تعلم ورياضة
اولـ ب

Differentiation

Differentiation Rules:

① **Derivative of a Constant Function:**

If f has the constant value $f(x) = C$,
then

$$\frac{df}{dx} = \frac{d}{dx}(C) = 0$$

② **Power Rule for Positive Integers:**

If n is a positive integer, then

$$\frac{d}{dx} x^n = n x^{n-1}$$

③ **Power Rule (General Version)**

If n is any real numbers then

$$\frac{d}{dx} x^n = n x^{n-1}$$

for all x where the powers x^n and x^{n-1} are defined.

Ex(1): Find the derivative of:

① $y = 6 \Rightarrow \frac{dy}{dx} = 0$

② $y = x^3 \Longrightarrow \frac{dy}{dx} = 3x^2$

③ $y = x^{\sqrt{4}} \Longrightarrow \frac{dy}{dx} = \sqrt{4}\, x^{\sqrt{4}-1}$

④ $y = \sqrt{x^{2+\pi}} \Longrightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\left(x^{2+\pi}\right)^{\frac{1}{2}}\right)$

$= \frac{d}{dx}\left(x^{\frac{2+\pi}{2}}\right) \Longrightarrow \frac{dy}{dx} = \frac{2+\pi}{2}\, x^{\frac{2+\pi}{2}-1}$

$\Rightarrow \frac{dy}{dx} = \left(1+\frac{\pi}{2}\right) x^{\frac{\pi}{2}} = \left(1+\frac{\pi}{2}\right)\sqrt{x^{\pi}}$

④ Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

Ex(2): Find the derivative of

$$y = 3x^2 \Longrightarrow \frac{dy}{dx} = 3 \cdot 2x = 6x$$

⑤ Derivative Sum Rule:

If u & v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u & v are both diff.

$$\boxed{\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}}$$

Ex(3): Find $\frac{dy}{dx}$ of $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

Sol: $\frac{dy}{dx} = 3x^2 + \frac{8}{3}x - 5$ #

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⑥ Derivative Product Rule:

If u & v are differentiable at x, then so is their product uv, and

$$\boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

Ex(4): Find $\frac{dy}{dx}$ of $y = (x^2+1)(x^3+3)$

Solution: From the Product Rule with $u = x^2+1$ and $v = x^3+3$, we find

$$\frac{d}{dx}\left((x^2+1)(x^3+3)\right) = (x^2+1)(3x^2) + (x^3+3)(2x)$$
$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x$$

$\Rightarrow \frac{dy}{dx} = 5x^4 + 3x^2 + 6x$ #

⑦ Derivative Quotient Rule:

If u and v are differentiable x and if $v(x) \neq 0$ then the quotient u/v is differentiable at x, and

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Ex(5): Find $\frac{dy}{dx}$ of $y = \frac{x^2-1}{x^3+1}$

Sol: $$\frac{dy}{dx} = \frac{(x^3+1)(2x) - (x^2-1)(3x^2)}{(x^3+1)^2}$$

$$= \frac{2x^4 + 2x - 3x^4 + 3x^2}{(x^3+1)^2} = \frac{-x^4 + 3x^2 + 2x}{(x^3+1)^2} \quad \#$$

Second and Higher-Order Derivatives.

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''.

So $\boxed{f'' = (f')'}$

The function f'' is called the second derivative of f because it is the derivative of the first derivative.

It is written in several ways:

The symbol $\underline{D}^2$ means the operation of differentiation is performed twice.

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2 f(x)$$

Ex(6): Find y'' of $y = x^6$

Sol: $y' = 6x^5 \Rightarrow y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4$

Thus $D^2(x^6) = 30x^4$. #

If y'' is diff., its derivative, $y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$, is the third derivative of y w.r.t x.
Then,

$$\boxed{y^{(n)} = \frac{d}{dx} y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y}$$

denoted the nth derivative of y w.r.t. x for any positive integer n.

Ex(7): The first four derivatives of $y = x^3 - 3x^2 + 2$ are

1. : $y' = 3x^2 - 6x$
2. : $y'' = 6x - 6$
3. : $y''' = 6$
4. : $y^{(4)} = 0$ #

H.W: Exercises (3.3).

Derivatives of Trigonometric Functions.

The Basic Trigonometric Functions are:

① $y = \sin x$, ④ $y = \cot x$

② $y = \cos x$, ⑤ $y = \sec x$

③ $y = \tan x$, ⑥ $y = \csc x$

Trigonometric Identities:

① $$\boxed{\cos^2 x + \sin^2 x = 1}$$

② $$\boxed{\begin{aligned} 1 + \tan^2 x &= \sec^2 x \\ 1 + \cot^2 x &= \csc^2 x \end{aligned}}$$

③ $$\boxed{\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B \end{aligned}}$$

④ $$\boxed{\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ \sin 2x &= 2 \sin x \cos x \end{aligned}}$$

⑤ $$\boxed{\begin{aligned} \cos^2 x &= \frac{1 + \cos 2x}{2} \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \end{aligned}}$$

Derivative of the sine function

Show that $\frac{d}{dx}(\sin x) = \cos x$

Sol: If $f(x) = \sin x$, then

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\sin x \cos \Delta x + \cos x \sin \Delta x) - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x(\cos \Delta x - 1) + \cos x \sin \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left(\sin x \cdot \frac{\cos \Delta x - 1}{\Delta x}\right) + \lim_{\Delta x \to 0}\left(\cos x \frac{\sin \Delta x}{\Delta x}\right)$$

$$= \sin x \cdot \underbrace{\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x}}_{\text{limit } 0} + \cos x \underbrace{\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}}_{\text{limit } 1}$$

$$= \sin x \cdot (0) + \cos x \cdot (1) = \cos x \ \#$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Ex(1): Find $\frac{dy}{dx}$ of

① $y = x^2 - \sin x$: $\frac{dy}{dx} = 2x - \cos x$

② $y = x^2 \sin x$: $\frac{dy}{dx} = x^2 \cos x + 2x \sin x$

③ $y = \frac{\sin x}{x}$: $\frac{dy}{dx} = \frac{x \cos x - \sin x}{x^2}$ #

The Derivative of the Trigonometric Functions.

① $\frac{d}{dx}(\sin u) = \cos u \frac{dy}{dx}$ (u is any function)

② $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

③ $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$

④ $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$

⑤ $\frac{d}{dx}(\sec u) = \sec u . \tan u \frac{du}{dx}$

⑥ $\frac{d}{dx}(\csc u) = -\csc u . \cot u \frac{du}{dx}$

Ex(2): Find $\frac{dy}{dx}$ of:

① $y = 5x + \cos x$: $\frac{dy}{dt} = 5 - \sin x$

② $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x(-\sin x) + \cos x.(\cos x)$$
$$= -\sin^2 x + \cos^2 x = \cos 2x \ \#$$

Ex(3): Find y'' if $y = \sec x$

Sol: $y = \sec x \Rightarrow y' = \sec x.\tan x$

$$\Rightarrow y'' = \sec x.\sec^2 x + \tan x(\sec x \tan x)$$
$$= \sec^3 x + \sec x \tan^2 x \quad \#$$

H.W: Exercises (3.5)