

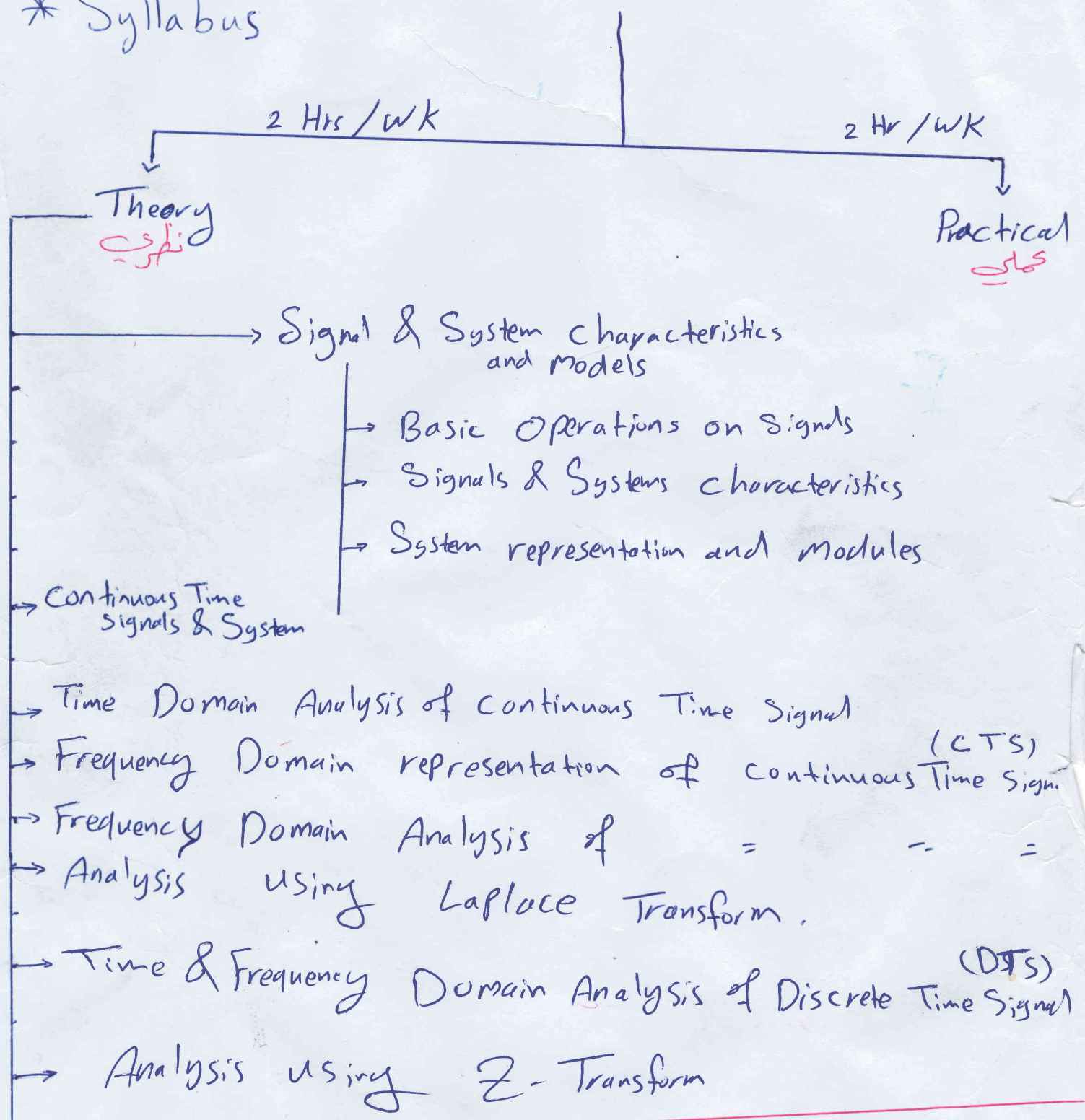
Signals & Systems

by
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code: CIE2305

units: 6

* Syllabus

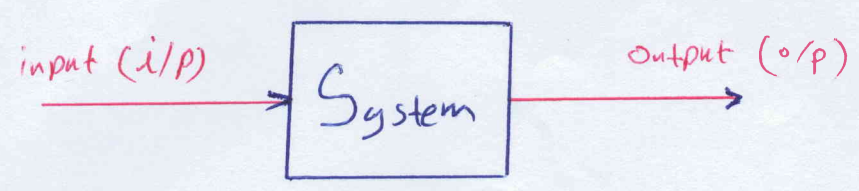


Text book

- ① "Introduction to Signal and System" by D.K. Lindner
- ② "Signal & System" by Carlson

* Signal :: defined as a function of one or more variables that conveys information on the nature of physical phenomena.

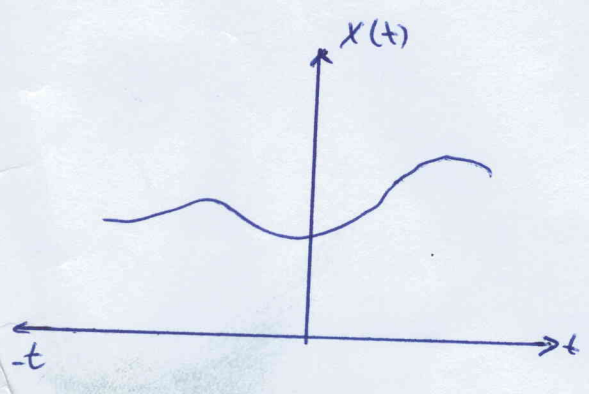
* System :: defined as entity that manipulates one or more signals to accomplish a function.



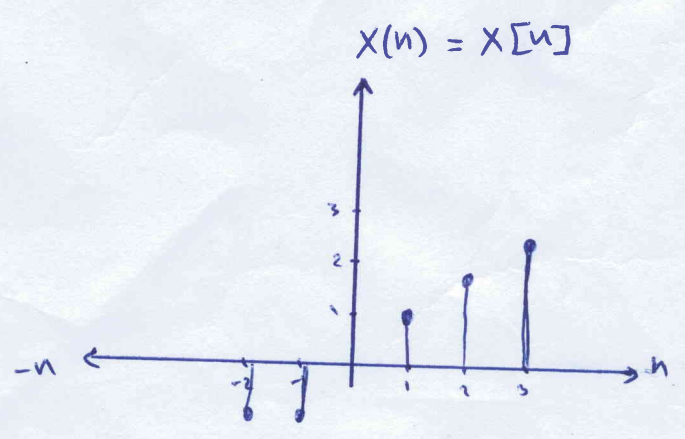
Classification of Signals

There are several methods to classify signals. The following are the most common:

1 - Continuous-time
 ↓
 Specified for every value of time (t)



&
 Discrete-time
 ↓
 Specified at discrete time intervals.



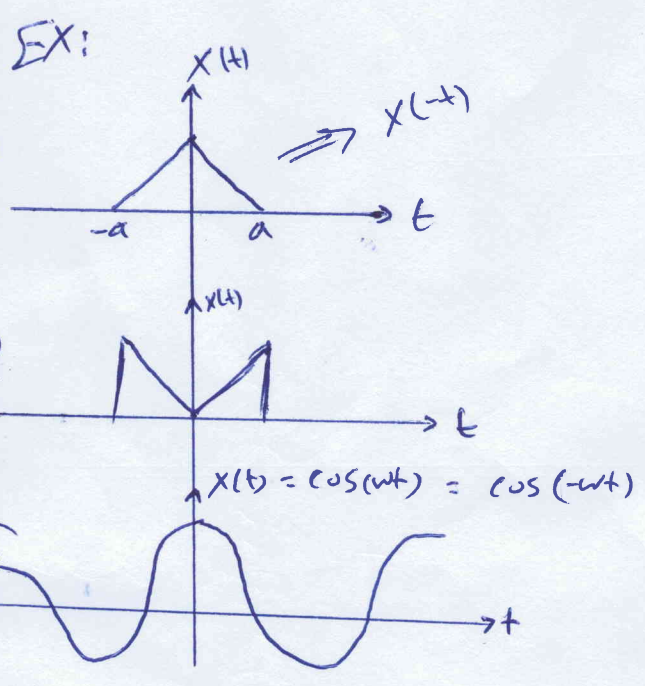
$n = 0, \pm 1, \pm 2, \dots$
 n - integer

2- Even & odd Signal

* Even Signals

- Remain identical under folding (reflection) operation.

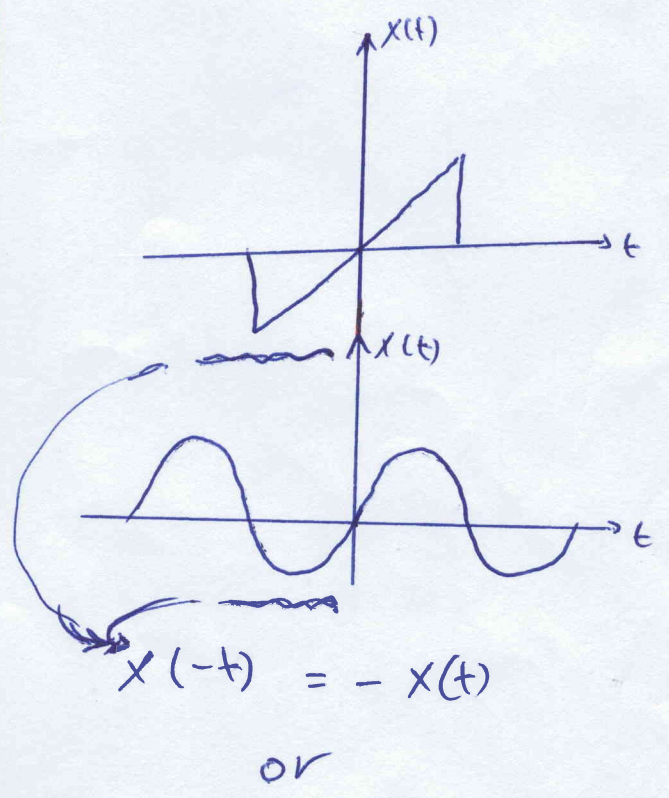
$X(t)$ $\xrightarrow{\text{Time Reverse}}$ $X(-t) = X(t)$



* Odd Signals

- Doesn't remain identical under reflection operation.

$X(t) \neq X(-t)$



~~***~~
if $x(t) = X(-t)$ for all t
then the signal is Even

if $x(t) = -X(-t)$ for all t , then the signal is odd

$\sin(-\theta) = -\sin(\theta)$

3- Periodic & non-periodic Signals (Aperiodic)

* Periodic Signal : is the signal that repeats itself after a regular interval of time.

if $x(t) = x(t-T)$ for all t \rightarrow periodic

The smallest value T is called period.

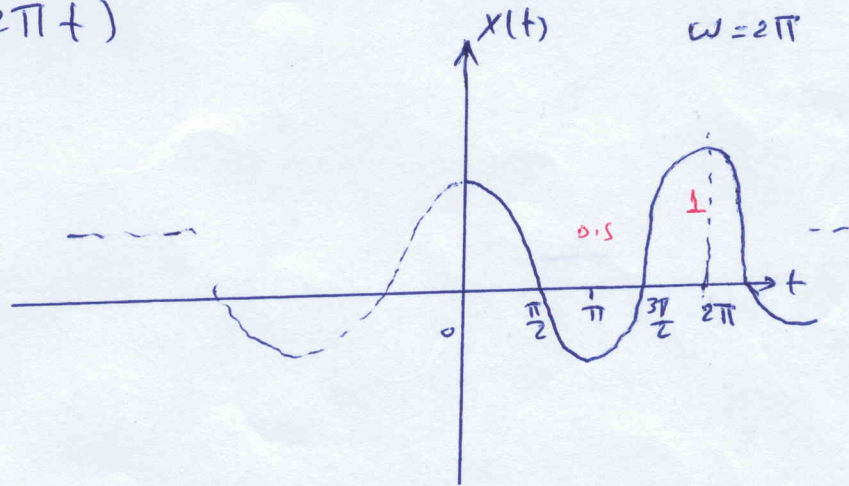
Ex₁

$$x(t) = \cos(2\pi t)$$

$$\omega = 2\pi f$$

$$f = \frac{1}{T}$$

$$2\pi = \frac{2\pi}{T} \rightarrow T = 1$$

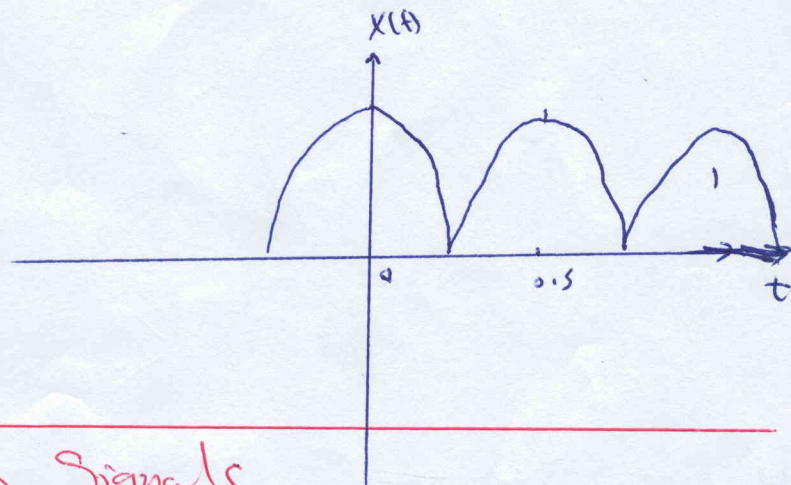


Ex₂ : $x(t) = \cos^2(2\pi t)$

Hint

$$T = 0.5$$

hint : \cos^2 كوساين



4- Deterministic & Random Signals

↓
لا تتغير عشوائياً مع الزمن

↓
تتغير عشوائياً مع الزمن

5. Energy and Power Signals

The Energy of Signal $x(t)$ is expressed as the integral of the squared magnitude of its values as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Aperiodic signals with finite energy are called Energy Signals

The ~~Energy~~ Energy of $x(t) = 3e^{-t}$, $t > 0$ is

$$E = \int_0^{\infty} |3e^{-t}|^2 dt = \boxed{\frac{9}{2}}$$

If the energy of a signal is infinite, then it may be possible to characterize it in terms of its average power. The average power is defined as

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

* For periodic signal, the Average Power over one period as

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

T is the period, periodic or aperiodic signals with finite average power are called power signals (sine and cosine).

For discrete signals

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2, \quad P = \lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{-N}^N x^2(n)$$

For periodic discrete signal the average power is:

$$P = \frac{1}{N} \sum_0^{N-1} x^2(n)$$

Summary

if $0 < E < \infty \Rightarrow$ energy signal

if $0 < P < \infty \Rightarrow$ power signal

Examples

1- Categorize each of the following signals as an Energy OR power signal, Find E & P.

$$x(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^1 t^2 dt + \int_1^2 (2-t)^2 dt$$

$$= \frac{2}{3} \text{ Joule}$$

$$P = 0$$

$$= \left. \frac{t^3}{3} \right|_0^1 + \left. \frac{(2-t)^3}{-3} \right|_1^2 = \left(\frac{1}{3} - 0 \right) + \left(0 - \left(-\frac{1}{3} \right) \right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Prove

2- $x(t) = 2 \cos \pi t$

$$\text{power} = (\text{Vr.m.s})^2 = \left(\frac{VP}{\sqrt{2}} \right)^2 = \left(\frac{2}{\sqrt{2}} \right)^2 = \frac{4}{2} = \frac{2}{1} \text{ Watt}$$

$$E = \infty$$

$$3. \quad x(n) = \begin{cases} n & 0 \leq n < 3 \\ 10-n & 5 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

This is discrete signal,

$$E = \sum_{-\infty}^{\infty} |x(n)|^2$$

$$E = 0^2 + 1^2 + 2^2 + 5^2 + 4^2 = \underline{\underline{46}} \text{ So, it Energy Signal}$$

$$P = 0$$

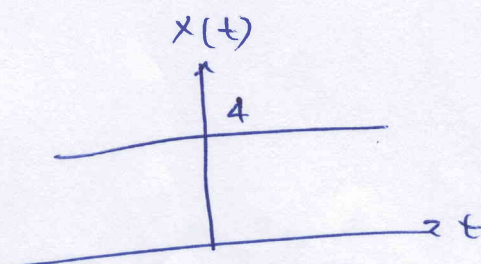
$$4. \quad x(t) = 4$$

$$P = V^2/R$$

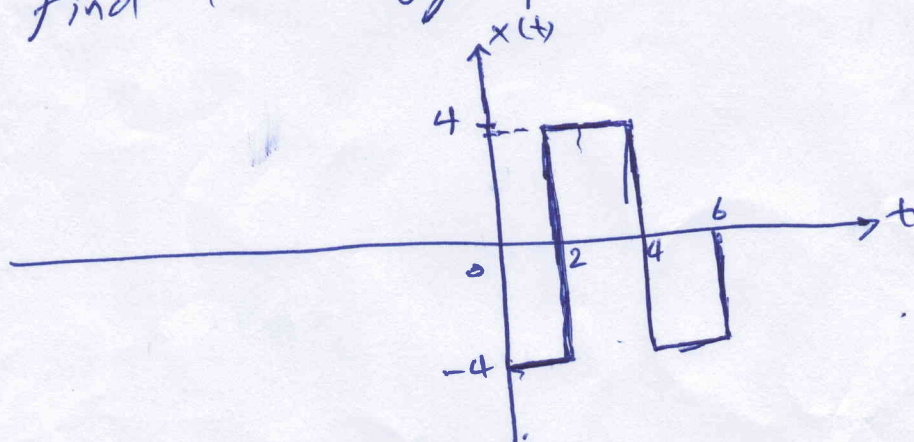
$$\text{assuming } R=1$$

$$P = V^2 = 16$$

$$E = \infty$$



How Find the Energy of the given Signal



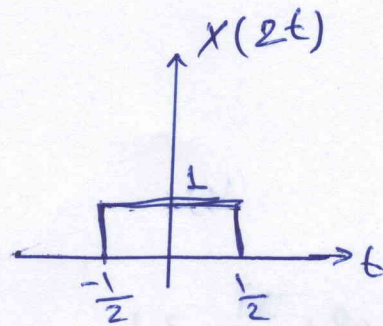
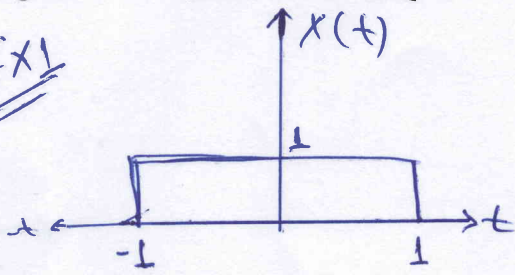
* Operations performed on the independent Variable ∞

Lecture #3

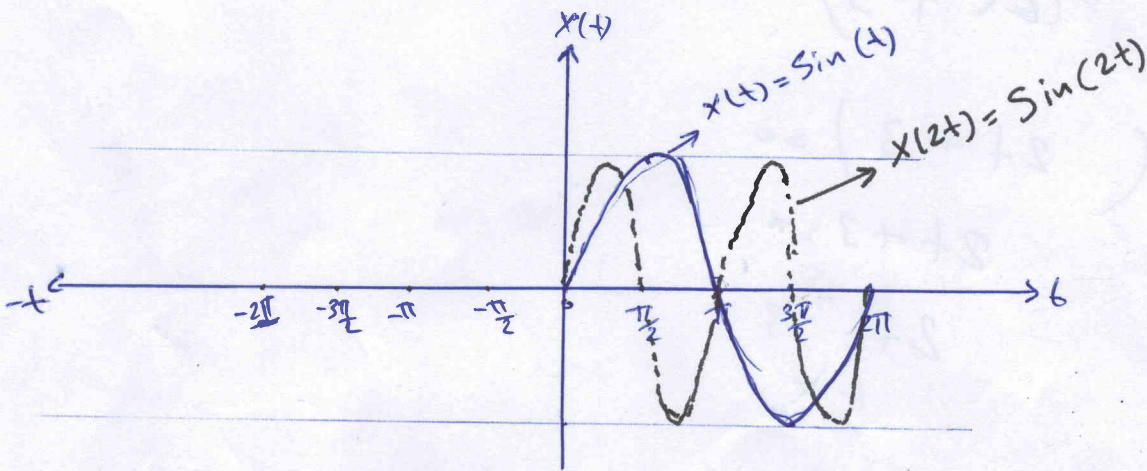
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1- Time Scaling ∞

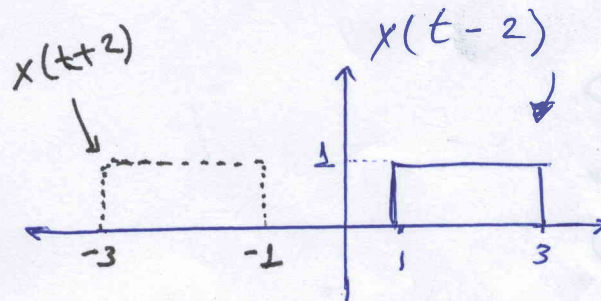
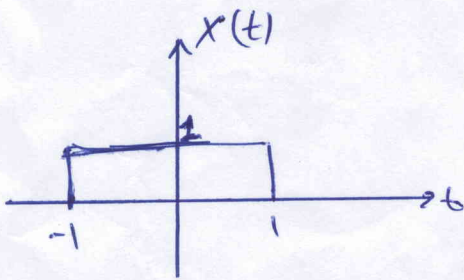
Ex1



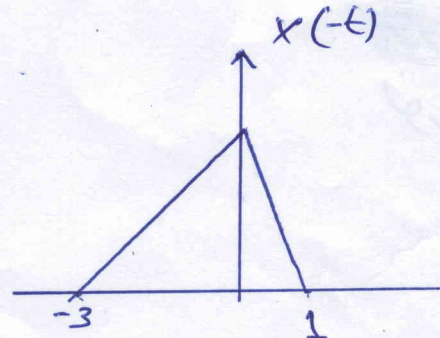
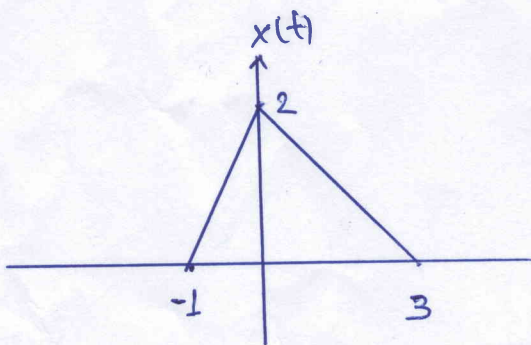
Ex2



2- Time Shifting

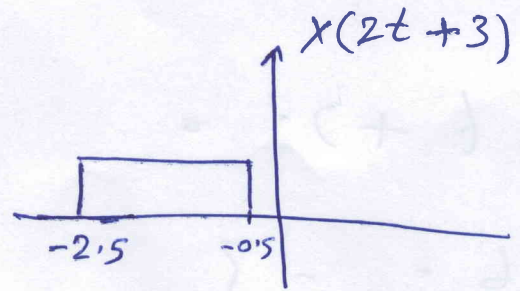
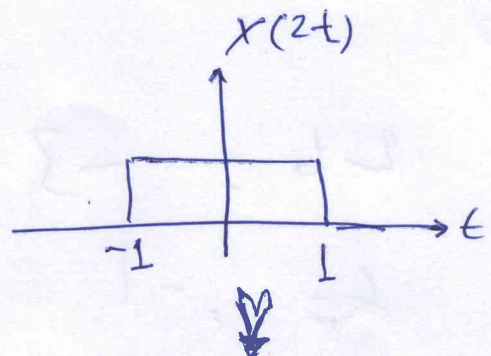
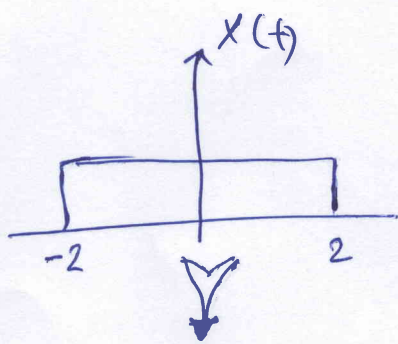


3- Reflection (time Reverse)

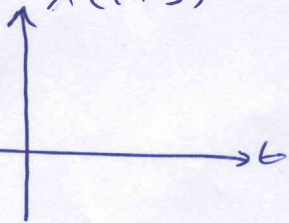


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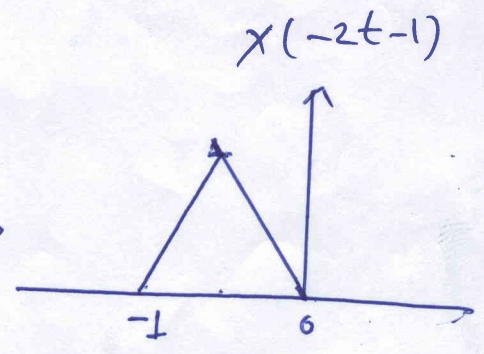
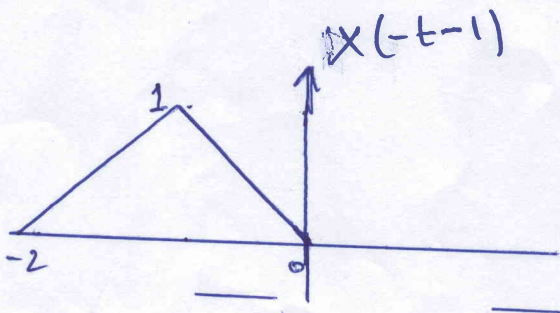
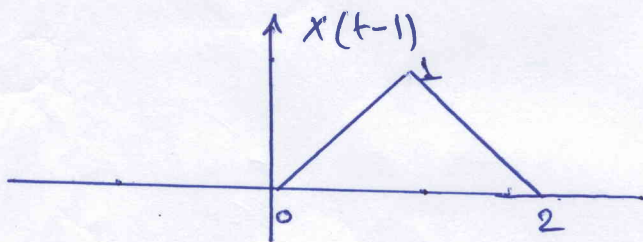
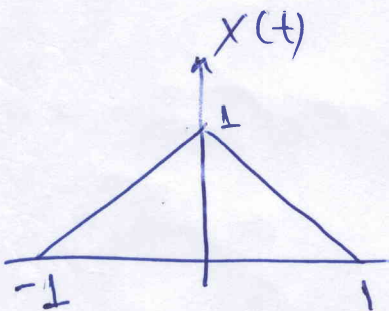
Example : Sketch $X(2t+3)$ if $X(t)$ is given as shown in the figure below :



$X(t+3)$



Example₂ : - Sketch $X(-2t-1)$, $X(t)$ is given below



* Some Elementary Signals ∞∞

by: Abdulbasit Sabawi

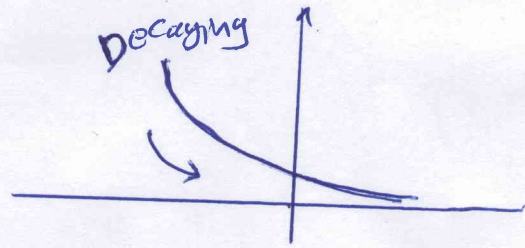
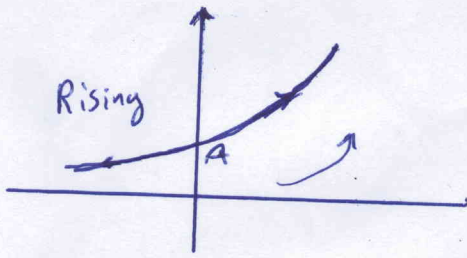
1. Exponential

$$x(t) = A e^{Bt}$$

A_{∞} is real & constant.

if B real and +ve \longrightarrow Increasing / rising

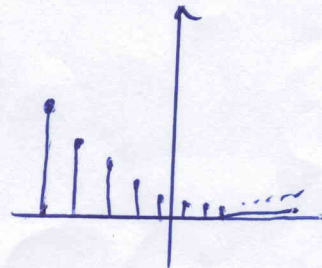
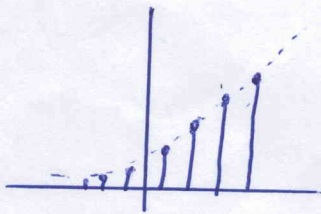
if B real and -ve \longrightarrow decaying



* Complex Exponential, if B Imaginary ($B = j\alpha$)

$$A e^{j\alpha t} = A (\cos \alpha t + j \sin \alpha t)$$

For discrete signal



2. Sinusoidal

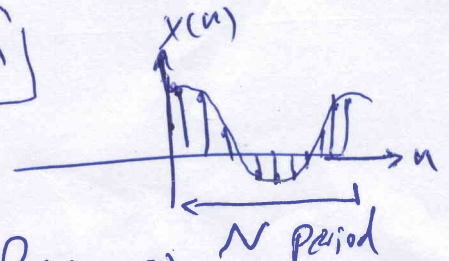
$$x(t) = A \cos(\omega t + \theta), \quad \omega = 2\pi f = \frac{2\pi}{T}$$

$$x(n) = A \cos(\Omega n + \theta)$$

$$x(n) = x(n+N)$$

$x(n)$ is periodic with N samples

~~$x(n) = A \cos(\Omega n + \theta)$~~



$$x(n+N) = A \cos(\Omega n + \Omega N + \theta)$$

$$\therefore \Omega N = 2\pi m$$

$m = \text{positive integer}$

$$\boxed{\frac{2\pi}{\Omega} = \frac{N}{m}}$$

$$= \frac{\text{integer}}{\text{integer}}$$

So it is Periodic

$$\text{EX}_1: x(n) = 2 \cos\left(\frac{8\pi n}{\Omega} + \frac{\pi}{4}\right)$$

$$\frac{2\pi}{\Omega} = \frac{2\pi}{8\pi} = \frac{1}{4} = \frac{\text{integer}}{\text{integer}} \rightarrow \text{Periodic}$$

$$\text{EX}_2: x(n) = 2 \sin(0.1\pi n)$$

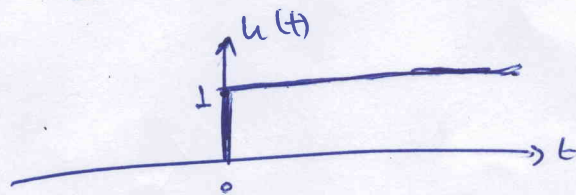
$$\frac{2\pi}{0.1\pi} = \frac{20}{1} \rightarrow \text{periodic}$$

$$\text{EX}_3: x(n) = \sin(n)$$

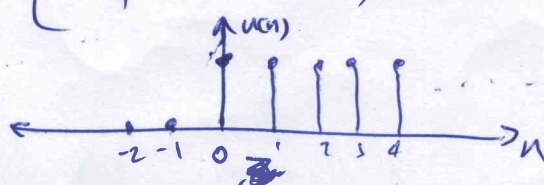
$$\frac{2\pi}{\Omega} = \frac{2\pi}{1} \text{ Non-periodic}$$

3- Unit step

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad \begin{matrix} t=0 \\ \text{no limit} \end{matrix}$$

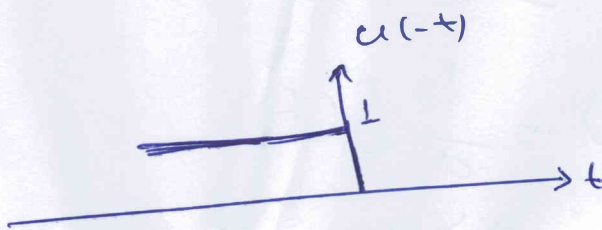


$$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n > 0 \end{cases}$$

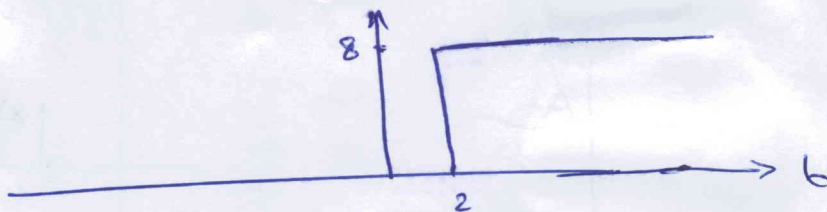


Example: Sketch the following signals.

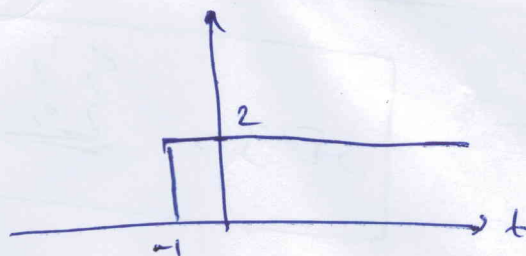
① $u(-t)$



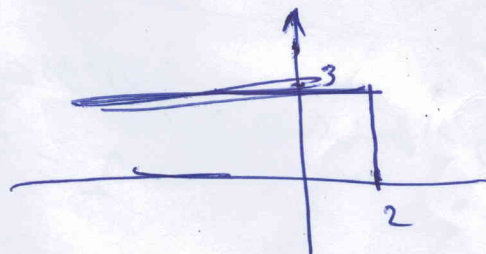
② $8u(t-2)$



③ $2u(t+1)$



④ $3u(-t+2)$

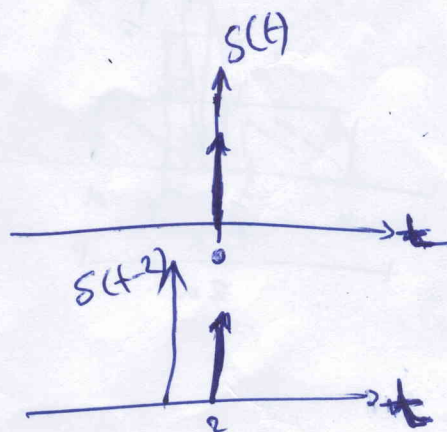


H.w = sketch $3[u(t+2) - u(t-2)]$

4 - Impulse (delta) function

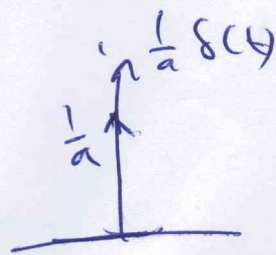
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\frac{d u(t)}{dt} = \delta(t)$$

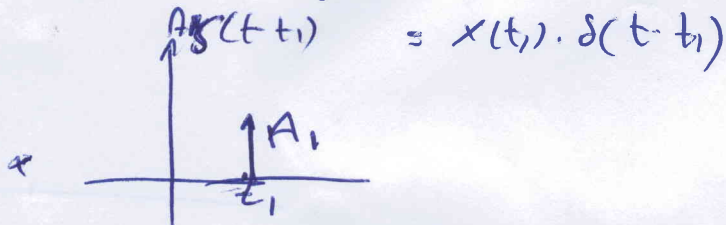
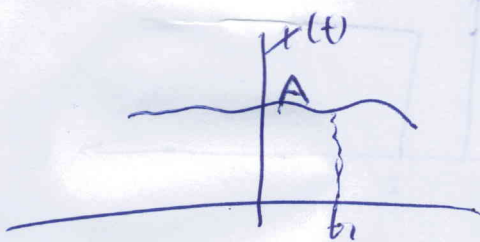


* $\delta(t) = \delta(-t)$ [Even]

* $\delta(at) = \frac{1}{|a|} \delta(t)$
 $a > 0$

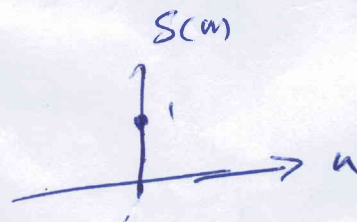


* $x(t) \cdot \delta(t - t_1) = x(t_1) \cdot \delta(t - t_1)$

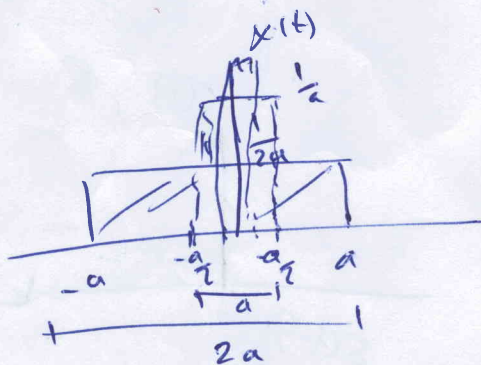


Discrete

$$S[n] = \sum_{n=0}^{\infty} \dots$$



Handwritten notes in Arabic script, possibly discussing the integral of a function over a symmetric interval.



$$\int_{-\infty}^{\infty} x(t) dt = \int_{-a}^a x(t) dt = 2a * \frac{1}{2a} = 1$$

$a * \frac{1}{a} = 1$

* Properties of Systems: \rightarrow Amplitude is finite.

1 - Stability (Bounded-input, bounded-output) (BIBO)

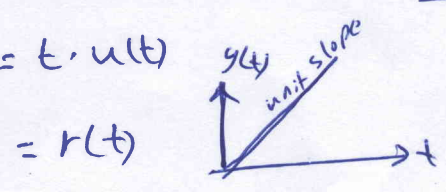
For a stable system output should be bounded for bounded input at each and every instant of time.

* Bounded signals for examples: dc , $\sin(t)$, $\cos(t)$, $u(t)$
 $x(t)=6$ $+1$ to -1 $+1$ to -1 0 or 1

Ex 1: $y(t) = t \cdot x(t)$

$x(t) \rightarrow \text{Sys.} \rightarrow y(t) = t \cdot x(t)$

$u(t) \rightarrow \text{Sys.} \rightarrow y(t) = t \cdot u(t)$



Ramp function
 $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

unstable because the output is not bounded

Ex 2: $y(t) = x(t) + 2$

$x(t) \rightarrow \text{Sys.} \rightarrow y(t) = x(t) + 2$

$dc=4 \rightarrow \text{System} \rightarrow y(t) = 4 + 2 = 6$ (bounded output)

Stable

2 - Memory: A system is called to be memory, if its output signal depends on past or future values of input signal. If its output depends only on present values of input signal, then the system is memoryless.

Ex 1: $y(t) = 2x(t) \rightarrow$ memoryless / static system

Ex 2: $y(t) = x(t) + x(t-1) \rightarrow$ memory / dynamic system.

3- Causality: If present value of the output depends only on the present and past values of the input, then the system is Causal. If the output depends on future values of the input it is non-causal.

Ex1: $y(t) = \underbrace{x(t)}_{\text{present}} \rightarrow \text{Causal}$

Ex2: $y(t) = \underbrace{x(t)}_{\text{present}} + \underbrace{x(t-1)}_{\text{past}} \rightarrow \text{Causal}$

Ex3: $y(t) = \underbrace{x(t+2)}_{\text{future}} \rightarrow \text{non-causal}$

Ex4: $y(t) = \underbrace{x(t)}_{\text{pre}} + \underbrace{x(t+1)}_{\text{fur}} + \underbrace{x(t-1)}_{\text{past}} \rightarrow \text{non-causal}$

4- Linearity:

Let $x_1 \rightarrow y_1$ & $x_2 \rightarrow y_2$, then

if $ax_1 + bx_2 \rightarrow ay_1 + by_2 \Rightarrow \text{linear}$

otherwise non linear

Ex: $y(t) = 2x(t) + 3$ is it linear?

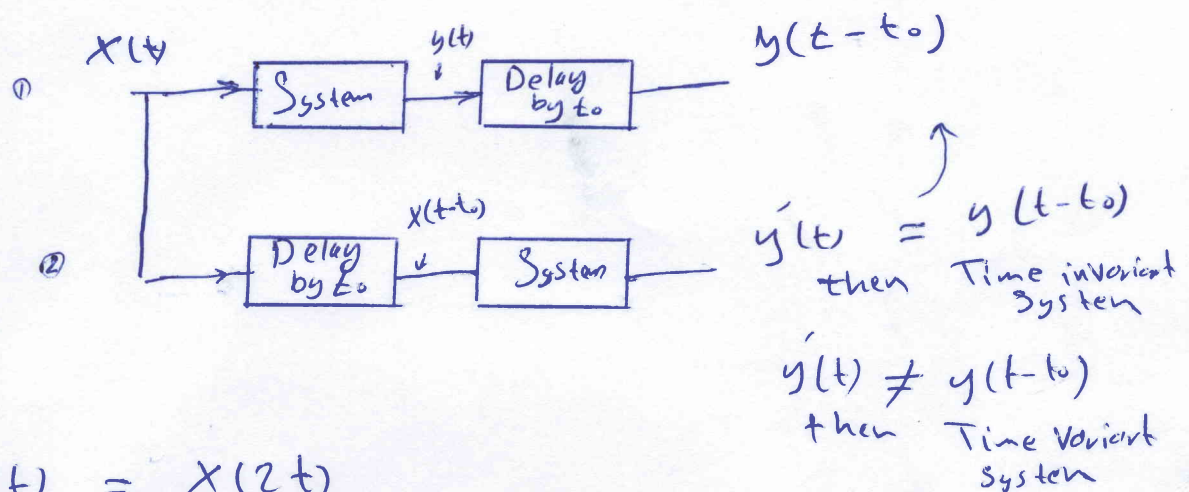
Let $x(t) = x_1(t) \rightarrow y_1(t) = 2x_1(t) + 3$

Let $x(t) = x_2(t) \rightarrow y_2(t) = 2x_2(t) + 3$

if $x(t) = ax_1(t) + bx_2(t) \rightarrow 2[ax_1 + bx_2] + 3 \stackrel{?}{=} \underline{ay_1 + by_2}$

$2ax_1(t) + 2bx_2(t) + 3 \stackrel{?}{=} 2ax_1(t) + 3a + 2bx_2(t) + 3$; No
Non-linear

5- Time Invariance : A System is said to be time-invariant if a time delay of the input signal leads to an identical time shift in the output signal. Otherwise, the system can be time variant



Ex 1: $y(t) = x(2t)$

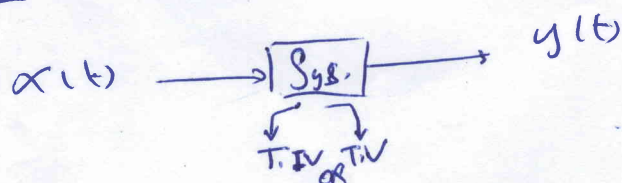
$x(t) \xrightarrow{\text{Scaling}} \boxed{\text{Sys.}} \rightarrow y(t) = x(2t)$

① $y(t) \xrightarrow{t_0} y(t-t_0) = x(2(t-t_0)) = \underline{x(2t-2t_0)}$

② $x(t) \xrightarrow{t_0} x(t-t_0) \xrightarrow{\text{Scaling}} \boxed{\text{System}} \rightarrow x(2t-t_0)$

So, $x(2t-2t_0) \neq x(2t-t_0)$, then the system is time variant.

Ex 2: $y(t) = 2 + x(t)$



$y(t) \xrightarrow{t_0} \underline{2 + x(t-t_0)}$

$x(t) \xrightarrow{t_0} x(t-t_0) \xrightarrow{\text{Sys.}} \underline{2 + x(t-t_0)}$, Same

So Time Invariant

H.W [$y(t) = \cos(t) x(t)$]

Examples : For the following Systems, check if it is Stable or not?

(1) $x(t) = \sin(\omega t)$

It is stable because $-1 \leq \sin(\omega t) \leq 1$ for all t

Example 2 : $y(t) = x(t) + 3x(t-d)$ is memory, causal?

* memory because it depends on $x(t-d)$

* causal depends on ~~future~~ past $x(t-d)$

Example 3 : Determine whether the System described by Equation $y(t) = \frac{1}{12}x(t) - \frac{5}{6}$ is Linear or Non-Linear?

Let $x(t) = x_1(t)$ then $y_1(t) = \frac{1}{12}x_1(t) - \frac{5}{6}$

Let $x(t) = x_2(t)$ then $y_2(t) = \frac{1}{12}x_2(t) - \frac{5}{6}$

$x(t) = ax_1(t) + bx_2(t)$ then :

$y(t) = \frac{1}{12}[ax_1(t) + bx_2(t)] - \frac{5}{6} \stackrel{?}{=} ay_1 + by_2$

$$ay_1 + by_2 = \frac{1}{12}ax_1(t) - \frac{5}{6}a + \frac{1}{12}bx_2(t) - \frac{5}{6}b$$

$$= \frac{1}{12}[ax_1(t) + bx_2(t)] - \frac{5}{6}[a+b]$$

∴ So Non-Linear ($ax_1 + bx_2 \neq ay_1 + by_2$)

Example 4: Is the System $y(t) = x(t+3) + 2 \int_{-\infty}^{0.5t} x(\tau) d\tau$ Causal, Linear?

$\therefore y(t)$ depends on $x(\overset{\text{future}}{t+3})$ so it is Non-causal.

$$\text{at } x = x_1 \rightarrow y_1(t) = x_1(t+3) + 2 \int_{-\infty}^{0.5t} x_1(\tau) d\tau$$

$$x = x_2 \rightarrow y_2(t) = x_2(t+3) + 2 \int_{-\infty}^{0.5t} x_2(\tau) d\tau$$

$$x = ax_1 + bx_2$$

$$y = [ax_1(t+3) + bx_2(t+3)] + 2 \int_{-\infty}^{0.5t} [ax_1(\tau) + bx_2(\tau)] d\tau$$

$$\stackrel{?}{=} ay_1 + by_2$$

$$a \left[x_1(t+3) + 2 \int_{-\infty}^{0.5t} x_1(\tau) d\tau \right] + b \left[x_2(t+3) + 2 \int_{-\infty}^{0.5t} x_2(\tau) d\tau \right]$$

So $ax_1 + bx_2 \stackrel{?}{=} ay_1 + by_2$ then the system is Linear

Example 5: Is the System below time variant or time-invariant? $y(t) = t x(t) + 4$

$$x(t) \rightarrow \boxed{\text{Sys.}} \rightarrow y(t)$$

$$y(t) \xrightarrow{t_0} \underbrace{(t-t_0) x(t-t_0) + 4}_{\neq}$$

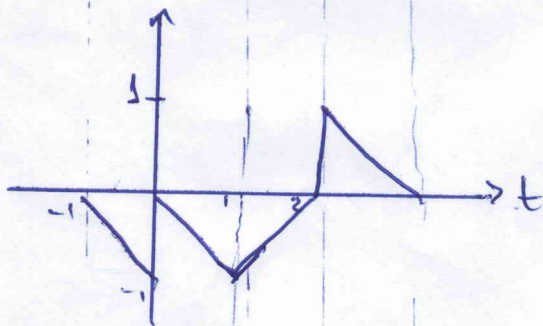
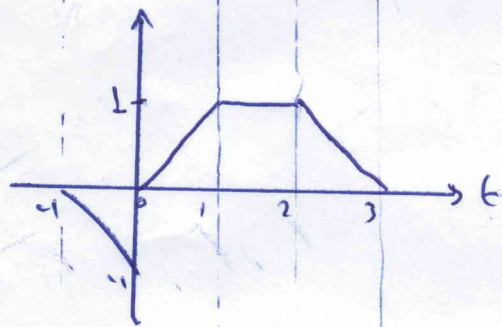
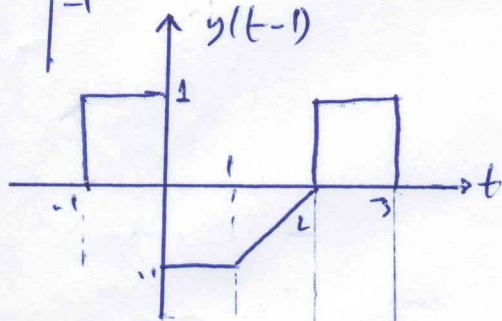
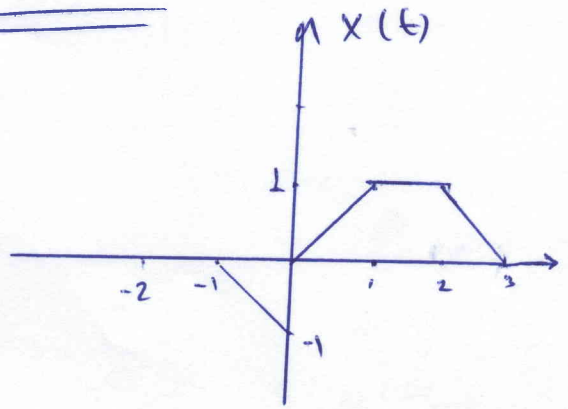
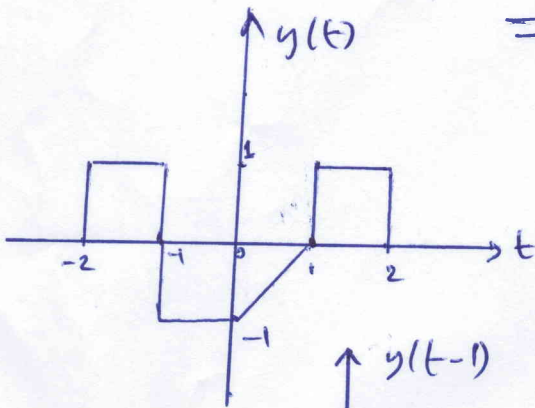
$$x(t) \xrightarrow{t_0} x(t-t_0) \rightarrow \boxed{\text{Sys.}} \rightarrow \underbrace{t x(t-t_0) + 4}$$

\therefore it is time variant System.

Example

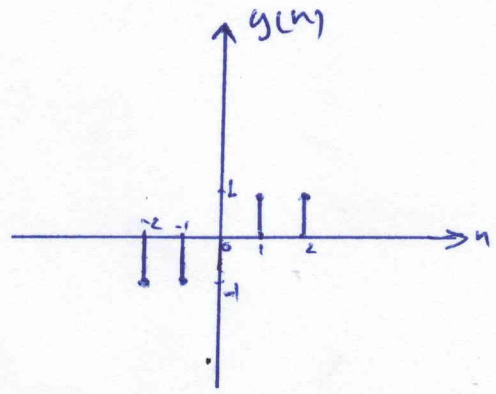
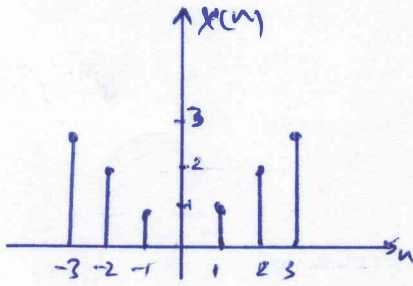
Let $x(t)$, $y(t)$ be given in the following figures,

Sketch : $x(t) \cdot y(t-1)$

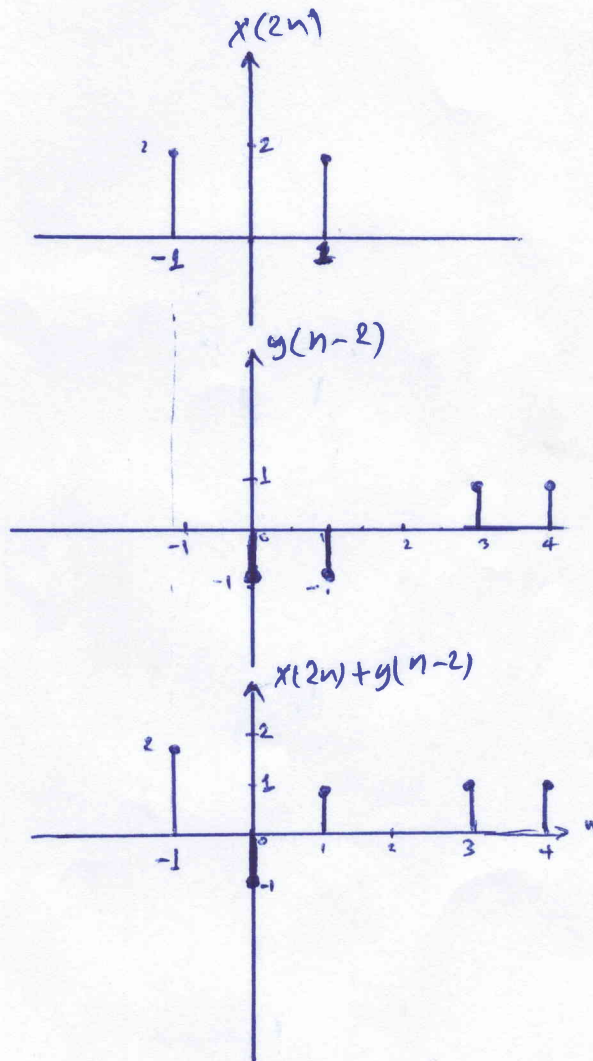


Example : Let $x(n]$, $y(n]$ are given below :

Sketch : $x(2n) + y(n-2)$



$2n = 0$
 $n = 0$



Ex^o The Discrete-time System shown below



Determine whether the System is a) memoryless, b) causal, c) Linear, d) Time-Invariant or e) stable.

Solution

a) the System input-output relation is given

$$y[n] = x[n-1]$$

Since the output value of $y(n)$ depends on the input value at $n-1$ (past), the System is not memoryless.

b) Since the output does not depend on the future input value, the System is causal.

9 Let $x = x_1 \longrightarrow y_1[n] = x_1[n-1]$

Let $x = x_2 \longrightarrow y_2[n] = x_2[n-1]$

$$x = ax_1 + bx_2 \longrightarrow y[n] = ax_1[n-1] + bx_2[n-1]$$

$$\stackrel{?}{=} ay_1 + by_2$$

$ax_1[n-1] + bx_2[n-1]$ it is Linear

d) $y[n] \xrightarrow{\text{no delay}} y[n-n_0] \longrightarrow x[n-n_0-1] = \underline{\underline{x[n-1-n_0]}}$

e) $x[n] \xrightarrow{\text{no delay}} x[n-n_0] \longrightarrow \boxed{\text{Sys}} \longrightarrow x[n-1-n_0]$ Time-invariant

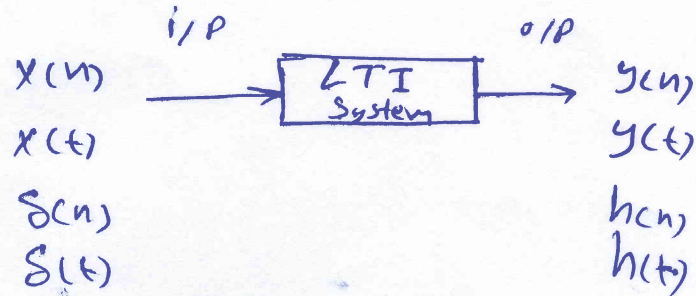
Since

$$|y[n]| = |x[n-1]| \leq k$$

if $|x[n]| \leq k$ for all n
the System is BIBO Stable.

* Time Domain Analysis

- Impulse Response: The impulse response $[h(t), h(n)]$ of a [Continuous, Discrete]-time LTI System is defined to be the response of the system when the input is $[\delta(t), \delta(n)]$.

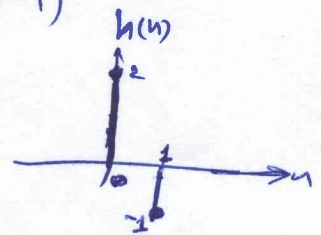
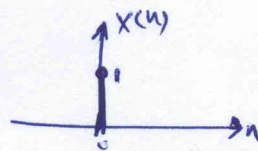


Ex: Find and sketch the impulse response of the system:

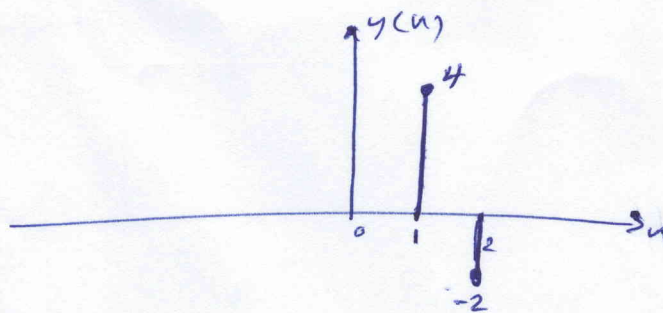
$$y(n] = 2x(n) - x(n-1]$$

$$x(n] = \delta(n), y(n) \rightarrow h(n)$$

$$h(n] = 2\delta(n) - \delta(n-1]$$



Ex: Consider the previous system as LTI system
 Find the output if $x(n] = 2\delta(n-1]$



It.w: For the same system above find the output if $x(n] = \delta(n) + 2\delta(n-1]$.

$$x(n)\delta(n) = x(0)\delta(n)$$

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

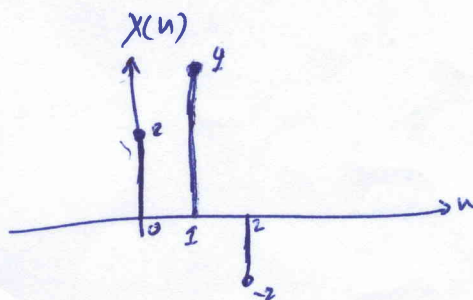
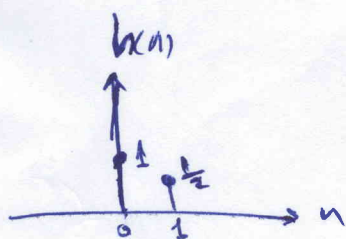
$$x(n) = \dots + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \dots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

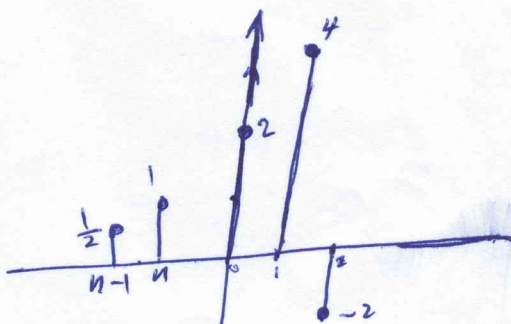
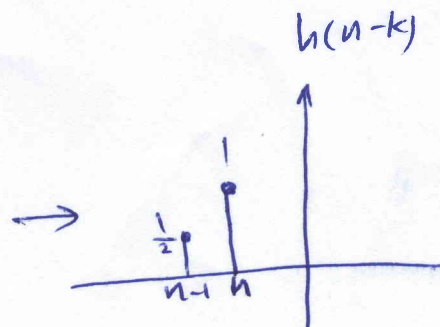
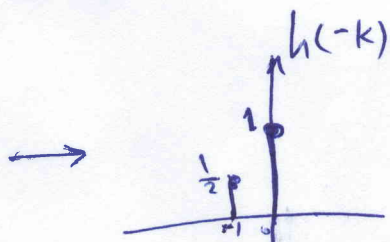
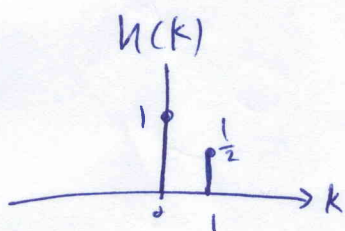
$$= x(n) \otimes h(n) \quad (\text{convolution})$$

EX: For the system of impulse response and input signal as shown
Find the output.



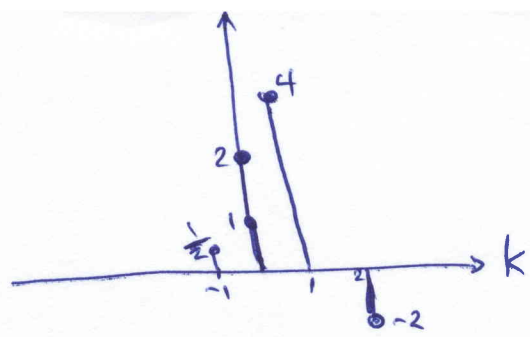
Sol

$$y(n) = x(n) \otimes h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

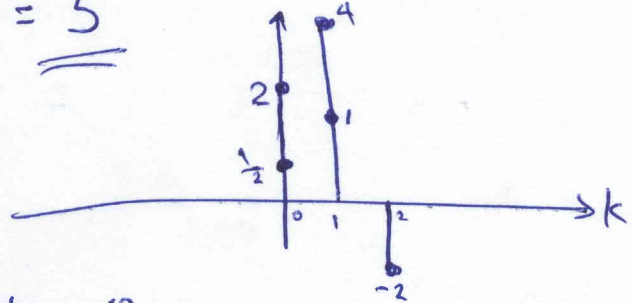


$$n < 0 \rightarrow y(n) = 0$$

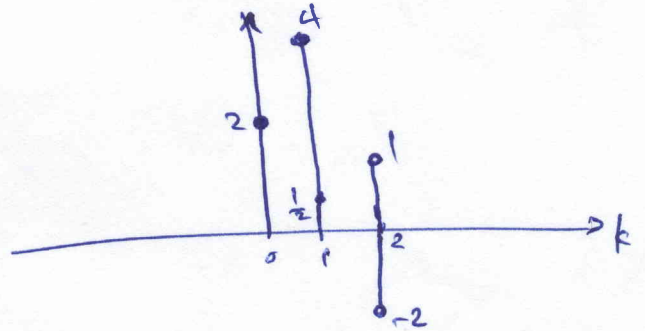
$$n = 0 \rightarrow y(n) = 2 * 1 = \underline{\underline{2}}$$



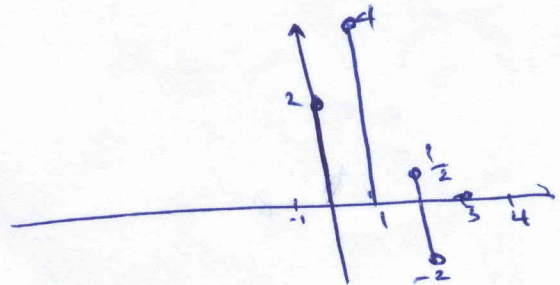
$$n = 1 \rightarrow y(n) = \frac{1}{2} * 2 + 4 * 1 = \underline{\underline{5}}$$



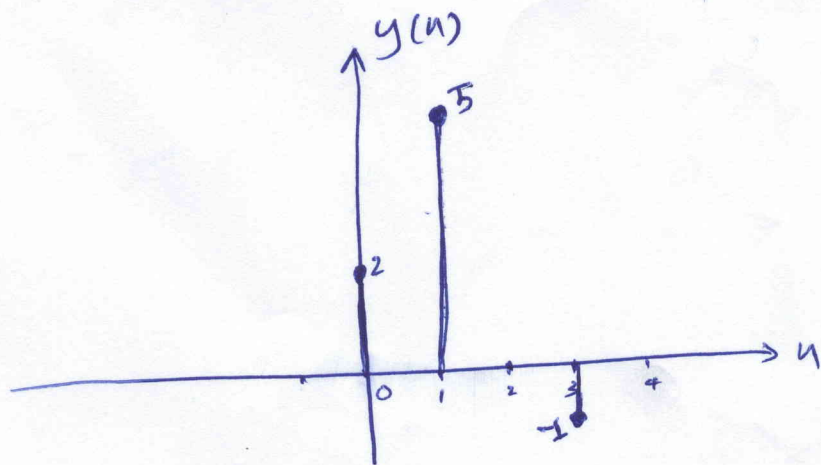
$$n = 2 \rightarrow y(n) = \frac{1}{2} * 4 + (-2) * 1 = \underline{\underline{0}}$$



$$n = 3 \rightarrow y(n) = \frac{1}{2} * -2 = \underline{\underline{-1}}$$



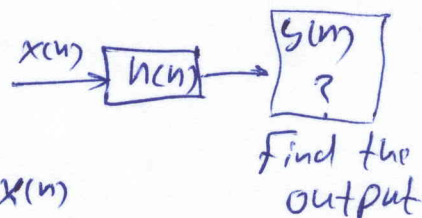
$$n \geq 4 \rightarrow y(n) = 0$$



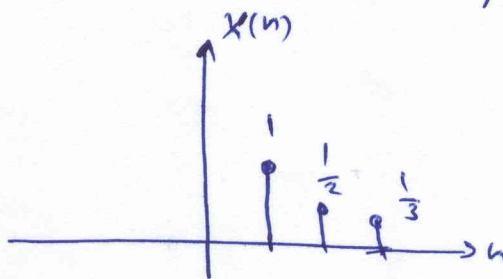
Example 2 For the System of impulse response and input signal

$$h(n) = n \quad 0 < n < 4$$

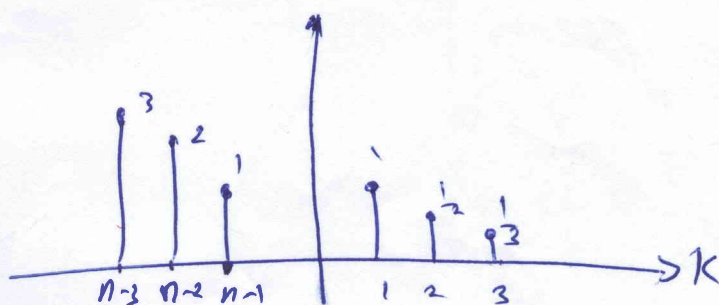
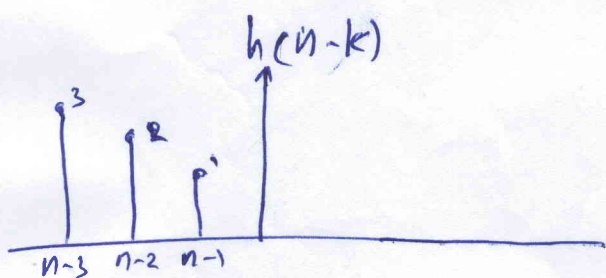
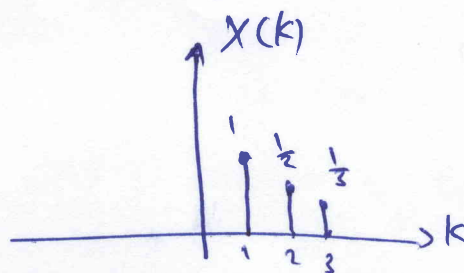
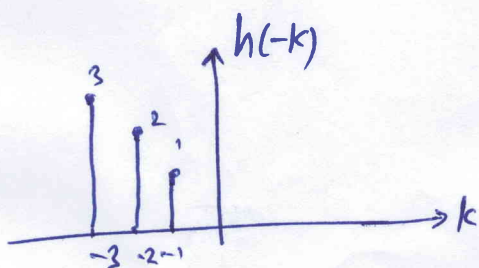
$$x(n) = \frac{1}{n} \quad 0 < n < 4$$



$$n = 1, 2, 3$$



$$y(n) = x(n) \otimes h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$



for $n-1 < 1 \Rightarrow n < 2 \Rightarrow y(n) = 0$

for $n-1 = 1 \Rightarrow n = 2 \Rightarrow y(n) = 1 * 1 = 1$

$n = 3 \Rightarrow y(n) = 2 * 1 + 1 * \frac{1}{2} = 2.5$

$n = 4 \Rightarrow y(n) = 1 * 3 + 2 * \frac{1}{2} + 1 * \frac{1}{3} = 4 \frac{1}{3}$

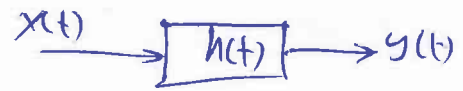
$n = 5 \Rightarrow y(n) = \frac{1}{3} * 2 + \frac{1}{2} * 3 = 2.166$

$n = 6 \Rightarrow y(n) = 3 * \frac{1}{3} = 1$

for $n \geq 7 \Rightarrow y(n) = 0$

For continuous LTI System, if the input is $x(t)$ and impulse response is $h(t)$ that the output $y(t)$ is

$$y(t) = x(t) * h(t)$$

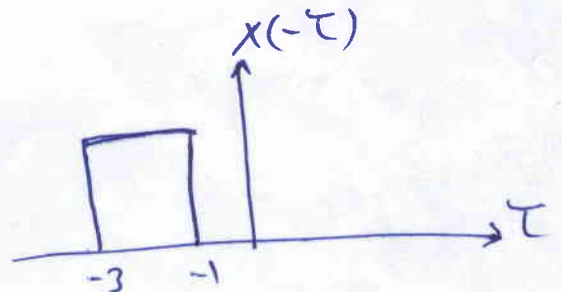
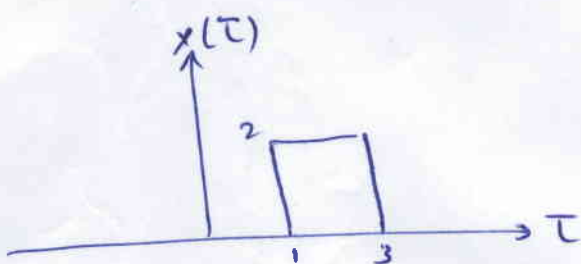
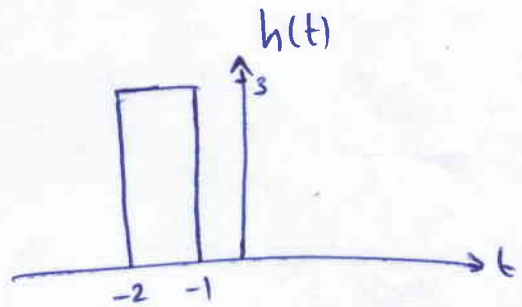
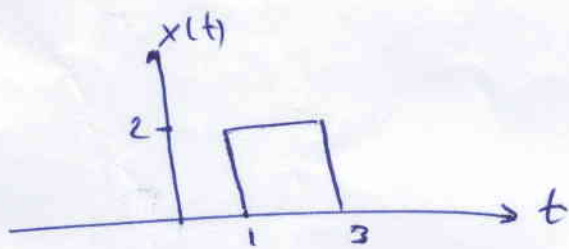


$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau$$

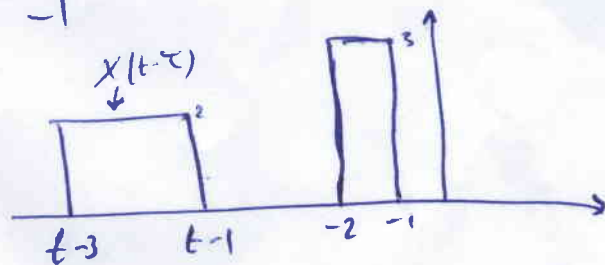
$\otimes \rightarrow$ this is called Convolution

Ex:

Find the output of the system whose impulse response and input signal are as shown



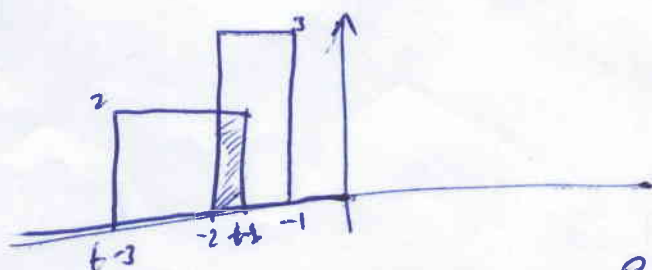
* for $t-1 \leq -2 \Rightarrow t \leq -1$
 $y(t) = 0$



* for $t-1 > -2$ & $t-1 \leq -1 \Rightarrow -1 \leq t \leq 0$

$$y(t) = \int_{-2}^{t-1} 2 \times 3 d\tau = 6\tau \Big|_{-2}^{t-1}$$

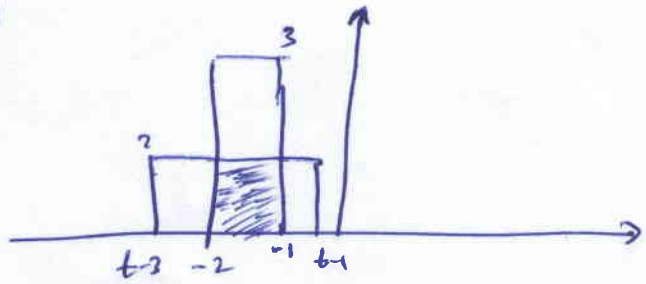
$$= 6(t-1) - 6(-2) = \underline{\underline{6t+6}}$$



* for $t-1 \geq -1$ & $t-3 \leq -2$
 $0 \leq t \leq 1$

$$y(t) = \int_{-2}^{-1} 2 \times 3 \, d\tau = 6\tau \Big|_{-2}^{-1}$$

$$= \underline{\underline{6}}$$

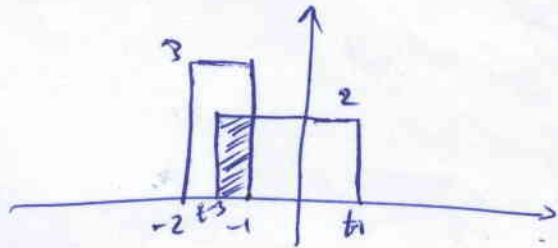


* for $1 \leq t \leq 2$ ($t-1 \geq 0$ & $t-3 \leq -1$)

$$y(t) = \int_{t-3}^{-1} 6 \, d\tau = 6\tau \Big|_{t-3}^{-1}$$

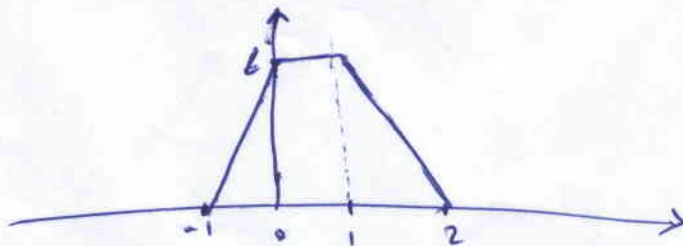
$$= -6 - 6t + 18 = \underline{\underline{12 - 6t}}$$

Signature
 2/2/2011

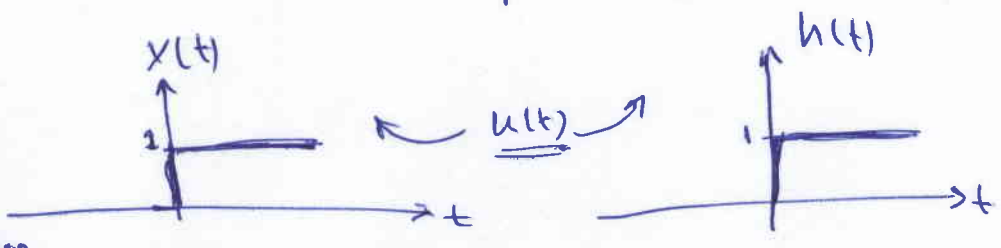


* for $t \geq 2$, $y(t) = 0$

The final output

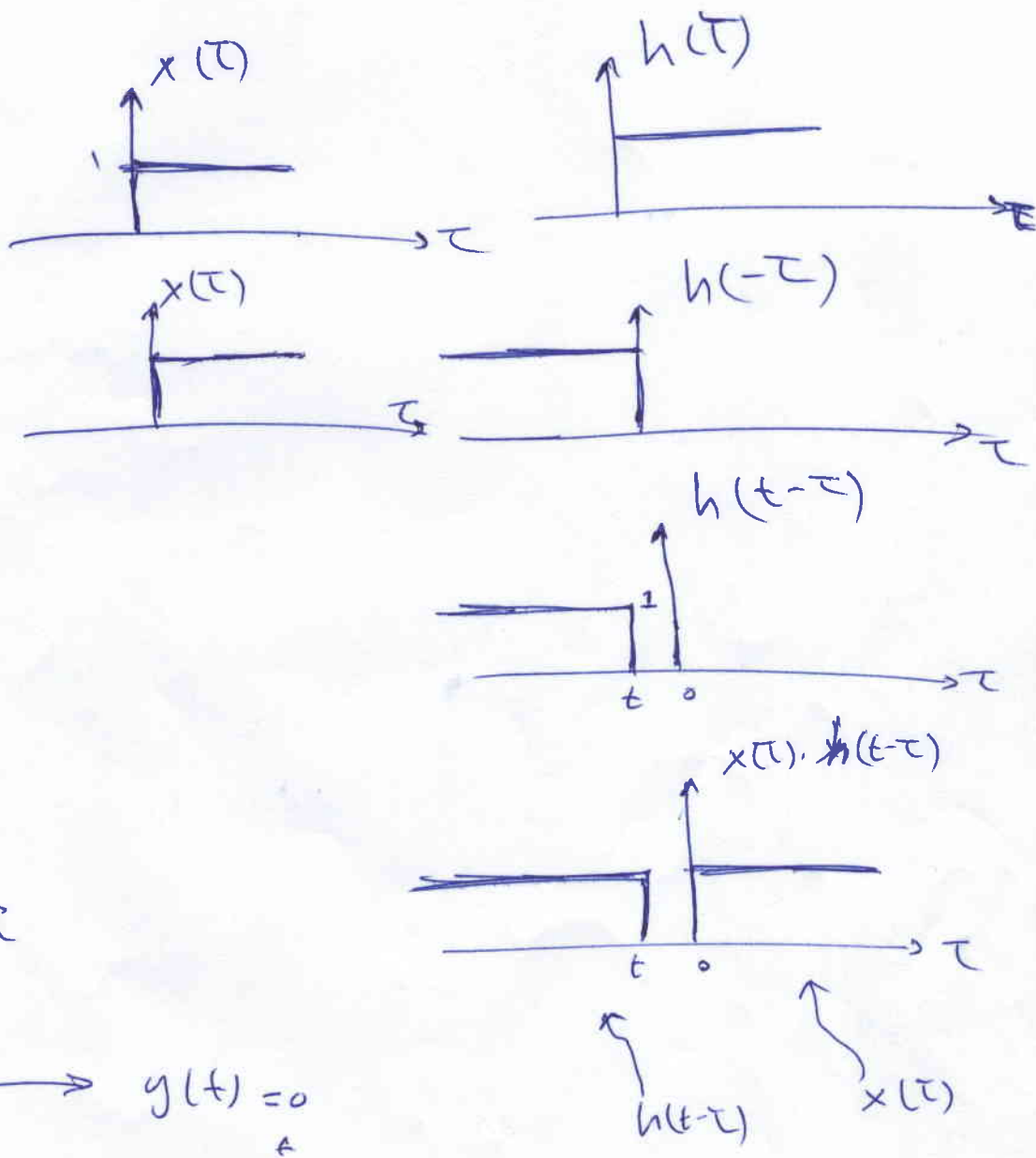


EX: Find the output $y(t)$ of an LTI System for the input and impulse response given:



Review with step

$x(t)$ $h(t)$
 $\downarrow t=\tau$ $\downarrow t=\tau$
 $x(\tau)$ $h(\tau)$
 fixed \downarrow Reflection
 $h(-\tau)$
 \downarrow shifting
 $h[-(\tau-t)] \rightarrow h(t-\tau)$
 \downarrow mul.
 $x(\tau) \cdot h(t-\tau)$
 \downarrow
 $\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$

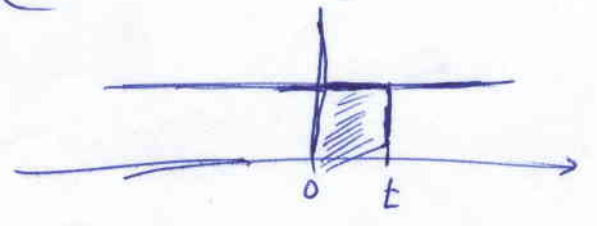


$t < 0 \rightarrow y(t) = 0$
 $t > 0 \rightarrow y(t) = \int_0^t 1 * 1 d\tau$

$y(t) = \tau \Big|_0^t = \underline{\underline{t}}$

$y(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases}$

$y(t) = r(t)$ ramp signal



Ex: The System below is formed by connecting two systems in cascade, the impulse response of the systems are given by $h_1(t)$ & $h_2(t)$

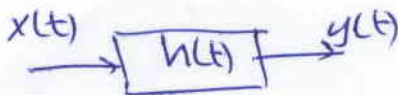


$$h_1(t) = u(t)$$

$$h_2(t) = e^{-at} u(t), a > 0$$

Find the impulse response $h(t)$ of the over all System

convolution



$$w(t) = x(t) \otimes h_1(t)$$

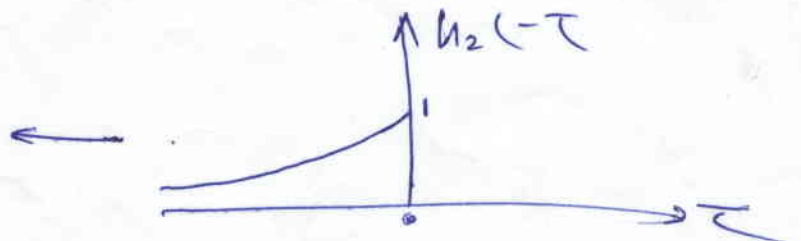
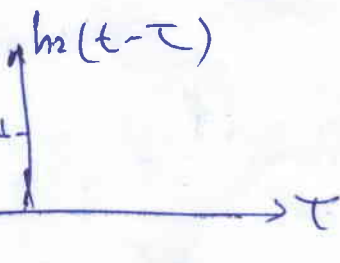
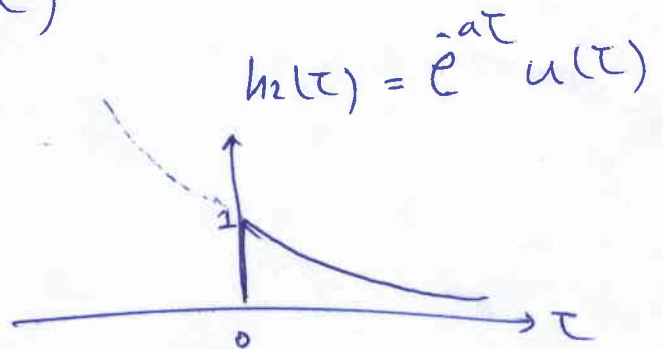
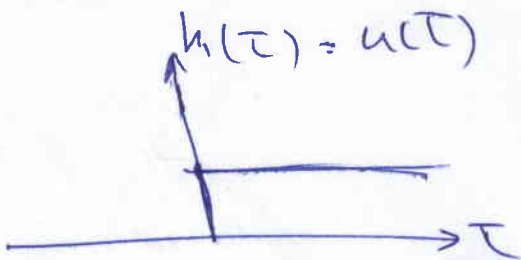
$$y(t) = w(t) \otimes h_2(t)$$

$$y(t) = x(t) \otimes \underline{h_1(t) \otimes h_2(t)}$$

$y(t) = x(t) \otimes h(t)$, then the impulse response of the overall

System is $\boxed{h(t) = h_1(t) \otimes h_2(t)}$

$$h(t) = \int_{-\infty}^{\infty} h_1(\tau) h_2(t-\tau) d\tau$$



$t < 0 \rightarrow h(t) = 0$ do not have overlap

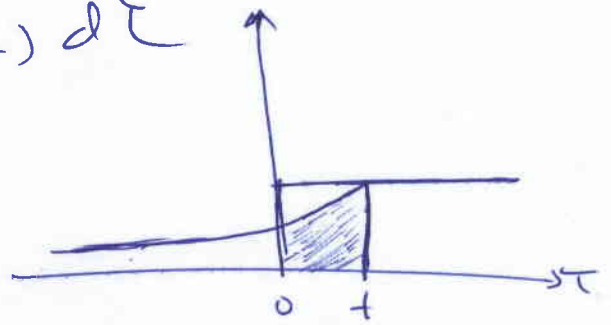
$$t > 0 \rightarrow h(t) = \int_0^t e^{-a\tau} u(\tau) d\tau$$

$$h(t) = \frac{e^{-a\tau}}{-a} \Big|_0^t$$

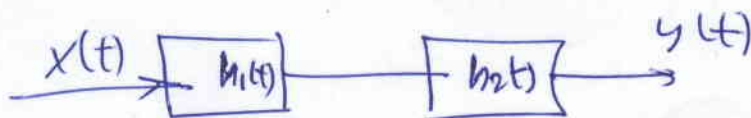
$$= \frac{-1}{a} (e^{-a\tau}) \Big|_0^t$$

$$= \frac{-1}{a} (e^{-at} - 1)$$

$$= \frac{1}{a} (1 - e^{-at}) u(t)$$

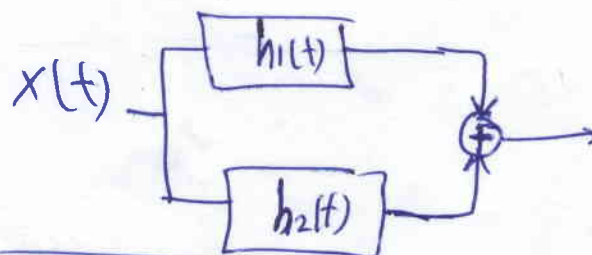


For
①



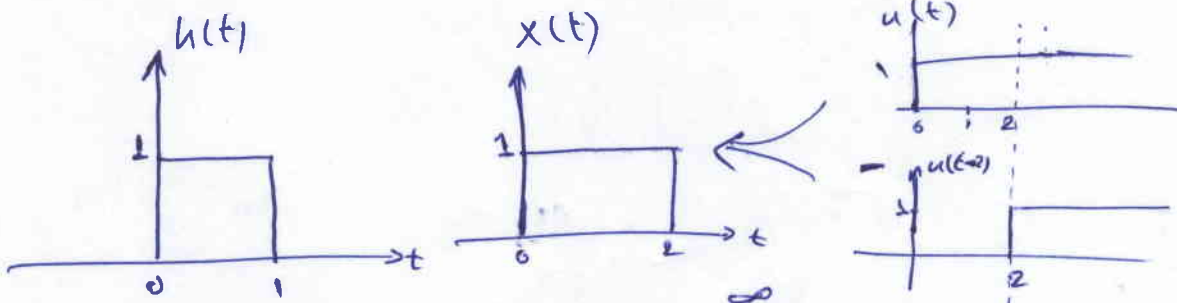
$$h_{\text{total}}(t) = h_1(t) * h_2(t)$$

②

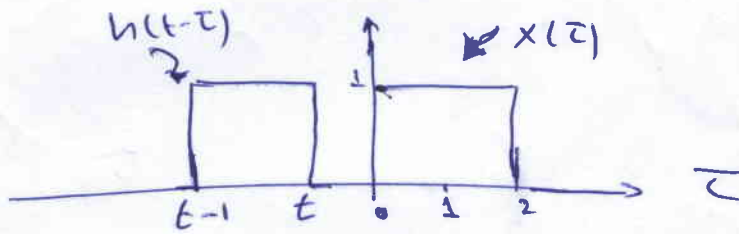
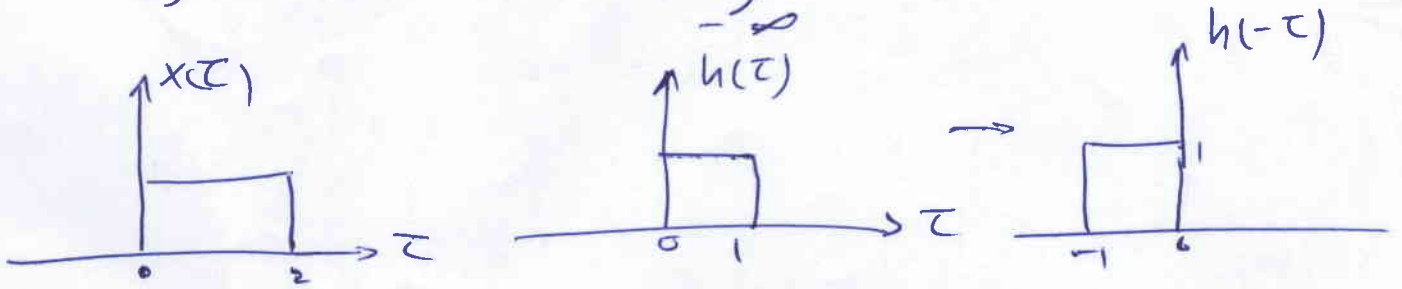


$$h_{\text{total}}(t) = h_1(t) + h_2(t)$$

EX: LTI System has impulse response $h(t)$ shown below, sketch output when input $x(t) = u(t) - u(t-2)$



$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$



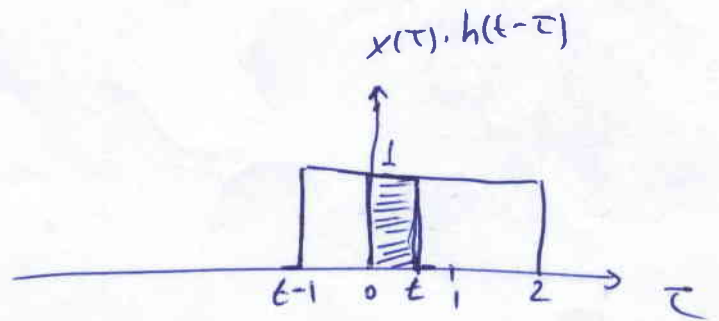
Case 1 : $t < 0 \rightarrow$ no overlap

$$y(t) = 0$$

Case 2 : $0 < t < 1$

$$y(t) = \int_0^t 1 \times 1 d\tau = \tau \Big|_0^t$$

$$y(t) = t$$

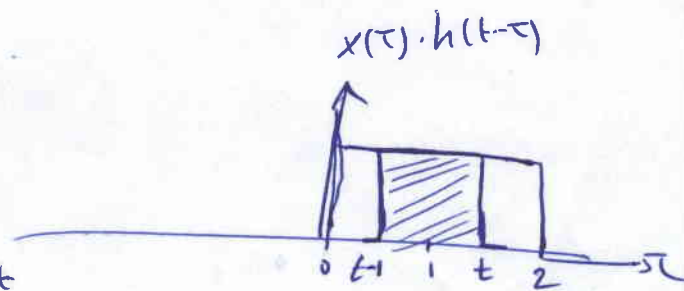


Case 3 : $1 \leq t \leq 2$

$$y(t) = \int_{t-1}^t 1 \times 1 d\tau = \tau \Big|_{t-1}^t$$

$$y(t) = t - (t-1) = t - t + 1$$

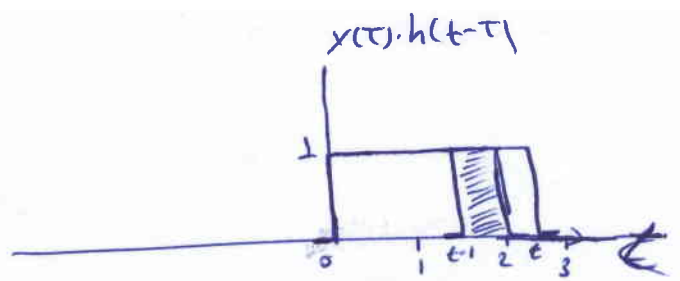
$$y(t) = 1$$



Case 4: $2 < t < 3$

$$y(t) = \int_{t-1}^2 1 d\tau = \tau \Big|_{t-1}^2$$

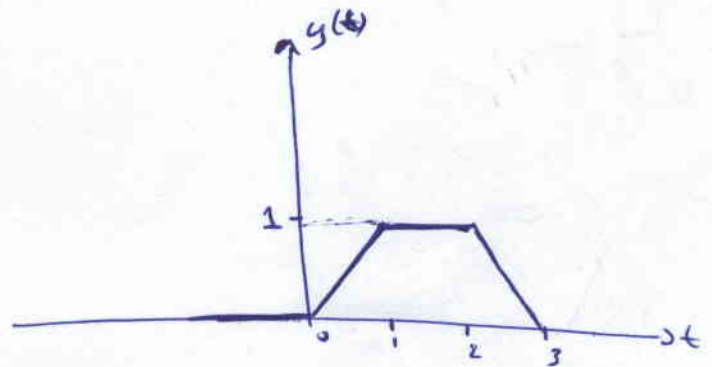
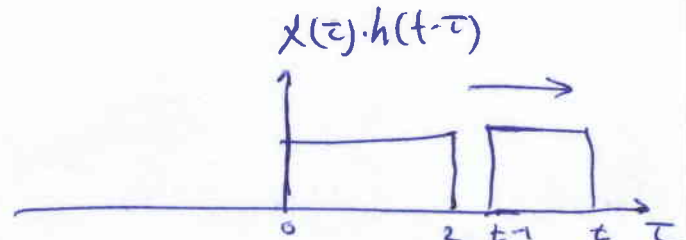
$$y(t) = 3 - t$$



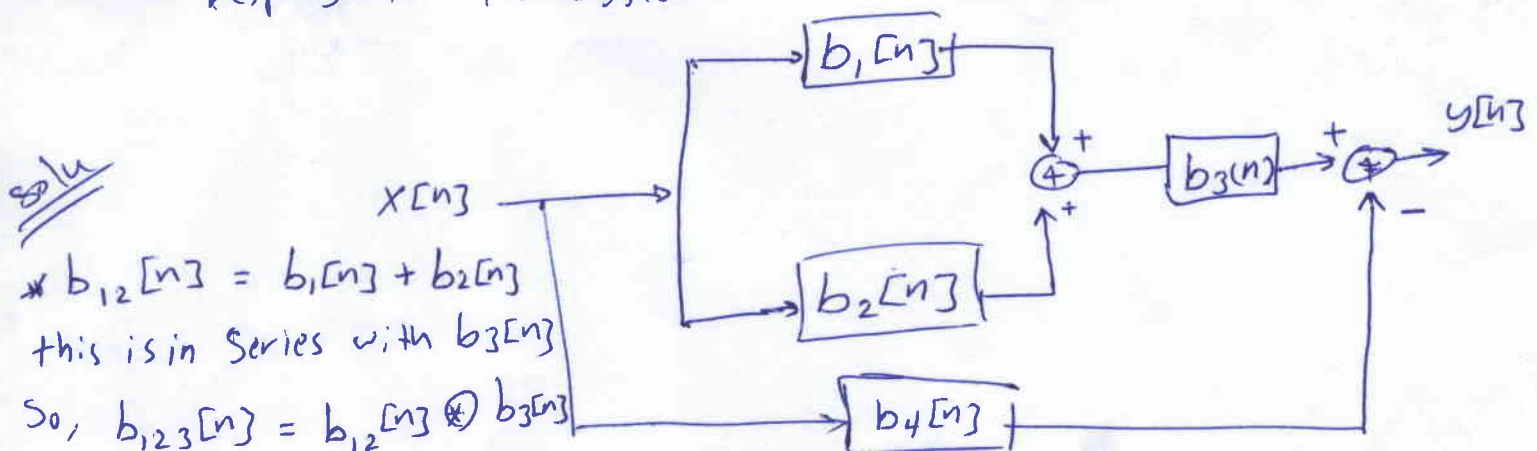
Case 5: $t > 3$

$$y(t) = 0$$

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & t > 3 \end{cases}$$



EX: Consider the inter connection of LTI Systems depicted in the figure below. Find the overall impulse response for the System.



Solu

$$* b_{1,2}[n] = b_1[n] + b_2[n]$$

this is in series with $b_3[n]$

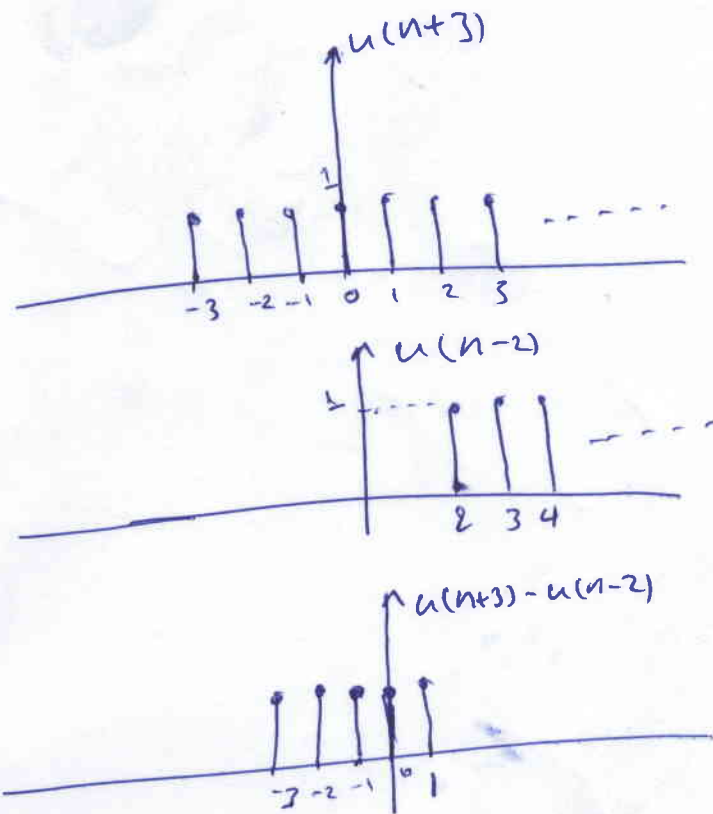
$$\text{So, } b_{1,2,3}[n] = b_{1,2}[n] * b_3[n]$$

$$b_{1,2,3}[n] = (b_1[n] + b_2[n]) * b_3[n]$$

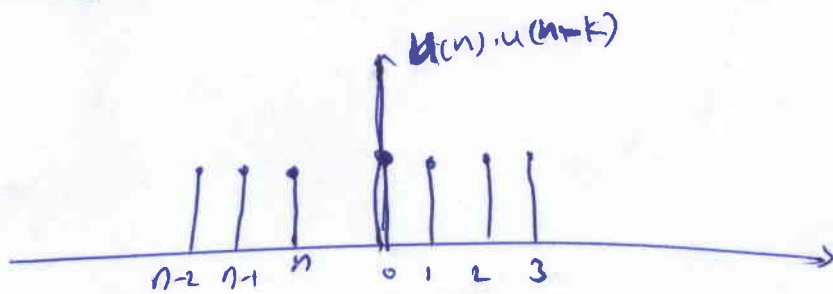
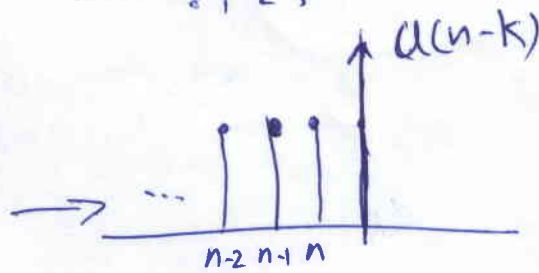
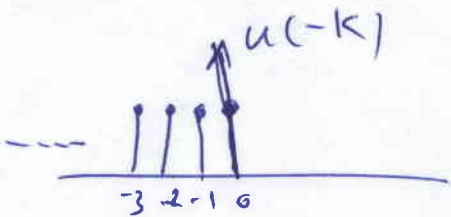
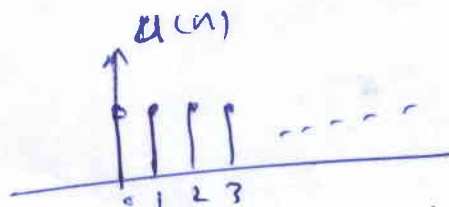
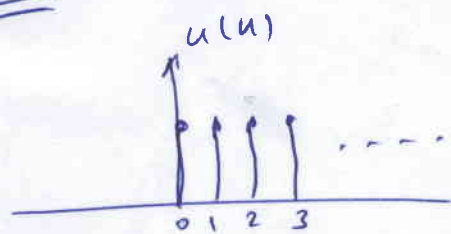
The upper branch is in parallel with the lower branch, ($b_4[n]$); hence the overall system impulse response is $b[n] = b_{1,2,3}[n] - b_4[n]$

$$\therefore b[n] = (b_1[n] + b_2[n]) * b_3[n] - b_4[n]$$

EX: sketch $u(n+3) - u(n-2)$



EX: find the convolution of $u(n) \otimes u(n)$



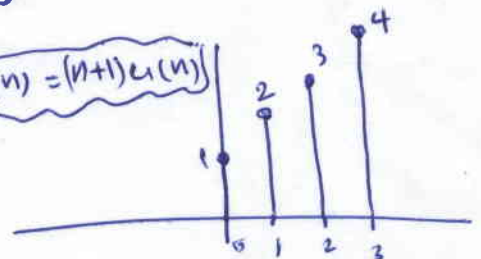
$n < 0 \rightarrow y(n) = 0$

$n = 0 \rightarrow y(n) = 1 \times 1 = 1$

$n = 1 \rightarrow y(n) = 1 \times 1 + 1 \times 1 = 2$

$n = 2 \rightarrow y(n) = 1 \times 1 + 1 \times 1 + 1 \times 1 = 3$

$y(n) = (n+1)u(n)$



* Fourier Series (FS) :-

A continuous periodic signal, $x(t)$ with period T is expressed as a sum of an infinite set of harmonically related sinusoids and a dc component in the FS.

$x(t) =$ dc (average value of $x(t)$) + Cosine terms + Sine terms

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \dots \\ + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \dots$$

The frequency of the fundamental or first harmonic is the frequency of the waveform under analysis.

That is $\omega_0 = 2\pi f_0$

$$f_0 = \frac{1}{T} \Rightarrow \omega_0 = 2\pi / T$$

f_0 : fundamental frequency

The frequency of the second harmonic is $2\omega_0$, that of the third harmonic is $3\omega_0$ and so on.

Hence, Fourier Series Equation has dc & Ac terms, So it can be simplified as :-

* Trigonometric Fourier Series :-

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

To find a_0 ::

$$\int_0^T x(t) dt = \int_0^T a_0 dt + \int_0^T a_1 \cos \omega_0 t dt + \int_0^T a_2 \cos 2\omega_0 t dt + \dots$$

$$+ \int_0^T b_1 \sin \omega_0 t dt + \int_0^T b_2 \sin 2\omega_0 t dt + \dots$$

$$\int_0^T x(t) dt = a_0 T$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt \quad \text{and so on}$$

Generally we can write a_0, a_n and b_n as shown below:

Fourier Coefficient

$$* a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$* a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega_0 t dt$$

$$* b_n = \frac{2}{T} \int_0^T x(t) \sin n\omega_0 t dt$$

if $x(t) \rightarrow$ Even Signal $\Rightarrow b_n = 0$ (when $x(t)$ is even there will be no sine term as sine is an odd signal and this means b_n is equal to zero)

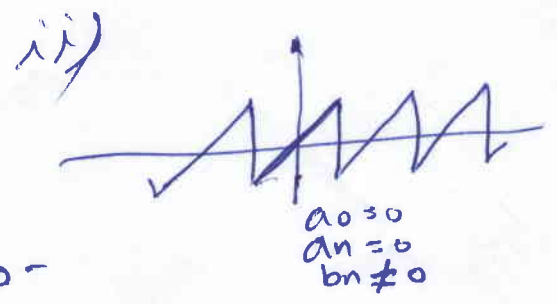
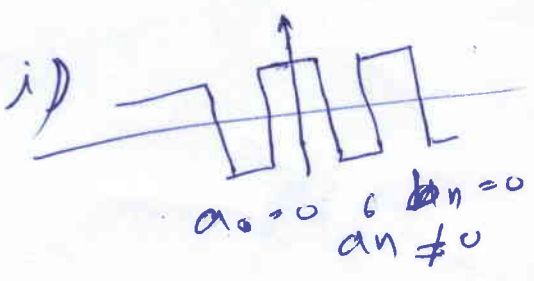
$$x(-t) = x(t)$$

if $x(t) \rightarrow$ Odd Signal $\Rightarrow a_n = 0$ (when $x(t)$ is odd there will be no cosine term as cosine is an even signal and this means a_n is equal to zero)

$$x(-t) = -x(t)$$

if $x(t) \rightarrow$ Symmetric about t-axis $\Rightarrow a_0 = 0$

Ex:

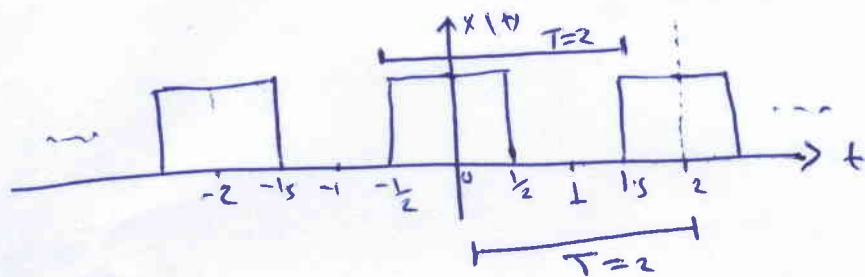


$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt = \frac{2}{T} \int_0^{\frac{T}{2}} x(t) dt$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos n\omega_0 t dt = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \cos n\omega_0 t dt$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin n\omega_0 t dt = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \sin n\omega_0 t dt$$

Ex: For the periodic signal $x(t)$ shown, find Fourier Series.



$T=2$

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt = \frac{2}{T} \int_0^{\frac{T}{2}} x(t) dt \\ &= \frac{2}{2} \int_0^1 x(t) dt = \int_0^1 1 dt + \int_0^0 0 \\ &= t \Big|_0^1 = \boxed{\frac{1}{2}} \end{aligned}$$

$b_n = 0$ because $x(t)$ is EVEN

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \cos n\omega_0 t dt = \frac{4}{2} \left[\int_0^1 1 \cdot \cos n\omega_0 t dt + \int_0^0 0 \cdot \cos n\omega_0 t dt \right]$$

$$= 2 \frac{\sin n\omega_0 t}{n\omega_0} \Big|_0^1$$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$$= \frac{2}{n\pi} \left[\sin n\pi \frac{1}{2} - \sin 0 \right] = \frac{2}{n\pi} \sin \frac{n\pi}{2}$$

$$\therefore x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{n\pi} \sin \frac{n\pi}{2} \right)}_{a_n} \cos n\pi t \quad - 31 -$$

EX: Find and sketch the spectrum up to 3rd harmonic for the previous example.

* What is the spectrum: It is a domain which shows how the signals change with frequency.

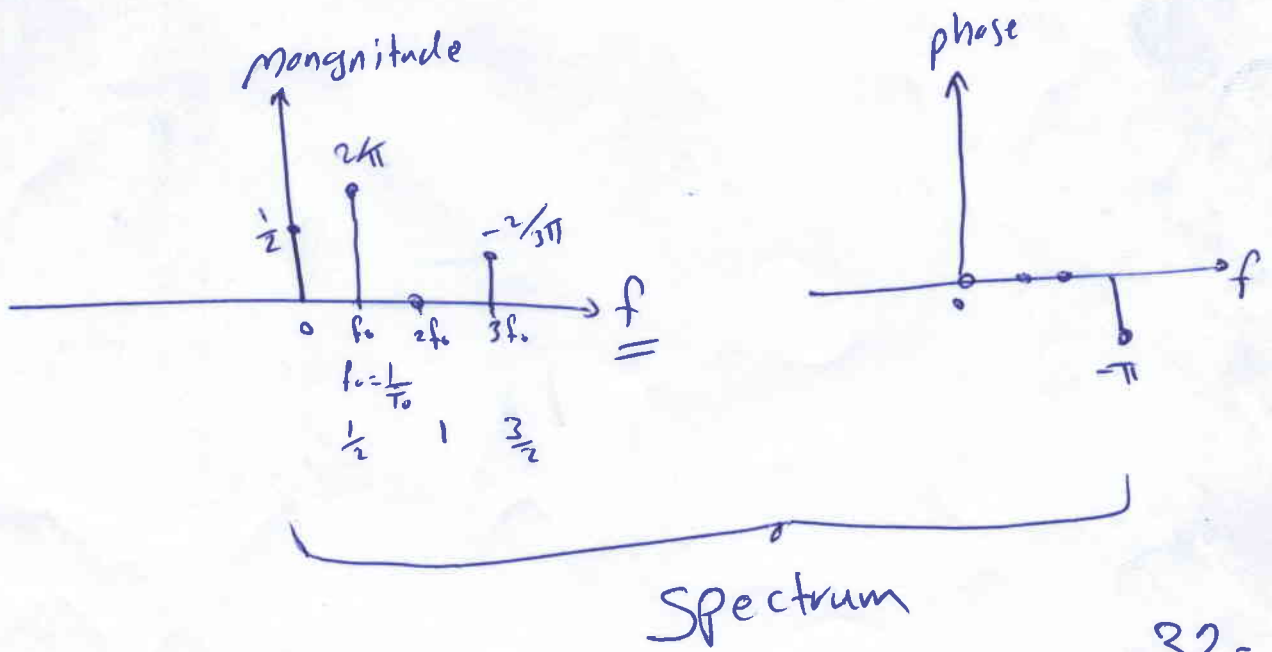
Spectrum has $\begin{cases} \text{Magnitude (Amplitude)} \Rightarrow A_n = \sqrt{a_n^2 + b_n^2} \\ \text{Phase} \Rightarrow \phi_n = -\tan^{-1} \frac{b_n}{a_n} \end{cases}$

Solution (for previous EX $a_0 = \frac{1}{2}, b_n = 0, a_n = \frac{2}{n\pi} \sin \frac{n\pi}{2}$)

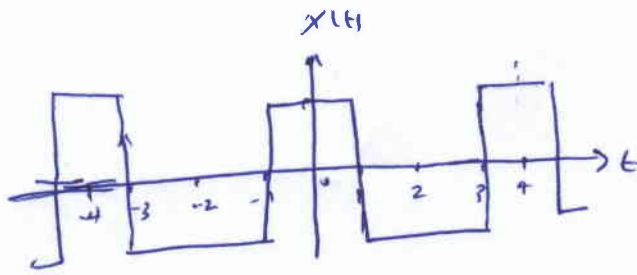
$n=0 \rightarrow a_0 = \frac{1}{2} \text{ \& } b_n = 0 \rightarrow \begin{cases} A_n = \sqrt{(\frac{1}{2})^2 + 0} = \boxed{\frac{1}{2}} \\ \phi_n = -\tan^{-1} \frac{0}{0.5} = \boxed{0} \end{cases}$

n	a_n	b_n	A_n	ϕ_n
Fundamental 1	$\frac{2}{\pi}$	0	$\frac{2}{\pi}$	0
2nd harmonic 2	0	0	0	0
3rd harmonic 3	$-\frac{2}{3\pi}$	0	$\frac{2}{3\pi}$	$-\pi$ (-180°)

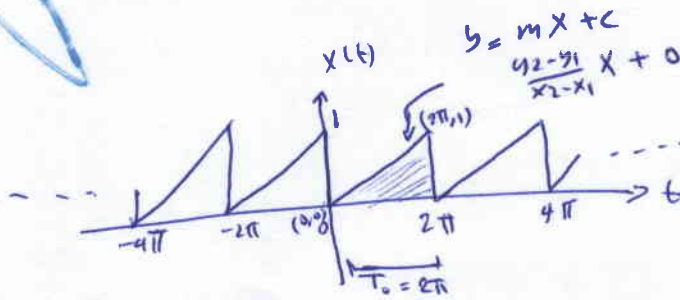
Note: ϕ_n is ϕ_n as per $\frac{b_n}{a_n}$



H.W : find Fourier Series of the following signal



EX 20 find Fourier Series for the following signal



$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{t}{2\pi} dt = \frac{1}{4\pi^2} \left[\frac{t^2}{2} \right]_0^{2\pi}$$

$$= \frac{1}{4\pi^2} (4\pi^2 - 0) = \boxed{\frac{1}{2}}$$

$$y = \frac{1-0}{2\pi-0} \cdot x \Rightarrow y = \frac{x}{2\pi}$$

$$x(t) = \frac{t}{2\pi}$$

$$a_n = \frac{2}{T_0} \int_0^T x(t) \cos n\omega_0 t dt$$

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} \frac{t}{2\pi} \cos nt dt$$

$$a_n = \frac{1}{2\pi^2} \int_0^{2\pi} \frac{t}{2} \cdot \overline{\cos nt} dt$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1 \text{ rad/sec}$$

SA
EX 20

~~Integration by parts~~

$$u dv = uv - \int v du$$

$$a_n = \frac{1}{2\pi^2} \left[t \cdot \frac{\sin nt}{n} - \int \frac{\sin nt}{n} \cdot 1 dt \right]$$

$$a_n = \frac{1}{2\pi^2} \left[t \cdot \frac{\sin t}{n} - \left[-\frac{\cos nt}{n^2} \right] \right]_{0}^{2\pi}$$

$$a_n = \frac{1}{2\pi^2} \left[t \cdot \frac{\sin t}{n} + \frac{\cos nt}{n^2} \right]_{0}^{2\pi}$$

$$a_n = \frac{1}{2\pi^2} \left[\frac{2\pi \sin n2\pi}{n} - \frac{0 \sin n(0)}{n} + \frac{\cos n2\pi}{n^2} - \frac{\cos n(0)}{n^2} \right]$$

$$a_n = \frac{1}{2\pi^2} \left[0 - 0 + \frac{1}{n^2} - \frac{1}{n^2} \right]$$

$$a_n = \frac{1}{2\pi^2} [0]$$

$$a_n = 0$$

$$b_n = -\frac{1}{n\pi}$$

$$So, x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{n\pi} \sin nt$$

$$x(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nt$$

Parseval's theorem

If the Fourier Series function is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

The Parseval's theorem is

$$P_T = \frac{1}{T} \int_0^T |x(t)|^2 dt = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Ex: For the periodic signal $x(t)$ is shown below find the total power & d.c power & percentage of power contained in harmonics up to 3rd.

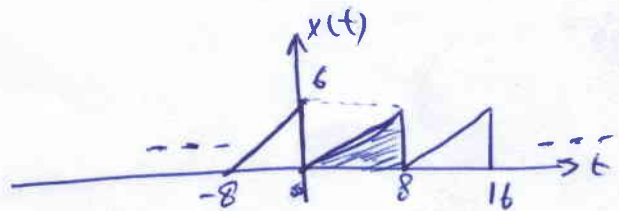
① The total power

$$P_T = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

$$P_T = \frac{1}{8} \int_0^8 \left| \frac{6}{8} t \right|^2 dt$$

$$= \frac{6^2}{8^3} \left[\frac{t^3}{3} \right]_0^8$$

$$= \frac{6^2 \cdot 8^3}{8^3 \cdot 3} = \frac{36}{3} = \boxed{12}$$



$$T = 8$$

$$x(t) = \frac{y_2 - y_1}{x_2 - x_1} t + 0$$

$$x(t) = \frac{6-0}{8-0} t$$

$$\boxed{x(t) = \frac{6}{8} t}$$

$$* P_{dc} = a_0^2$$

$$a_0 = \frac{1}{8} \int_0^8 \frac{6}{8} t \, dt$$

$$a_0 = \frac{6}{8^2} \left[\frac{t^2}{2} \right]_0^8$$

$$a_0 = \frac{6 \times 8^2}{8^2 \times 2} = \frac{6}{2} = \boxed{3}$$

$$\boxed{a_0^2 = 9 = P_{dc}}$$

by: Abdulbasit Saibawi

$$* a_n = \frac{2}{T} \int_0^T x(t) \cos n\omega_0 t \, dt$$

$$= \frac{1}{4} \int_0^8 \frac{6^3}{8^4} \frac{t}{u} \frac{\cos n\omega_0 t}{dv}$$

$$= \frac{3}{4 \times 4} \left[t \cdot \frac{\sin n\omega_0 t}{n\omega_0} - \int \frac{\sin n\omega_0 t}{n\omega_0} \cdot 1 \, dt \right]_0^8$$

$$= \frac{3}{16} \left[t \cdot \frac{\sin n \frac{\pi}{4} t}{\frac{n\pi}{4}} + \frac{\cos n \frac{\pi}{4} t}{\left(\frac{n\pi}{4}\right)^2} \right]_0^8$$

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T} \\ &= \frac{2\pi}{8} \\ &= \frac{\pi}{4} \end{aligned}$$

$$= \frac{3 \times 4}{16 \times n\pi} \left[t \sin n \frac{\pi}{4} t + \frac{\cos n \frac{\pi}{4} t}{\frac{n\pi}{4}} \right]_0^8$$

$$\begin{aligned} \sin(2\pi) &= 0 \\ \sin(0) &= 0 \\ \cos(n\pi) &= 1 \\ \cos(0) &= 1 \end{aligned}$$

$$\boxed{a_n = 0}$$

$$\begin{aligned}
 b_n &= \frac{1 \times 3}{4 \times 4} \int_0^8 \underbrace{t}_u \cdot \underbrace{\sin n \omega_0 t}_{dv} dt \\
 &= \frac{3}{16} \left[-t \cdot \frac{\cos n \omega_0 t}{n \omega_0} + \int \frac{\cos n \omega_0 t}{n \omega_0} dt \right] \\
 &= \frac{3}{16} \left[-t \cdot \frac{\cos n \omega_0 t}{n \omega_0} + \frac{\sin n \omega_0 t}{(n \omega_0)^2} \right] \Bigg|_0^8 \\
 &= \frac{3}{16 \times n \omega_0} \left[-t \cdot \cos n \omega_0 t + \frac{\sin n \omega_0 t}{n \omega_0} \right] \Bigg|_0^8 \\
 &= \frac{3}{16 \times n \times \frac{\pi}{4}} \left[-t \cdot \cos n \frac{\pi}{4} t + \frac{\sin \frac{\pi}{4} t}{\frac{n \pi}{4}} \right] \Bigg|_0^8 \\
 &= \frac{4 \times 3}{16 \times n \times \pi} [-8] = \boxed{-\frac{6}{n \pi}}
 \end{aligned}$$

$$P_n = \frac{1}{2} (a_n^2 + b_n^2)$$

$$= \frac{1}{2} b_n^2$$

$$\left\{ b_n = \frac{-6}{n \pi} \right\}$$

$$P_1 = \frac{1}{2} \left(\frac{-6}{\pi} \right)^2$$

$$P_2 = \frac{1}{2} \left(\frac{-3}{\pi} \right)^2$$

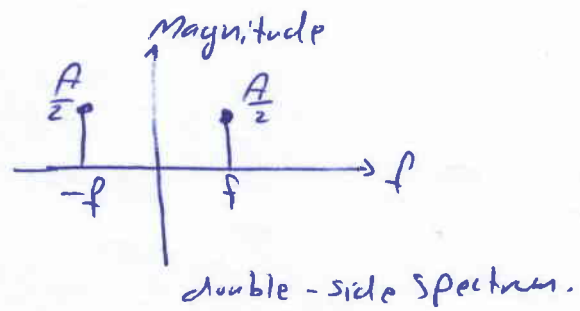
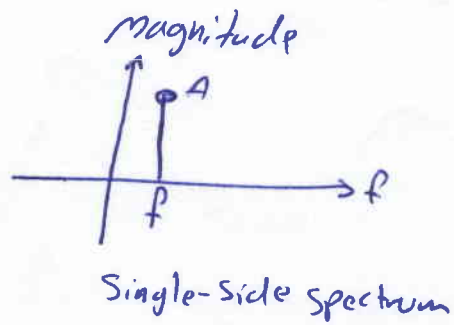
$$P_3 = \frac{1}{2} \left(\frac{-2}{\pi} \right)^2$$

$$P_{\text{up to 3rd}} = P_0 + P_1 + P_2 + P_3 = 11.48$$

$$\% = \frac{P_{\text{up to 3rd}}}{P_T} = \frac{11.48}{12} \times 100 = \boxed{95.6\%}$$

(37)

Ex: Sketch the single-side and double-side spectrum of the signal $x(t) = A \cos(2\pi ft)$



* Complex Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{jn\omega t} + e^{-jn\omega t}}{2} + b_n \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j} \right]$$

* $\boxed{\frac{1}{j} = -j}$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(\frac{1}{2} e^{jn\omega_0 t} (a_n - j b_n) + \frac{1}{2} e^{-jn\omega_0 t} (a_n + j b_n) \right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left(C_n e^{jn\omega_0 t} + C_{-n} e^{-jn\omega_0 t} \right)$$

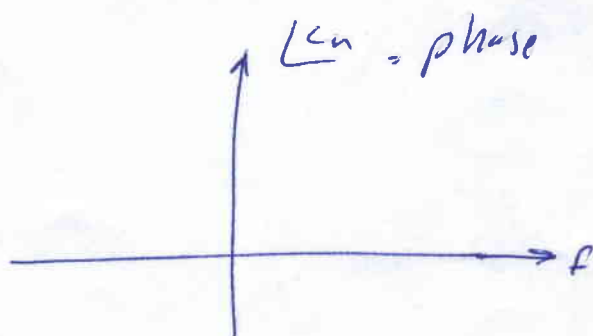
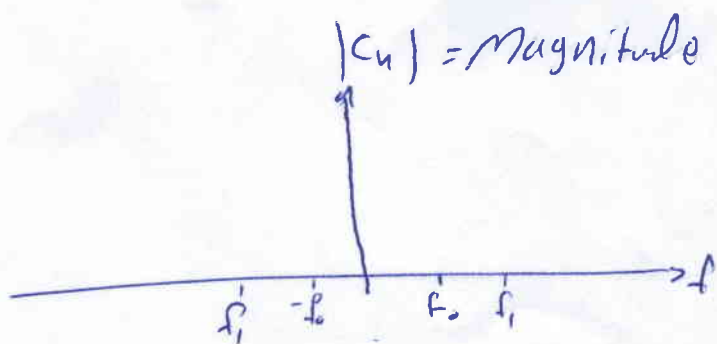
* $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

* $C_0 = a_0 = \frac{1}{T} \int_0^T x(t) dt$

* $C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$

$$C_n = \frac{a_n - j b_n}{2}$$

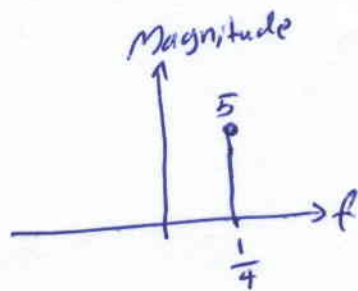
$$C_{-n} = \frac{a_n + j b_n}{2}$$



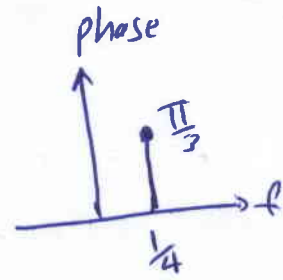
EX: Sketch single side and double side spectrum

$$x(t) = 5 \cos\left(\frac{\pi t}{2} + \frac{\pi}{3}\right)$$

* Single-side



6



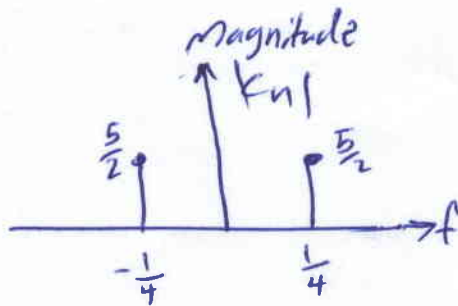
* double-side

$$C_n = \frac{1}{2} (a_n - j b_n)$$

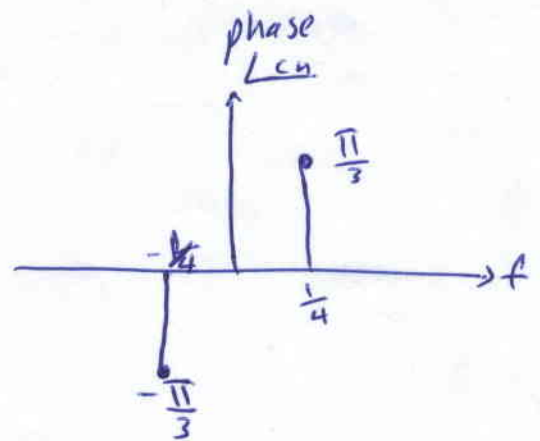
$x(t)$ is Even so $b_n = 0$

$$C_n = \frac{1}{2} a_n$$

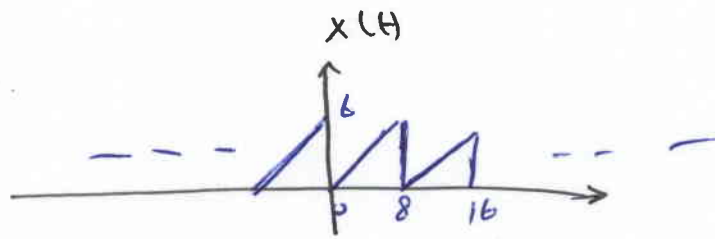
$$C_1 = \frac{a_1}{2} = \frac{5}{2}$$



6



EX: Sketch double-side spectrum of the periodic signal shown:-



$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$T = 8, \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$C_n = \frac{1}{8} \int_0^8 \frac{6}{8}t e^{-jn\frac{\pi}{4}t} dt$$

$$\boxed{u dv = uv - \int v du}$$

$$= \frac{6}{8 \times 8} \left[t \frac{e^{-jn\frac{\pi}{4}t}}{-jn\frac{\pi}{4}} - \int_0^8 \frac{e^{-jn\frac{\pi}{4}t}}{-jn\frac{\pi}{4}} dt \right]$$

$$= \frac{j6 \times 4}{8 \times 8 \times n\pi} \left[t e^{-jn\frac{\pi}{4}t} - \frac{e^{-jn\frac{\pi}{4}t}}{-jn\frac{\pi}{4}} \right]_0^8$$

$$= \frac{j6 \times 4}{8 \times 8 \times n\pi} \left[8 \frac{e^{-jn2\pi}}{e^{-j2\pi n}} + \frac{e^{-j2n\pi}}{\frac{jn\pi}{4}} - \frac{1}{jn\frac{\pi}{4}} \right]$$

$e^{-j2\pi n} = \cos 2\pi n - j \sin 2\pi n = 1$

$$C_n = \frac{8 \times j6 \times 4}{8 \times 8 \times n\pi} = \boxed{\frac{j3}{n\pi}}$$

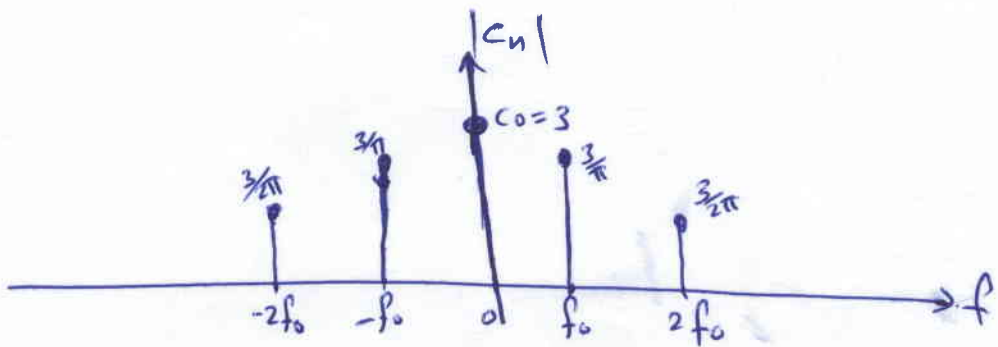
$$C_0 = \frac{1}{T} \int_0^8 x(t) dt = \boxed{3}$$

$$C_n = \frac{j3}{n\pi} \quad \text{So, the magnitude of } |C_n| = \sqrt{(Re)^2 + (Im)^2}$$

$$|C_n| = \sqrt{(0)^2 + \left(\frac{3}{n\pi}\right)^2} \Rightarrow |C_n| = \frac{3}{n\pi}$$

$n = \text{integer}, 1, 2, 3, \dots$

$$C_1 = \frac{3}{\pi} \quad \& \quad C_2 = \frac{3}{2\pi} \quad \& \quad \text{and so on}$$



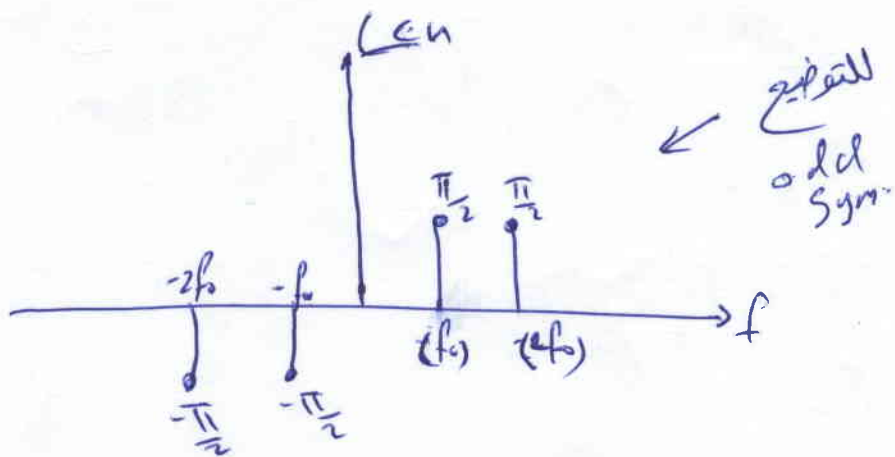
We can notice that the magnitude \rightarrow **Even Symmetry**
 and the phase $\angle C_n$ is **odd Symmetry**

* notice that

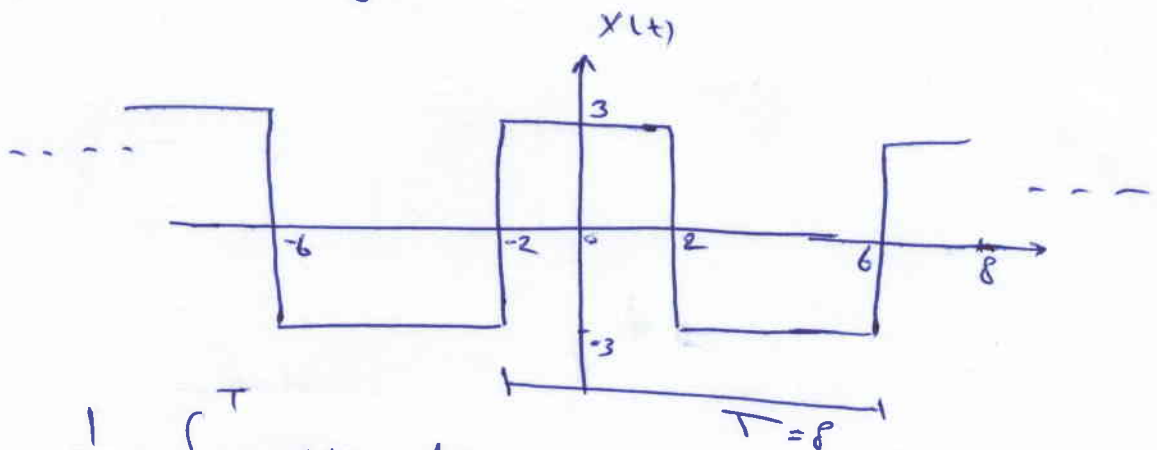
$$C_n = j \frac{3}{n\pi}$$

$$j = e^{j\frac{\pi}{2}}$$

$$\angle C_n = \frac{\pi}{2}$$



EX: Use complex Fourier Series to find the spectrum of the following signal.



$$C_0 = \frac{1}{T} \int_0^T x(t) dt$$

[Handwritten signature]

$$T = 8 \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$C_0 = \frac{1}{8} \left[\int_{-2}^2 3 dt + \int_2^6 -3 dt \right]$$

$$C_0 = \frac{3}{8} \left[t \Big|_{-2}^2 + -t \Big|_2^6 \right]$$

$$= \frac{3}{8} [2 + 2 - 6 + 2] = \boxed{0}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$C_n = \frac{1}{8} \left[\int_{-2}^2 3 e^{-jn\omega_0 t} dt + \int_2^6 -3 e^{-jn\omega_0 t} dt \right]$$

$$\omega_0 = \frac{\pi}{4}$$

$$C_n = \frac{3}{8} \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \Big|_{-2}^2 + \frac{-e^{-jn\omega_0 t}}{-jn\omega_0} \Big|_2^6 \right]$$

$$= \frac{3}{-jn\omega_0 \times 8} \left[e^{-j2n\omega_0} - e^{j2n\omega_0} - e^{-j6n\omega_0} + e^{-j2n\omega_0} \right]$$

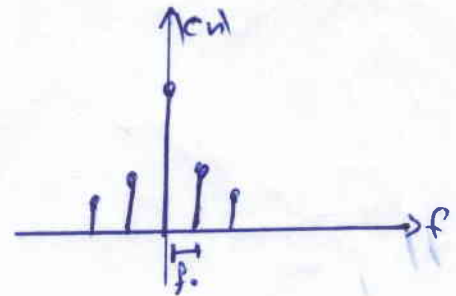
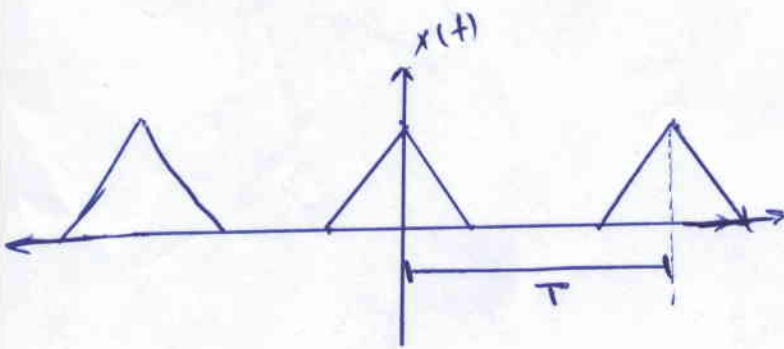
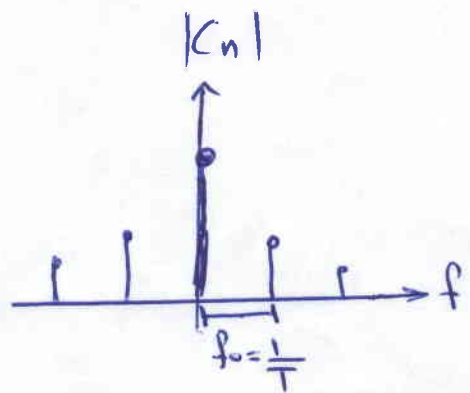
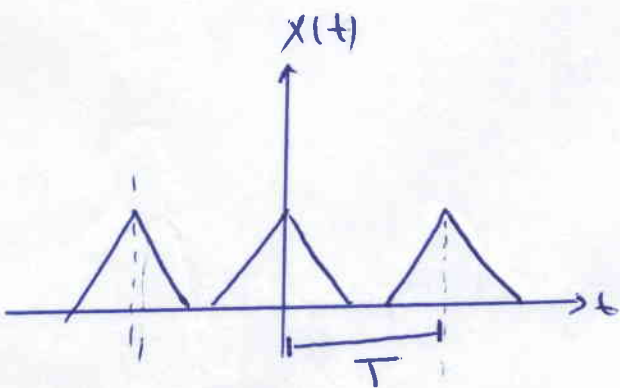
$$= \frac{3}{-j2n\pi} \left[\underline{e^{-jn\frac{\pi}{2}}} - e^{jn\frac{\pi}{2}} - e^{-jn\frac{3\pi}{2}} + \underline{e^{jn\frac{\pi}{2}}} \right]$$

* Fourier Transforms ::

Let $x(t)$ be periodic signal of period T

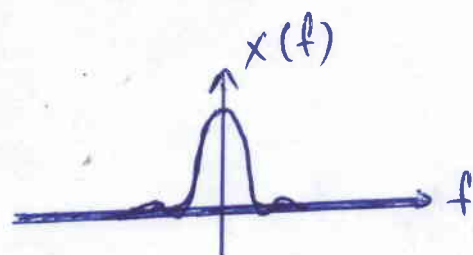
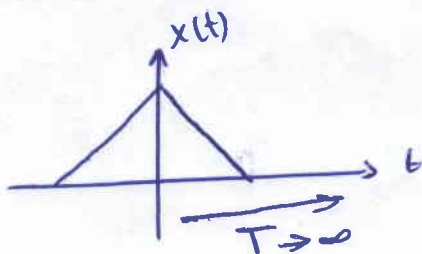
$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$$

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi n f_0 t} dt$$



We can noticed that if (T) becomes larger, the frequency (f) becomes closer

as $T \rightarrow \infty$ (non-periodic signal) $n f_0 \rightarrow f$



as $T \rightarrow \infty$

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi f t} dt$$

$$T \cdot c_n = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Fourier
Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{+j2\pi f t} df$$

Inverse
Fourier
Transform

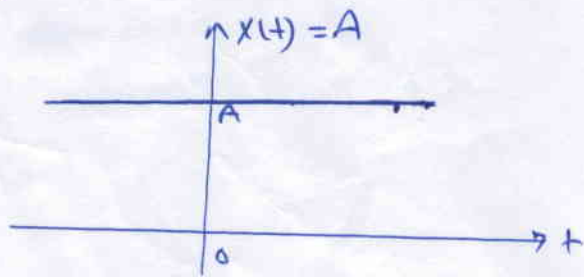
Q/ why do we care about Fourier Transform (FT) ?

Sol/ physically, it will tell us the frequency components of any non-periodic signal and many signals are composed of many different frequencies. In other words, Fourier Transform is mathematical tool used for frequency analysis of signals.

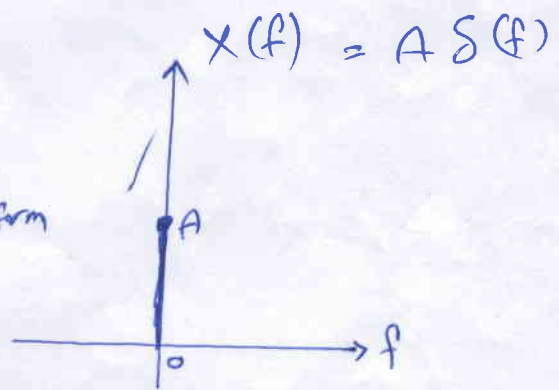
* Determine the Fourier Transform of the following

1. DC Signal
2. Cosine Signal
3. Rectangular pulse
4. Multiplication
5. Convolution
6. Periodic Signal (e.g.: rectangular pulses)
7. Sampled Signal
8. Periodic and Sampled Signal

① DC Signal

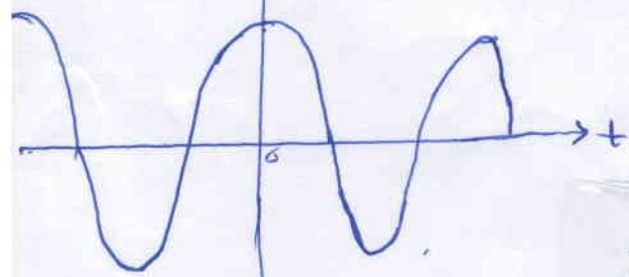


FT
Fourier Transform

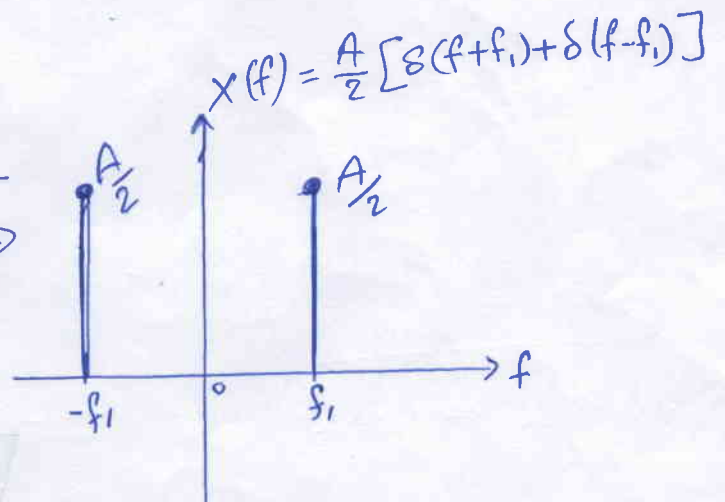


② Cosine Signal

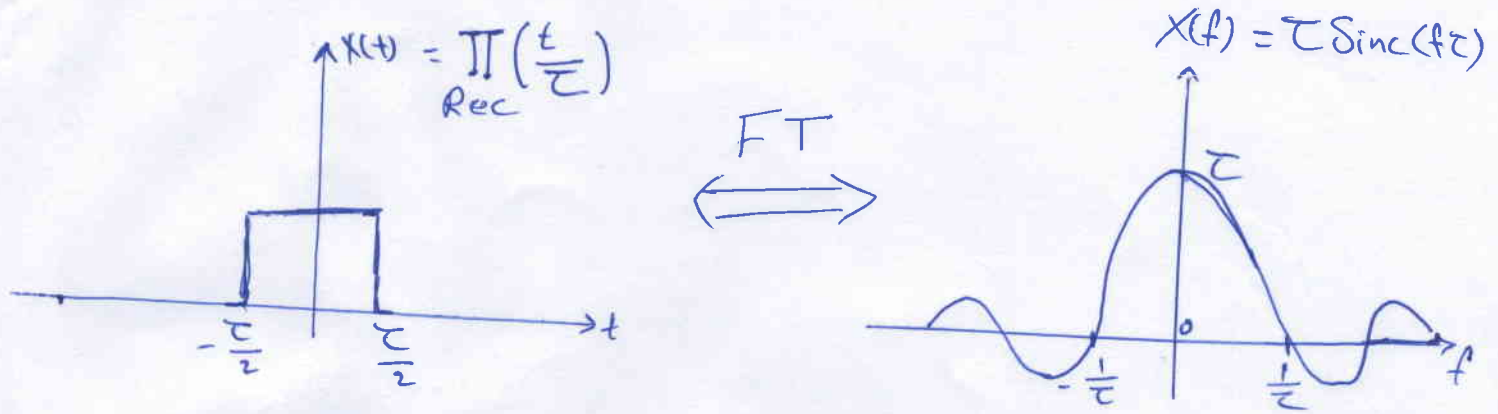
$$x(t) = A \cos(2\pi f_1 t)$$



FT



3) - Rectangular pulse



4) - Multiplication

Let $X(f) = FT[x(t)]$, $Y(f) = FT[y(t)]$

$$FT[x(t) \cdot y(t)] = X(f) \otimes Y(f)$$

\uparrow Multiplication in Time domain = Convolution in Freq. domain
 \uparrow

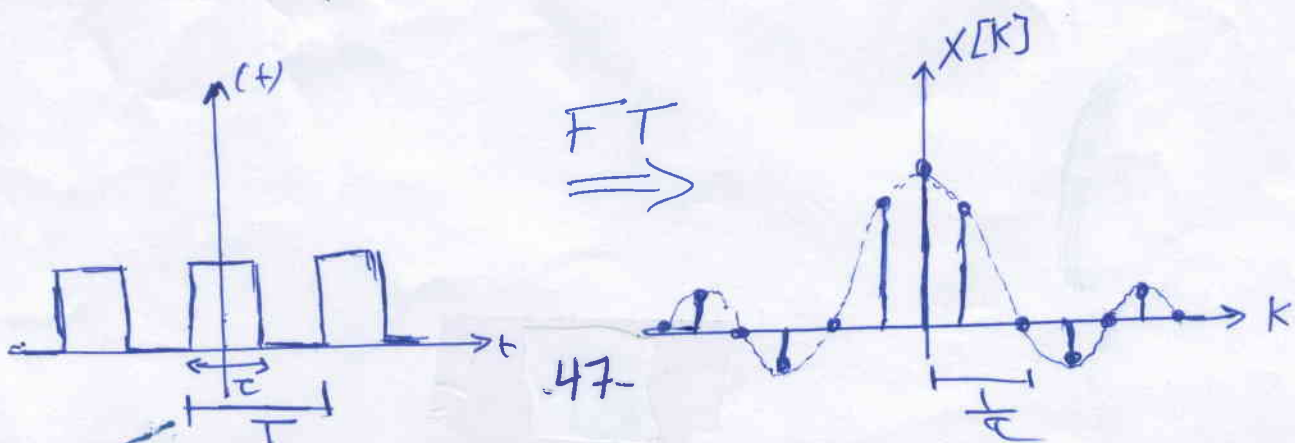
5) - Convolution

Let $X(f) = FT[x(t)]$, $Y(f) = FT[y(t)]$

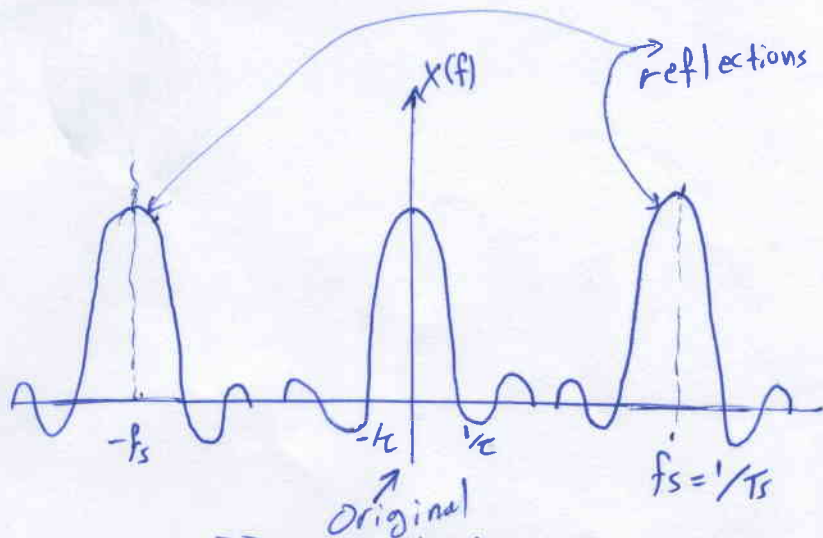
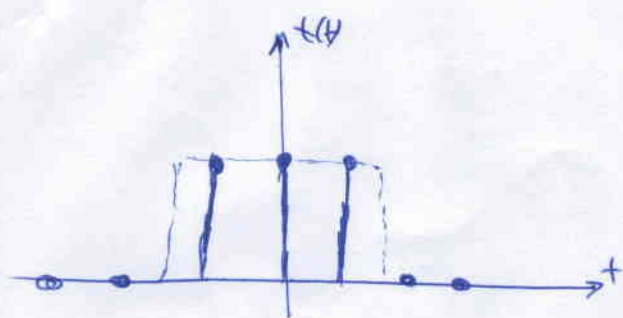
$$FT[x(t) \otimes y(t)] = X(f) \cdot Y(f)$$

\uparrow Convolution in Time domain = Multiplication in Freq. domain
 \uparrow

6) Periodic Signal



7) - Sampled Signal



$X(f)$ is periodic i.e. original spectrum is repeated every f_s and these are known as "Reflections".

8) H.W

* OBSERVATIONS *

Time domain

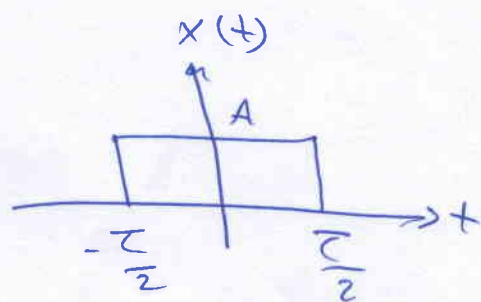
1. periodic
2. Discrete
3. periodic + Discrete
4. Non-periodic
5. Non-discrete (continuous)

Freq. Domain

- Discrete
 Periodic
 Discrete + periodic
 Non-discrete (continuous)
 Non-periodic

Ex: Find Fourier transform of the signal below.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$



$$= \int_{-\tau/2}^{\tau/2} A e^{-j2\pi ft} dt$$

$$= A \left[\frac{e^{-j2\pi ft}}{-j2\pi f} \right]_{-\tau/2}^{\tau/2}$$

$$= A \left[e^{-j2\pi f \frac{\tau}{2}} - e^{+j2\pi f \frac{\tau}{2}} \right]$$

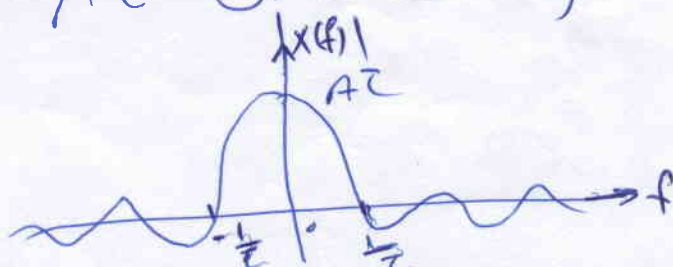
$$= \frac{A}{-j\pi f} \left[\frac{e^{-j\pi f \tau} - e^{j\pi f \tau}}{2j} \right]$$

$$= \frac{A}{\pi f} \left[\frac{e^{j\pi f \tau} - e^{-j\pi f \tau}}{2j} \right]$$

$$= \frac{A}{\pi f} \sin \pi f \tau$$

$$= A \tau \frac{\sin(\pi f \tau)}{\pi f \tau}$$

$$= A \tau \text{Sinc}(\pi f \tau)$$



Hint:
 $\text{Sinc}(x) = \frac{\sin(x)}{x}$
 $\text{Sinc}(\pi x) = \frac{\sin(\pi x)}{\pi x}$

EX: Find Fourier transform of $x(t) = e^{-at} u(t)$, $a > 0$
 then sketch the spectrum.

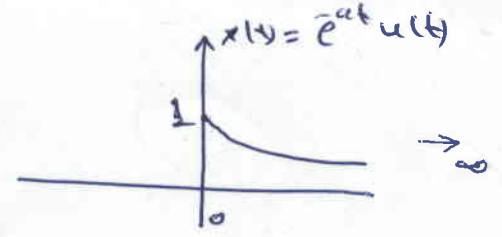
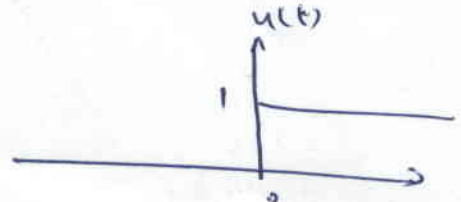
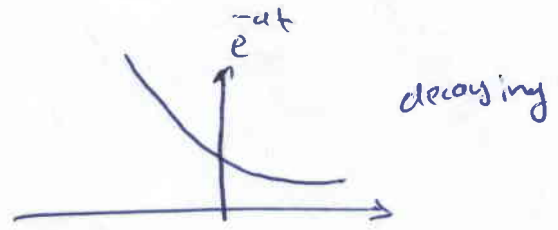
Fourier

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$= \int_0^{\infty} e^{-at} * 1 * e^{-j2\pi ft} dt$$

$$= \int_0^{\infty} e^{-(a + j2\pi f)t} dt$$

$$= \frac{e^{-(a + j2\pi f)t}}{-(a + j2\pi f)} \Big|_0^{\infty}$$



hint: $e^{-\infty} = 0$
 $e^0 = 1$

$$X(f) = \frac{e^{-\infty}}{-(a + j2\pi f)} - \frac{e^0}{-(a + j2\pi f)}$$

Signature

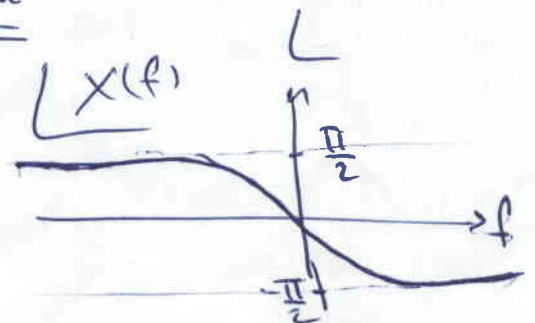
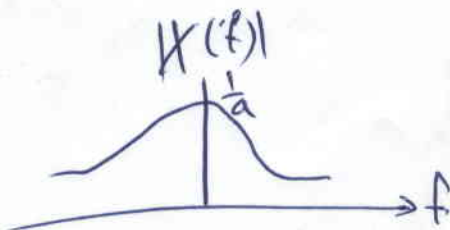
$$X(f) = \frac{1}{a + j2\pi f} = \frac{1}{a + j\omega}$$

Magnitude

$$|X(f)| = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}}$$

Phase

$$\angle X(f) = -\tan^{-1} \frac{2\pi f}{a}$$



EX: Find Fourier Transform of $x(t) = \delta(t)$, then sketch its spectrum.

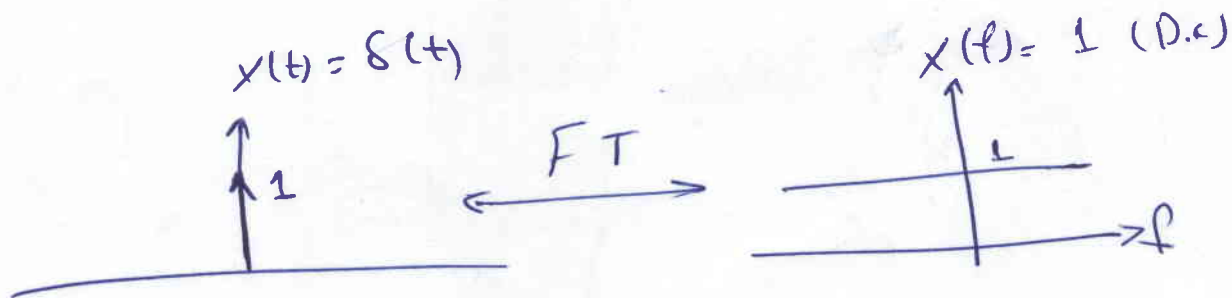
$$x(t) = \delta(t)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$X(f) = \int_0^{\infty} \delta(t) e^{\frac{-j\omega(0)}{1}} dt$$

$$X(f) = 1 \int_0^{\infty} \delta(t) dt$$
$$= 1 \quad (\text{D.C.})$$



EX: Find Fourier Transform of $x(t) = \delta(t-a)$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t-a) e^{-j2\pi ft} dt$$

$$X(f) = e^{-j2\pi fa}$$

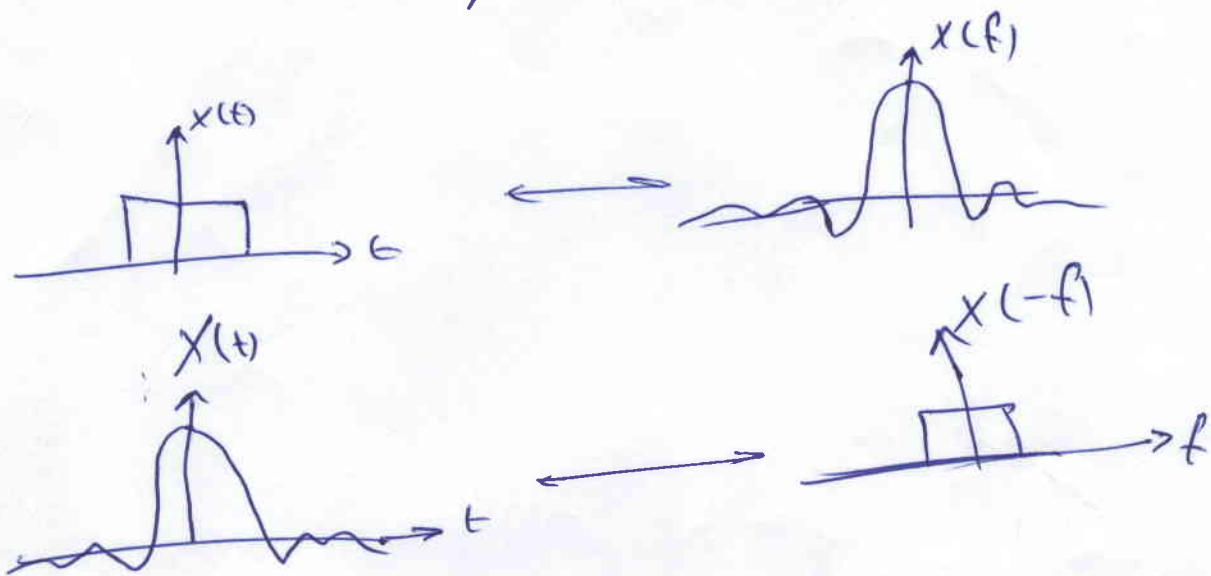
So, the Fourier transform of shifted impulse is a complex exponential.

* Properties of Fourier Transform ::

1- Linearity :: if $x(t) \rightarrow X(f)$
and $y(t) \rightarrow Y(f)$
then $ax(t) + by(t) \leftrightarrow aX(f) + bY(f)$

2- Scaling :: if $x(t) \leftrightarrow X(f)$
then $x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

3- Duality : if $x(t) \leftrightarrow X(f)$
 then $X(t) \leftrightarrow x(-f)$



4- Time shift : if $x(t) \leftrightarrow X(f)$
 then $x(t-t_0) \leftrightarrow X(f) e^{-j2\pi f t_0}$

5- Frequency shift : - if $x(t) \leftrightarrow X(f)$
 $x(t) e^{j2\pi f_0 t} \leftrightarrow X(f-f_0)$

6- Time differentiation : if $x(t) \leftrightarrow X(f)$
 then $\frac{d^n x(t)}{dt^n} \leftrightarrow (j2\pi f)^n X(f)$

7. Multiplication and Convolution properties

$$\begin{aligned} \text{if } f_1(t) &\xrightarrow{FT} F_1(f) \\ f_2(t) &\xrightarrow{FT} F_2(f) \end{aligned}$$

* Then multiplication property states that

$$x_1(t) \cdot f_2(t) \xrightarrow{F.T} X_1(f) \otimes X_2(f)$$

* Convolution property states that

$$x_1(t) \overset{\text{conv}}{\otimes} x_2(t) \longleftrightarrow X_1(f) \cdot X_2(f)$$

Examples: find fourier transform of

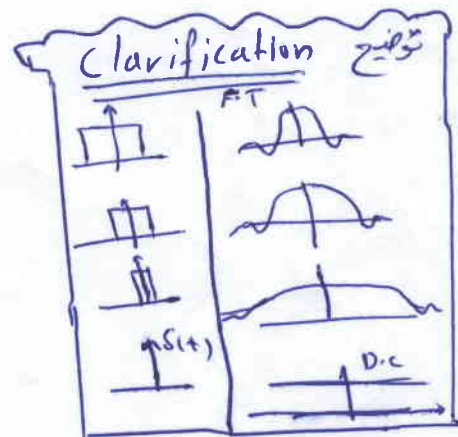
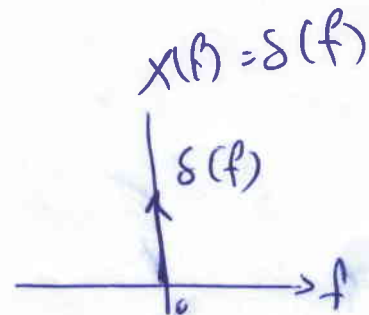
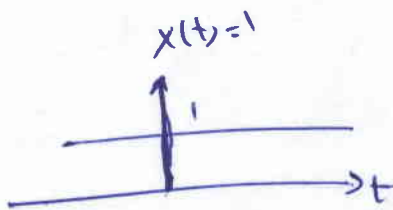
① $x(t) = 1$

$$\delta(t) \longleftrightarrow 1$$

Duality

$$1 \longleftrightarrow \delta(-f) = \delta(f)$$

$$X(t) \longrightarrow x(-f)$$



② - $3\delta(t-2)$

$\delta(t) \leftrightarrow 1$

$3\delta(t) \leftrightarrow 3 * 1 = 3$ (Linearity)

$3\delta(t-2) \leftrightarrow 3 e^{-j2\pi f 2}$ ← t_0

$x(t-t_0) \leftrightarrow X(f) e^{-j2\pi f t_0}$ time-shift

$3\delta(t-2) \rightarrow 3 e^{-j4\pi f}$

③ - e^{j6t}

$e^{j\omega_0 t} = 1 \cdot e^{j\omega_0 t} = 1 \cdot e^{j2\pi f_0 t}$

$\omega_0 = 2\pi f_0$
 $f_0 = \frac{\omega_0}{2\pi}$

$1 \leftrightarrow \delta(t)$

$x(t) e^{j2\pi f_0 t} \leftrightarrow X(f-f_0)$ freq. shift

$1 \cdot e^{j2\pi f_0 t} \leftrightarrow \delta(f - \frac{\omega_0}{2\pi})$

So if we have e^{j6t} ?? Same above ↑ ∞

$\omega = 6$
 $\omega = 2\pi f_0$
 $f_0 = \frac{6}{2\pi}$

$1 \cdot e^{j6t} \leftrightarrow \delta(f - f_0)$
 $\delta(f - \frac{6}{2\pi})$
 $\delta(f - \frac{3}{\pi})$

④ - $5 \text{rect}\left(\frac{3t}{2}\right)$

$\therefore f(t) = 5 \text{rect}\left(\frac{3t}{2}\right)$

$$\boxed{A \text{rect} \frac{t}{T} \longleftrightarrow AT \text{Sinc} f T}$$

Let $A=5$ $T=1$ and $\boxed{\frac{3}{2}}$ is time scaling

$\therefore 5 \text{rect} \frac{3}{2}t \longleftrightarrow \frac{5}{3/2} \text{Sinc} \left(\frac{f}{3/2}\right)$

$X(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$ Scaling property of FT.

5) $\cos \omega t$

$$\cos \omega t = \frac{1}{2} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

$$= \frac{1}{2} e^{j2\pi f_0 t} + \frac{1}{2} e^{-j2\pi f_0 t}$$

$$= \frac{1}{2} [\delta(f-f_0) + \delta(f+f_0)]$$

$$1 \longleftrightarrow \delta(t)$$

$$\frac{1}{2} * 1 \longleftrightarrow \frac{1}{2} \delta(t)$$

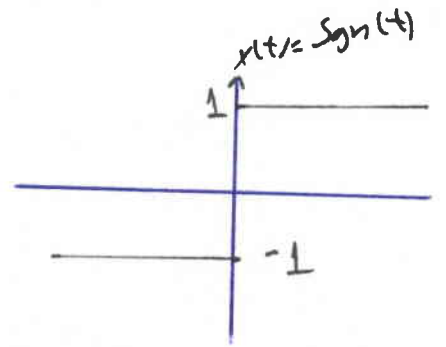
$$\frac{1}{2} \delta(t) e^{j2\pi f_0 t} \longleftrightarrow \frac{1}{2} \delta(f-f_0)$$

$$\frac{1}{2} \delta(t) e^{-j2\pi f_0 t} \longleftrightarrow \frac{1}{2} \delta(f+f_0)$$

$$\boxed{H.\omega = F.T [\text{Sin} \omega t]}$$

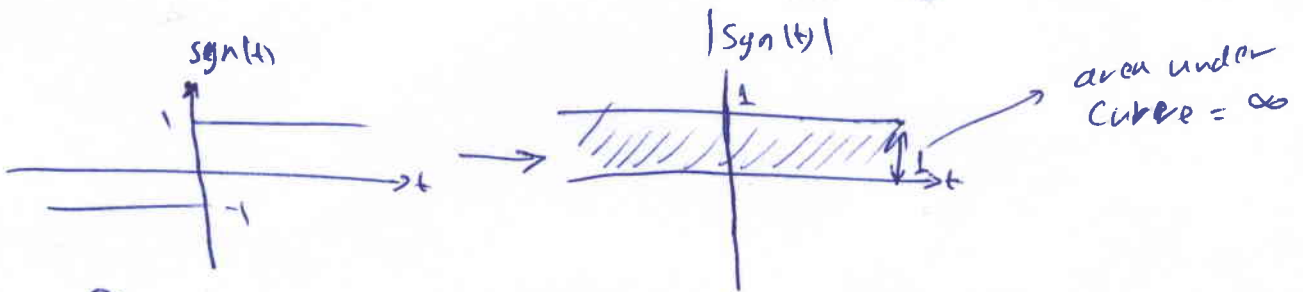
6) Signum function $\text{Sgn}(t) :$

$$\text{Sgn}(t) = \begin{cases} -1 & , t < 0 \\ 0 & , t = 0 \\ 1 & , t > 0 \end{cases}$$

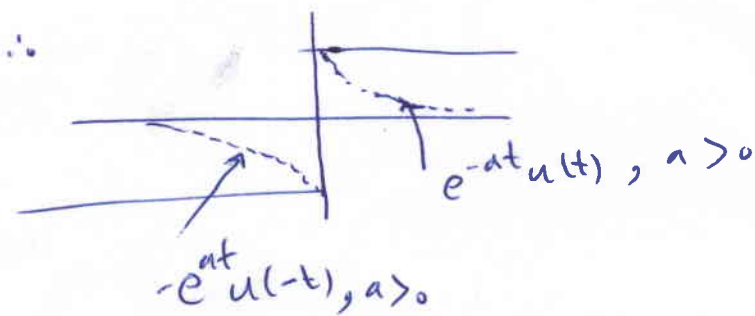


لايجاد تكامل اي اشارة (بعد قابلية للتكامل ام لا) يجب ان توضع للشروط

$$\int_{-\infty}^{\infty} |\text{Sgn}(t)| dt < \infty \text{ (finite)}$$



$$\therefore \int_{-\infty}^{\infty} |\text{Sgn}(t)| dt = 1 \times \infty = \infty$$



$$u(t) = \lim_{a \rightarrow 0} e^{-at} u(t)$$

$$u(-t) = \lim_{a \rightarrow 0} e^{at} u(-t)$$

$$\text{Sgn}(t) = u(t) - u(-t)$$

$$x(t) = \lim_{a \rightarrow 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

$$X(\omega) = \lim_{a \rightarrow 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right]$$

$$= \lim_{a \rightarrow 0} \frac{a-j\omega - a-j\omega}{a^2 - j^2\omega^2}$$

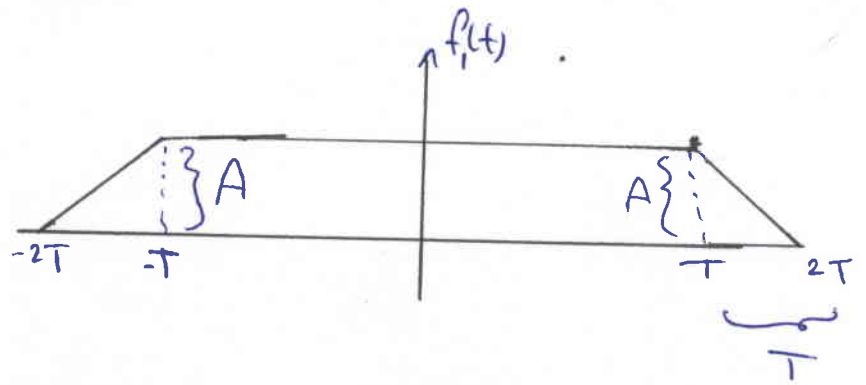
$$= \lim_{a \rightarrow 0} \frac{-2j\omega}{a^2 + \omega^2} = \frac{-2j\omega}{\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

Summary

$$\text{Sgn}(t) \iff \frac{2}{j\omega}$$

8) Hint

find $f_1'(t)$ then $F_1(f)$



9) $Z(t) = \underbrace{e^{-t} u(t)}_{Z_1(t)} \otimes^{\text{conv}} \underbrace{e^{-2t} u(t)}_{Z_2(t)}$

$$Z_1(f) \longleftrightarrow \frac{1}{1 + j2\pi f}$$

$$Z_2(f) \longrightarrow \frac{1}{2 + j2\pi f}$$

$$\begin{aligned} \text{F.T} [Z_1(t) \otimes Z_2(t)] &= Z_1(f) \cdot Z_2(f) \quad \leftarrow \text{multiplication} \\ &= \frac{1}{1 + j2\pi f} \cdot \frac{1}{2 + j2\pi f} \end{aligned}$$

10) $\text{F.T} [\cos^2 \omega_0 t]$

$$\cos^2 \omega_0 t = \text{F.T} \left[\frac{1}{2} + \frac{\cos 2\omega_0 t}{2} \right]$$

$$= \frac{1}{2} \delta(f) + k \quad \text{where } k = \frac{\cos 2\omega_0 t}{2}$$

$$\text{F.T} \left[\frac{\cos 2\omega_0 t}{2} \right] = \text{F.T} \left[\frac{e^{+j4\pi f_0 t} + e^{-j4\pi f_0 t}}{4} \right]$$

$$= \frac{1}{2} \delta(f) + \frac{1}{4} \delta(f - 2f_0) + \frac{1}{4} \delta(f + 2f_0)$$

Ex: Find Fourier Transform of $u(t)$

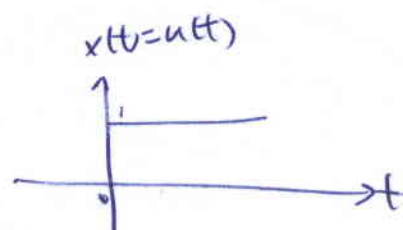
$$x(t) = u(t)$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$X(f) = \frac{e^{-j2\pi ft}}{-j2\pi f} \Big|_0^{\infty}$$

$$= \frac{1}{-j2\pi f} (e^{-j\infty} - e^{-j0})$$

$$e^{-j\infty} = \infty$$



So it cannot be continue in FT Integration way.

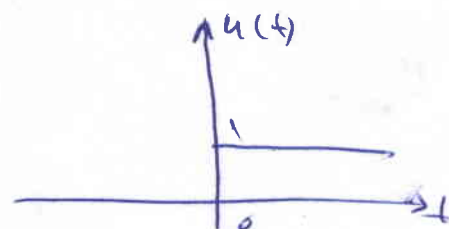
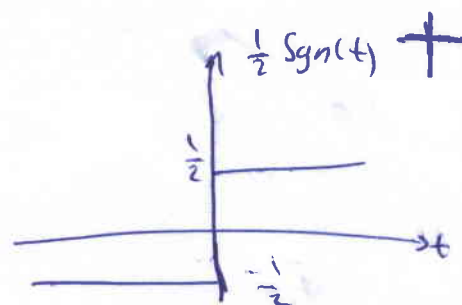
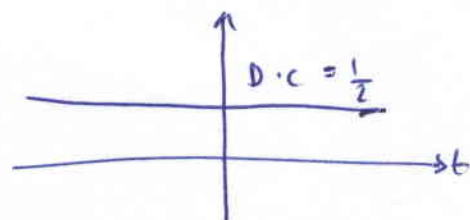
$$x(t) = u(t)$$

$$x(t) = \frac{1}{2} + \frac{1}{2} \text{Sgn}(t)$$



$$X(f) = \frac{1}{2} \delta(f) + \frac{1}{j2\pi f}$$

$$\boxed{u(t) \longleftrightarrow \frac{1}{2} \delta(f) + \frac{1}{j2\pi f}}$$



Laplace Transforms :-

$$\begin{array}{ccc}
 x(t) & \xrightarrow{\boxed{h(t)}} & y(t) \\
 \downarrow \text{(FT)} & & \downarrow \text{(FT)} \\
 \text{Fourier Transform} & & \text{Fourier Transform} \\
 & & y(t) = x(t) \otimes h(t) \\
 & & Y(f) = X(f) \cdot H(f) \\
 & & Y(s) = X(s) \cdot H(s)
 \end{array}$$

The process of transforming the signal from time-domain $x(t)$ to s -domain is called Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad (\text{Bilateral Laplace Transform})$$

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt \quad (\text{Unilateral Laplace Transform})$$

Laplace Transform is general case of Fourier Transform

where $s = \sigma + j\omega$ (complex number)

when $\sigma = 0 \rightarrow \boxed{s = j\omega}$

* Symbols

$$\begin{array}{ccc}
 \int x(t) & \xrightarrow{\text{Laplace } x(t)} & x(t) \rightarrow X(s) \\
 \int X(s) & \xrightarrow{\text{Laplace Inverse}} &
 \end{array}$$

- 60 -

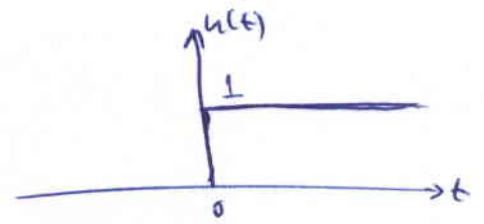
EX1: Find Laplace Transform of $x(t) = u(t)$

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

$$= \int_0^{\infty} 1 \cdot e^{-st} dt$$

$$= \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s}$$

$$= -\frac{1}{-s} = \frac{1}{s}$$



$$\mathcal{L}\{u(t)\} \longleftrightarrow \frac{1}{s}$$

EX2: Find Laplace Transform of $x(t) = e^{at} u(t)$ a : constant

$$x(t) = e^{at} u(t)$$

$$X(s) = \int_0^{\infty} e^{at} e^{-st} dt$$

$$X(s) = \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^{\infty}$$

$$= 0 - \frac{1}{-(s-a)} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at} u(t)\} = \frac{1}{s-a}$$

$$\sin at u(t) \longleftrightarrow \frac{a}{s^2 + a^2}$$

$$\cos at u(t) \longleftrightarrow \frac{s}{s^2 + a^2}$$

EX3: Find Laplace inverse of $X(s) = \frac{1}{(s-a)(s-b)}$

$$\frac{1}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$$

$$1 = A(s-b) + B(s-a)$$

when $s = b \rightarrow A \cancel{(s-b)}^{\rightarrow 0}$

$$1 = B(b-a) \rightarrow B = \frac{1}{b-a}$$

when $s = a \rightarrow B \cancel{(s-a)}^{\rightarrow 0}$

$$1 = A(a-b) \rightarrow A = \frac{1}{a-b}$$

$$= \frac{1}{a-b} \left[\frac{1}{s-a} - \frac{1}{s-b} \right]$$

$$= \frac{1}{a-b} \left[e^{at} u(t) - e^{bt} u(t) \right]$$

-62-

* Laplace Transform Properties :


د. محمد العبد

1 - Linearity

$$\text{if } x(t) \leftrightarrow X(s) \text{ \& } y(t) \leftrightarrow Y(s)$$

$$\text{then } a x(t) + b y(t) \leftrightarrow a X(s) + b Y(s)$$

2 - Scaling

$$\text{if } x(t) \leftrightarrow X(s)$$

$$\text{then } x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$$

3 - Time delay

$$\text{if } x(t) \leftrightarrow X(s)$$

$$\text{then } x(t-t_0) \leftrightarrow X(s) e^{-s t_0}$$

4 - Shift (shifting)

$$\text{if } x(t) \leftrightarrow X(s)$$

$$e^{-at} x(t) \leftrightarrow X(s+a)$$

5 - Multiplication by t :

$$t x(t) \leftrightarrow - \frac{dX(s)}{ds}$$

$$t^n x(t) \leftrightarrow (-1)^n \frac{d^n X(s)}{ds^n}$$

6- Differentiation

$$\text{if } x(t) \longleftrightarrow X(s)$$

$$\frac{d^1 x(t)}{dt} \longleftrightarrow sX(s) - x(0)$$

$$\frac{d^2 x(t)}{dt^2} \longleftrightarrow s^2 X(s) - sx(0) - x'(0)$$

$$\frac{d^3 x(t)}{dt^3} \longleftrightarrow s^3 X(s) - s^2 x(0) - sx'(0) - x''(0)$$

7- Convolution & Multiplication

$$x(t) \otimes y(t) \longleftrightarrow X(s) \cdot Y(s)$$

$$x(t) \cdot y(t) \longleftrightarrow X(s) \otimes Y(s)$$

8- Initial value

$$\lim_{s \rightarrow \infty} sX(s) = \lim_{t \rightarrow 0^+} x(t)$$

by: Abdel Hossit Saadani

9- Final Value

$$\lim_{s \rightarrow 0} sX(s) = \lim_{t \rightarrow \infty} x(t)$$

Examples: find Laplace Transform of the following signals.

① $t u(t)$

$$u(t) \leftrightarrow \frac{1}{s}$$

$$t u(t) \leftrightarrow -\frac{d}{ds} \left(\frac{1}{s} \right) = \frac{1}{s^2}$$

② $t^2 u(t)$

$$t u(t) \leftrightarrow \frac{1}{s^2}$$

$$t^2 u(t) \leftrightarrow -\frac{d}{ds} \left(\frac{1}{s^2} \right) = \frac{2}{s^3}$$

In general $t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$
--

③ $t^2 e^{-2t} u(t)$

$$t^2 u(t) \leftrightarrow \frac{2}{s^3}$$

$$e^{-2t} t^2 u(t) \leftrightarrow \frac{2}{(s+2)^3}$$

④ $t \cos 2t u(t)$

$$\cos 2t u(t) \leftrightarrow \frac{s}{s^2+4}$$

$$t \cos 2t u(t) \leftrightarrow -\frac{d}{ds} \left(\frac{s}{s^2+4} \right)$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

5)

$$x(t) = \begin{array}{c} \text{graph of } x(t) \\ \text{a rectangular pulse from } t=0 \text{ to } t=2 \text{ with height } 1 \end{array}$$

$$x(t) = u(t) - u(t-2)$$

$$\downarrow \qquad \qquad \downarrow$$

$$\frac{1}{s} - \frac{1}{s} e^{-2s}$$

$$b) e^{-at} \sin \omega_0 t u(t)$$

$$\sin \omega_0 t u(t) \longleftrightarrow \frac{\omega_0}{s^2 + \omega_0^2}$$

$$e^{-at} \sin \omega_0 t u(t) \longleftrightarrow \frac{\omega_0}{(s+a)^2 + \omega_0^2}$$

$$\text{EX: Find } \mathcal{L}^{-1} \frac{e^{-3s}}{s^3}$$

$$(\mathcal{L}^{-1} = \text{laplace inverse})$$

$$t^2 u(t) \longleftrightarrow \frac{2}{s^3}$$

$$\frac{1}{s^3} \longleftrightarrow \frac{1}{2} t^2 u(t)$$

$$\frac{1}{s^3} e^{-3s} \longleftrightarrow \frac{1}{2} (t-3)^2 u(t-3)$$

Example: Find The initial and Final values for

$$X(s) = \frac{1}{s(s+1)}$$

$$\begin{aligned} * \text{Initial value} &= \lim_{s \rightarrow \infty} s X(s) \\ &= \lim_{s \rightarrow \infty} \frac{s}{s(s+1)} \\ &= \frac{1}{\infty+1} = \boxed{0} \end{aligned}$$

OR: $x(t) = \mathcal{L}^{-1} X(s)$

$$\begin{aligned} &= \mathcal{L}^{-1} \frac{1}{s(s+1)} \\ &= \mathcal{L}^{-1} \left(\frac{A}{s} + \frac{B}{s+1} \right) \\ &= \mathcal{L}^{-1} \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \\ &= \mathcal{L}^{-1} \left[\frac{1}{s(s+1)} = \frac{A(s+1) + Bs}{s(s+1)} \right] \end{aligned}$$

$$A(s+1) + Bs = 1$$

$$\text{when } s=0 \rightarrow A=1$$

$$s=-1 \rightarrow B=-1$$

$$\mathcal{L}^{-1} \left(\frac{1}{s} - \frac{1}{s+1} \right) = u(t) - e^{-t} u(t)$$

$$\begin{aligned} \text{initial value} &= \lim_{t \rightarrow 0^+} x(t) = u(0^+) - e^{-0} u(0) \\ &= 1 - 1 = \boxed{0} \end{aligned}$$

* Final value

$$\begin{aligned}\text{final value} &= \lim_{s \rightarrow 0} s X(s) \\ &= \frac{s}{s(s+1)} = \frac{1}{0+1} = \boxed{1}\end{aligned}$$

OR: $\text{final value} = \lim_{t \rightarrow \infty} x(t)$

$$\begin{aligned}&= u(\infty) - e^{-\infty} u(0) \\ &= 1 - 0 = \boxed{1}\end{aligned}$$

Example: Solve $\left(\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 3y = 0 \right)$ $y(0) = 3$
 $y'(0) = 1$

$$s^2 Y(s) - s y(0) - y'(0) + 4s Y(s) - 4y(0) + 3Y(s) = 0$$

$$Y(s) [s^2 + 4s + 3] - 3s - 1 - 12 = 0$$

$$Y(s) = \frac{3s + 13}{s^2 + 4s + 3} = \frac{3s + 13}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$3s + 13 = A(s+3) + B(s+1)$$

when $s = -1 \rightarrow 2A = 10 \rightarrow \boxed{A = 5}$

$s = -3 \rightarrow -2B = 4 \rightarrow \boxed{B = -2}$

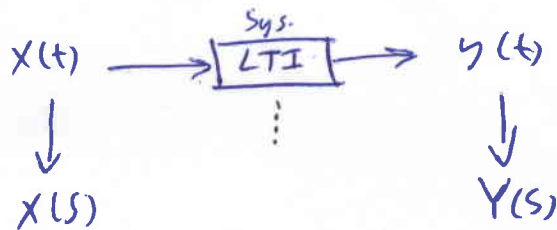
$$\int Y(s) = \frac{5}{s+1} - \frac{2}{s+3}$$

$$y(t) = (5e^{-t} - 2e^{-3t})u(t)$$

* Transfer Function ($H(s)$)

Definition \Rightarrow Transfer function is defined as the ratio of Laplace transform of o/p to the Laplace transform of i/p, when all the initial conditions are assumed to be zero.

$$h(t) \xrightarrow{\text{Laplace}} H(s)$$
$$\xleftarrow{\text{inverse}}$$



$$H(s) = \frac{Y(s)}{X(s)} \Big|_{\text{initial conditions} = 0}$$

LTIV (Linear + Time Invariant)

When the input of Linear System (i/p) = 0 \Rightarrow the output (o/p) of the Linear System also will become (o/p = 0). When don't assume initial conditions to be zero, this property $i/p = 0 \Rightarrow o/p = 0$ becomes violated. Then the system will become not-linear and so the system become not LTIV, hence we cannot define the Transfer function $H(s)$.

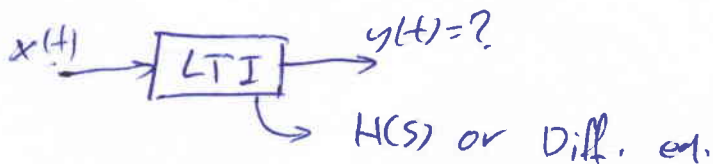
Example: A continuous time LTI system is described by

$$\frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 3y(t) = 2 \frac{dx(t)}{dt} + 4x(t)$$

The response $y(t)$ of the system for the input $x(t) = e^{-2t} u(t)$ is _____

o/p i/p

Sol:



$$y(t) \iff Y(s) = H(s) \cdot X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$\frac{d^2 y(t)}{dt^2} + 4 \frac{dy(t)}{dt} + 3y(t) = 2 \frac{dx(t)}{dt} + 4x(t)$$

$$s^2 Y(s) + 4s Y(s) + 3Y(s) = 2s X(s) + 4X(s)$$

$$Y(s) [s^2 + 4s + 3] = X(s) [2s + 4]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s + 4}{s^2 + 4s + 3} = \frac{2(s+2)}{(s+1)(s+3)}$$

$$x(t) = e^{-2t} u(t) \xrightarrow{\text{Laplace}} X(s) = \frac{1}{s+2}$$

$$Y(s) = H(s) \cdot X(s) = \frac{2(s+2)}{(s+1)(s+3)} \cdot \frac{1}{s+2} = \frac{2}{(s+1)(s+3)}$$

$$\frac{2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$\begin{pmatrix} A = 1 \\ B = -1 \end{pmatrix}$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+3} \rightarrow y(t) = e^{-t} u(t) - e^{-3t} u(t)$$

70-

H.W: for the following equation

$$\frac{d^2 y(t)}{dt^2} - \frac{dy(t)}{dt} - 6y(t) = x(t) \quad , \quad \underline{x(t) = e^{-t} u(t)}$$

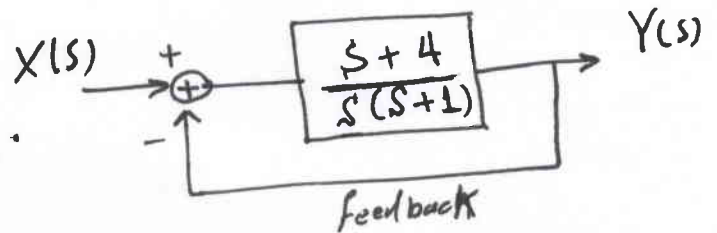
① Find the Transfer function

② Find the output if $y(0) = 0$ & $y'(0) = 5$.

EX: For the system shown:

① Find $H(s)$?

② for $x(t) = e^{-4t} u(t)$, find the output.



Sol:

① $\left\{ \begin{array}{l} X(s) + (-Y(s)) = \text{input} \\ (\text{Input}) \cdot H(s) = \overset{\text{output}}{Y(s)} \end{array} \right\} \underline{\text{hint}}$

$$\therefore [X(s) - Y(s)] \cdot \left[\frac{s+4}{s(s+1)} \right] = Y(s)$$

$$X(s) \frac{s+4}{s(s+1)} = Y(s) \left[1 + \frac{s+4}{s(s+1)} \right]$$

$$\boxed{\frac{Y(s)}{X(s)} = \frac{s+4}{s^2 + 2s + 4} = H(s)}$$

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$$\textcircled{2} - X(s) = \frac{1}{s+4} \quad (x(t) = e^{-4t} u(t))$$

$$Y(s) = X(s) \cdot H(s) = \frac{1}{(s+4)} \cdot \frac{(s+4)}{s^2+2s+4}$$

$$Y(s) = \frac{1}{s^2+2s+4}$$

$$y(t) = \int^{-1} Y(s)$$

$$= \int^{-1} \frac{1}{s^2+2s+4}$$

$$= \int^{-1} \frac{1}{(s+1)^2+3}$$

↑ change this shift in s-domain to exp in time domain.

$$= e^{-t} \int^{-1} \frac{1}{s^2+3}$$

$$= e^{-t} \int^{-1} \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{s^2+(\sqrt{3})^2}$$

hint₁: 0

$$\left. \begin{array}{l} \text{shift property} \\ e^{-at} x(t) = X(s+a) \end{array} \right\}$$

hint₂:

$$\left. \sin at = \frac{a}{s^2+a^2} \right\}$$

$$y(t) = \frac{1}{\sqrt{3}} e^{-t} \sin \sqrt{3} t u(t)$$

Example: The Transfer function of a linear system is:

$$H(s) = \frac{1}{s+a}$$

- ① Find the differential Equation.
- ② For input $x(t) = t u(t)$, find the output assuming zero initial conditions.

Sol: ①

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+a}$$

$$sY(s) + aY(s) = X(s)$$

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

② $x(t) = t u(t)$

$$X(s) = \frac{1}{s^2} \quad \& \quad H(s) = \frac{1}{s+a}$$

$$Y(s) = X(s) \cdot H(s) = \frac{1}{s^2(s+a)}$$

$$y(t) = \int \frac{1}{s^2(s+a)} = \int \left[\frac{A}{s+a} + \frac{B}{s} + \frac{C}{s^2} \right]$$

$$y(t) = \int \frac{1}{a^2} \left[\frac{a}{s^2} - \frac{1}{s} + \frac{1}{s+a} \right]$$

$$y(t) = \frac{1}{a^2} [at - 1 + e^{-at}] u(t)$$

* Z-Transform

Z-transform is a convenient tool in designing analyzing and representing discrete signals & systems.

It plays in discrete signals, the same role as Laplace transform plays in continuous signals.

Z-transform has wide applications such as stability and frequency response.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Bilateral Z-transform

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

unilateral Z-transform

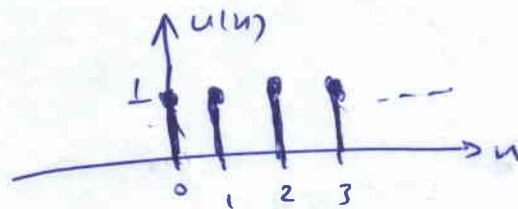
Ex: Find Z-transform of the following signals:

① - $x(n) = \delta(n)$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \delta(n) z^{-n} = \delta(0) z^0 = \underline{\underline{1}}$$

② - $x(n) = u(n)$

$$X(z) = \sum_{n=0}^{\infty} u(n) z^{-n}$$



$$= 1 * z^0 + 1 * z^{-1} + 1 * z^{-2} + 1 * z^{-3} \dots$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \dots$$

$$= \frac{1}{1 - z^{-1}}$$

$$= \frac{1}{1 - z^{-1}}$$

← $X(z)$ $u(n)$

* Properties of z-transform

1- Linearity

$$a f_1(n) + b f_2(n) \leftrightarrow a F_1(z) + b F_2(z)$$

2- Shifting (delay)

$$f(n-m) \leftrightarrow z^{-m} F(z)$$

3- Shifting (Advance)

$$f(n+1) \leftrightarrow z [F(z) - f(0)]$$

$$f(n+2) \leftrightarrow z [F(z) - f(0)] - z f(1)$$

$$f(n+3) \leftrightarrow z [F(z) - f(0)] - z f(1) - z f(2)$$

4- Convolution & Multiplication

$$x(n) \otimes y(n) \leftrightarrow X(z) \cdot Y(z)$$

$$x(n) \cdot y(n) \leftrightarrow X(z) \otimes Y(z)$$

5- Multiplication by n

$$n f(n) \leftrightarrow -z \frac{dF(z)}{dz}$$

6- Multiplication by a^n

$$a^n f(n) \xrightarrow{\text{Example:}} \left[a^n u(n) \leftrightarrow \frac{z}{z-a} \right]$$

$$a^n f(n) \leftrightarrow F\left(\frac{z}{a}\right)$$

Ex Find Z-transform of $u(n)$

① - $u(n-m)$

$$u(n) \longleftrightarrow \frac{z}{z-1} \quad \left(\frac{1}{1-z^{-1}} * \frac{z}{z} \right)$$

$$u(n-m) \longleftrightarrow z^{-m} \frac{z}{z-1}$$

1/1

② $n u(n)$

$$u(n) \longleftrightarrow \frac{z}{z-1} \quad \checkmark f(z)$$

$$n u(n) \longleftrightarrow -z \frac{d f(z)}{d z} \quad \leftarrow \frac{z}{z-1}$$

$$-z \frac{d}{d z} \frac{z}{z-1} = -z \frac{(z-1) * 1 - z * 1}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

3) $a^n u(n) \longleftrightarrow F\left(\frac{z}{a}\right)$

$$u(n) \longleftrightarrow \frac{z}{z-1}$$

$$a^n u(n) \longleftrightarrow \frac{z/a}{z/a - 1} = \frac{z}{z-a}$$

Ex 26 Find Z-transform of $x(n]$ sequence:

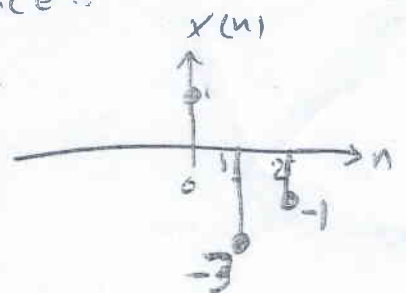
$$x(n) = \{1, -3, -1\}$$

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= 1 + (-3)z^{-1} + (-1)z^{-2}$$

$$= 1 - 3z^{-1} - z^{-2}$$

$\xleftrightarrow{\text{n-domain equation}}$



$$= \delta(n) - 3\delta(n-1) - \delta(n-2)$$

Ex₃: Find the inverse of Z-transform-

$$X(z) = 2 + 3z^{-4}$$

$$x(n) = 2\delta(n) + 3\delta(n-4)$$

Ex₄: Find the inverse of $X(z) = \frac{z}{z^2 - 6z + 8}$

$$\frac{X(z)}{z} = \frac{1}{z^2 - 6z + 8} \Rightarrow$$

$$\frac{X(z)}{z} = \frac{1}{(z-2)(z-4)} = \frac{A}{z-2} + \frac{B}{z-4}$$

$$A = (z-2) \frac{1}{(z-2)(z-4)} \Big|_{z=2} = \frac{1}{2-4} = \boxed{-\frac{1}{2}}$$

$$B = (z-4) \frac{1}{(z-2)(z-4)} \Big|_{z=4} = \frac{1}{4-2} = \boxed{\frac{1}{2}}$$

$$\frac{X(z)}{z} = \frac{-\frac{1}{2}}{z-2} + \frac{\frac{1}{2}}{z-4}$$

$$X(z) = -\frac{1}{2} \frac{z}{z-2} + \frac{1}{2} \frac{z}{z-4}$$

$$u(n) \longrightarrow \frac{z}{z-1}$$

$$a^n u(n) \longrightarrow \frac{z}{z-a}$$

$$= -\frac{1}{2} (2)^n u(n) + \frac{1}{2} (4)^n u(n) \rightarrow$$

Ex: Find $y(n)$ as function of n when $x(n) = \delta(n)$ using Z transform of the following equation.

$$y(n) - 4y(n-1] + 3y(n-2) = x(n)$$

$$Y(z) - 4Y(z)z^{-1} + 3Y(z)z^{-2} = X(z)$$

$$Y(z) [1 - 4z^{-1} + 3z^{-2}] = 1$$

$$Y(z) = \frac{1}{1 - 4z^{-1} + 3z^{-2}} \times \frac{z^2}{z^2}$$

$$Y(z) = \frac{z^2}{z^2 - 4z + 3}$$

$$\frac{Y(z)}{z} = \frac{z}{z^2 - 4z + 3} \Rightarrow \frac{Y(z)}{z} = \frac{z}{(z-3)(z-1)}$$

$$A = \frac{3}{2}$$

$$B = -\frac{1}{2}$$

$$\frac{Y(z)}{z} = \frac{A}{z-3} + \frac{B}{z-1}$$

$$A = (z-3) \frac{z}{(z-3)(z-1)} \Big|_{z=3} = \frac{3}{3-1} = \frac{3}{2}$$

$$B = (z-1) \frac{z}{(z-1)(z-3)} \Big|_{z=1} = \frac{1}{1-3} = -\frac{1}{2}$$

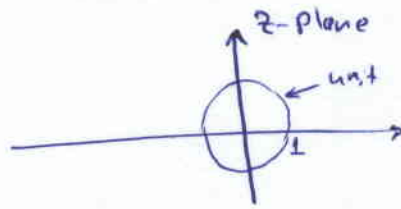
$$\frac{Y(z)}{z} = \frac{3/2}{z-3} + \frac{-1/2}{z-1}$$

$$Y(z) = \frac{3}{2} \frac{z}{z-3} - \frac{1}{2} \frac{z}{z-1}$$

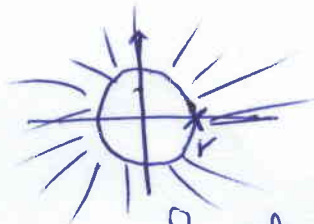
$$\rightarrow y(n) = \frac{3}{2} (3)^n u(n) - \frac{1}{2} u(n)$$

* Region of convergence (ROC) of Z-transform.

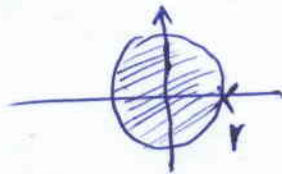
① - The ROC in Z-transform is indicated as a unit circle



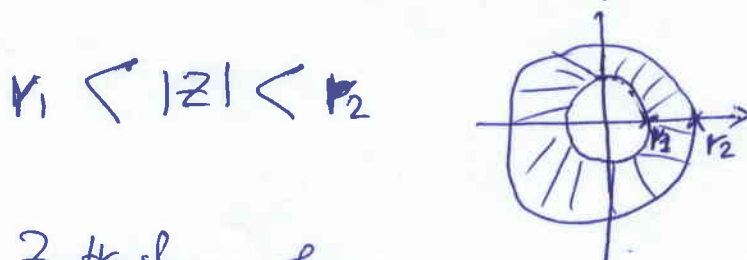
② If $x(n)$ is a right-sided sequence & if $|z| = r$ in ROC, then all finite values of z for which $|z| > r$ will also be in ROC.



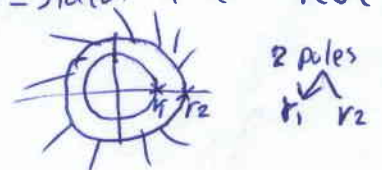
③ If $x(n)$ is left-sided sequence & if $|z| = r$ in ROC, then all finite values of z for which $|z| < r$ will also be in ROC.



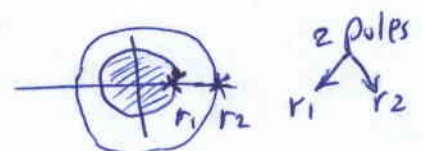
④ If $x(n)$ is two-sided signal & if $|z| = r$, circle is in ROC, then the ROC will contain Ring in z-plane that includes



⑤ If the Z-transform of $x(n)$ is rational & Right-sided then ROC is z-plane outside the outermost pole.



⑥ If the Z-transform of $x(n)$ is rational & Left-sided then ROC is z-plane innermost pole



note: if Roc includes $|z|=1$ ($r=1$), then
 DTFT EXISTS \implies converges

* Pole - Zero Diagram

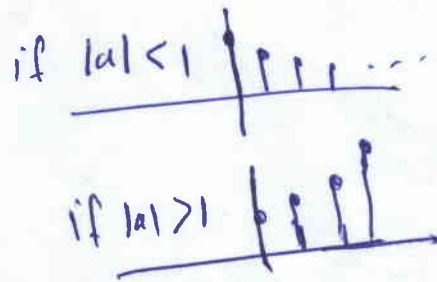
$$X(z) = \frac{N(z)}{D(z)} \quad \left. \begin{array}{l} \text{polynomials in } z \\ \text{polynomials in } z \end{array} \right\}$$

$N(z) = 0 \rightarrow X(z) = 0$ "Zeros" $\xrightarrow{\text{arrow}} \oplus$
 $D(z) = 0 \rightarrow X(z) = \infty$ "poles" $\xrightarrow{\text{arrow}} *$

EX₁: (Right-sided Exponential) $x[n] = a^n u[n]$

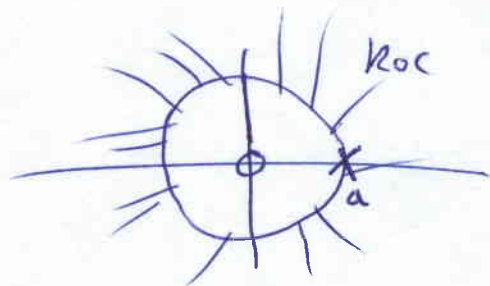
$$u[n] \xrightarrow{z.T} \frac{z}{z-1}$$

$$a^n u[n] \xrightarrow{z.T} \frac{z}{z-a}$$



$$X(z) = \frac{z}{z-a} \rightarrow \begin{cases} \text{zero} = 0 \\ \text{pole} = a \end{cases}$$

Roc, $|z| > a$



EX₂: Left-sided Exponential, $x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n} = \sum_{-\infty}^{\infty} -a^n z^{-n}$$

$$X(z) = \sum_{-\infty}^{-1} -\left(\frac{a}{z}\right)^n = \sum_{n=1}^{\infty} -\left(\frac{z}{a}\right)^n$$



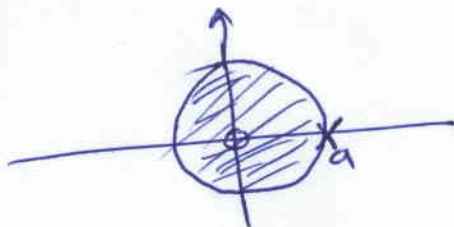
$$= \frac{1}{1 - \frac{z}{a}} + 1 \implies \frac{-a}{a-z} + \frac{a-z}{a-z} = \frac{-z}{a-z} = \boxed{\frac{z}{z-a}}$$

- Ro -

Notice that the z-transform of Right-sided Exponential looks similar to the one of left sided Exp. But actually they are not the same, Check the ROC of 1st case it is opposite of ROC in 2nd case.

$$|z| < a$$

left-sided



EX 3: $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

$$\frac{z}{z - \frac{1}{2}} + \frac{z}{z + \frac{1}{3}} = \frac{2z^2 - \frac{1}{6}z}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

$$X(z) = \frac{2z(z - \frac{1}{2})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

ROC:

$$\frac{z}{z - \frac{1}{2}}$$

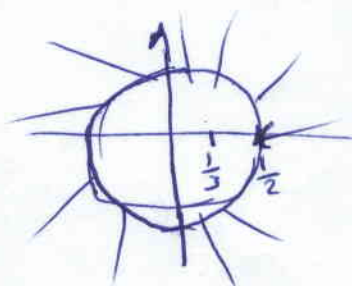
ROC ↓

$$|z| > \frac{1}{2}$$

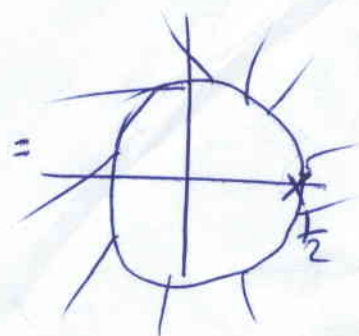
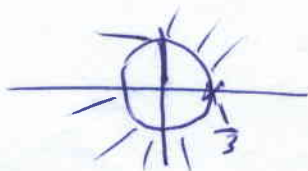
$$+ \frac{z}{z + \frac{1}{3}}$$

ROC ↓

$$|z| > \frac{1}{3}$$

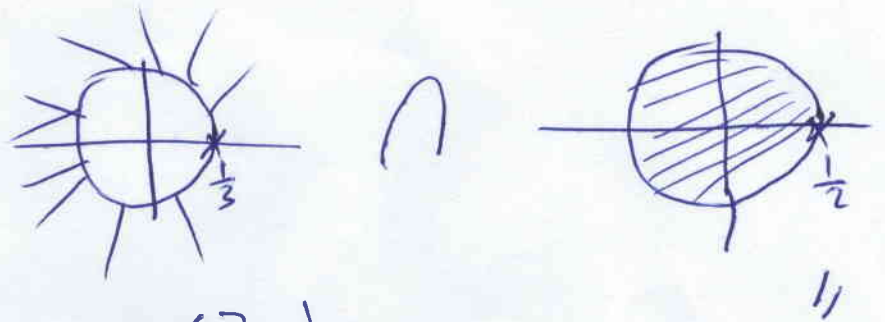


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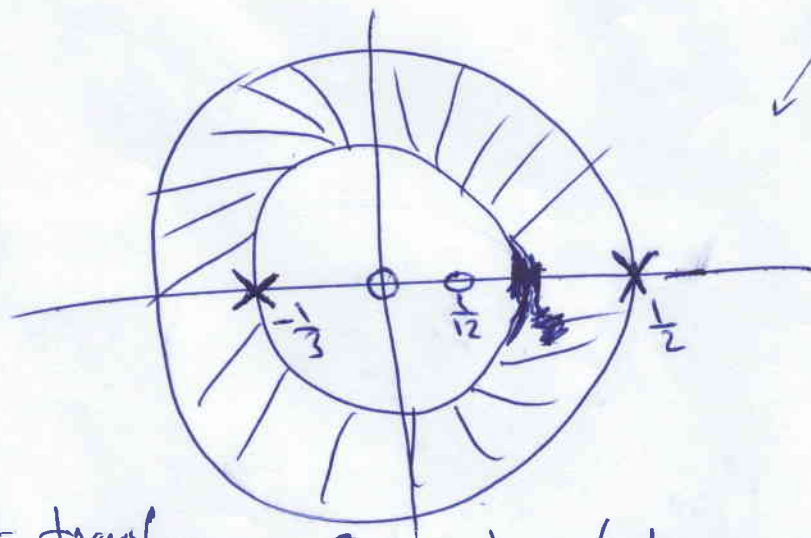


$$E: X_{40} \quad x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-1)$$

$$\text{Roc: } |z| > \frac{1}{3} \cap \text{Roc: } |z| < \frac{1}{2}$$



$$X(z) = \frac{2z(z - \frac{1}{2})}{(z + \frac{1}{3})(z - \frac{1}{2})}$$



* Inverse z -transform can be taken at least by Four different methods.

- ✓ ① by Inspection
- ✓ ② partial fraction Expansion
- ③ power Series Expansion
- ④ Contour Integration.