نظم مرحلة اولى

رياضيات

تكملة المشتقات

Transcendental Functions

Inverse Functions and Their Derivatives:

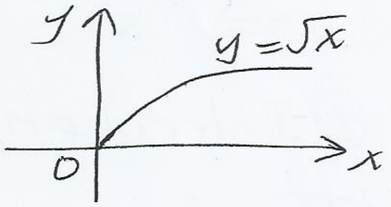
Definition(1): (One-to-One Functions)

A function $f(x)$ is <u>one-to-one</u> on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D

$$\boxed{f(x_1) \neq f(x_2) \Leftrightarrow x_1 \neq x_2}$$

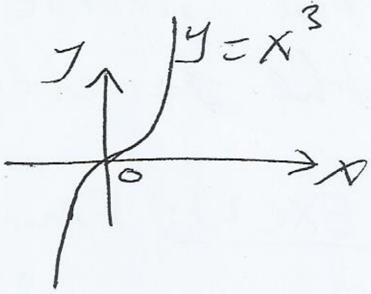
Ex(1): $y = \sqrt{x}$

is one-to-one



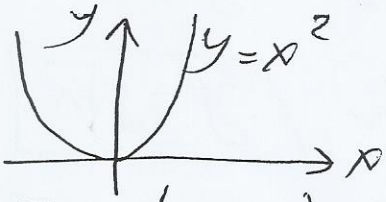
Ex(2): $y = x^3$

is one-to-one



Ex(3): $y = x^2$

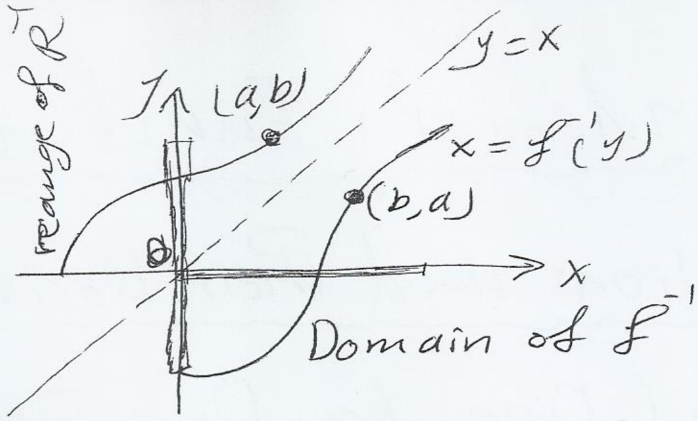
is not one-to-one



Definition(2): (Inverse functions)

Suppose that f is a one-to-one function on a domain D with range R. The <u>inverse function</u> f^{-1} is defined by $f^{-1}(b) = a$ if $f(a) = b$.

The domain of f^{-1} is R and the range of f^{-1} is D.



Ex(1): Find the inverse of $y=\frac{1}{2}x+1$, expressed as a function of x.

Sol: ① Solve for x in terms of y:

$$y = \frac{1}{2}x + 1$$
$$\Rightarrow 2y = x + 2$$
$$\Rightarrow x = 2y - 2$$

② Interchange x and y: $\boxed{y = 2x-2}$

The inverse of function $f(x)=\frac{1}{2}x+1$ is the function $\boxed{f^{-1}(x) = 2x-2}$

EX(2): Find $f^{-1}(x)$ of the function $y=x^2$, $x\geq 0$

① $y = x^2 \Rightarrow \sqrt{y} = \sqrt{x^2} \Rightarrow \sqrt{y} = |x| = x$

② $\boxed{y = \sqrt{x}} \Rightarrow \therefore f^{-1}(x) = \sqrt{x}$

Derivatives of Inverses of Differentiable Functions.

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{2}x+1\right) = \frac{1}{2}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx}(2x-2) = 2$$

Natural Logarithms

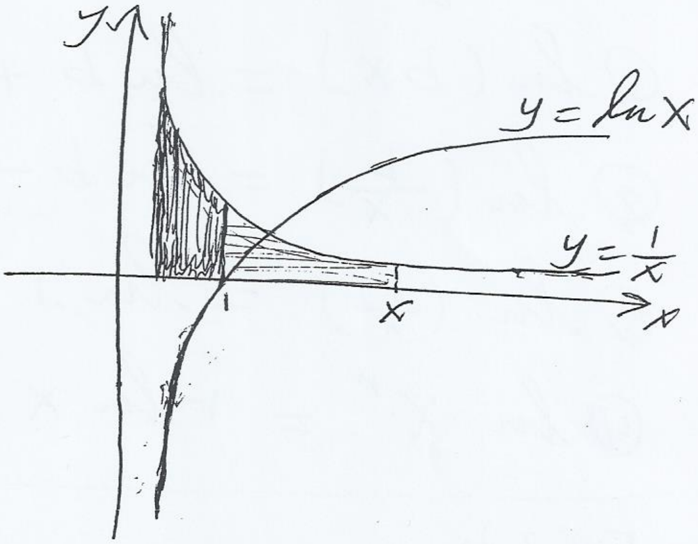
Definition (1): (Natural Logarithm)

The natural logarithm is the function given by

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt \quad , x > 0}$$

x	ln x
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$



Definition (2):

The number e is that number in the domain of the natural logarithm satisfying

$$\boxed{\ln e = 1}$$

The Derivative of $y = \ln x$:

$$\boxed{\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad , \quad u > 0}$$

Ex(1):

(a) $\frac{d}{dx}(\ln 2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$

(b) $\frac{d}{dx}(\ln(x^2+3)) = \frac{1}{x^2+3} \cdot (2x) = \frac{2x}{x^2+3}$

Properties of Logarithms:

For any numbers $b > 0$ & $x > 0$, then

① $\ln(bx) = \ln b + \ln x$

② $\ln\left(\frac{b}{x}\right) = \ln b - \ln x$

③ $\ln\left(\frac{1}{x}\right) = \ln 1 - \ln x = 0 - \ln x = -\ln x$

④ $\ln x^r = r \ln x$, For r rational

Ex(2):

(a) $\ln(4) + \ln \sin x = \ln(4 \sin x)$

(b) $\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$

(c) $\ln \frac{1}{8} = -\ln 8 = -\ln 2^3 = -3 \ln 2$

The Integral $\int \frac{1}{u}\,du$:

$$\boxed{\int \frac{1}{u}\,du = \ln|u| + C}$$

EX(3): Evaluate

$$\int_0^2 \frac{2x}{x^2-5}\,dx$$

Sol: Let $u = x^2 - 5 \Rightarrow du = 2x\,dx$

$\Rightarrow \quad u(0) = 0^2 - 5 = -5 \quad \& \quad u(2) = 2^2 - 5 = -1$

$$\int_0^2 \frac{2x\,dx}{x^2-5} = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$

$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$

The Integral of $\tan x$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Let $u = \cos x \Rightarrow du = -\sin x\,dx$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln|\cos x|^{-1} + C = \ln\frac{1}{|\cos x|} + C$$

$$= \ln|\sec x| + C$$

The Integral of $\sec x$:

$$\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \ln|\sec x + \tan x| + C$$

Logarithm Differentiation

Ex(1): Find $\frac{dy}{dx}$ if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{(x-1)} , \quad x > 1$$

Sol: $\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{(x-1)}$

$$\Rightarrow \ln y = \ln(x^2+1)(x+3)^{1/2} - \ln(x-1)$$

$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1)$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}(2x) + \frac{1}{2}\frac{1}{x+3} - \frac{1}{(x-1)}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{(x-1)}\right]$$

$$= \left(\frac{(x^2+1)(x+3)^{1/2}}{(x-1)}\right)\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right)$$

H.W: Exercises (7.2)