

College of Electronic Engineering
Department of Electronic Engineering

Mathematics

EE 1203

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Reference text book: Thomas Calculus

Integration of Rational Functions by Partial Fractions

How to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called partial fractions, which are easily integrated.

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3},$$

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. See Example 3 of this section.
- *We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x .
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Distinct Linear Factors

Evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx$$

using partial fractions.

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A , B , and C we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining

$$\begin{aligned} \text{Coefficient of } x^2: \quad & A + B + C = 1 \\ \text{Coefficient of } x^1: \quad & 4A + 2B = 4 \\ \text{Coefficient of } x^0: \quad & 3A - 3B - C = 1 \end{aligned}$$

There are several ways for solving such a system of linear equations for the unknowns A , B , and C , including elimination of variables, or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx &= \int \left[\frac{3}{4} \frac{1}{x - 1} + \frac{1}{2} \frac{1}{x + 1} - \frac{1}{4} \frac{1}{x + 3} \right] dx \\ &= \frac{3}{4} \ln |x - 1| + \frac{1}{2} \ln |x + 1| - \frac{1}{4} \ln |x + 3| + K, \end{aligned}$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2 A Repeated Linear Factor

Evaluate

$$\int \frac{6x + 7}{(x + 2)^2} dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

$$6x + 7 = A(x + 2) + B \quad \text{Multiply both sides by } (x + 2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2} dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) dx \\ &= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} dx \\ &= 6 \ln |x + 2| + 5(x + 2)^{-1} + C \end{aligned}$$

■

EXAMPLE 3 Integrating an Improper Fraction

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x} \\ 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx &= \int 2x dx + \int \frac{5x - 3}{x^2 - 2x - 3} dx \\ &= \int 2x dx + \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx \\ &= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C. \end{aligned}$$

A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.

EXAMPLE 4 Integrating with an Irreducible Quadratic Factor in the Denominator

Evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx$$

using partial fractions.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \quad (2)$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Equating coefficients of like terms gives

$$\begin{aligned}\text{Coefficients of } x^3: & \quad 0 = A + C \\ \text{Coefficients of } x^2: & \quad 0 = -2A + B - C + D \\ \text{Coefficients of } x^1: & \quad -2 = A - 2B + C \\ \text{Coefficients of } x^0: & \quad 4 = B - C + D\end{aligned}$$

We solve these equations simultaneously to find the values of A , B , C , and D :

$$\begin{aligned}-4 &= -2A, & A &= 2 & \text{Subtract fourth equation from second.} \\ C &= -A = -2 & & & \text{From the first equation} \\ B &= 1 & & & A = 2 \text{ and } C = -2 \text{ in third equation.} \\ D &= 4 - B + C = 1. & & & \text{From the fourth equation}\end{aligned}$$

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \ln(x^2 + 1) + \tan^{-1}x - 2 \ln|x - 1| - \frac{1}{x - 1} + C. \quad \blacksquare\end{aligned}$$

EXAMPLE 5 A Repeated Irreducible Quadratic Factor

Evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}$$

Multiplying by $x(x^2 + 1)^2$, we have

$$\begin{aligned} 1 &= A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x \\ &= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex \\ &= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

Solving this system gives $A = 1$, $B = -1$, $C = 0$, $D = -1$, and $E = 0$. Thus,

$$\begin{aligned} \int \frac{dx}{x(x^2 + 1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2 + 1} + \frac{-x}{(x^2 + 1)^2} \right] dx \\ &= \int \frac{dx}{x} - \int \frac{x dx}{x^2 + 1} - \int \frac{x dx}{(x^2 + 1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \int \frac{du}{u^2} && \begin{array}{l} u = x^2 + 1, \\ du = 2x dx \end{array} \\ &= \ln |x| - \frac{1}{2} \ln |u| + \frac{1}{2u} + K \\ &= \ln |x| - \frac{1}{2} \ln (x^2 + 1) + \frac{1}{2(x^2 + 1)} + K \\ &= \ln \frac{|x|}{\sqrt{x^2 + 1}} + \frac{1}{2(x^2 + 1)} + K. \end{aligned}$$

■

The Heaviside “Cover-up” Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

Heaviside Method

1. Write the quotient with $g(x)$ factored:

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2) \cdots (x - r_n)}.$$

2. Cover the factors $(x - r_i)$ of $g(x)$ one at a time, each time replacing all the uncovered x 's by the number r_i . This gives a number A_i for each root r_i :

$$\begin{aligned} A_1 &= \frac{f(r_1)}{(r_1 - r_2) \cdots (r_1 - r_n)} \\ A_2 &= \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3) \cdots (r_2 - r_n)} \\ &\vdots \\ A_n &= \frac{f(r_n)}{(r_n - r_1)(r_n - r_2) \cdots (r_n - r_{n-1})}. \end{aligned}$$

3. Write the partial-fraction expansion of $f(x)/g(x)$ as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

EXAMPLE 6 Using the Heaviside Method

Find A , B , and C in the partial-fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \quad (3)$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A :

$$\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,$$

$$A = 1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} \quad (4)$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\boxed{(x - 1)} (1 - 2)(1 - 3)} = \frac{2}{(-1)(-2)} = 1.$$

\uparrow
 Cover

Similarly, we find the value of B in Equation (3) by covering the factor $(x - 2)$ in Equation (4) and evaluating the rest at $x = 2$:

$$B = \frac{(2)^2 + 1}{(2 - 1) \boxed{(x - 2)} (2 - 3)} = \frac{5}{(1)(-1)} = -5.$$

\uparrow
 Cover

Finally, C is found by covering the $(x - 3)$ in Equation (4) and evaluating the rest at $x = 3$:

$$C = \frac{(3)^2 + 1}{(3 - 1)(3 - 2) \boxed{(x - 3)}} = \frac{10}{(2)(1)} = 5. \quad \blacksquare$$

\uparrow
 Cover

EXAMPLE 7 Integrating with the Heaviside Method

Evaluate

$$\int \frac{x + 4}{x^3 + 3x^2 - 10x} dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x + 4}{x^3 + 3x^2 - 10x} = \frac{x + 4}{x(x - 2)(x + 5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0 + 4}{\boxed{x} (0 - 2)(0 + 5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

\uparrow
 Cover

$$A_2 = \frac{2 + 4}{2 \boxed{(x - 2)} (2 + 5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

\uparrow
 Cover

$$A_3 = \frac{-5 + 4}{(-5)(-5 - 2) \boxed{(x + 5)}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

\uparrow
 Cover

Therefore,

$$\frac{x + 4}{x(x - 2)(x + 5)} = -\frac{2}{5x} + \frac{3}{7(x - 2)} - \frac{1}{35(x + 5)},$$

and

$$\int \frac{x + 4}{x(x - 2)(x + 5)} dx = -\frac{2}{5} \ln |x| + \frac{3}{7} \ln |x - 2| - \frac{1}{35} \ln |x + 5| + C. \quad \blacksquare$$

Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions.

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2} \quad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 m is Odd

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution

$$\begin{aligned}
 \int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
 &= \int (1 - \cos^2 x) \cos^2 x (-d(\cos x)) \\
 &= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
 &= \int (u^4 - u^2) \, du \\
 &= \frac{u^5}{5} - \frac{u^3}{3} + C \\
 &= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C. \quad \blacksquare
 \end{aligned}$$

EXAMPLE 2 m is Even and n is Odd

Evaluate

$$\int \cos^5 x \, dx.$$

Solution

$$\begin{aligned}
 \int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 d(\sin x) && m = 0 \\
 &= \int (1 - u^2)^2 \, du && u = \sin x \\
 &= \int (1 - 2u^2 + u^4) \, du \\
 &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C. \quad \blacksquare
 \end{aligned}$$

EXAMPLE 3 m and n are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution

$$\begin{aligned} \int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{8} \int (1 - \cos 2x)(1 + 2 \cos 2x + \cos^2 2x) \, dx \\ &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx \\ &= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx \right]. \end{aligned}$$

For the term involving $\cos^2 2x$ we use

$$\begin{aligned} \int \cos^2 2x \, dx &= \frac{1}{2} \int (1 + \cos 4x) \, dx \\ &= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right). \end{aligned}$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\begin{aligned} \int \cos^3 2x \, dx &= \int (1 - \sin^2 2x) \cos 2x \, dx \\ &= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right). \end{aligned}$$

$u = \sin 2x,$
 $du = 2 \cos 2x \, dx$
Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C. \quad \blacksquare$$

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx.$$

Solution To eliminate the square root we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Therefore,

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx &= \int_0^{\pi/4} \sqrt{2 \cos^2 2x} \, dx = \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} \, dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx = \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx && \begin{array}{l} \cos 2x \geq 0 \\ \text{on } [0, \pi/4] \end{array} \\ &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} [1 - 0] = \frac{\sqrt{2}}{2}. \quad \blacksquare \end{aligned}$$

Integrals of Powers of $\tan x$ and $\sec x$

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x \, dx.$$

Solution

$$\begin{aligned} \int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx. \end{aligned}$$

In the first integral, we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$



EXAMPLE 6 Evaluate

$$\int \sec^3 x \, dx.$$

Solution We integrate by parts, using

$$u = \sec x, \quad dv = \sec^2 x \, dx, \quad v = \tan x, \quad du = \sec x \tan x \, dx.$$

Then

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x \, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx && \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx. \end{aligned}$$

Combining the two secant-cubed integrals gives

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

and

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C. \quad \blacksquare$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \quad \int \cos mx \cos nx \, dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2} [\cos (m - n)x - \cos (m + n)x], \quad (3)$$

$$\sin mx \cos nx = \frac{1}{2} [\sin (m - n)x + \sin (m + n)x], \quad (4)$$

$$\cos mx \cos nx = \frac{1}{2} [\cos (m - n)x + \cos (m + n)x]. \quad (5)$$

These come from the angle sum formulas for the sine and cosine functions (Section 1.6). They give functions whose antiderivatives are easily found.

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin (-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \end{aligned}$$



Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

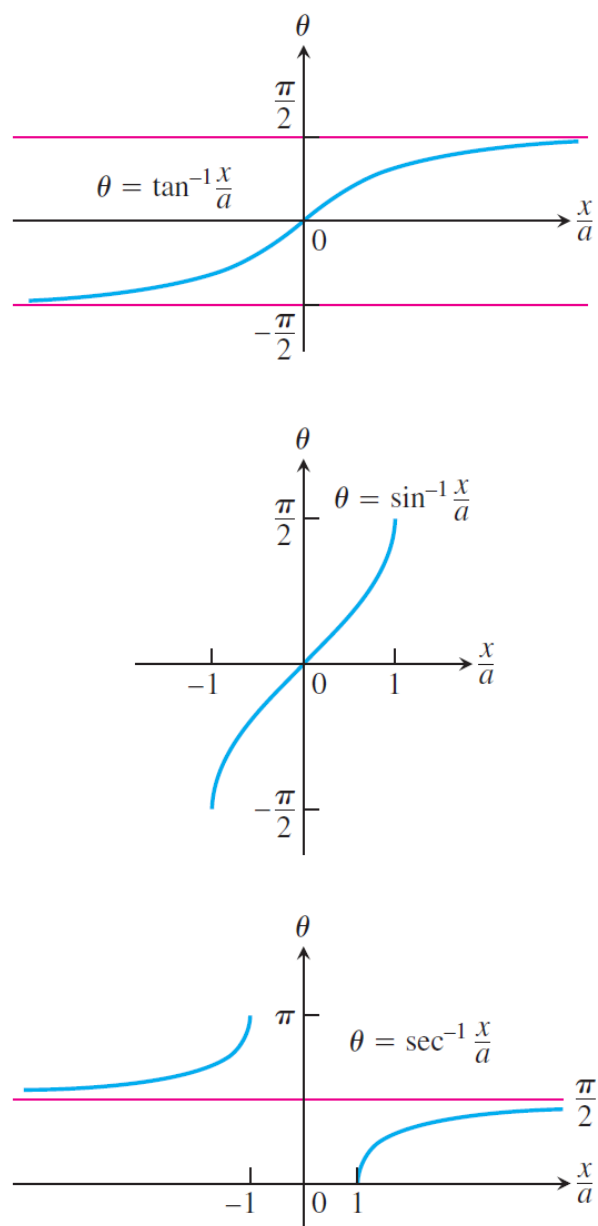


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a , graphed as functions of x/a .

Three Basic Substitutions

The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. They come from the reference right triangles in Figure 8.2.

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

With $x = a \sec \theta$,

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

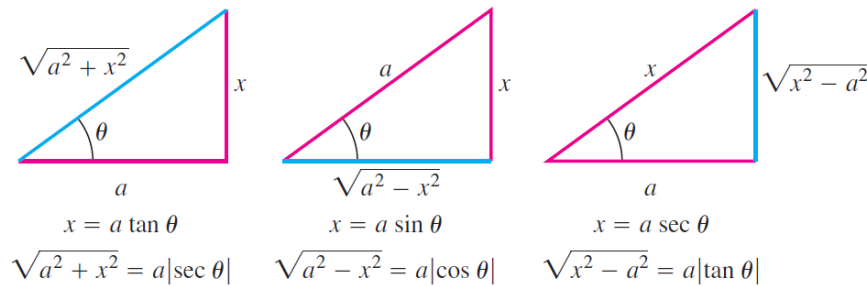


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a \tan \theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a \sin \theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a \sec \theta$.

As we know from Section 7.7, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a \tan \theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},$$

$$x = a \sec \theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \leq \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \geq 1, \\ \frac{\pi}{2} < \theta \leq \pi & \text{if } \frac{x}{a} \leq -1. \end{cases}$$

To simplify calculations with the substitution $x = a \sec \theta$, we will restrict its use to integrals in which $x/a \geq 1$. This will place θ in $[0, \pi/2)$ and make $\tan \theta \geq 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta| = a \tan \theta$, free of absolute values, provided $a > 0$.

EXAMPLE 1 Using the Substitution $x = a \tan \theta$

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2 \tan \theta, \quad dx = 2 \sec^2 \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{4 + x^2}} &= \int \frac{2 \sec^2 \theta d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta d\theta}{|\sec \theta|} && \sqrt{\sec^2 \theta} = |\sec \theta| \\ &= \int \sec \theta d\theta && \sec \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C && \text{From Fig. 8.4} \\ &= \ln |\sqrt{4 + x^2} + x| + C'. && \text{Taking } C' = C - \ln 2 \end{aligned}$$

Notice how we expressed $\ln |\sec \theta + \tan \theta|$ in terms of x : We drew a reference triangle for the original substitution $x = 2 \tan \theta$ (Figure 8.4) and read the ratios from the triangle. ■

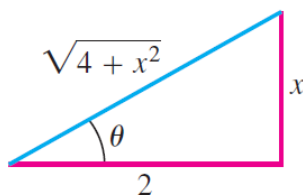


FIGURE 8.4 Reference triangle for $x = 2 \tan \theta$ (Example 1):

$$\tan \theta = \frac{x}{2}$$

and

$$\sec \theta = \frac{\sqrt{4 + x^2}}{2}.$$

EXAMPLE 2 Using the Substitution $x = a \sin \theta$

Evaluate

$$\int \frac{x^2 dx}{\sqrt{9 - x^2}}.$$

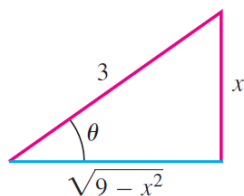
Solution We set

$$x = 3 \sin \theta, \quad dx = 3 \cos \theta d\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9 \sin^2 \theta = 9(1 - \sin^2 \theta) = 9 \cos^2 \theta.$$

Then

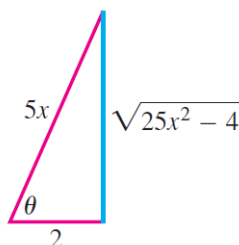
$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{9 - x^2}} &= \int \frac{9 \sin^2 \theta \cdot 3 \cos \theta d\theta}{|3 \cos \theta|} \\ &= 9 \int \sin^2 \theta d\theta && \cos \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ &= 9 \int \frac{1 - \cos 2\theta}{2} d\theta \\ &= \frac{9}{2} \left(\theta - \frac{\sin 2\theta}{2} \right) + C \\ &= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C && \sin 2\theta = 2 \sin \theta \cos \theta \\ &= \frac{9}{2} \left(\sin^{-1} \frac{x}{3} - \frac{x}{3} \cdot \frac{\sqrt{9 - x^2}}{3} \right) + C && \text{Fig. 8.5} \\ &= \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{x}{2} \sqrt{9 - x^2} + C. \end{aligned}$$

**FIGURE 8.5** Reference triangle for $x = 3 \sin \theta$ (Example 2):

$$\sin \theta = \frac{x}{3}$$

and

$$\cos \theta = \frac{\sqrt{9 - x^2}}{3}.$$

**FIGURE 8.6** If $x = (2/5)\sec \theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 3).

EXAMPLE 3 Using the Substitution $x = a \sec \theta$

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\begin{aligned} \sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\ &= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2} \end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$\begin{aligned} x &= \frac{2}{5} \sec \theta, & dx &= \frac{2}{5} \sec \theta \tan \theta d\theta, & 0 < \theta < \frac{\pi}{2} \\ x^2 - \left(\frac{2}{5}\right)^2 &= \frac{4}{25} \sec^2 \theta - \frac{4}{25} \\ &= \frac{4}{25} (\sec^2 \theta - 1) = \frac{4}{25} \tan^2 \theta \\ \sqrt{x^2 - \left(\frac{2}{5}\right)^2} &= \frac{2}{5} |\tan \theta| = \frac{2}{5} \tan \theta. \end{aligned}$$

$\tan \theta > 0$ for
 $0 < \theta < \pi/2$

With these substitutions, we have

$$\begin{aligned} \int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5) \sec \theta \tan \theta d\theta}{5 \cdot (2/5) \tan \theta} \\ &= \frac{1}{5} \int \sec \theta d\theta = \frac{1}{5} \ln |\sec \theta + \tan \theta| + C \\ &= \frac{1}{5} \ln \left| \frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2} \right| + C. \end{aligned}$$

Fig. 8.6

