Complex Numbers

A complex number z is an ordered pair (x, y) of real numbers x and y, written

$$\boxed{z = (x, y)}$$

x is called the real part and y the imaginary part of z, written

$$\boxed{x = \operatorname{Re} z} \qquad \boxed{y = \operatorname{Im} z}$$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

$(0, 1)$ is called the imaginary unit and is denoted by i,

$$\boxed{i = (0, 1)} \qquad (1)$$

"Addition, Multiplication, Notation $z = x + iy$"

Addition of two complex numbers

$$z_1 = (x_1, y_1) \text{ and } z_2 = (x_2, y_2)$$

is defined by

$$\boxed{z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)} \quad (2)$$

Multiplication is defined by

$$\boxed{z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)} \quad (3)$$

The complex numbers "extend" the reals, and we can write

$$(x, 0) = x$$

Similarly, for any real y

$$(0, y) = iy.$$

In practice, complex numbers $z = (x, y)$ are written $\boxed{z = x + iy}$ (4)

If $x = 0$, then $z = iy$ and is called pure imaginary.

$$\boxed{i^2 = -1}$$ (5)

$i^2 = i \cdot i = (0,1)(0,1) = (0-1, 0) = (-1,0) = -1$

- For <u>addition</u>

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2)$$

- For <u>multiplication</u>

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_1) + i(x_1 y_2 + x_2 y_1)$$

<u>Example (1):</u>

Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$.

Then $Re\, z_1 = 8$ & $Im\, z_1 = 3$,

$Re\, z_2 = 9$ & $Im\, z_2 = -2$.

and

$$z_1 + z_2 = (8 + 3i) + (9 - 2i)$$
$$= (8 + 9) + i(3 - 2) = 17 + i,$$

$$z_1 z_2 = (8 + 3i)(9 - 2i)$$
$$= (72 + 6) + i(-16 + 27) = 78 + 11i.$$

« Subtraction, Division »

Subtraction of two complex numbers

$$z_1 - z_2 = (x_1 + iy) - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

The quotient

$$z = \frac{z_1}{z_2} , (z_2 \neq 0) \Rightarrow \boxed{z = \frac{z_1}{z_2} = x + iy}$$

$$z = \frac{z_1}{z_2} = \frac{(x_1 + iy_1)}{(x_2 + iy_2)}$$

$$= \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

$$= x + iy$$

where $x = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}$ & $y = \dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$

EX(2): Let $z_1 = 8 + 3i$ & $z_2 = 9 - 2i$

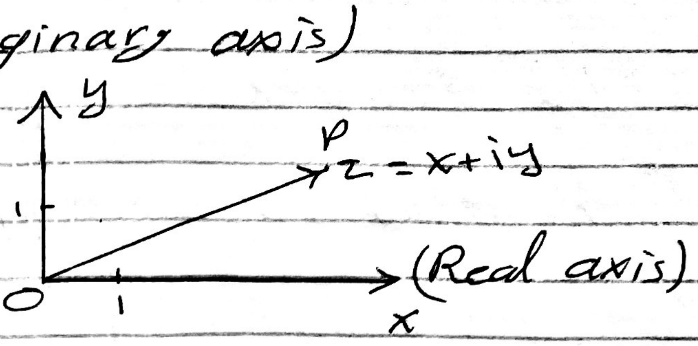
$$z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$$

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{(72 - 6) + i(16 + 27)}{81 + 4}$$

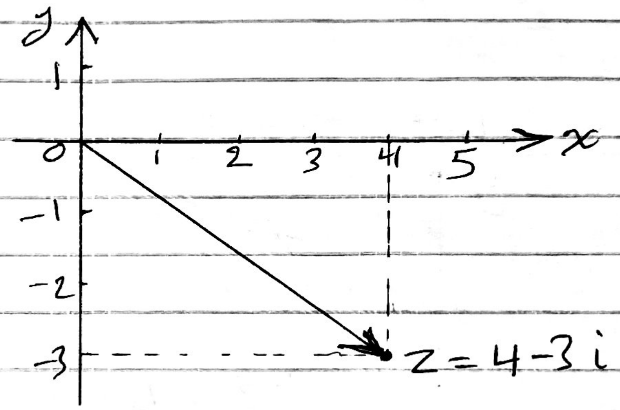
$$= \frac{66 + 43i}{85} = \frac{66}{85} + i\frac{43}{85}$$

Complex Plane:

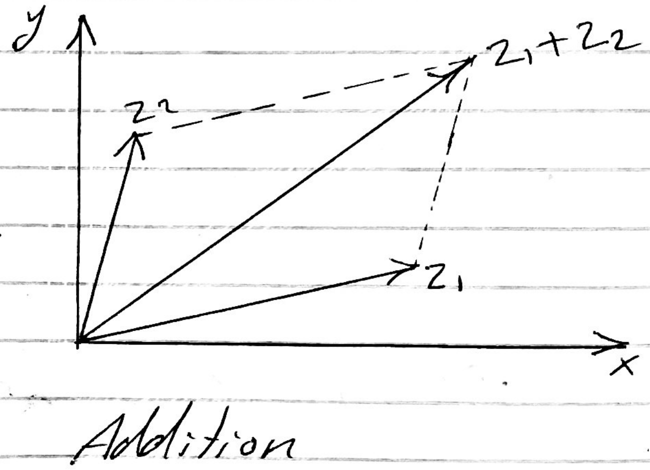
Cartesian coordinate system

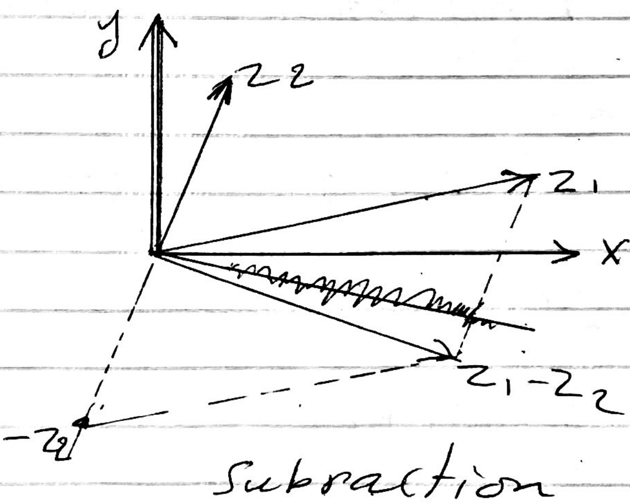


Ex(3): Let $z = 4-3i$



Addition and Subtraction

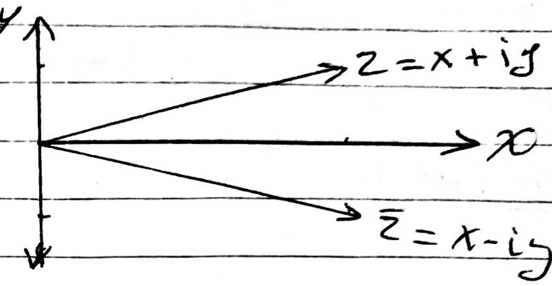




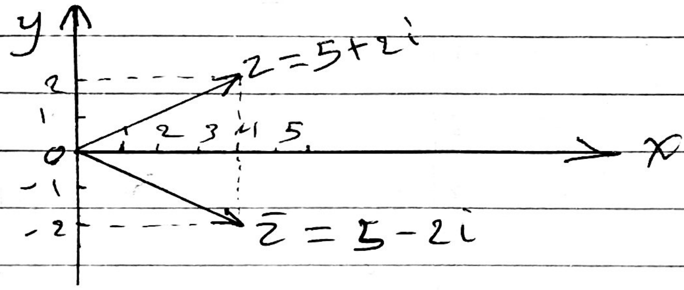
Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\boxed{\bar{z} = x - iy}$$



Ex(4): Let $z = 5 + 2i \Rightarrow \bar{z} = 5 - 2i$



Important Formulas:

$$\operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}) \quad \& \quad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z})$$

Properties:

① $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

② $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$

③ $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$

④ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Polar form of Complex Numbers

Powers and Roots

The polar coordinates r, θ defined by

$$\boxed{x = r\cos\theta \quad , \quad y = r\sin\theta} \qquad (1)$$

Then, $z = x + iy$ in polar form

$$\boxed{z = r(\cos\theta + i\sin\theta)} \qquad (2)$$

$$\boxed{|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}} \qquad (3)$$

where $|z|$ is the distance of the point z from the origion.

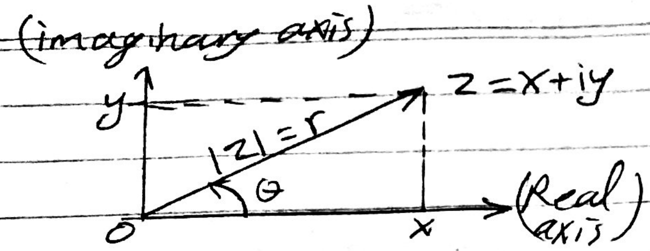
Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2.

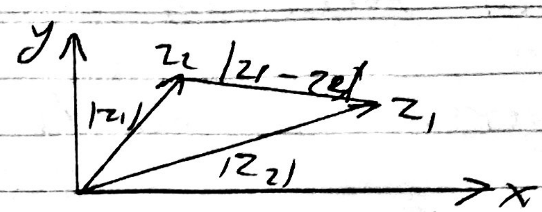
θ is called the argument of z and is denoted by

$$\boxed{\arg z}$$

Thus

$$\boxed{\theta = \arg z = \arctan \frac{y}{x}} \qquad (4)$$





Ex(5):

Multiplication and Division in Polar Form.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplication:

$$z_1 z_2 = r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right]$$

$$\boxed{z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]}$$

$$\boxed{\arg(z_1 z_2) = \arg z_1 + \arg z_2}$$

Division:

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]}$$

$$\boxed{\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2}$$

Integer Powers.

De Moivre's formula

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$$

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

Roots:

$$\boxed{w = \sqrt[n]{z}}$$

is called nth root of z, $(n = 1, 2, \ldots)$

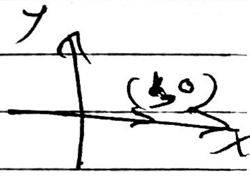
$$\boxed{w = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)}$$

$k = 0, 1, 2, \ldots, n-1$ & $n = 1, 2, 3, \ldots$

Ex(1): Find the roots of $z = \sqrt[3]{1}$

Sol: $n = 3$, $z = 1 \Rightarrow x = 1$ & $y = 0$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{1} = \tan^{-1}0$



$\boxed{\theta = 0}$, $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 0^2} = 1$

$k = 0, 1, 2$

$$w_0 = \sqrt[3]{1}\left(\cos\frac{0+2\pi(0)}{3} + i\sin\frac{0+2\pi(0)}{3}\right)$$

$$= \cos 0 + i\sin 0 = 1 \Rightarrow \boxed{w_0 = 1}$$

$$w_1 = 1\left(\cos\frac{0+2\pi}{3} + i\sin\frac{0+2\pi}{3}\right)$$

$$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} =$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \Rightarrow \boxed{w_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

$$w_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \Rightarrow \boxed{w_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

Roots are

$w_1, w_2\ w_3$

$\Rightarrow 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

H.W:

Find and plot all roots

① $w = \sqrt[4]{1}$

② $w = \sqrt[3]{\underline{1+i}} \rightarrow z$

③ $w = \sqrt[n]{1}$