The Ratio Test

Let $\sum a_n$ be a series with positive terms and suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho ,$$

Then,

(a) the series converges if $\rho < 0$,

(b) the series diverges if $\rho > 1$ or ρ is infinite,

(c) the test is inconclusive (failed) if $\rho = 1$

Ex(1): Investigate the convergence of the series.

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$$

Sol: $a_n = \dfrac{2^n + 5}{3^n}$

$$a_{n+1} = \frac{2^{n+1} + 5}{3^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1} + 5}{3^{n+1}}}{\frac{2^n + 5}{3^n}} = \frac{2^{n+1} + 5}{3^{n+1}} \cdot \frac{3^n}{2^n + 5}$$

$$= \frac{2^{n+1} + 5}{\cancel{3^n} \cdot 3} \cdot \frac{\cancel{3^n}}{2^n + 5} = \frac{2^{n+1} + 5}{3(2^n + 5)}$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{2^n}\left(2+\frac{5}{2^n}\right)}{3\cancel{2^n}\left(1+\frac{5}{2^n}\right)} = \frac{2+\frac{5}{2^n}}{3\left(1+\frac{5}{2^n}\right)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2+\frac{5}{2^n}}{3\left(1+\frac{5}{2^n}\right)}$$

$$= \frac{2+\frac{5}{\infty}}{3\left(1+\frac{5}{\infty}\right)} = \frac{2+0}{3(1+0)} = \frac{2}{3}$$

$\Rightarrow \therefore \rho = \frac{2}{3} < 1$

$\therefore$ the series $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ <u>converges</u>.

Now, we can find the sum of the series because the series $\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$ is convergence.

$$\sum_{n=0}^{\infty} \frac{2^n+5}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2^n}{3^n} + \frac{5}{3^n}\right)$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{3^n} + \sum_{n=0}^{\infty} \frac{5}{3^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n}$$

geometric series
$a=1$ & $r=\frac{2}{3}$

$$= \left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\dots+\left(\frac{2}{3}\right)^n+\dots\right) +$$

$$\left(5+\frac{5}{3}+\frac{5}{3^2}+\frac{5}{3^3}+\dots+\frac{5}{3^n}+\dots\right)$$

geometric series
$a=5$ & $r=\frac{1}{3}$

$$= \frac{1}{1-\frac{2}{3}} + \frac{5}{1-\frac{1}{3}} = \frac{21}{2}$$

Ex (2): Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$$

Sol:

$$a_n = \frac{(2n)!}{n!\,n!} \quad , a_{n+1} = \frac{(2(n+1))!}{(n+1)!\,(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}}$$

$$= \frac{(2n+2)!}{(n+1)!\,(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$$

Note: $5! = 5 * 4 * 3 * 2 * 1$

$7! = 7 * 6 * 5 * 4 * 3 * 2 * 1$

$= 7 * 6 * 5! = 7 * 6!$

Now, $n! = n(n-1)(n-2) * \cdots * 3 * 2 * 1$

$(n+1)! = (n+1)(n)(n-1)(n-2) \cdots 3 * 2 * 1$

$= (n+1)\,n!$

$(2n+2)! = (2n+2)(2n+1)(2n)!$

$\Rightarrow$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!\,n!}{(2n)!}$$

$$= \frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)\cancel{n!}\,(n+1)\cancel{n!}} \cdot \frac{\cancel{n!\,n!}}{\cancel{(2n)!}}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= \frac{2n(1+\frac{1}{n}) \cdot 2n(1+\frac{1}{2n})}{[n(1+\frac{1}{n})]^2}$$

$$= \frac{4\cancel{n^2}\cancel{(1+\frac{1}{n})}(1+\frac{1}{2n})}{\cancel{n^2}(1+\frac{1}{n})^{\cancel{2}}}$$

$$= \frac{4(1+\frac{1}{2n})}{(1+\frac{1}{n})}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4(1+\frac{1}{2n})}{(1+\frac{1}{n})}$$

$$= \frac{4(1+\frac{1}{\infty})}{(1+\frac{1}{\infty})} = \frac{4(1+0)}{(1+0)}$$

$$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4 > 1$$

$\therefore$ The series diverges because $\rho = 4$ is greater than 1.

Ex(3): Investigate the convergence of the series

$$\sum_{n=1}^{\infty} \frac{4^n n!\, n!}{(2n)!}$$

Sol: $a_n = \dfrac{4^n n!\, n!}{(2n)!}$

$$a_{n+1} = \frac{4^{n+1}(n+1)!\,(n+1)!}{(2(n+1))!}$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}(n+1)!\,(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{4^n n!\, n!}$$

$$= \frac{\cancel{4^n}\cdot 4(n+1)\cancel{n!}\cdot(n+1)\cancel{n!}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{4^n}\,\cancel{n!}\,\cancel{n!}}$$

$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{\cancel{4}\cdot \cancel{n}(1+\frac{1}{n})\cancel{n}(1+\frac{1}{n})}{\cancel{2n}(1+\frac{1}{n})\cdot \cancel{2n}(1+\frac{1}{2n})}$$

$$= \frac{(1+\frac{1}{n})}{(1+\frac{1}{2n})}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})}{(1+\frac{1}{2n})} = \frac{1+\frac{1}{\infty}}{1+\frac{1}{\infty}}$$

$= \dfrac{1+0}{1+0} = 1 \Rightarrow$ the test is fail.

Ex(4): Use the ratio test to determine if the series is convergent or divergent.

$$1/2 + 2/4 + 3/8 + 4/16 + \cdots$$

Sol: $a_n = \dfrac{n}{2^n}$, $a_{n+1} = \dfrac{n+1}{2^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$$

$$= \frac{n+1}{\cancel{2^n} \cdot 2} \cdot \frac{\cancel{2^n}}{n} = \frac{n+1}{2n}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left(\frac{n+1}{2n} \right)$$ By using the L-Hòpetal

$$= \lim_{n \to \infty} \frac{1}{2} = \frac{1}{2}$$

$$\therefore \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1$$

Since $r < 1$, the series is convergent.

Ex(5): Test for convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Sol: Now, we use the ratio test to determine if the series convergence or divergence.

$$a_n = \frac{n^2}{2^n} \quad \text{and} \quad a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{(n+1)^2}{2n^2}$$

$$= \frac{n^2+2n+1}{2n^2} = \frac{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{2\cancel{n^2}}$$

$$= \frac{\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{2} = \frac{1}{2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$$

$$= \frac{1}{2} \lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) = \frac{1}{2}\cdot\left(1+\frac{2}{\infty}+\frac{1}{\infty}\right)$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}(1+0+0) = \frac{1}{2} < 1$$

Since this ratio is less than 1, the series converges.