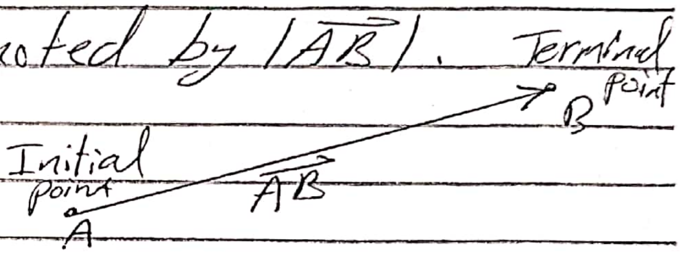
Vectors

The vector represented by the directed line segment

$\overrightarrow{AB}$

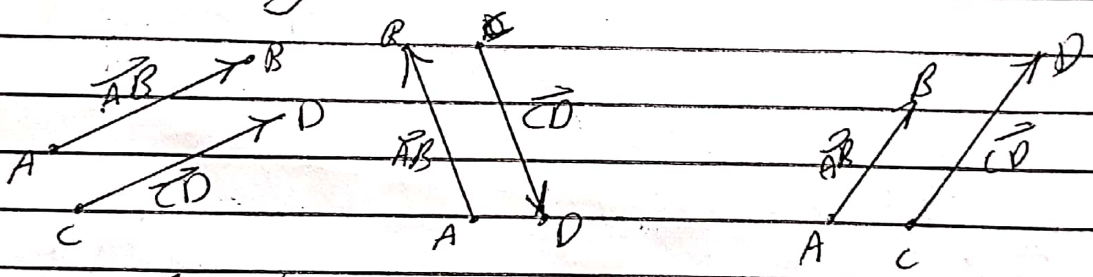
has initial point A and terimial point B

and its length is denoted by $|\overrightarrow{AB}|$.

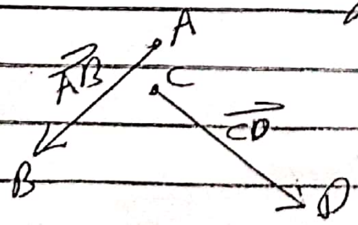


Equality of Vectors:

Two vectors are equal if they have the same length and direction.



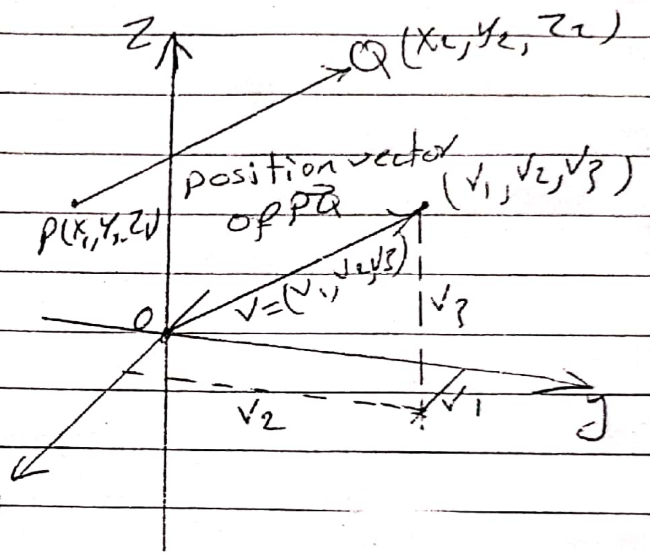
Equal vectors $\overrightarrow{AB} = \overrightarrow{CD}$	Vectors having the same length but different direction.	Vectors having the same direction but different length



Vector having different length and different direction

Position vector

The position vector v of a point (v_1, v_2, v_3) is the vector with the origin $(0,0,0)$ as the initial point and (v_1, v_2, v_3) as the terminal point



- If v is a two-dimensional vector in the plane equal to the vector with initial point as the origin and terminal point (v_1, v_2),

then the component form of v is

$$v = (v_1, v_2)$$

- If v is a three-dimensional vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the component form of v is

$$v = (v_1, v_2, v_3).$$

$$\vec{V} = \overrightarrow{PQ}$$

$$P(x_1, y_1, z_1) \longrightarrow Q(x_2, y_2, z_2)$$

$$\vec{V} = (V_1, V_2, V_3)$$

where, $V_1 = x_2 - x_1$, $V_2 = y_2 - y_1$, $V_3 = z_2 - z_1$

$$\vec{V} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

The magnitude or length of the vector $\vec{V} = \overrightarrow{PQ}$ is the nonnegative number

$$|\vec{V}| = \sqrt{V_1^2 + V_2^2 + V_3^2}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Ex(1): Find the

(a) Component form

(b) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$

Sol: (a) The component form of $\overrightarrow{PQ}$ is

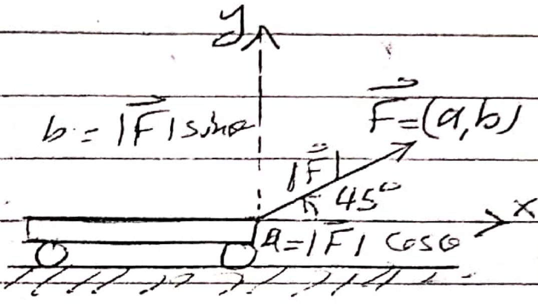
$$\overrightarrow{PQ} = \vec{V} = (V_1, V_2, V_3)$$

$$\left.\begin{array}{l} V_1 = x_2 - x_1 = -5 - (-3) = -2 \\ V_2 = y_2 - y_1 = 2 - 4 = -2 \\ V_3 = z_2 - z_1 = 2 - 1 = 1 \end{array}\right\} \Rightarrow V = (-2, -2, 1)$$

(b) The length of $\vec{V} = \overrightarrow{PQ}$ is

$$|\vec{V}| = \sqrt{V_1^2 + V_2^2 + V_3^2} = \sqrt{(-2)^2 + (-2)^2 + 1^2} = 3$$

Ex(2): A small cart is being pulled along a smoth horizontal floor with an 80-N force $\vec{F}$ making a 45° angle to the floor. What is the effective force moving the cart forward?



Sol: $\vec{F} = (a, b)$

$a = |\vec{F}| \cos\theta$

$= |\vec{F}| \cos 45^\circ$

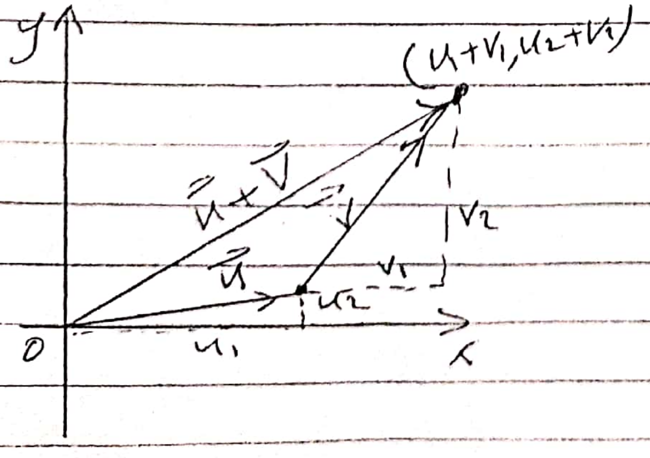
$= (80)\left(\frac{1}{\sqrt{2}}\right) \approx 56.57\ N$

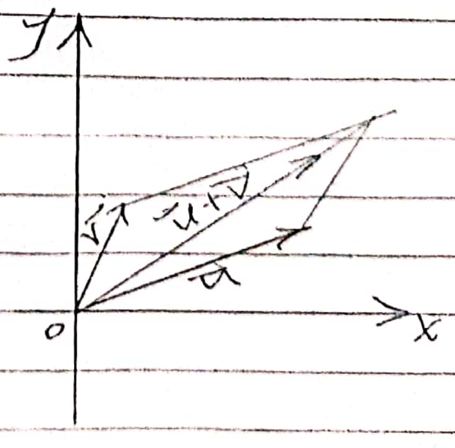
Vector Algebra Operations

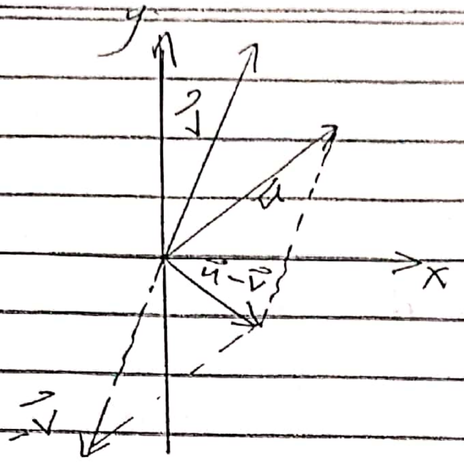
Let $u = (u_1, u_2, u_3)$ & $v = (v_1, v_2, v_3)$ be vectors with k a scalar.

Addition : $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

Scalar multiplication : $ku = (ku_1, ku_2, ku_3)$







EX(3): Let $\vec{u} = (-1,3,1)$ and $\vec{v} = (4,7,0)$. Find the components of

ⓐ $2\vec{u}+3\vec{v}$ ⓑ $\vec{u}-\vec{v}$ ⓒ $\left|\frac{1}{2}\vec{u}\right|$

Sol:

ⓐ $2\vec{u}+3\vec{v} = 2(-1,3,1)+3(4,7,0)$
$= (-2,6,2)+(12,21,0) = (10,27,2)$

ⓑ $\vec{u}-\vec{v} = (-1,3,1)-(4,7,0)$
$= (-1-4,\ 3-7,\ 1-0) = (-5,-4,1)$

ⓒ $\left|\frac{1}{2}\vec{u}\right| = \left|\frac{1}{2}(-1,3,1)\right| = \left|\left(-\frac{1}{2},\frac{3}{2},\frac{1}{2}\right)\right|$
$= \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}$

Properties of Vector Operations

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and a, b scalars.

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + 0 = \vec{u}$
4. $\vec{u} + (-\vec{u}) = 0$
5. $0\vec{u} = 0$
6. $1\vec{u} = u$
7. $a(b\vec{u}) = (ab)\vec{u}$
8. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
9. $(a+b)\vec{u} = a\vec{u} + b\vec{u}$.

Unit Vector:

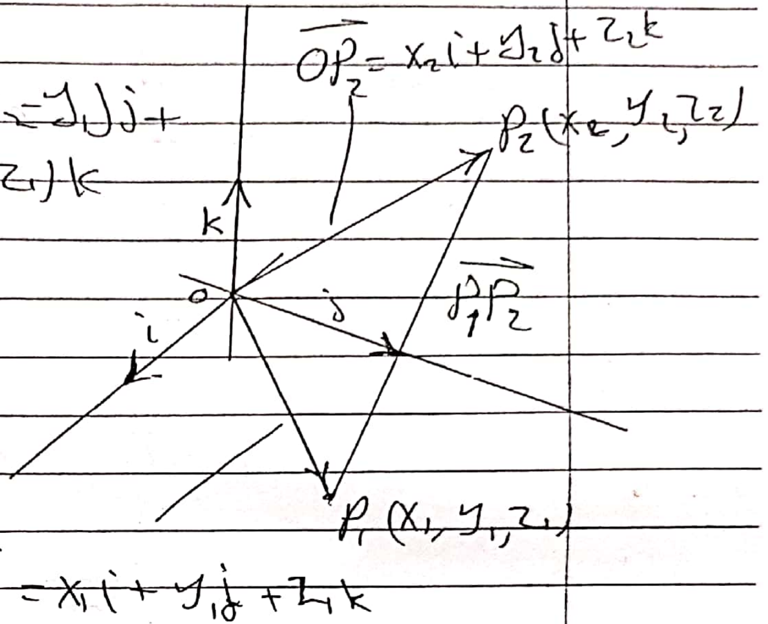
A vector $\vec{v}$ of length 1 is called a unit vector.

The standard unit vectors are

$i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, k)$

$$\vec{V} = (v_1, v_2, v_3)$$
$$= (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3)$$
$$= v_1(1, 0, 0) + v_2(0, 1, 0) + v_3(0, 0, 1)$$
$$= v_1 i + v_2 j + v_3 k$$

$$\overrightarrow{P_1P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$



The length $|V|$ is not zero and $v \neq 0$, and

$$\left|\frac{1}{|V|} V\right| = \frac{1}{|V|}|V|$$
$$= 1$$

$\frac{V}{|V|}$ is a unit vector in the direction of V,

called the direction of the nonzero vector V.

Ex(4): Find a unit vector $\vec{u}$ in the direction of the of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Sol: $\vec{u} = \dfrac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|}$

$$\overrightarrow{P_1P_2} = (3-1)i + (2-0)j + (0-1)k$$

$$= 2i + 2j - k$$

$$|\overrightarrow{P_1P_2}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9}$$

$$= 3$$

$$\vec{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2i + 2j - k}{3}$$

$$= \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

The unit vector $\vec{u}$ is the direction of $\overrightarrow{P_1P_2}$

EX(5): If $v = 3i - 4j$ is a velocity vector, express v as a product of its speed times a unit vector in the direction of motion.

Sol: Speed is the magnitude (length) of v:

$$|v| = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = 5$$

$$\frac{v}{|v|} = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$$v = 3i - 4j = 5\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

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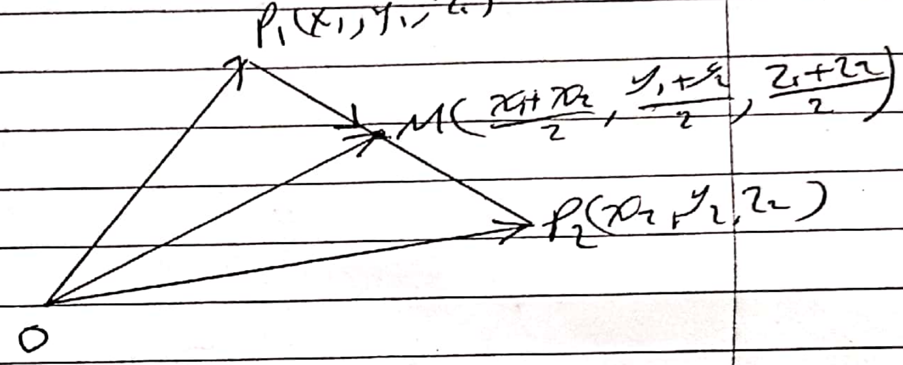
Midpoint of a Line Segment

The midpoint M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$



Ex(6): The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right) = (5, 1, 2)$$

The Dot Product

The dot product $\vec{u}\cdot\vec{v}$ ("$\vec{u}$ dot $\vec{v}$") of vectors $\vec{u}=(u_1, u_2, u_3)$ & $\vec{v}=(v_1, v_2, v_3)$ is

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3}$$

Ex(1): Find $\vec{u}\cdot\vec{v}$

(a) $\vec{u}=(1,-2,-1)$ & $\vec{v}=(-6,2,-3)$

$$\vec{u}\cdot\vec{v} = (1)(-6) + (-2)(2) + (-1)(-3)$$
$$= -6 - 4 + 3 = -7$$

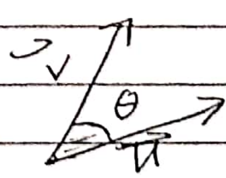
(b) $\vec{u} = \frac{1}{2}i + 3j + k)$ and $\vec{v} = 4i - j + 2k$

$$\vec{u}\cdot\vec{v} = (\tfrac{1}{2})(4) + (3)(-1) + (1)(2)$$
$$= 2 - 3 + 2 = 1 \quad \#$$

Angle Between Two Vectors

The angle between two nonzero vectors $u=(u_1, u_2, u_3)$ & $v=(v_1, v_2, v_3)$ is given by

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$



Ex(2): Find the angle between
$\vec{u} = i - 2j - 2k$ and $\vec{v} = 6i + 3j + 2k$

Sol: $\vec{u} \cdot \vec{v} = (1)(6) + (-2)(3) + (-2)(2)$
$= -4$

$|\vec{u}| = \sqrt{1^2 + (-2)^2 + (-2)^2} = 3$

$|\vec{v}| = \sqrt{6^2 + 3^2 + 2^2} = 7$

$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$

$= \cos^{-1}\left(\frac{-4}{(3)(7)}\right) = \cos^{-1}\left(\frac{-4}{21}\right)$

$\simeq 1.76$ #

Perpendicular (Orthogonal) vectors

Vectors $\vec{u}$ & $\vec{v}$ are orthogonal if and only if

$$\boxed{\vec{u} \cdot \vec{v} = 0}$$

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Properties

If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are any vectors and c is a scalar, then

① $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

② $(c\vec{u})\cdot\vec{v} = u\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v})$

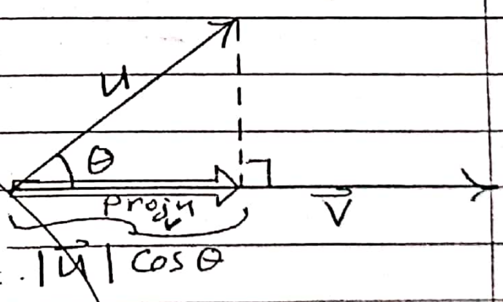
③ $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

④ $\vec{u}\cdot\vec{u} = |u|^2$

⑤ $0\cdot\vec{u} = 0$

Vector Projections

$\text{length} = |\vec{u}|\cos\theta$



$\text{Proj}_{\vec{v}}\vec{u} = (|\vec{u}|\cos\theta)\dfrac{\vec{v}}{|\vec{v}|}$

$= (|\vec{u}|\,\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|})\,\dfrac{\vec{v}}{|\vec{v}|}$

$= \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\dfrac{\vec{v}}{|\vec{v}|} = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$

uinte victor $\frac{\bar{V}}{|\bar{V}|}$ = يحدد الاتجاه

$u \angle_{\theta} v$ $u \cdot v = |u||v|\cos\theta$ / $u \times v = |u||v|\sin\theta$

عمودي → $u \perp v \Leftrightarrow u \cdot v = 0$, $\theta = \frac{\pi}{2}$

توازي → $u // v \Leftrightarrow \vec{u} \times \vec{v} = 0$, $\theta = 0$

The Cross Product

$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\,n$$

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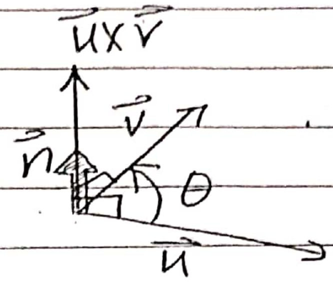


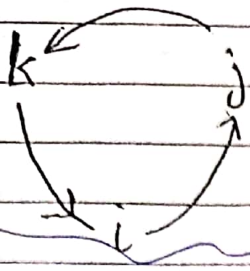
where $\vec{n}$ is the unit (normal) vector. ~~that~~

$\boxed{\text{Nonzero vectors } \vec{u} \text{ and } \vec{v} \text{ are parallel if and only if } \vec{u} \times \vec{v} = 0}$

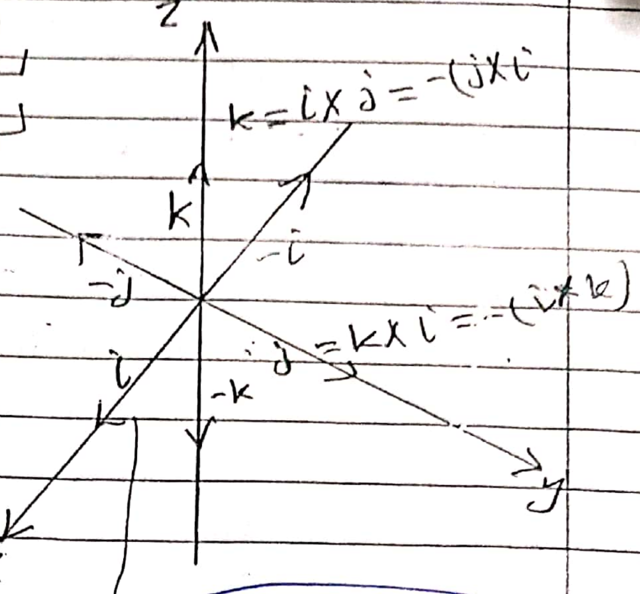
Properties of the cross Product

If u, v and w are any vectors and r, s are scalars, then

1. $(ru) \times (sv) = (rs)(\vec{u} \times \vec{v})$
2. $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(v + w) \times u = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
5. $0 \times \vec{u} = 0$
6. $u \times (v \times w) = (u \cdot w)\,v - (u \cdot v)\,w$



i j k



$i \times j = -(j \times i) = k$

$j \times k = -(k \times j) = i$

$k \times i = -(i \times k) = j$

$i = j \times k = -(k \times j)$

$j \times i = -k$

$k \times j = -i$

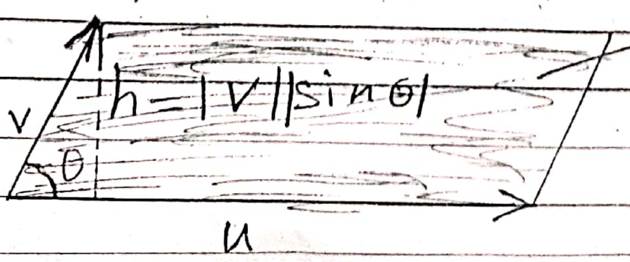
$i \times k = -j$

$$\boxed{\begin{aligned} i \cdot i &= j \cdot j = k \cdot k = 1 \\ i \cdot j &= j \cdot k = k \cdot i = 0 \end{aligned}}$$

$$\boxed{i \times i = j \times j = k \times k = 0}$$

$\boxed{|\vec{u} \times \vec{v}|}$ Is the Area of a Parallelogram

$$\boxed{\begin{aligned} |\vec{u} \times \vec{v}| &= |\vec{u}|\,|\vec{v}|\,|\sin\theta|\,|n| \\ &= |\vec{u}|\,|\vec{v}|\,\sin\theta \end{aligned}}$$



Area = base × height
$= |\vec{u}|\,|\vec{v}|\,|\sin\theta|$
$= |\vec{u} \times \vec{v}|$

Calculating Cross Product as a Determinant.

If $u = u_1 i + u_2 j + u_3 k$ and $\vec{v} = v_1 i + v_2 j + v_3 k$ then

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Ex(1): Find $u \times v$ and $v \times u$ if
$\vec{u} = 2i + j + k$ and
$\vec{v} = -4i + 3j + k$

Sol:

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}$$

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$$= i \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}$$

$$= -2i - 6j + 10k$$

$$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}) = 2i + 6j - 10k$$

Ex(2): Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$ and $R(-1, 1, 2)$.

Sol:

$$\begin{matrix} -2 & 0 & 0 & 3 \\ 0 & 7 & 7 & -4 \end{matrix}$$

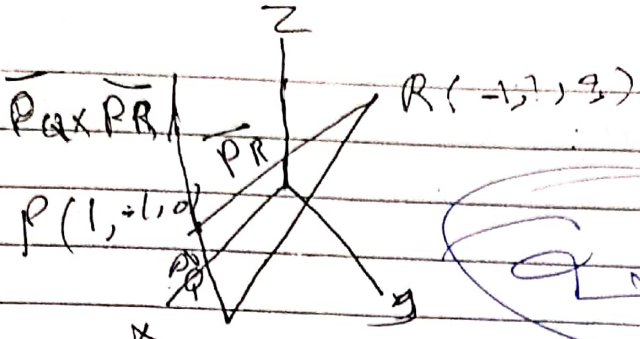
$$\begin{matrix} -2 & 3 \\ 0 & -4 \end{matrix}$$

$(-14-0)-(6-21)+6-3$

$-14-(-21)+3$

$-14+21+3$

$10 \quad 24$



$$\overrightarrow{PQ} = (x_2-x_1)i + (y_2-y_1)j + (z_2-z_1)k$$

$$= (2-1)i + (1-(-1))j + (-1-0)k$$

$$= i + 2j - k$$

$$\overline{PR} = (-1-1)i + (1-(-1))j + (2-0)k$$

$$= -2i + 2j + 2k$$

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$$\overline{PQ}\times\overline{PR} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= i\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} - j\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + k\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}$$

$4-(-4)$

$$= 6i + 0 + 6k = \frac{1}{2}(6i + 6k)$$

$\frac{1}{2}(6i+6k)$

$(3i+3k)$

$\boxed{3i+3k}$