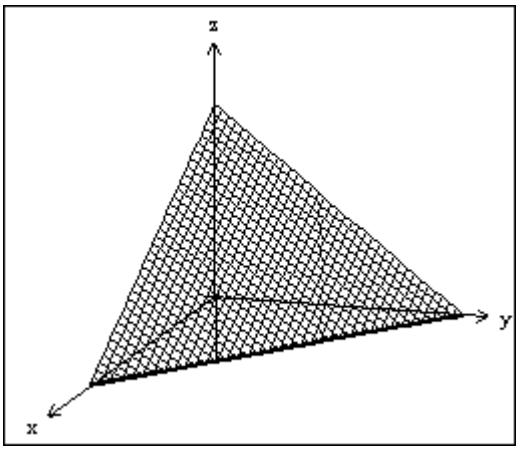
$$Mass=\int_0^2\int_0^{\frac{6-3x}{2}}\int_0^{6-3x-2y}x\ \ dzdydx\ =3$$

Practice:

1. Compute the triple integral of $f(x,y,z)=4ye^{x}\sin(z)$ over the volume Q bounded by $0 \le x \le 1$, $-2 \le y \le 2$, $0 \le z \le \pi$.

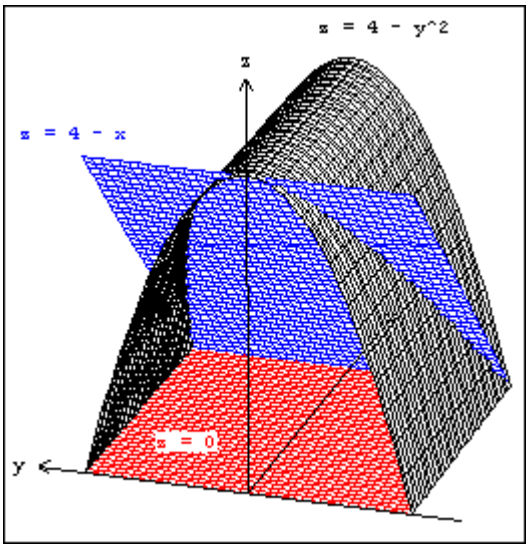
Practice:

2. Compute the triple integral of $f(x,y,z)=24xyz$ over the volume Q bounded by the plane $2x + 3y + 4z = 12$ and the planes $x = 0$, $y = 0$, and $z = 0$.



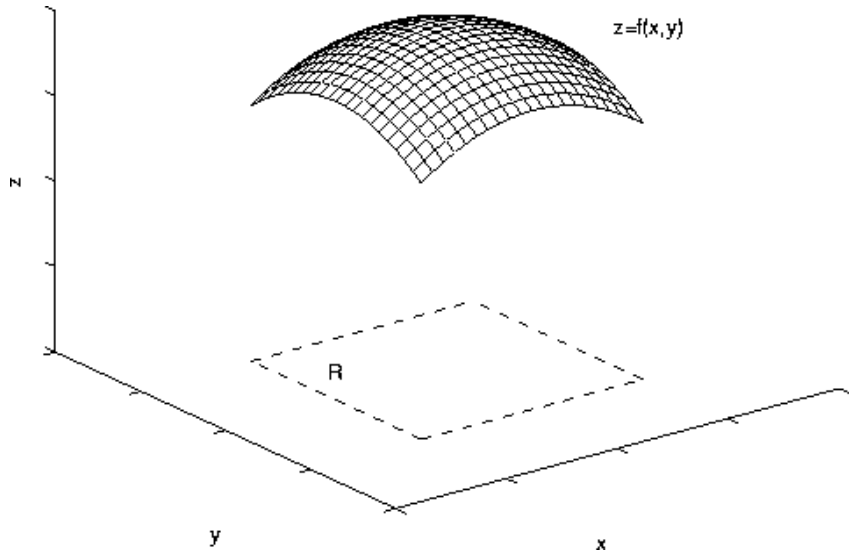
Practice:

3. Find the volume of the solid formed by the intersection of the surfaces $z=4-y^2$, $x+z=4$, $x = 0$, and $z = 0$.



Surface Area

Let f(x ,y) be a differentiable function. As we have seen, z = f(x ,y) defines a surface in xyz-space. In some applications, it necessary to know the surface area of the surface above some region R in the xy-plane. See the figure.



The formula for the surface area is

$$\int\int_R \sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\ \ dA$$

This is a double integral.

Example 1:
What is the surface area of the plane z = 2x + 3y above the rectangle with
$-1 \le x \le 2$ and $0 \le y \le 2$?

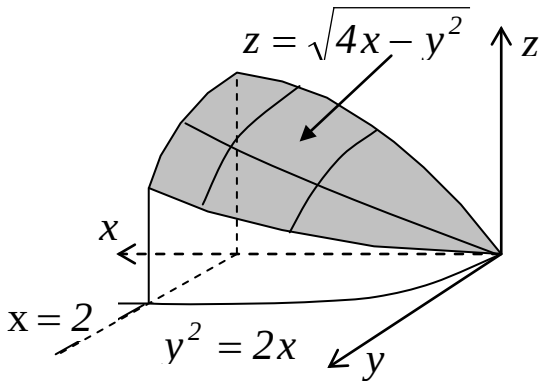
Solution
In this case $f_x = 2$ and $f_y = 3$. Applying the above formula, the surface area S is given by

$$S = \int_{-1}^{2}\int_{0}^{2}\sqrt{14}dydx = \int_{0}^{2}\int_{-1}^{2}\sqrt{14}dxdy.$$

= 6*sqrt(14).

Example 2:

Calculate the surface area of part of the paraboloid $y^2 + z^2 = 4x$ enclosed between the cylinder $y^2 = 2x$ and the plane $x = 2$.



The paraboloid given is symmetric with respect to the plane $z = 0$ and its upper part is described by the equation $z = \sqrt{4x - y^2}$, so the partial derivatives are:

$$\frac{\partial z}{\partial x} = \frac{2}{\sqrt{4x - y^2}}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{4x - y^2}},$$

$$\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2} = \sqrt{1 + \frac{4 + y^2}{4x - y^2}} = 2\sqrt{\frac{x+1}{4x - y^2}}.$$

$$S = 4\iint_D 2\sqrt{\frac{x+1}{4x - y^2}}\,dx\,dy$$

$$S = 8\iint_D \sqrt{\frac{x+1}{4x - y^2}} = 8\int_0^2 \sqrt{x+1}\,dx \int_0^{\sqrt{2x}} \frac{dy}{\sqrt{4x - y^2}} =$$

$$\int_0^2 \sqrt{x+1}\left(\arcsin \frac{y}{2\sqrt{x}}\Big|_0^{\sqrt{2x}} \right) dx = 8\int_0^2 \sqrt{x+1}\,\frac{\pi}{4}\,dx =$$

$$= \frac{4\pi}{3}(x+1)^{3/2}\Big|_0^2$$

$$= \frac{4\pi}{3}\left(3\sqrt{3} - 1\right)$$