Infinite Sequences and Series

10.1

Sequences: A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order.

- Each of a_1, a_2, a_3 and so on represents a number.
- These are the terms of the sequence.

Example: $2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$

The first term $a_1 = 2$
" second " $a_2 = 4$
⋮
" nth " $a_n = 2n$.

- The integer n is called the index of a_n.

Example: Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n}, \quad b_n = (-1)^{n+1}\frac{1}{n}, \quad c_n = \frac{n-1}{n}$$

$$d_n = (-1)^{n+1}$$

or by listing terms

$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \cdots, \sqrt{n}, \cdots\}$

$\{b_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots, (-1)^{n+1}\frac{1}{n}, \cdots\}$

$\{c_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, \frac{n-1}{n}, \cdots\}$

$\{d_n\} = \{1, -1, 1, -1, 1, -1, \cdots, (-1)^{n+1}, \cdots\}$

<u>Notation</u>: The sequence $\{a_1, a_2, a_3, \cdots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

<u>Example</u>: $\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$ #

<u>Note that</u>: There are two ways to represent sequences <u>graphically</u>.

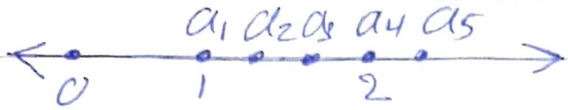
- The <u>first</u> few points from $a_1, a_2, a_3, \cdots, a_n, \cdots$ on the real axis.
- The <u>second</u> method shows the graph of the function defining the sequence.

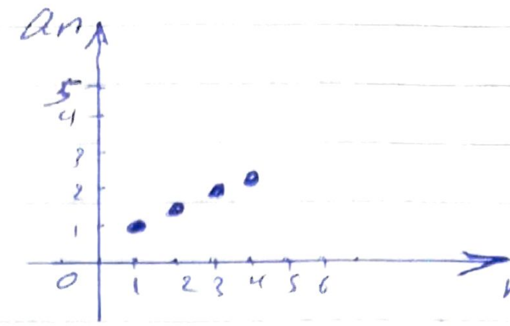
The function is defined only on integer inputs, and the graph consists of some points in the xy-plane located at

$$(1, a_1), (2, a_2), (3, a_3), \cdots, (n, a_n), \cdots$$

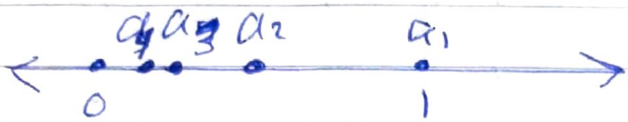
Example:

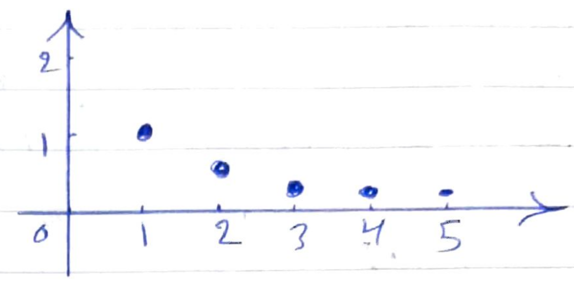
1) $a_n = \sqrt{n}$





2) $a_n = \frac{1}{n}$





Convergence and Divergence:

Definition: The sequence $\{a_n\}$ has a limit L and we write

$$\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

If $\lim_{n \to \infty} a_n$ exists ($\lim_{n \to \infty} a_n = L$), then the sequence $\{a_n\}$ converges. Otherwise, the sequence $\{a_n\}$ diverges.

Ex(1): Check the sequence is convergent or divergent.

1) $a_n = \frac{1}{n}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = \frac{1}{\infty} = 0 \Rightarrow \{a_n\} \text{ is convergent.}$$

E

2. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$ $(a_n = n)$

$\Rightarrow \{a_n\} = \{n\}$ is divergent.

Example (2): Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1} = 1, -1$

The limit has two values 1 & -1 and this is not correct (impossible).

Then the sequence $a_n = (-1)^{n+1}$ is divergent.

Limit Laws for Sequences.

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and k is a constant, then

① $\lim_{n\to\infty} (a_n + b_n) = A + B$ $\left(\lim_{n\to\infty} a_n = A \text{ and } \lim_{n\to\infty} b_n = B\right)$

② $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

③ $\lim_{n\to\infty} k a_n = k \lim_{n\to\infty} a_n$

④ $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

⑤ $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

Ex(3): Which of the sequence $\{a_n\}$ converge, and which diverge? find the limit of each convergent sequence.

① $a_n = -\frac{1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(-\frac{1}{n}\right) = -\lim_{n\to\infty}\frac{1}{n} = 0$ #

convergence

② $a_n = \frac{n-1}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n-1}{n} = \frac{\infty}{\infty}$

We can change the equation $\frac{n-1}{n}$ in the following way:

$\Rightarrow \frac{n-1}{n} = 1 - \frac{1}{n}$

$$\begin{array}{r|l} & 1 \\ n & n-1 \\ & \mp n \\ & 0 \ -1 \end{array}$$

$\Rightarrow \lim_{n\to\infty}\frac{n-1}{n} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)$

$= \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n}$ #

$= 1 - 0 = 1 \Rightarrow$ converges

③ $a_n = \frac{5}{n^2}$

$\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n^2} = 5 \cdot \lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right)$

$= 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot (0) \cdot (0) = 5 \cdot 0$

$= 0 \Longrightarrow$ converges.

④ $a_n = \frac{4-7n^6}{n^6+3} \Rightarrow \lim_{n\to\infty}\frac{4-7n^6}{n^6+3} = \lim_{n\to\infty}\frac{n^6\left(\frac{4}{n^6}-7\right)}{n^6\left(1+\frac{3}{n^6}\right)}$

$= \frac{0-7}{1+0} = -7 \Rightarrow$ converges.

Using L'Hôpital's Rule

Ex(1): Is $a_n = \frac{\ln n}{n}$ converges or diverges?

Sol: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$

Now, we can use the L'Hôpital's Rule

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{\frac{1}{\infty}}{1} = \frac{0}{1} = 0$$

$\therefore a_n = \frac{\ln n}{n}$ is convergence.

ملاحظة:
قاعدة لوبيتال نستخدمها في حالة لدينا متتابعة a_n كسرية وعندما نعوض ∞ او الصفر بعد اخذ النهاية (limit) وكانت النتيجة $\frac{\infty}{\infty}$ أو $\frac{0}{0}$. في هذه الحالة نستخدم طريقة لوبيتال.

نشتق البسط ، نشتق المقام

$$\frac{d}{dn}(\ln n) = \frac{1}{n} \quad \text{and} \quad \frac{d}{dn}(n) = 1$$

Ex(2): $a_n = \frac{n+1}{n-1}$

Sol: $\lim\limits_{n\to\infty} \frac{n+1}{n-1} = \lim\limits_{n\to\infty} \frac{(1)+1}{(1)-1} = \lim\limits_{n\to\infty} 1$

$= 1 \Rightarrow$ converges.

(ملاحظة: $\lim\limits_{n\to\infty} \frac{n+1}{n-1} = \frac{\infty}{\infty}$)

Q15: $1, -4, 9, -16, 25$

$$a_n = (-1)^{n+1} \cdot n^2 \quad , n = 1, 2, 3, 4, \dots$$

Q17: $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \dots$

$$a_n = \frac{2^{n-1}}{3(n+2)} \quad , n = 1, 2, 3, 4, \dots$$

~~**Q18:**~~ $\cancel{-\frac{3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}, \dots}$

$$\cancel{a_n = \frac{(-1)^n}{n(n+1)}}$$

Q13: $a_n = (-1)^{n+1} \quad , n = 1, 2, 3, \dots$

Q23: $\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}, \dots$

$$a_n = \frac{3n+2}{n!} \quad , n = 1, 2, 3, 4, \dots$$

10.2

Infinite Series

Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an <u>infinite series.</u>

The number a_n is the <u>nth term</u> of the series. The sequence $\{S_n\}$ defined by

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

$$\vdots$$

is the <u>sequence of partial sums</u> of the series, the number S_n being the <u>nth partial</u> sum.

Geometric Series

Geometric series are series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a & r are fixed real numbers and $a \neq 0$.

The series can also be written as

$$\sum_{n=0}^{\infty} ar^n.$$

r is the ratio can be positive, as in

Ex(1): $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$

$\Rightarrow r = 1/2$ and $a = 1$

or negative, as in

Ex(2): $1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots,$

$\Rightarrow r = -1/3$ and $a = 1$.

Convergence and Divergence.

- If $|r| < 1$, the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

converges to $\dfrac{a}{1-r}$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad , |r| < 1 .$$

- If $|r| \geq 1$, the series diverges.

__Ex(3):__ The series

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} =$$

$$\frac{(1/9)}{1 - (1/3)} = \frac{1}{6} .$$

is the geometric series with

$$a = \frac{1}{9} \text{ and } r = \frac{1}{3} \qquad \#$$

__Ex(4):__ The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -\frac{1}{4}$

$|r| = \left|-\frac{1}{4}\right| = \frac{1}{4} < 1 \Rightarrow$

It converges to $\frac{a}{1-r} = \frac{5}{1+\frac{1}{4}} = 4$

Ex(5):

Express the repeating decimal

$5.232323\ldots$ as the ratio of two integers.

Sol:

$5.232323\ldots = 5 + 0.232323\ldots$

$$= 5 + 0.23 + 0.0023 + 0.000023 + \cdots$$

$$= 5 + \frac{23}{100} + \frac{23}{10000} + \frac{23}{1000000} + \cdots$$

$$= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots$$

$$= 5 + \frac{23}{100}\underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)}_{\text{geometric series}}$$

$a = 1$ & $r = \frac{1}{100}$

$$= 5 + \frac{23}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = 5 + \frac{23}{100}\left(\frac{1}{0.99}\right)$$

$$= \frac{518}{99} \quad \#$$

H.W.

(Exercises 10.1)

~~1~~ 2 3 ~~4~~ ~~5~~ ~~6~~

~~7~~ ~~8~~

13 15 17 ~~18~~ 23

27 ~~28~~ 29 ~~30~~ 31 ~~33~~ ~~34~~

~~49~~ ~~60~~ ~~71~~ ~~77~~

~~10.2~~: P. 546

Ex(1) , Ex(2) , Ex(4)

P. 548 غير داخل

10.2: ~~1~~ 2 3 7 19 ~~20~~

10.5: Ex(1) a, b, c

1 5 17 32

14.5 : 7 8 Gradient ∇f

$f(x,y,z) =$ — divergent $\vec{\nabla} \cdot f$ (div f)

curl $\vec{\nabla} \times f$.