Matrices

A matrix is a rectangular array of number (or functions) enclosed in brackets. These no.s (or functions) are called entries or elements of the matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad \text{rows}$$

$(m \times n)$ size of A

columns

In general, $A = [a_{ij}]_{m \times n}$

where $i = 1, 2, 3, \ldots, m$ and
$j = 1, 2, 3, \ldots, n$

Square Matrix

If $m = n$, we call A an $n \times n$ square matrix

Main Diagonal

The main diagonal containing the entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ is called the main diagonal or principal diagonal.

Vectors

A vector is a matrix with only one row or one column.

$a = [a_1 \; a_2 \; --- \; a_n]$ is row vector

$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ column vector

Equality of Matrices:

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal, written

$$A = B,$$

if and only if they have the same size and the corresponding entries are equal, that is,

$a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on.

Ex(1): Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$,

then,

$$A = B \Leftrightarrow \begin{matrix} a_{11} = 4, & a_{12} = 0 \\ a_{21} = 3, & a_{21} = -1 \end{matrix}$$

Addition and Scalar Multiplication of Matrices and vectors

Addition of Matrices

The sum of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size is written

$$A + B = [a_{ij} + b_{ij}]$$

EX(2): If

$$A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

Scalar Multiplication

The product of any $m \times n$ matrix $A = [a_{ij}]$ and any scalar c (number c) is written cA and is the $m \times n$ matrix $cA = [ca_{ij}]$ obtained by multiplying each element of A by c.

EX(3): $A = \begin{bmatrix} 3 & 0.7 \\ 1 & 0 \\ -2 & 3 \end{bmatrix}$ and $c = -1$

$$\Rightarrow cA = \begin{bmatrix} -3 & -0.7 \\ -1 & 0 \\ 2 & -3 \end{bmatrix}$$

Rules for Matrix Addition and Scalar Multiplication

1. $A + B = B + A$
2. $(A+B) + C = A + (B+C)$
3. $A + 0 = A$
4. $A + (-A) = 0$

And,

1. $c(A+B) = cA + cB$
2. $(c+k)A = cA + kA$
3. $c(kA) = (ck)A$
4. $1A = A$

Matrix Multiplication

The product $C = AB$ of an $m \times n$ matrix A and $r \times p$ matrix B is defined if and only if $r = n$ and is then the $m \times p$ matrix

$$C = [c_{ij}] \quad \text{with}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj},$$

$$i = 1, 2, \ldots, m$$
$$j = 1, 2, \ldots, p$$

$$\underset{[m \times n]}{A} \; \underset{[n \times p]}{B} = \underset{[m \times p]}{C}$$

$r = n$

For instance,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}_{(4 \times 3)} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{(3 \times 2)}$$

$m = 4$, $n = 3$, $p = 2$

$$AB = C \quad (A_{[4 \times 3]} B_{[3 \times 3]} = C_{[4 \times 3]})$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}$$

Ex(4): If

$$A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 3 & 0 \\ -4 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 3 & 0 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 25 & -7 \\ 0 & -6 \\ -29 & 14 \end{bmatrix}$$

$(3\times -2)+(5\times 0)+(-1\times 1) = -7$ $\quad (3\times 2)+(5\times 3)+(-1\times -4) = 25$ $\quad (-6\times 2)+(-3\times 3)+(2\times -4) = -29$

$(4\times -2)+(0\times 0)+(2\times 1) = -6$ $\quad (-6\times -2)+(-3\times 0)+2\times(1) = 14$

$(4\times 2)+(0\times 3)+(2\times -4) = 0$

Ex(5): If

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$AB = \underset{(2\times 2)}{\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}} \underset{(2\times 1)}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} = \underset{(2\times 1)}{\begin{bmatrix} 4.3 + 2.5 \\ 1.3 + 8.5 \end{bmatrix}} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

$\boxed{AB \neq BA}$ not commutative

Properties of Matrix Multiplication

1. $(kA)B = k(AB) = A(kB)$
2. $A(BC) = (AB)C$
3. $(A+B)C = AC + BC$
4. $C(A+B) = CA + CB$

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Transposition of Matrices and Vectors

The transpose of $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix A^T (read A transpose) that has the first row of A as its first column, th second row of A as its second column, and so on.

Thus, the transpose of A is $A = [a_{ji}]$, written

$$A^T = [a_{ji}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix}$$

Rules of transposition are

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(cA)^T = cA^T$

④ $(AB)^T = B^T A^T$

Ex(6): If

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

Symmetric and Skew-Symmetric Matrices

Symmetri matrices are square matrices whose

$$A^T = A \text{, thus } a_{ij} = a_{ji}$$

Skew-symmetric matrices are square matrices whose

$$A^T = -A \text{, thus } a_{ij} = -a_{ji},\ a_{ii} = 0$$

Ex(7): Symmetric Matric

$$A = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$$

Ex(8): Skew-Symmetric Matrix

$$B = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

$a_{ii} = 0$

Triangular Matrices (Upper triangular matrices (UTM))

U.T.M are square matrices that have nonzero entries only on and above the main diagonal, whereas any entry below the diagonal must be zero.

Ex(9): U.T.M, $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$

Lower triangular Matrices (L.T.M)

L.T.M can have nonzero entries only on and below the main diagonal.

Any entry on the main diagonal of the triangular matrix may be zero or not.

Ex(10): L.T.M,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

Diagonal Matrices:

These are square matrices that can have nonzero entries only on the main diagonal. Any entry above or below the main diagonal must be zero.

Unit Matrix (Identity Matrix)

A diagonal matrix whose entries on the main diagonal are all 1, is called a unit matrix (or identity matrix) and is denoted by

$$I$$

$$\Rightarrow \quad AI = IA = A$$

Ex(11):

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse of a Matrix

The inverse of an $n \times n$ matrix $A = [a_{ij}]$ is denoted by A^{-1} and is an $n \times n$ matrix s.t.

$$A A^{-1} = A^{-1} A = I$$

where I is the $n \times n$ unit matrix.

- If A has an inverse, then A is called a nonsingular matrix.
- If A has no inverse, then A is called a singular matrix.
- If A has an inverse, the inverse is unique.

Determinants

A determinant of order n is a scalar associated with an $n \times n$ matrix $A = [a_{ij}]$, and is denoted by

$$D = \det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

For $n = 1$, this determinant is defined by
$$D = a_{11}$$

Second-Order Determinant

A determinant of second order is denoted and defined by

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Third-Order Determinant

A determinant of third order can be defined by

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Ex(11): If $A = \begin{bmatrix} 3 & -7 \\ 0 & 2 \end{bmatrix}$

$$\Rightarrow \det A = |A| = \begin{vmatrix} 3 & -7 \\ 0 & 2 \end{vmatrix} = (3)(2) - (-7)(0)$$
$$= 6 - 0 = 6$$

Ex(12): If a matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$,

then $\det A = |A| = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix}$

$$= (1)\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ 6 & 4 \end{vmatrix} = (1)(12-0) -$$
$$(2)(6-0) - (12-0) = 12 - 12 - 12 = -12$$

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Linear Systems of Equations

A linear system of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (1)$$

Matrix Form of the Linear System (1)

$$\boxed{Ax = b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If $b_1, b_2, \ldots, b_m$ are zero then (1) is called a homogenous system. If at least one b_i, $i = 1, 2, \ldots, m$ is not zero, then (1) is called a nonhomogenous system.

Ex(13): Linear System

$$\left.\begin{aligned} 4x_1 + 3x_2 &= 12 \\ 2x_1 + 5x_2 &= -8 \end{aligned}\right\} \quad A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{and } b = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

Solution of Linear Systems (Cramer's Rule)

- Cramer's rule for solving linear systems of two equations in two unknowns.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Let $D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

and

$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$

$\Rightarrow x_1 = \dfrac{D_1}{D}$ and $x_2 = \dfrac{D_2}{D}$, $D \neq 0$

Ex(1): Solve the linear system of equations

$$4x_1 + 3x_2 = 12$$
$$2x_1 + 5x_2 = -8$$

By using Cramer's rule

Sol: $D = \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = (4)(5) - (3)(2) = 20 - 6 = 14$

$D_1 = \begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix} = (12)(5) - (3)(-8) = 60 + 24 = 84$

$D_2 = \begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix} = (4)(-8) - (12)(2) = -32 - 24 = -56$

$\Rightarrow x_1 = \dfrac{D_1}{D} = \dfrac{84}{14} = 6$ and $x_2 = \dfrac{D_2}{D} = \dfrac{-56}{14} = -4$

$\therefore$ The solution is $x_1 = 6$ & $x_2 = -4$ #

Cramer's rule for solving linear systems of three equations.

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = b_1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = b_2$$
$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = b_3$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$\Longrightarrow$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad X_3 = \frac{D_3}{D}$$

Ex(2): Solve the linear system of equations

$$\boxed{\begin{aligned} 2X_1 - 6X_2 + X_3 &= 2 \\ X_2 + x_3 &= 1 \\ x_1 - X_2 - X_3 &= 0 \end{aligned}}$$

by using the Cramer's rule.

Sol:

$$\det D = \begin{vmatrix} 2 & -6 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + 0 + (1)\begin{vmatrix} -6 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 0 + 0 + (-7) = -7 \Rightarrow \boxed{D = -7}$$

$$D_1 = \begin{vmatrix} 2 & -6 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - (1)\begin{vmatrix} -6 & 1 \\ -1 & -1 \end{vmatrix} + 0$$

$$= (2)(-1+1) - (6-(-1)) + 0 = -7$$

$$\Rightarrow \boxed{D_1 = -7}$$

$$D_2 = \begin{vmatrix} 2 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + 0 + (1)\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -2+1 = 1$$

$$\Rightarrow \boxed{D_2 = -1}$$

$$D_3 = \begin{vmatrix} 2 & -6 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 2(1) + 0 + (-8) = -6 \Rightarrow \boxed{D_3 = -6}$$

$$\Rightarrow x_1 = \frac{D_1}{D} = \frac{-7}{-7} = 1, \quad x_2 = \frac{D_2}{D} = \frac{-1}{-7} = \frac{1}{7}, \quad x_3 = \frac{D_3}{D} = \frac{-6}{-7} \quad \frac{-6}{7}$$

H.W: Use Cramer's rule to solve each of the following systems of equations:

① $2X_1 + 2X_2 = 7$
$8X_1 + X_2 = -2$

② $-8X_1 + 6X_2 = 4$
$3X_1 + 2X_2 = 6$

③ $2X_1 \quad - X_3 = 1$
$2X_1 + 4X_2 - X_3 = 0$
$x_1 - 8X_2 - 3X_3 = -2$

$A^{-1} = \frac{1}{\det(A)} [C_{ij}]^T$

Inverse of a Matrix → By Determinants
→ Gauss - Jordan Meth

Inverse of a Matrix by determinants

The inverse of a nonsingular $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\boxed{A^{-1} = \frac{1}{\det(A)} [C_{ij}]^T} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & & C_{n2} \\ \vdots & \vdots & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

In particular, the inverse of

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \longrightarrow \text{is } A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{(3)(4)-(2)(1)} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} \Rightarrow = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1}A = I \quad \begin{bmatrix} \frac{4}{10} & \frac{-1}{10} \\ \frac{-2}{10} & \frac{3}{10} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} =$$

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