# Gradient, Divergence, and Curl

The operators named in the title are built out of the **del operator**

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

(It is also called **nabla**. That always sounded goofy to me, so I will call it "del".)

Del is a formal vector; it has components, but those components have partial derivative operators ($\frac{\partial}{\partial x}$ and so on) which want to be fed functions to differentiate. So as a general rule, when you multiply FOO by del, the partial derivative operators differentiate FOO as opposed to multiplying.

First, suppose f is a function. $\nabla f$ is the **gradient** of f, sometimes denoted $\operatorname{grad} f$ .

---

**Example.** Compute the gradient of $f(x, y, z) = xye^{y^2z}$ .

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = i\left(ye^{y^2z}\right) + j\left(xe^{y^2z} + 2xy^2ze^{y^2z}\right) + k\left(xy^3e^{y^2z}\right).$$

You can also write this as

$$\nabla f = \left(ye^{y^2z}, xe^{y^2z} + 2xy^2ze^{y^2z}, xy^3e^{y^2z}\right). \quad \square$$

---

$\nabla$ is a vector; what kinds of things can you do with a vector? Well, you can take the dot product of the vector with another vector. If you do that with $\nabla$ , you obtain the **divergence**:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}.$$

For this to make sense, $\vec{F}$ should be a **vector field**; since the dot product of two vectors is a number, $\operatorname{div} \vec{F}$ should be a number (that is, a numerical function).

---

**Example.** Compute the divergence of $\vec{F} = (x^2 + y)i + (y^2 - z)j + (z^2 + x)k$ .

$$\operatorname{div} \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left((x^2+y)i + (y^2 - z)j + (z^2+x)k\right) =$$

$$\frac{\partial}{\partial x}(x^2+y) + \frac{\partial}{\partial y}(y^2 - z) + \frac{\partial}{\partial z}(z^2+x) = 2x + 2y + 2z. \quad \square$$

---

What is the divergence of a vector field? If you think of the field as the velocity field of a fluid flowing in three dimensions, then $\nabla \cdot \vec{F} = 0$ means the fluid is **incompressible** --- for any closed region, the amount of fluid flowing in through the boundary equals the amount flowing out. This result follows from the **Divergence Theorem**, one of the big theorems of vector integral calculus.

You can take the **cross product** of two 3-dimensional vectors; if you do this with $\nabla$ , you obtain the **curl** of a vector field:

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}.$$

More specfically, suppose $\vec{F} = (F_1, F_2, F_3)$ . Then

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

The cross product of two vectors is a vector, so curl takes a vector field to another vector field.

---

**Example.** Compute the curl of $\vec{F} = (x^2+y)i + (y^2 - z)j + (z^2+x)k$ .

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y & y^2-z & z^2+x \end{vmatrix}.$$

As usual, I'll expand the determinant by cofactors of the first row:

$$i\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2-z & z^2+x \end{vmatrix} - j\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x^2+y & z^2+x \end{vmatrix} + k\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x^2+y & y^2-z \end{vmatrix} =$$

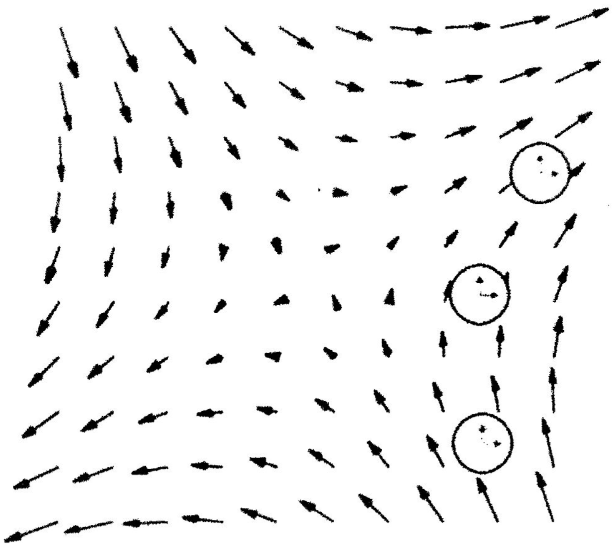
$$i\left(\frac{\partial}{\partial y}(z^2+x) - \frac{\partial}{\partial z}(y^2-z)\right) - j\left(\frac{\partial}{\partial x}(z^2+x) - \frac{\partial}{\partial z}(x^2+y)\right) + k\left(\frac{\partial}{\partial x}(y^2-z) - \frac{\partial}{\partial y}(x^2+y)\right) =$$

$$(0-(-1))\vec{i}-(1-0)\vec{j}+(0-1)\vec{k}=\vec{i}-\vec{j}+\vec{k}=(1,-1,1). \quad \square$$

إلى هنا

---

What is the curl of a vector field? To make it easier to visualize, suppose $\vec{F}$ is the velocity field for a fluid flow *in the plane* (so the z component is 0).



Drop a marked float into the flow and let it be carried along by the fluid. Then $\operatorname{curl}\vec{F}=\vec{0}$ means, roughly, that the float doesn't rotate as it moves.

The old name for $\operatorname{curl}\vec{F}$ was $\operatorname{rot}\vec{F}$, the **rotation** of $\vec{F}$. A field with $\operatorname{rot}\vec{F}=\vec{0}$ is said to be **irrotational** (and this terminology is still used, even though rot has been replaced by curl).

There are many identities involving div, grad, and curl; here is an important one.

---

**Example.** Show that $\operatorname{curl}\operatorname{grad}f=\vec{0}$.

$$\operatorname{curl}\operatorname{grad}f=\nabla\times\nabla f=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}.$$