1

Signals & Systems

by:
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code: CIE2305
units: 6

* Syllabus

2 Hrs / WK → Theory نظري
2 Hr / WK → Practical عملي

Theory:

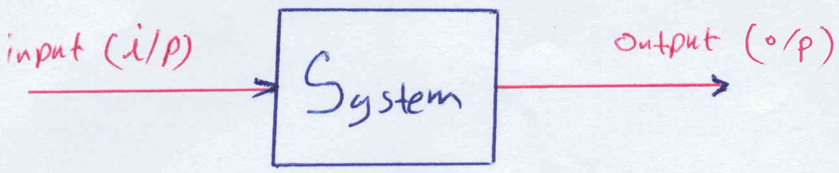
- Signal & System characteristics and models
  - Basic Operations on Signals
  - Signals & Systems characteristics
  - System representation and modules
- Continuous Time signals & System
- Time Domain Analysis of Continuous Time Signal
- Frequency Domain representation of Continuous Time Signal (CTS)
- Frequency Domain Analysis of " " "
- Analysis using Laplace Transform.
- Time & Frequency Domain Analysis of Discrete Time Signal (DTS)
- Analysis using Z-Transform

Text book

1. "Introduction to Signal and System" by D.K. Lindner
2. "Signal & System" by Carlson

* **Signal** : defined as a function of one or more variables that conveys information on the nature of physical phenomens.

* **System** : defined as entity that manipulates one or more signals to accomplish a function.



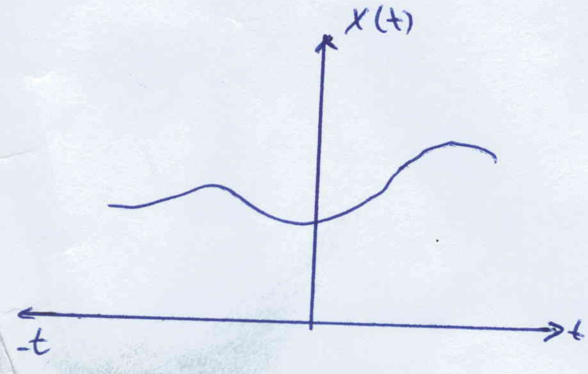
Classification of Signals

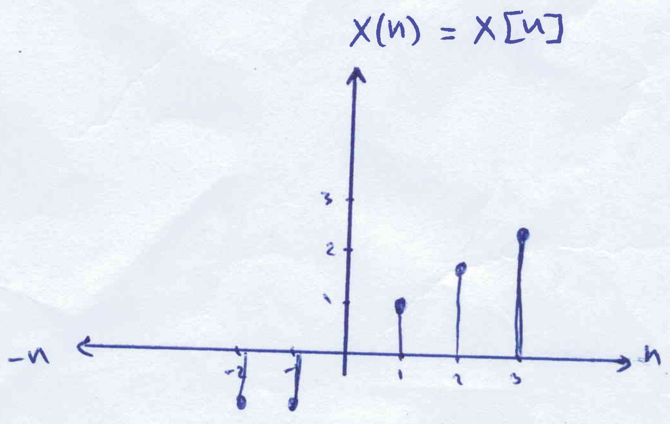
There are several methods to classify signals. The following are the most common:

1 - Continuous - time & Discrete - time

Continuous - time: specified for every value of time (t)

Discrete - time: specified at discrete time intervals.





$n = 0, \pm 1, \pm 2 \ldots$

n - integer

2- Even & odd Signal

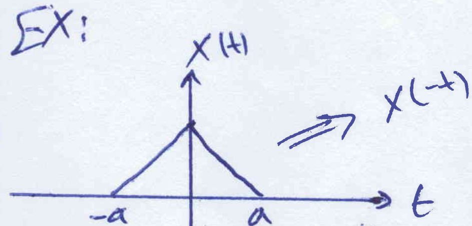
* Even Signals

- Remain identical under folding (Reflection) operation.

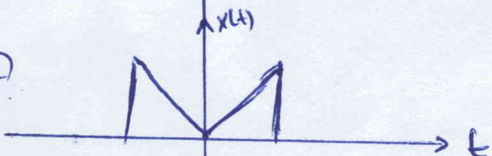
$$x(t) \xrightarrow{\text{Time Reverse}} \boxed{x(-t) = x(t)}$$

EX:

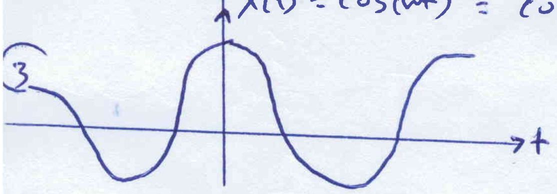
①



②



③ $x(t) = \cos(\omega t) = \cos(-\omega t)$



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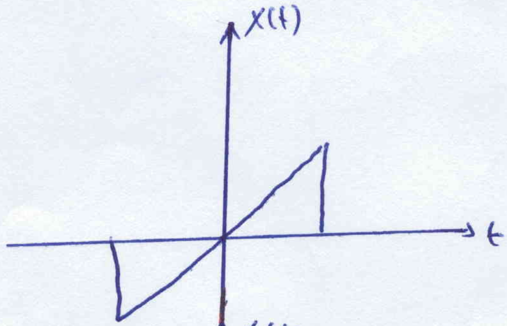
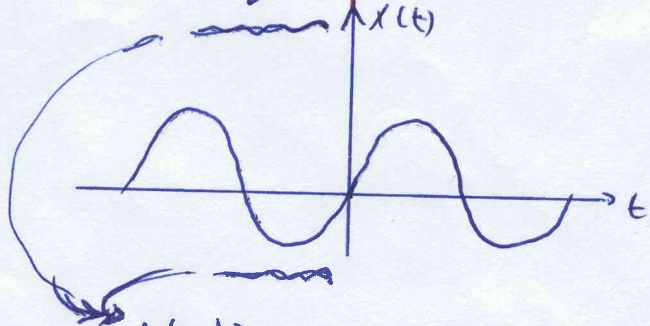
if $x(t) = x(-t)$ for all t then the Signal is Even

* Odd Signals

- Doesn't remain identical under reflection operation.

$$x(t) \neq x(-t)$$





$$x(-t) = -x(t)$$

or

if $x(t) = -x(-t)$ for all t, then the Signal is odd

$$\sin(-\theta) = -\sin(\theta)$$

3- Periodic & non-periodic Signals (Aperiodic)

* Periodic Signal : is the Signal that repeats itself after a regular interval of time.

if $x(t) = x(t-T)$ for all (t) $\longrightarrow$ periodic

The smallest value T is called <u>period</u>.

<u>Ex 1</u>:

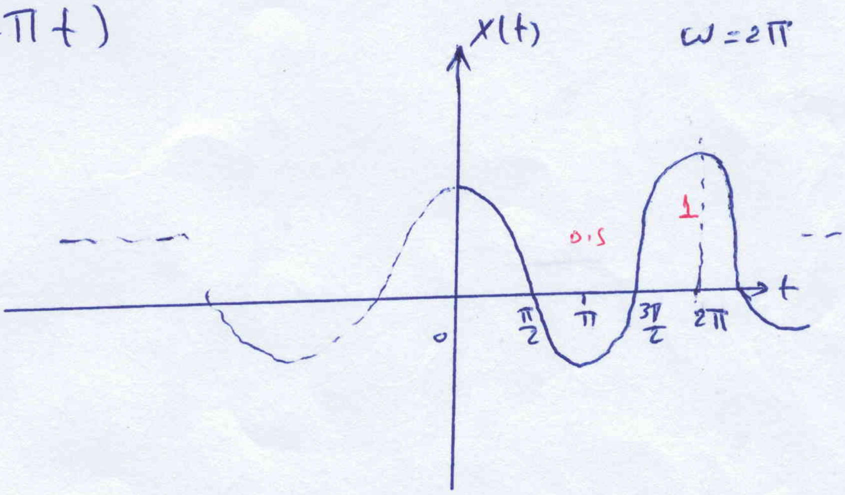
$$x(t) = \cos(2\pi t)$$

$\omega = 2\pi$

$\omega = 2\pi f$

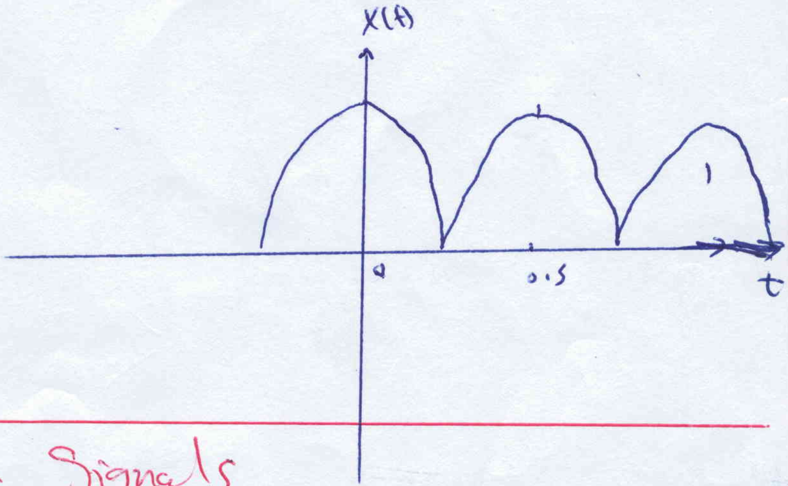
$f = \frac{1}{T}$

$2\pi = \frac{2\pi}{T} \rightarrow \boxed{T = 1}$



Ex 2 : $x(t) = \cos^2(2\pi t)$

<u>H.W</u> $\boxed{T = 0.5}$



hint: cosine الـ قواسين

4- Deterministic & Random Signals

Deterministic ↓ لا تتغير عشوائياً مع الزمن

Random ↓ تتغير عشوائياً مع الزمن

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Lecture #2

2/11/2017

5- Energy and Power Signals

The Energy of Signal $x(t)$ is expressed as the integral of the squared magnitude of its values as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Aperiodic Signals with finite energy are called Energy Signals

The Energy of $x(t) = 3e^{-t}, \; t > 0$ is

$$E = \int_{0}^{\infty} |3e^{-t}|^2 \, dt = \boxed{\frac{9}{2}}$$

If the energy of a Signal is infinite, then it may be possible to characterize it in terms of its average power. The avarage power is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt$$

* For periodic Signal, the Avarage Power over one period as

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt.$$

T is the period, Periodic or aperiodic Signals with finite Avarage power are called Power Signal Ex (Sine and Cosine).

For discrete Signals

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad , \quad P = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N}^{N} x^2(n)$$

For periodic discrete Signal the average power is:

$$P = \frac{1}{N} \sum_{0}^{N-1} x^2(n)$$

Summary

if $0 < E < \infty \Longrightarrow$ energy Signal

if $0 < P < \infty \Longrightarrow$ Power Signal

Examples

1- Categorize each of the following Signals as an Energy OR power Signal, Find E & P.

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^1 t^2 dt + \int_1^2 (2-t)^2 dt$$

$$\boxed{= \frac{2}{3} \text{ Joule}}$$

Prove

$$= \frac{t^3}{3}\Big|_0^1 + \frac{(2-t)^3}{-3}\Big|_1^2 = \left(\frac{1}{3} - 0\right) + \left(0 - \left(\frac{-1}{3}\right)\right)$$

$$= \frac{1}{3} + \frac{1}{3} \boxed{= \frac{2}{3}}$$

$$\boxed{P = 0}$$

2- $x(t) = 2\cos \pi t$

$$\text{power} = (V_{r.m.s})^2 = \left(\frac{V_P}{\sqrt{2}}\right)^2 = \left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = \boxed{2 \text{ Watts}}$$

$$\boxed{E = \infty}$$

3- $x(n) = \begin{cases} n & 0 \leq n < 3 \\ 10-n & 5 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$

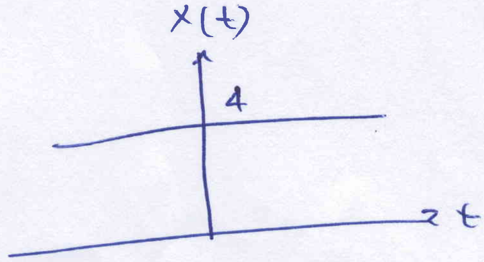
This is discrete Signal,

$$E = \sum_{-\infty}^{\infty} |x(n)|^2$$

$E = 0^2 + 1^2 + 2^2 + 5^2 + 4^2 = \underline{46}$ So, it Energy Signal

$\boxed{P = 0}$

4- $x(t) = 4$

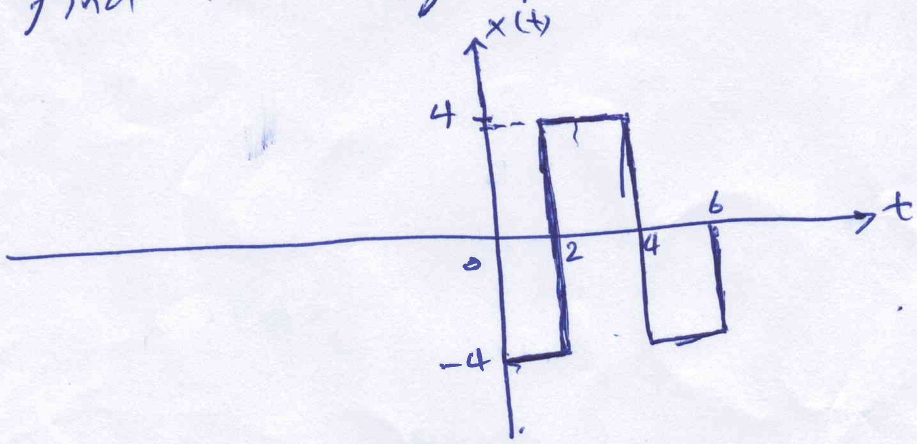


$P = V^2/R$ $\boxed{\text{assuming } R = 1}$

$P = V^2 = 16$

$\boxed{E = \infty}$

H.W Find the Energy of the given Signal

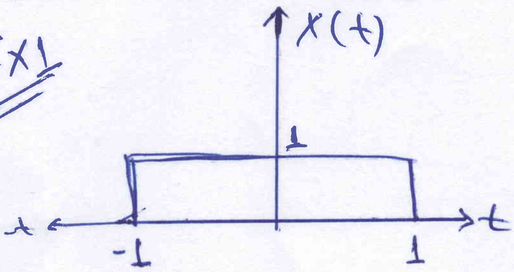


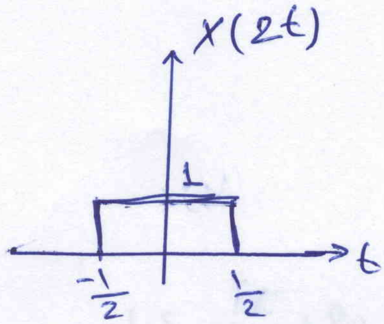
* Operations performed on the independent Variable :

Lecture #3

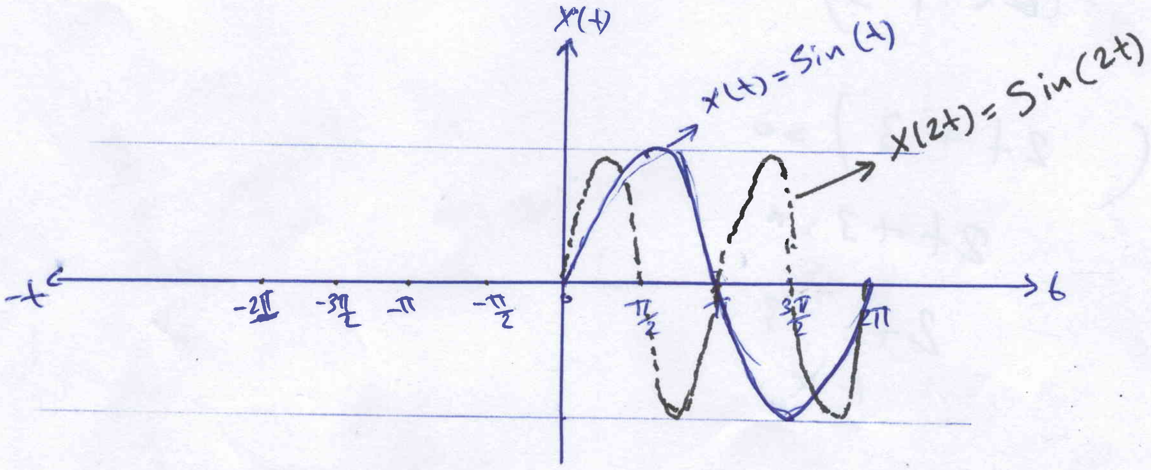
1- Time Scaling :

Ex1

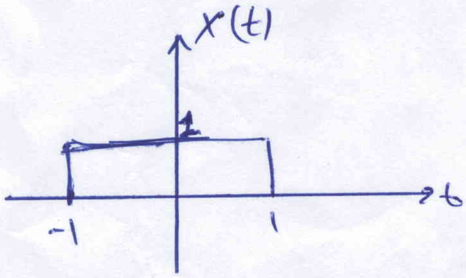


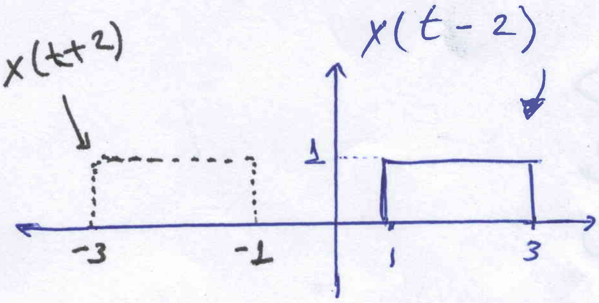


Ex2

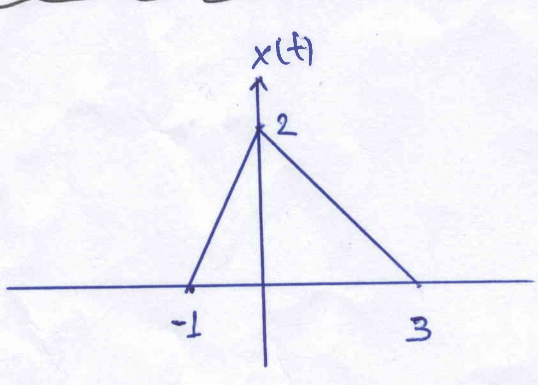


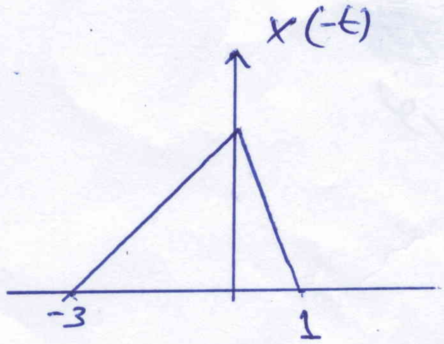
2- Time Shifting



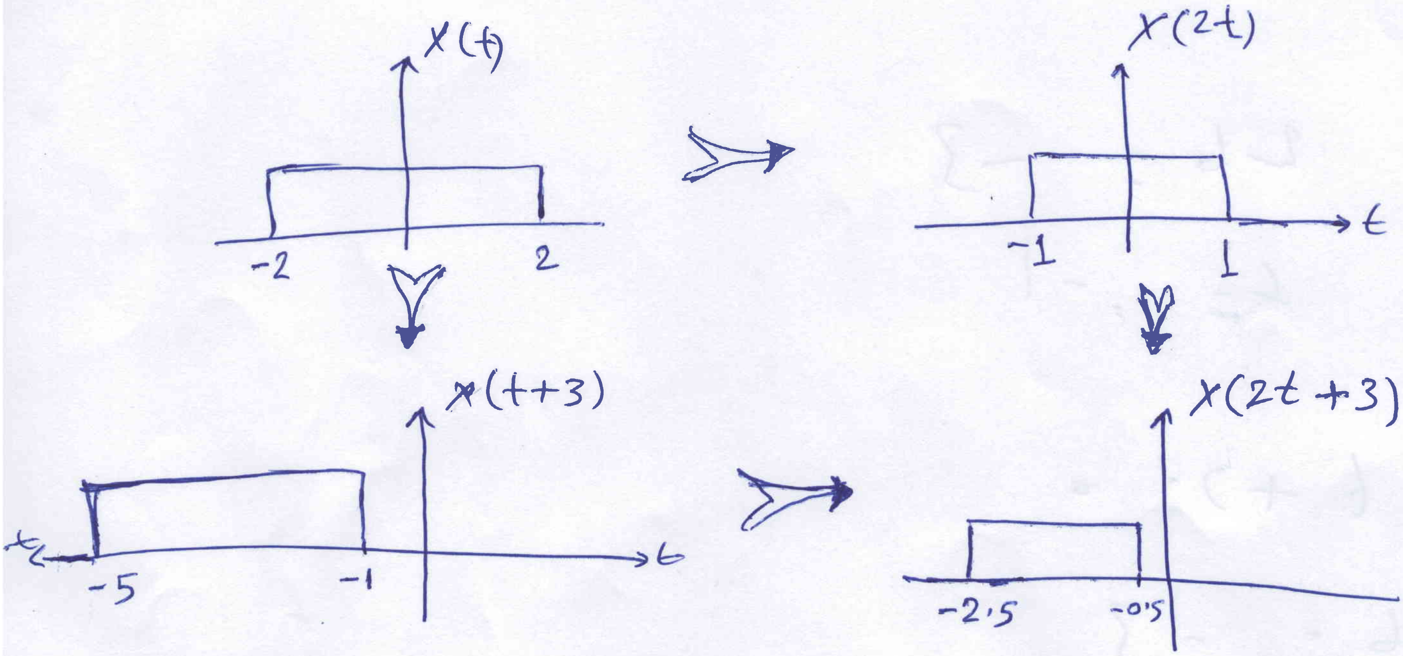


3- Reflection (time Reverse)

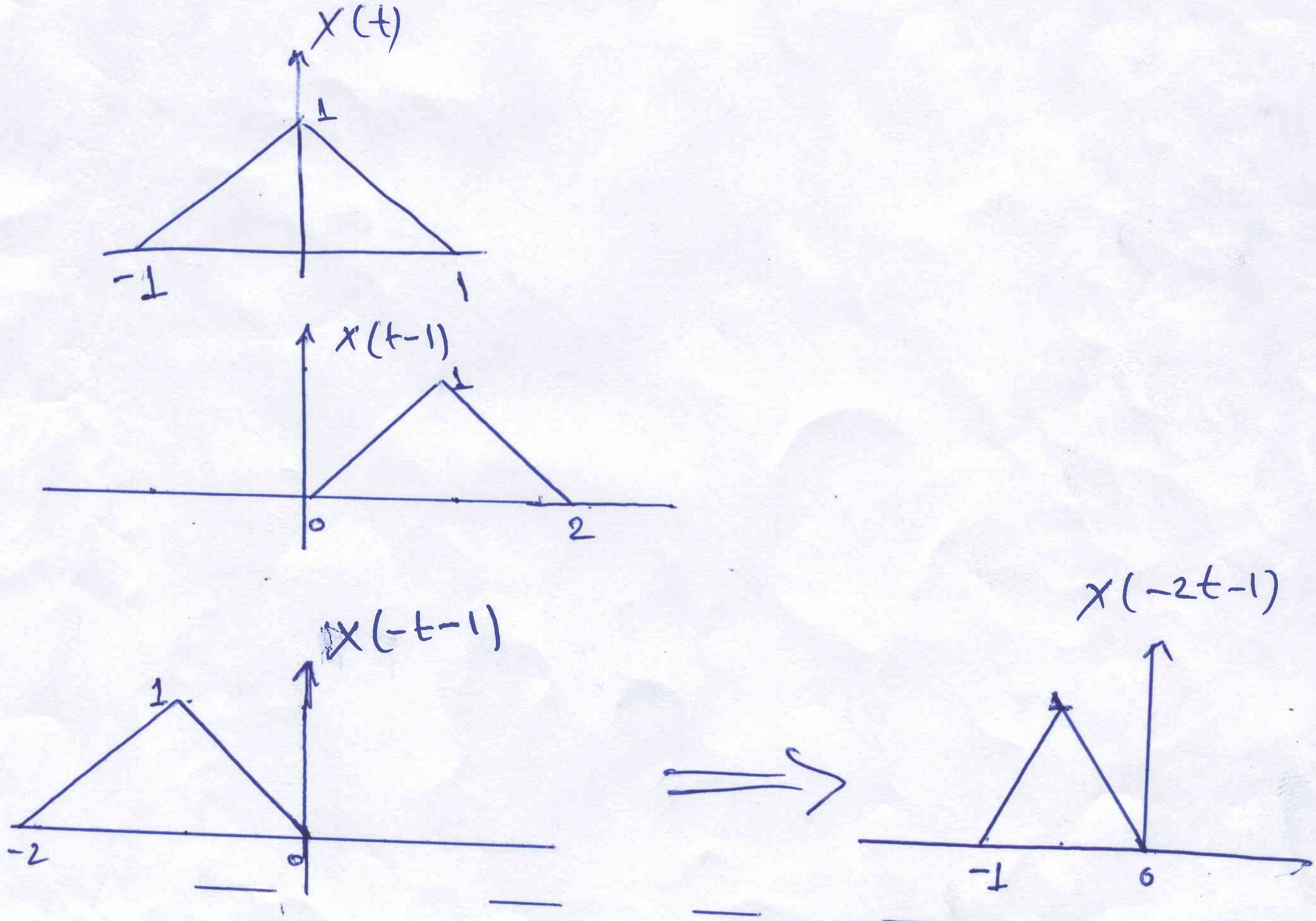




Example : Sketch $x(2t+3)$ if $x(t)$ is given as shown in the figure below :.



Eample$_2$:- Sketch $x(-2t-1)$, $x(t)$ is given below



* Some Elementary Signals :-

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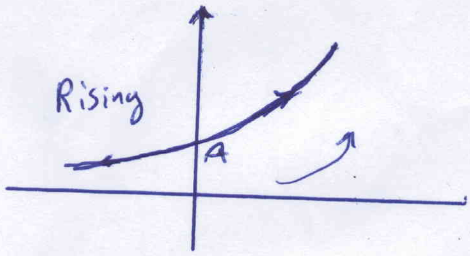
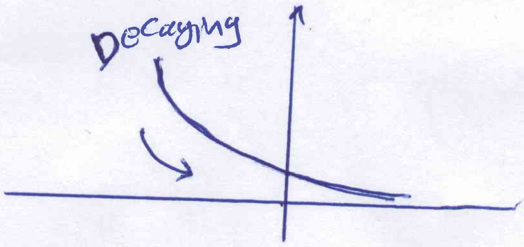
1- Exponential

$x(t) = A e^{Bt}$ A : is real & constant.

if B real and $\underline{\underline{+ve}}$ $\longrightarrow$ Increasing / rising

if B real and $\underline{\underline{-ve}}$ $\longrightarrow$ decaying

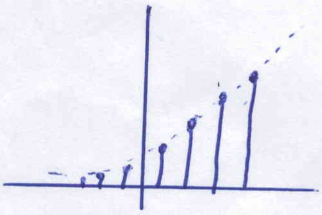


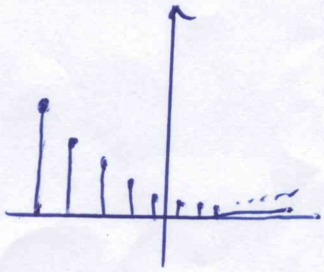


* Complex Exponential, if B Imaginary $(B = j\alpha)$

$$A e^{j\alpha t} = A(\cos\alpha t + j\sin\alpha t)$$

For discrete Signal





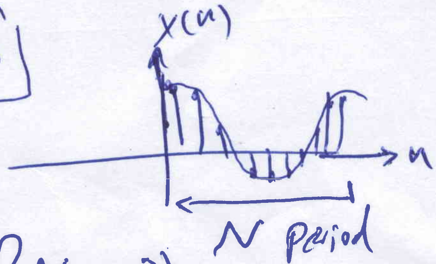
2- Sinusoidal

$x(t) = A\cos(\omega t + \theta)$, $\omega = 2\pi f = \dfrac{2\pi}{T}$, $f = \dfrac{1}{T}$

$x(n) = A\cos(\Omega n + \theta)$

$x(n) = x(n + N)$, $\boxed{x(n) \text{ is periodic with } N \text{ samples}}$

$\cancel{A\cos(\Omega n)}$



$x(n+N) = A\cos(\Omega n + \Omega N + \theta)$

$\therefore \Omega N = 2\pi m$ m - positive integer

$$\boxed{\frac{2\pi}{\Omega} = \frac{N}{m}} = \frac{\text{integer}}{\text{integer}}$$

So it is **Periodic**

Ex_1: $x(n) = 2\cos\left(\underbrace{8\pi}_{\Omega} n + \frac{\pi}{4}\right)$

$$\frac{2\pi}{\Omega} = \frac{2\pi}{8\pi} = \frac{1}{4} = \frac{\text{integer}}{\text{integer}} \rightarrow \text{periodic}$$

Ex_2: $x(n) = 2\sin(0.1\pi n)$

$$\frac{2\pi}{0.1\pi} = \frac{20}{1} \rightarrow \text{periodic}$$

Ex_3: $x(n) = \sin(n)$

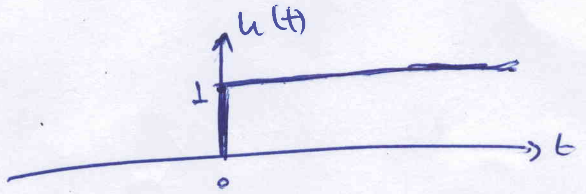
$$\frac{2\pi}{\Omega} = \frac{2\pi}{1} \quad \text{non-periodic}$$

3- Unit step

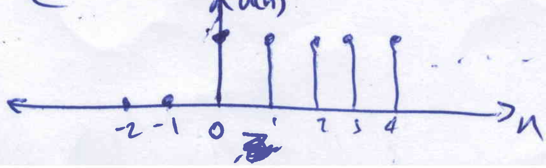
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

t = 0

no Limit

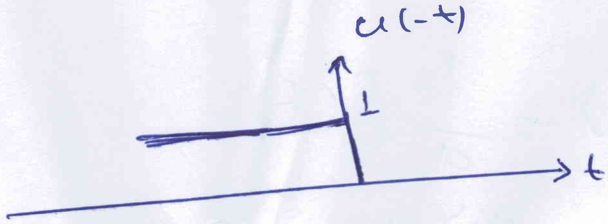


$$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n > 0 \end{cases}$$

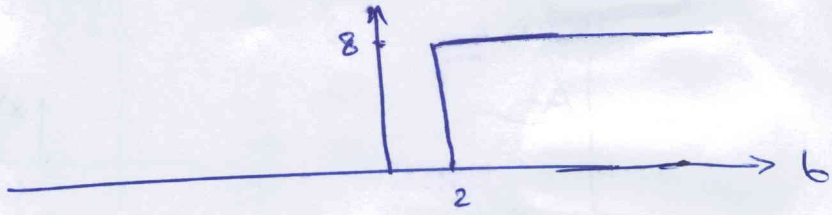


Example: Sketch the following signals.

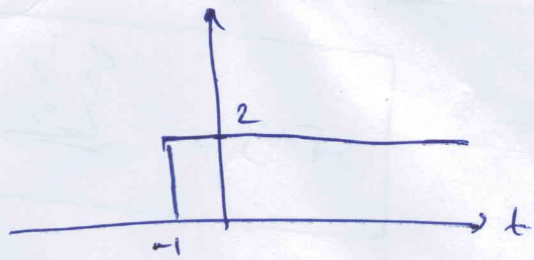
① $u(-t)$



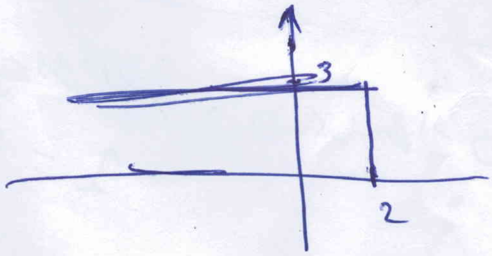
② $8u(t-2)$



③ $2u(t+1)$



④ $3u(-t+2)$

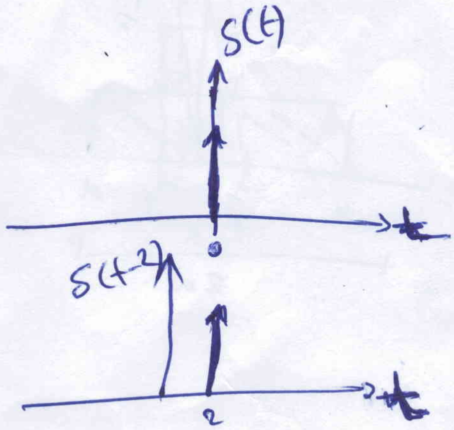


H.w: sketch $3[u(t+2) - u(t-2)]$

4- Impluse (delta) function

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

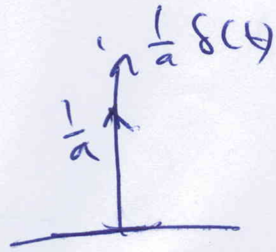
$$\boxed{\frac{du(t)}{dt} = \delta(t)}$$



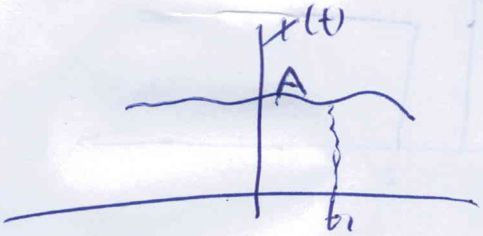
* $\delta(t) = \delta(-t)$ [even]

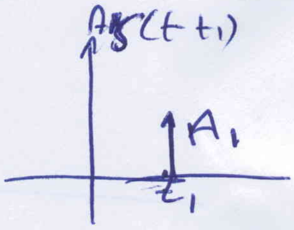
* $\delta(at) = \frac{1}{a}\delta(t)$

$a > 0$



* $x(t) \cdot \delta(t - t_1) = x(t_1) \cdot \delta(t - t_1)$

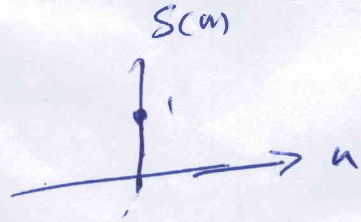




$= x(t_1) \cdot \delta(t - t_1)$

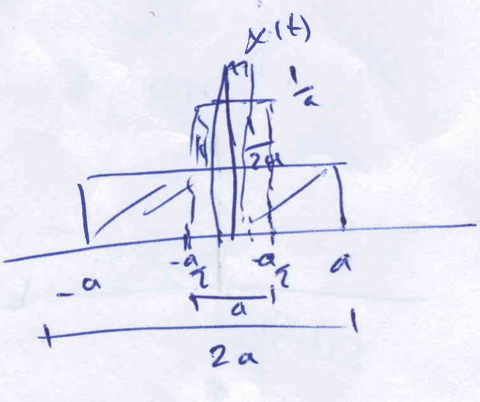
Discrete

$$\boxed{\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}}$$



دالة مستطيلة عرضها يقترب من الصفر وارتفاعها من الـ ∞

ومساحتها = 1



$$\int_{-\infty}^{\infty} x(t) = \int_{-a}^{a} x(t) = 2a * \frac{1}{2a} = 1$$

$a \cdot \frac{1}{a} = 1$

* Properties of Systems :

$-\infty$ to ∞ → Amplitude is finite.

1 - Stability (Bounded - input, bounded - output) (BIBO)

For a stable system output should be bounded for bounded input at each and every instant of time.

* Bounded Signals for examples ; dc , $\sin(t)$, $\cos(t)$, $u(t)$

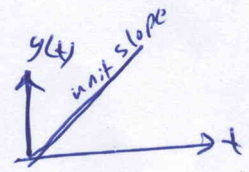
$x(t)=6$ $+1$ to -1 $+1$ to -1 0 OR 1

Ex1 : $y(t) = t \cdot x(t)$

$x(t) \rightarrow$ Sys. $\rightarrow y(t) = t \cdot x(t)$ $\Longrightarrow$ unstable because the o/p not ~~bounded~~ bounded

$u(t) \rightarrow$ Sys. $\rightarrow y(t) = t \cdot u(t)$

$= r(t)$



Ramp function

$$x(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Ex2 : $y(t) = x(t) + 2$

$x(t) \rightarrow$ Sys. $\rightarrow y(t) = x(t) + 2$ $\Longrightarrow$ Stable

$dc = 4 \rightarrow$ System $\rightarrow y(t) = 4 + 2 = 6$ (bounded output

2 - Memory :

A System is called to be *Memory*, if its output Signal depends on Past OR future values of input Signal. If its output depends only on present values of input signal, then the System is <u>memoryless</u>.

Ex1 : $y(t) = 2x(t)$ $\longrightarrow$ memoryless / static System

Ex2 : $y(t) = x(t) + x(t-1)$ $\longrightarrow$ memory / dynamic system.

3- Causality:
If present value of the output depends only on the present and past values of the input, then the system is Causal. If the output depends on future values of the input it is non-causal.

Ex1: $\underbrace{y(t) = x(t)}_{\text{present}} \longrightarrow$ Causal

Ex2: $y(t) = \underbrace{x(t)}_{\text{present}} + \underbrace{x(t-1)}_{\text{past}} \longrightarrow$ Causal

Ex3: $y(t) = \underbrace{x(t+2)}_{\text{future}} \longrightarrow$ non-causal

Ex4: $y(t) = \underbrace{x(t)}_{\text{pre.}} + \underbrace{x(t+1)}_{\text{fut.}} + \underbrace{x(t-1)}_{\text{past}} \longrightarrow$ Non-causal.

4- Linearity:
let $x_1 \longrightarrow y_1$ & $x_2 \longrightarrow y_2$, then

if $a x_1 + b x_2 \longrightarrow a y_1 + b y_2 \Longrightarrow$ linear

otherwise non linear

Ex: $y(t) = 2x(t) + 3$ is it linear?

Let $x(t) = x_1(t) \longrightarrow y_1(t) = 2x_1(t) + 3$

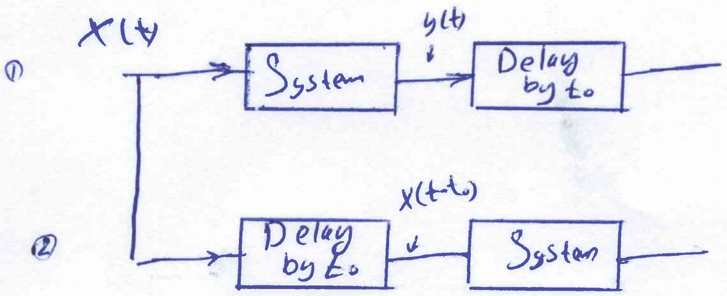
Let $x(t) = x_2(t) \longrightarrow y_2(t) = 2x_2(t) + 3$

if $x(t) = a x_1(t) + b x_2(t) \longrightarrow 2[a x_1 + b x_2] + 3 \overset{?}{=} a y_1 + b y_2$

$2a x_1(t) + 2b x_2(t) + 3 \overset{?}{=} 2a x_1(t) + 3a + 2b x_2(t) + 3b$: No

Non-linear

5- Time Invariance : A System is said to be time-invariant if a time delay of the input signal leads to an identical time shift in the output signal. Otherwise, the System can be time variant



$y(t-t_0)$

$y'(t) = y(t-t_0)$ then Time invariant System

$y'(t) \neq y(t-t_0)$ then Time Variant System

Ex1: $y(t) = x(2t)$

$x(t) \longrightarrow \boxed{\underset{}{\overset{\text{scaling}}{\text{Sys.}}}} \longrightarrow y(t) = x(2t)$

① $y(t) \xrightarrow{t_0} y(t-t_0) = x(2(t-t_0)) = x(2t-2t_0)$

② $x(t) \xrightarrow{t_0} x(t-t_0) \rightarrow \boxed{\overset{\text{scaling}}{\text{System}}} \rightarrow x(2t-t_0)$

So, $x(2t-2t_0) \neq x(2t-t_0)$, then the System is time Variant.

Ex2: $2 + x(t)$

$x(t) \longrightarrow \boxed{\text{Sys.}} \longrightarrow y(t)$

T.IV OR T.V

$y(t) \xrightarrow{t_0} 2 + x(t-t_0)$

$x(t) \xrightarrow{t_0} x(t-t_0) \longrightarrow \boxed{\text{Sys.}} \longrightarrow 2 + x(t-t_0)$, Same

So Time Invariant

H.W [$y(t) = \cos(t)\, x(t)$]

Examples : For the following Systems, check if it is Stable or not?

① $x(t) = \sin(\omega t)$

it is stable because $-1 \le \sin\omega t \le 1$ for all t

Example 2 : $y(t) = x(t) + 3x(t-d)$ is memory, causal?

* memory because it depends on $x(t-d)$
* causal depends on ~~future~~ past $x(t-d)$

Example 3 : Determine whether the System described by Equation $y(t) = \frac{1}{12}x(t) - \frac{5}{6}$ is Linear or non-Linear?

Let $x(t) = x_1(t)$ then $y_1(t) = \frac{1}{12}x_1(t) - \frac{5}{6}$

Let $x(t) = x_2(t)$ then $y_2(t) = \frac{1}{12}x_2(t) - \frac{5}{6}$

$x(t) = ax_1(t) + bx_2(t)$ then:

$$y(t) = \frac{1}{12}[ax_1(t) + bx_2(t)] - \frac{5}{6} \stackrel{?}{=} ay_1 + by_2$$

$$ay_1 + by_2 = \frac{1}{12}a\,x_1(t) - \frac{5}{6}a + \frac{1}{12}bx_2(t) - \frac{5}{6}b$$

$$= \frac{1}{12}[ax_1(t) + bx_2(t)] - \frac{5}{6}[a+b]$$

$\therefore$ So non-Linear $(ax_1 + bx_2 \neq ay_1 + by_2)$

Example 4: Is the System $y(t) = x(t+3) + 2\int_{-\infty}^{0.5t} x(\tau)\,d\tau$ Causal, Linear?

$\because$ $y(t)$ depends on $x(\underbrace{t+3}_{\text{future}})$ so it is Non-Causal.

at $x = x_1 \rightarrow y_1(t) = x_1(t+3) + 2\int_{-\infty}^{0.5t} x_1(\tau)\,d\tau$

$x = x_2 \rightarrow y_2(t) = x_2(t+3) + 2\int_{-\infty}^{0.5t} x_2(\tau)\,d\tau$

$x = a x_1 + b x_2$

$$y = [a x_1(t+3) + b x_2(t+3)] + 2\int_{-\infty}^{0.5t} [a x_1(\tau) + b x_2(\tau)]\,d\tau$$

$$\stackrel{?}{=} a y_1 + b y_2$$

$$a\left[x_1(t+3) + 2\int_{-\infty}^{0.5t} x_1(\tau)\,d\tau\right] + b\left[x_2(t+3) + 2\int_{-\infty}^{0.5t} x_2(\tau)\,d\tau\right]$$

So $a x_1 + b x_2 \stackrel{\checkmark}{=} a y_1 + b y_2$ then the System is Linear

Example 5: Is the System below time variant or time-invariant? $y(t) = t\,x(t) + 4$

$x(t) \longrightarrow \boxed{\text{Sys.}} \longrightarrow y(t)$

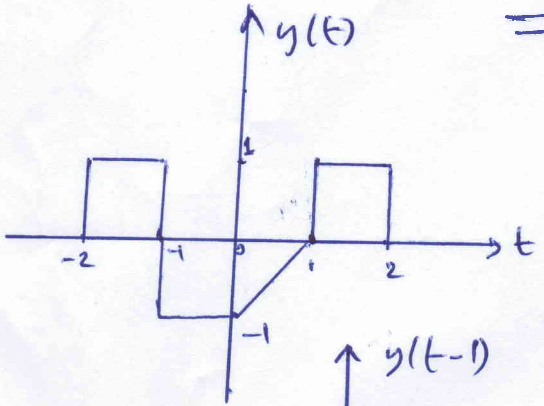
$y(t) \xrightarrow{t_0} (t-t_0)\,x(t-t_0) + 4$ $\neq$

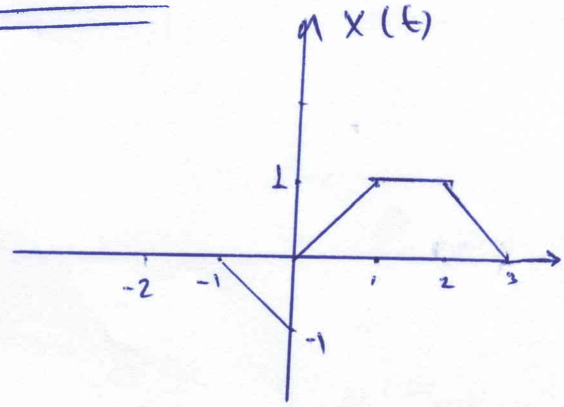
$x(t) \xrightarrow{t_0} x(t-t_0) \longrightarrow \boxed{\text{Sys}} \longrightarrow t\,x(t-t_0) + 4$

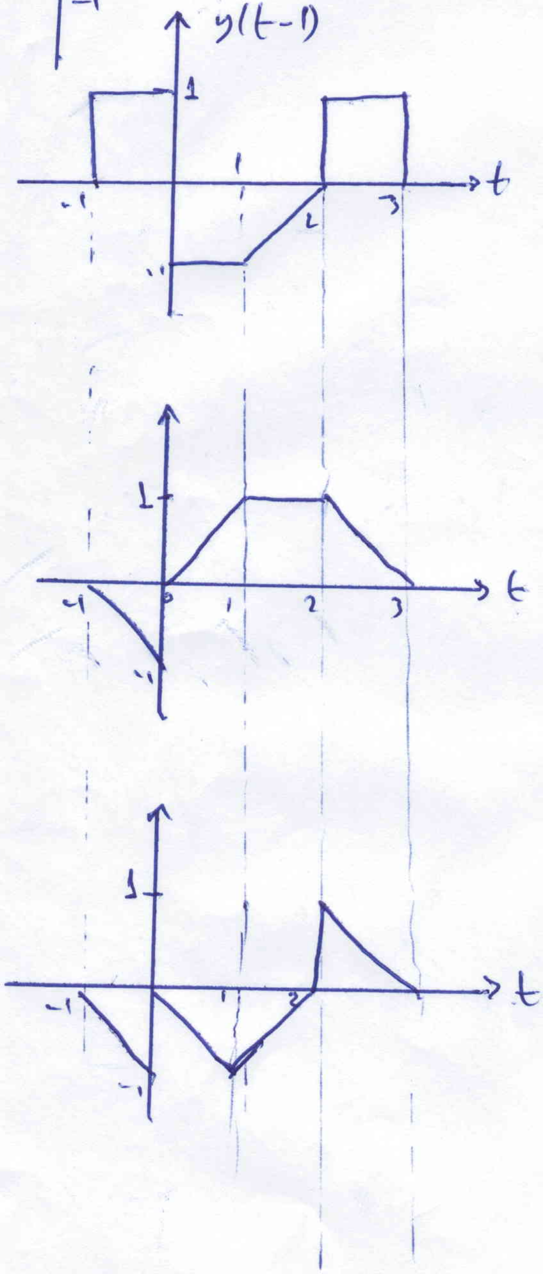
So it is time variant System.

Example Let $x(t)$, $y(t)$ be given in the following figures,

Sketch : $x(t) \cdot y(t-1)$

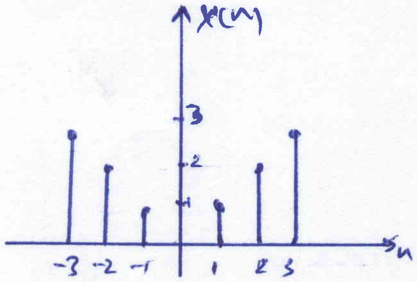


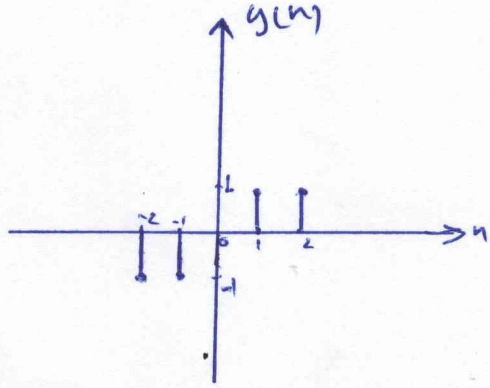


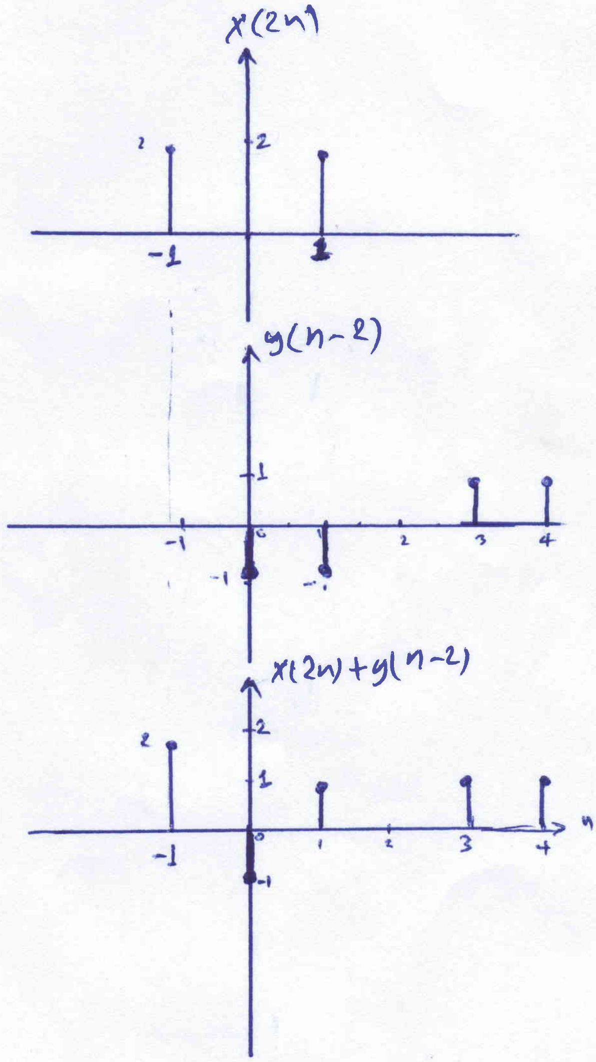


Example : Let $x(n)$, $y(n)$ are given below :

Sketch : $x(2n) + y(n-2)$







Ex: The Discrete-time System shown below

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{unit}\\ \text{delay}\end{matrix}} \longrightarrow y[n] = x[n-1]$$

Determine whether the System is a) memoryless, b) Causal c) Linear, d) time-invariant or e) stable.

<u>Solution</u>

a) the System input-output relation is given

$$y[n] = x[n-1]$$

Since the output value of $y(n)$ depends on the input value at <u>n-1</u> (past), the System is not Memory less.

b) Since the output does not depends on the future input value, the System is Causal.

c)

Let $x = x_1 \longrightarrow y_1[n] = x_1[n-1]$

Let $x = x_2 \longrightarrow y_2[n] = x_2[n-1]$

$x = a x_1 + b x_2 \longrightarrow y[n] = a x_1[n-1] + b x_2[n-1]$

$\stackrel{?}{=} a y_1 + b y_2$

$a x_1[n-1] + b x_2[n-1]$ it is Linear

d) $y[n] \xrightarrow[\text{delay}]{n_o} y[n-n_o] \longrightarrow x[n-n_o-1] = \underline{x[n-1-n_o]}$

$x[n] \xrightarrow[\text{delay}]{n_o} x[n-n_o] \longrightarrow \boxed{\text{Sys}} \longrightarrow x[n-1-n_o]$

Time-invariant

e)

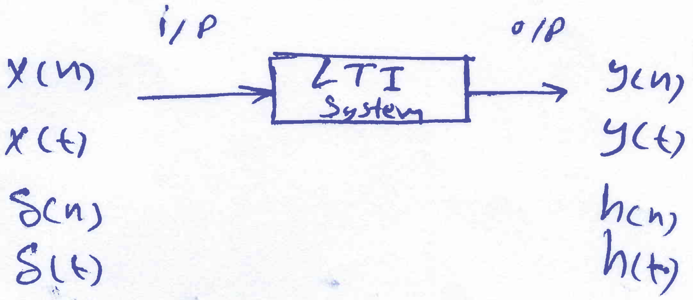
Since $|y[n]| = |x[n-1]| \leq k$

if $|x[n]| \leq k$ for all n

the System is BIBO Stable.

* Time Domain Analysis :

- Impulse Response : The impluse response [$h(t)$, $h(n)$] of a [Continuous, Discrete]-time LTI System is defined to be the response of the System when the input is [$\delta(t)$, $\delta(n)$].

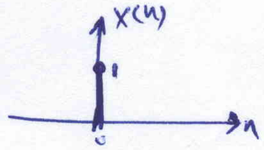


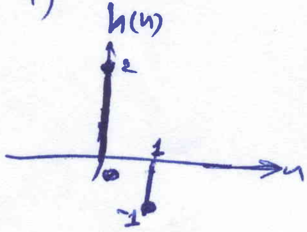
Ex: Find and sketch the impulse response of the System:

$$y(n) = 2x(n) - x(n-1)$$

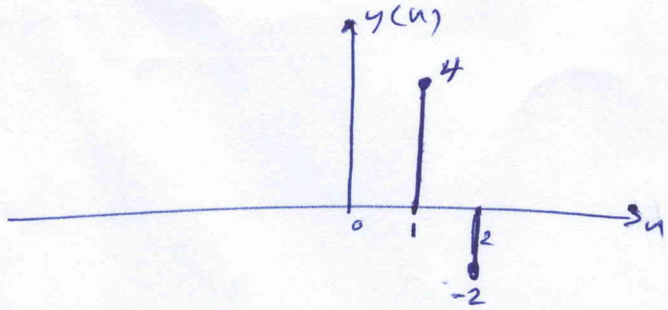
$x(n) = \delta(n)$, $y(n) \rightarrow h(n)$

$h(n) = 2\delta(n) - \delta(n-1)$





Ex: Consider the previous System as LTI System. Find the output if $x(n) = 2\delta(n-1)$



H.W: for the Same System above find the output if $x(n) = \delta(n) + 2\delta(n-1)$.

$$x(n)\delta(n) = x(0)\,\delta(n)$$

$$x(n)\,\delta(n-k) = x(k)\,\delta(n-k)$$

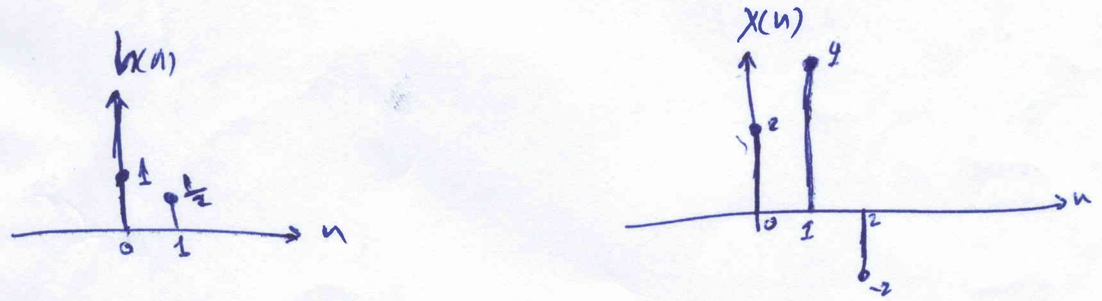
$$x(n) = \cdots + x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + \cdots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)\,h(k)$$

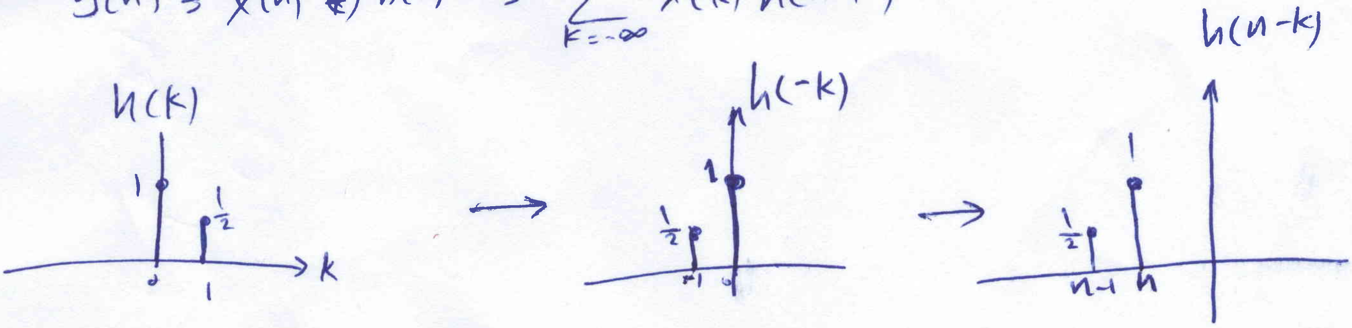
$$= x(n) \circledast h(n) \quad (\text{convolution})$$

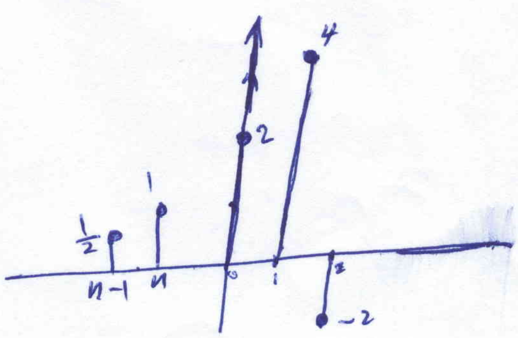
EX: For the System of implse response and input Signal as shown, Find the output.



Sol

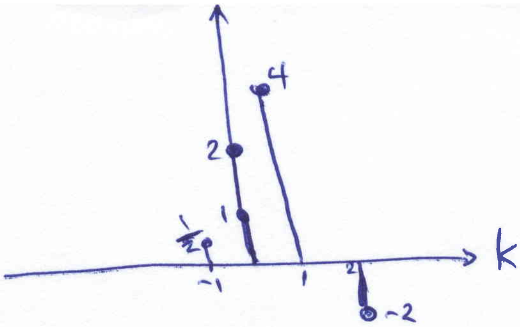
$$y(n) = x(n) \circledast h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$



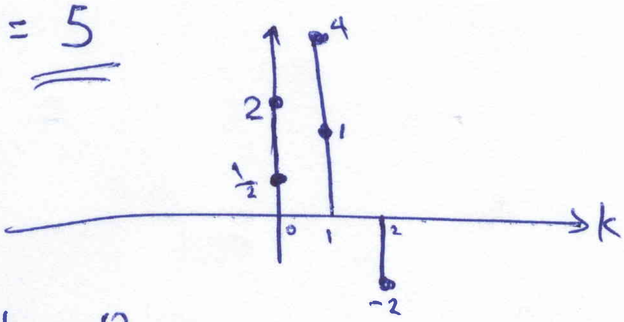


$n < 0 \longrightarrow y(n) = 0$

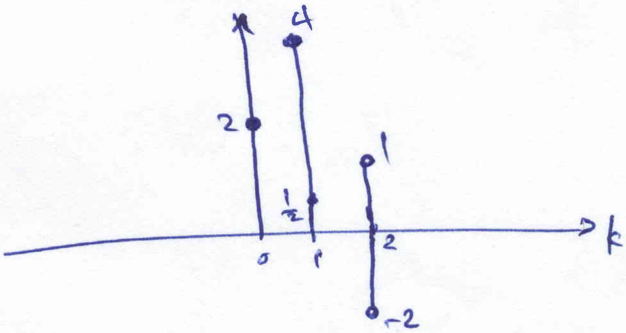
$n = 0 \longrightarrow y(n) = 2 * 1 = \underline{\underline{2}}$



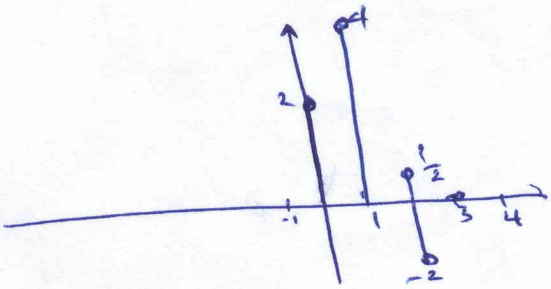
$n = 1 \longrightarrow y(n) = \frac{1}{2} * 2 + 4 * 1 = \underline{\underline{5}}$



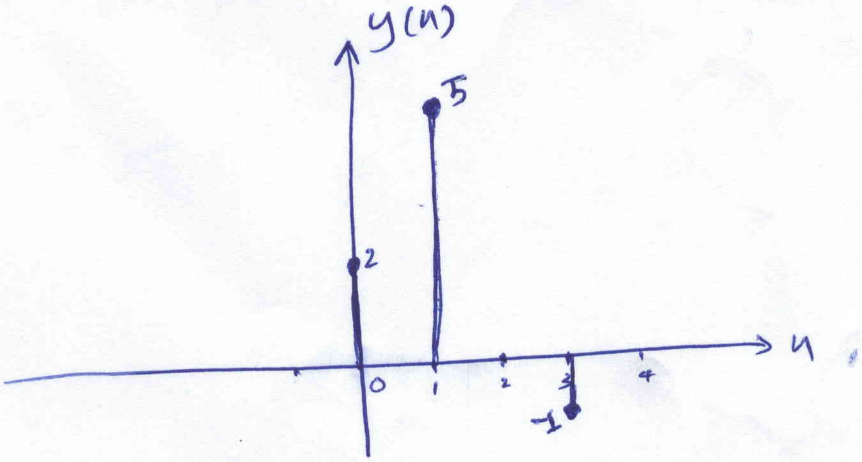
$n = 2 \longrightarrow y(n) = \frac{1}{2} * 4 + (-2) * 1 = \underline{\underline{0}}$



$n = 3 \longrightarrow y(n) = \frac{1}{2} * -2 = \underline{\underline{-1}}$



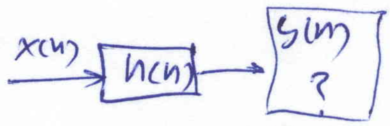
$n \geq 4 \longrightarrow y(n) = 0$



Example 2: For the System of impluse response and input Signal:

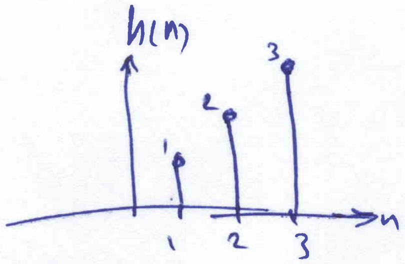
$$h(n) = n \qquad 0 < n < 4$$

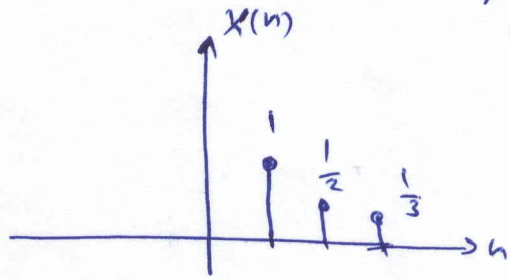
$$x(n) = \frac{1}{n} \qquad 0 < n < 4$$



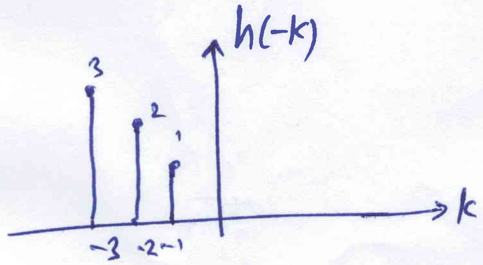
Find the output

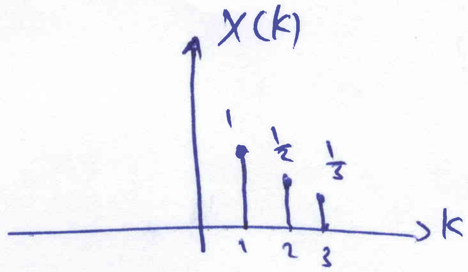
$n = 1, 2, 3$

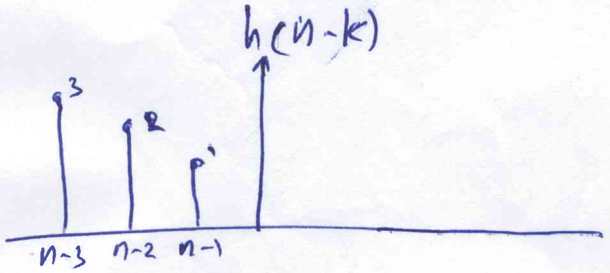


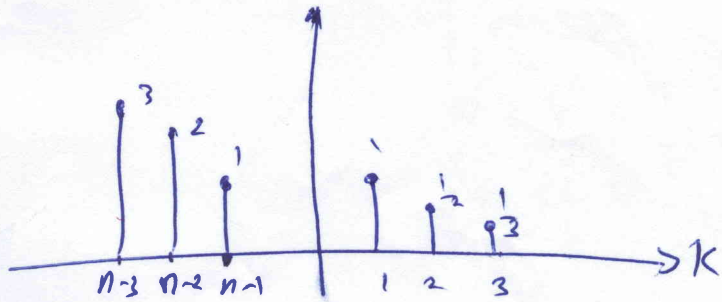


$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$









for $n-1 < 1 \Rightarrow n < 2 \Rightarrow y(n) = 0$

for $n-1 = 1 \Rightarrow n = 2 \Rightarrow y(n) = 1*1 = 1$

$n = 3 \Rightarrow y(n) = 2*1 + 1*\frac{1}{2} = 2.5$

$n = 4 \Rightarrow y(n) = 1*3 + 2*\frac{1}{2} + 1*\frac{1}{3} = 4\frac{1}{3}$

$n = 5 \Rightarrow y(n) = \frac{1}{3}*2 + \frac{1}{2}*3 = 2.166$

$n = 6 \Rightarrow y(n) = 3*\frac{1}{3} = 1$

for $n \geq 7 \Rightarrow y(n) = 0$

For continuous LTI System, if the input is $x(t)$ and impluse response is $h(t)$ that the output $y(t)$ is go

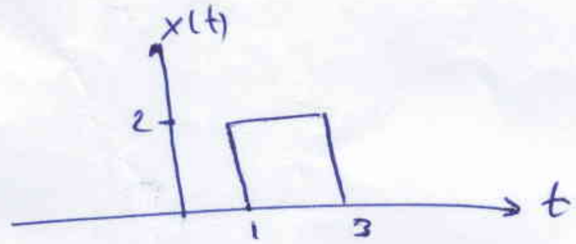
$$y(t) = x(t) \circledast h(t) \qquad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

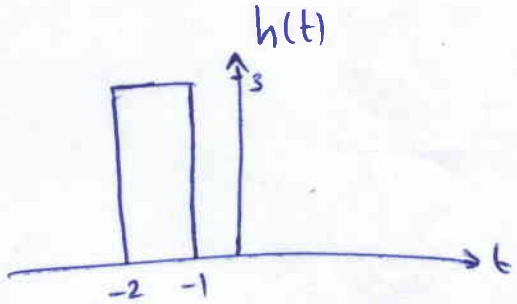
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

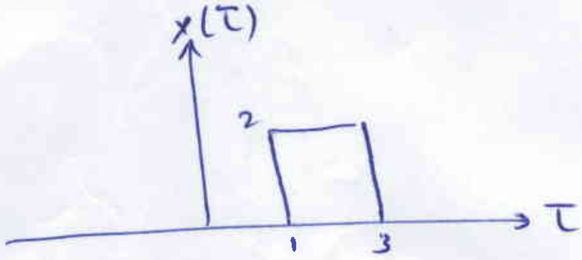
$\circledast$ → this is called <u>Convolution</u>

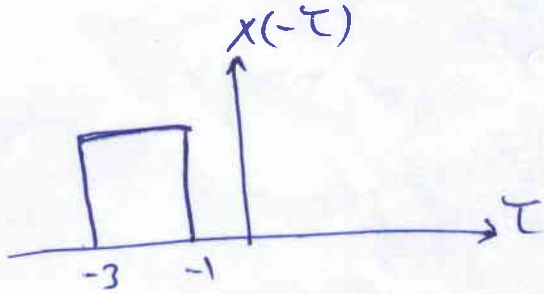
EX:

Find the output of the System whose impluse response and input signal are as shown



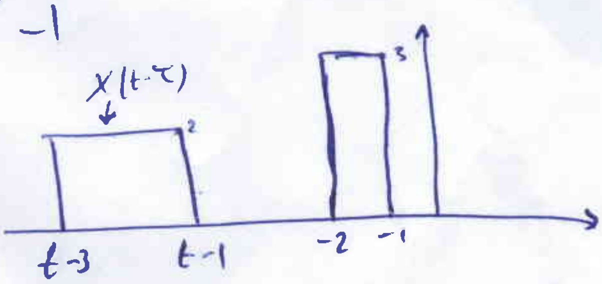






* for $t-1 \le -2 \Rightarrow t \le -1$

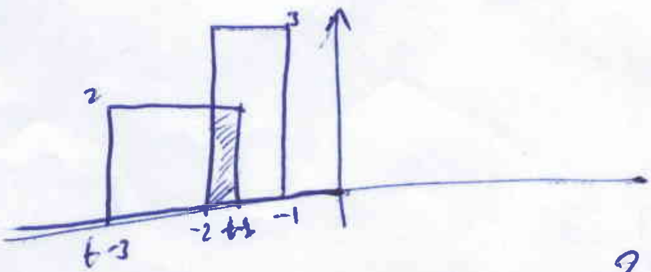
$y(t) = 0$



* for $t-1 \geq -2$ & $t-1 \le -1 \Rightarrow -1 \le t \le 0$

$$y(t) = \int_{-2}^{t-1} 2\times 3\, d\tau = 6\tau \Big|_{-2}^{t-1}$$

$$= 6(t-1) - 6(-2) = \underline{\underline{6t+6}}$$

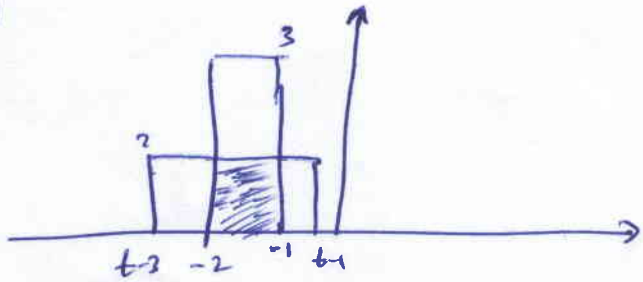


* for $t-1 \geq -1$ & $t-3 \leq -2$

$0 \leq t \leq 1$

$$y(t) = \int_{-2}^{-1} 2*3 \, d\tau = 6\tau \Big|_{-2}^{-1}$$

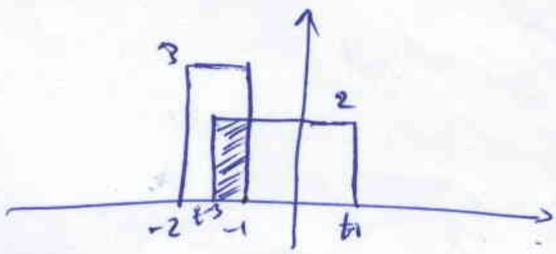
$$= \underline{\underline{6}}$$



* for $1 \leq t \leq 2$ $(t-1 \geq 0$ & $t-3 \leq -1)$

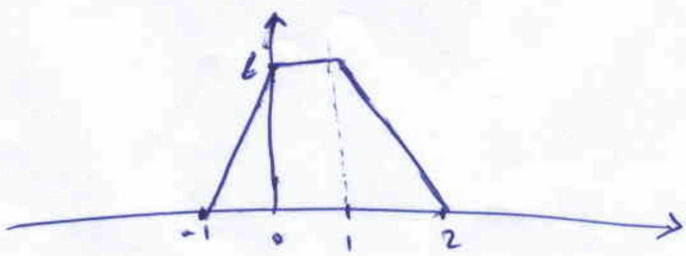
$$y(t) = \int_{t-3}^{-1} 6 \, d\tau = 6\tau \Big|_{t-3}^{-1}$$

$$= -6 - 6t + 18 = \underline{\underline{12 - 6t}}$$

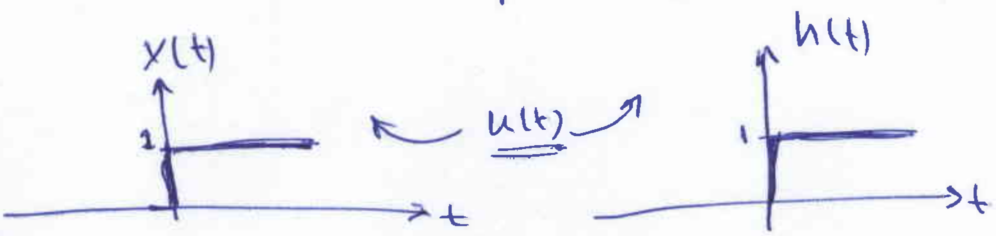


* for $t \geq 2$, $y(t) = 0$

the final output



EX: Find the output $y(t)$ of an LTI System for the input and impulse response given:

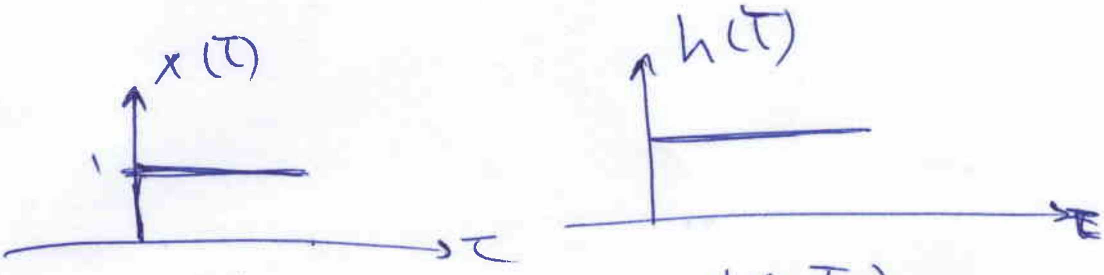


Review with step

$x(t)$ $h(t)$

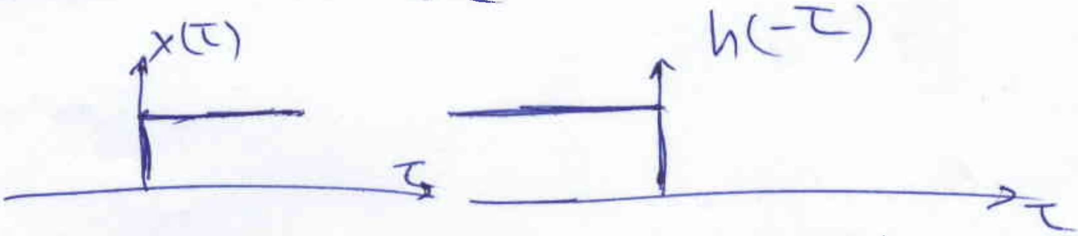
$\downarrow t=\tau$ $\downarrow t=\tau$

$x(\tau)$ fixed $h(\tau)$



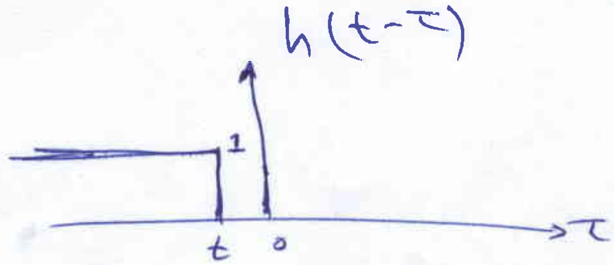
$\downarrow$ Reflection

$h(-\tau)$



$h[-(\tau-t)]$ $\downarrow$ Shifting

$h(t-\tau)$

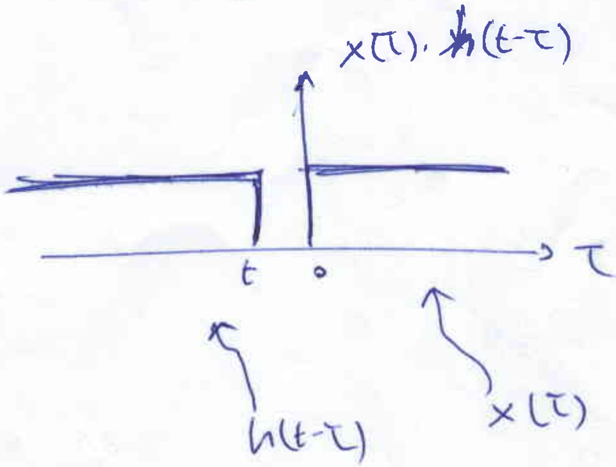


$\downarrow$ Mul.

$x(\tau)\cdot h(t-\tau)$

$\downarrow$

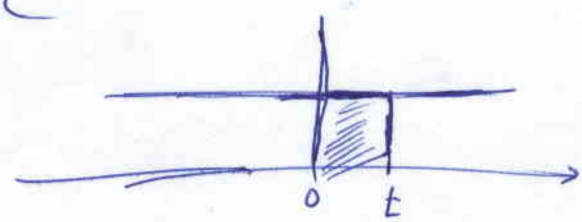
$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$



$t < 0 \longrightarrow y(t) = 0$

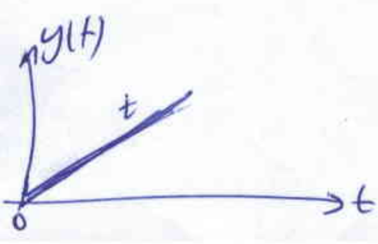
$t > 0 \longrightarrow y(t) = \int_0^t 1 * 1 \, d\tau$

$y(t) = \tau \Big|_0^t = \underline{\underline{t}}$

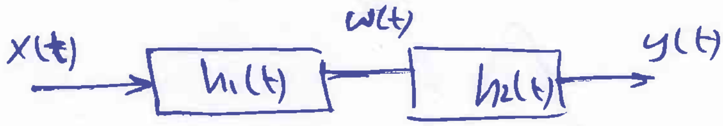


$$y(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases}$$

$y(t) = r(t)$ ramp Signal $\longrightarrow$



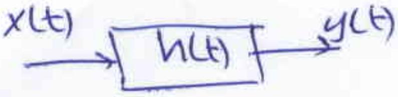
EX: The System below is formed by connecting two Systems in cascade, the impulse response of the systems are given by $h_1(t)$ & $h_2(t)$



$h_1(t) = u(t)$

$h_2(t) = e^{-at}\, u(t), \; a>0$

Find the impulse response $h(t)$ of the over all System



convolution

$w(t) = x(t) \circledast h_1(t)$

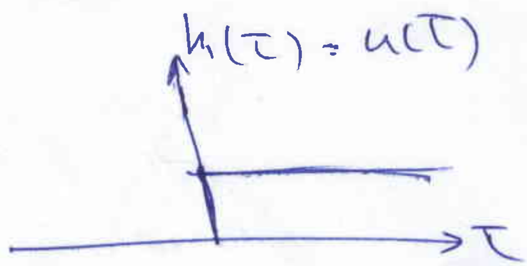
$y(t) = w(t) \circledast h_2(t)$

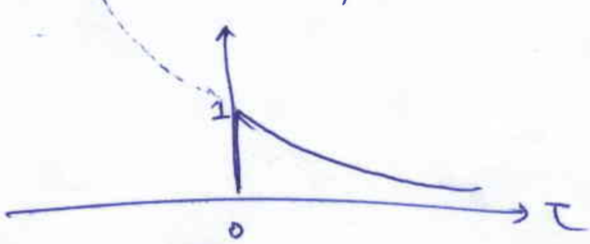
$y(t) = x(t) \circledast \underline{h_1(t) \circledast h_2(t)}$

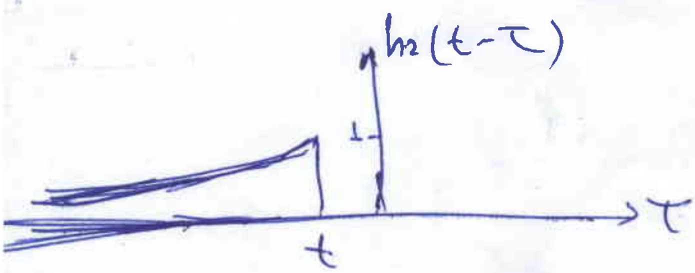
$y(t) = x(t) \circledast h(t)$, then the impulse response of the overall System is $\boxed{h(t) = h_1(t) \circledast h_2(t)}$

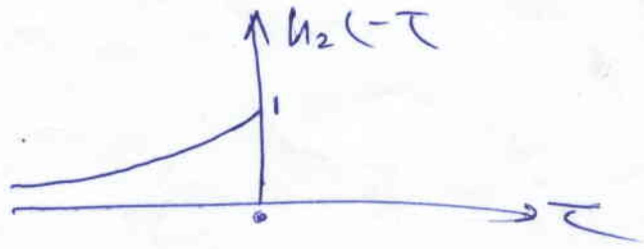
$$h(t) = \int_{-\infty}^{\infty} h_1(\tau)\, h_2(t-\tau)$$

$h_2(\tau) = e^{-a\tau}\, u(\tau)$



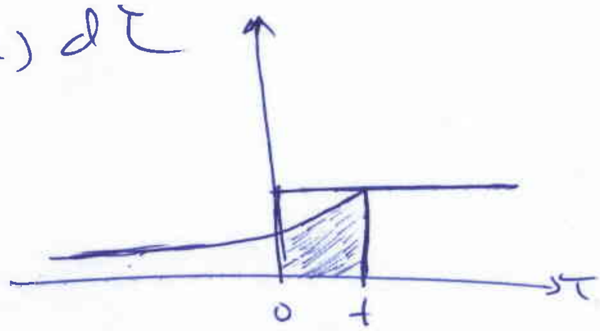






$t < 0 \longrightarrow h(t) = 0$ do not have overlap

$t > 0 \rightarrow h(t) = \int_0^t e^{-a\tau} u(\tau)\, d\tau$



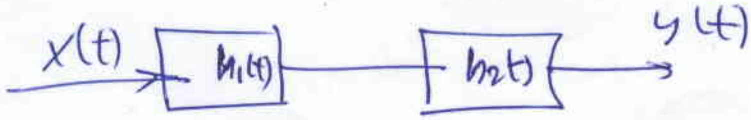
$$h(t) = \left. \frac{e^{-a\tau}}{-a} \right|_0^t$$

$$= \frac{-1}{a} \left. \left(e^{-a\tau}\right) \right|_0^t$$

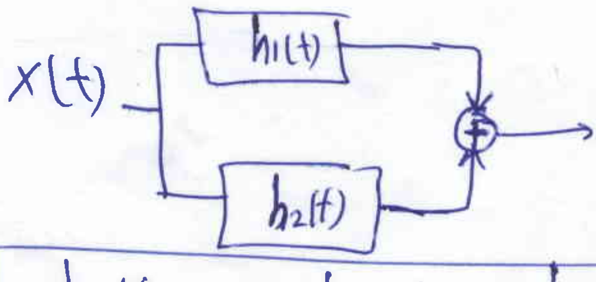
$$= \frac{-1}{a} \left(e^{-at} - 1\right)$$

$$= \frac{1}{a} \left(1 - e^{-at}\right) u(t)$$

For

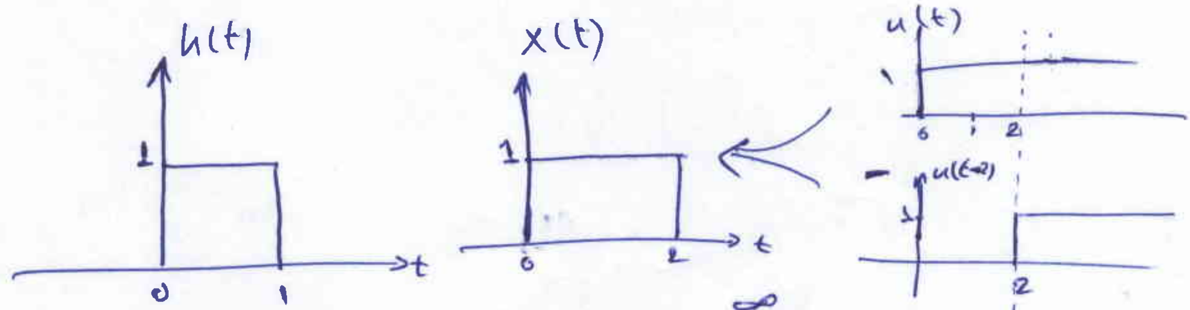
① 

$$\boxed{h(t)_{total} = h_1(t) \circledast h_2(t)}$$

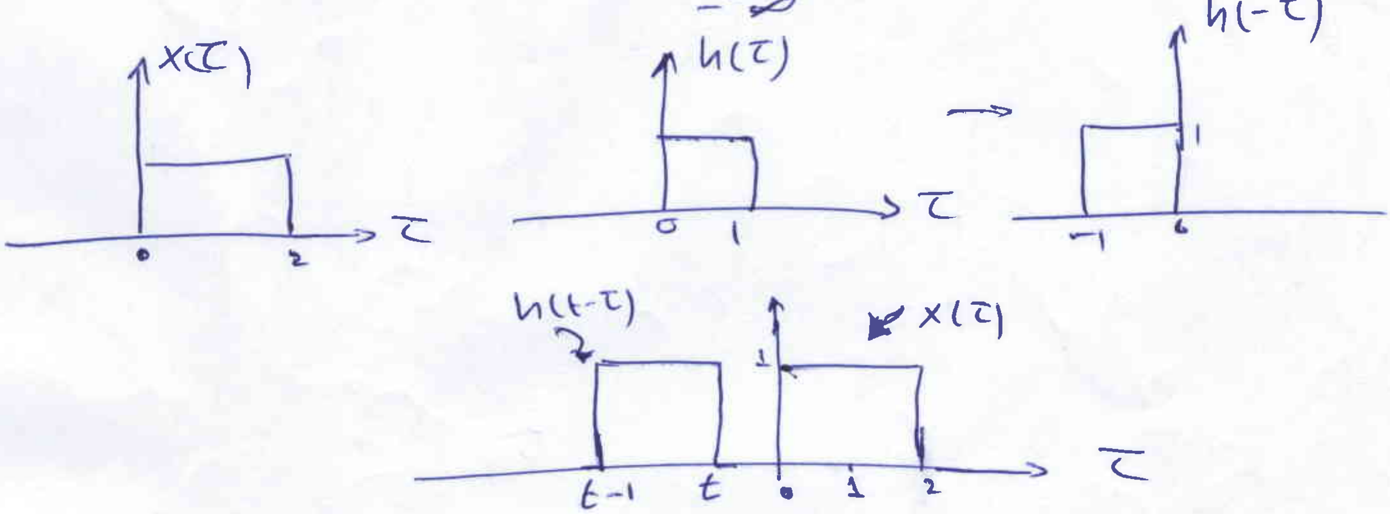
② 

$$\boxed{h(t)_{total} = h_1(t) + h_2(t)}$$

EX: LTI System has impulse response h(t) shown below, sketch output when input $x(t) = u(t) - u(t-2)$



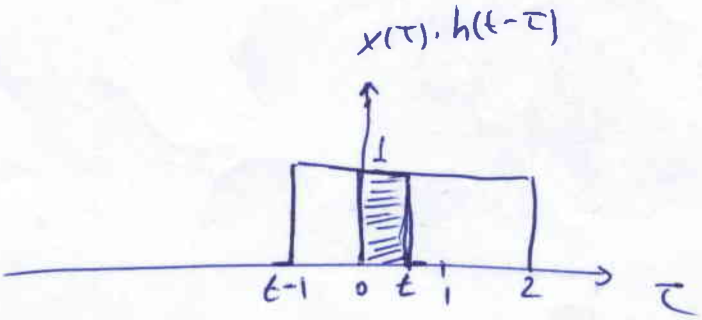
$$y(t) = x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$



Case 1 : $t < 0 \rightarrow$ no overlap

$\boxed{y(t) = 0}$

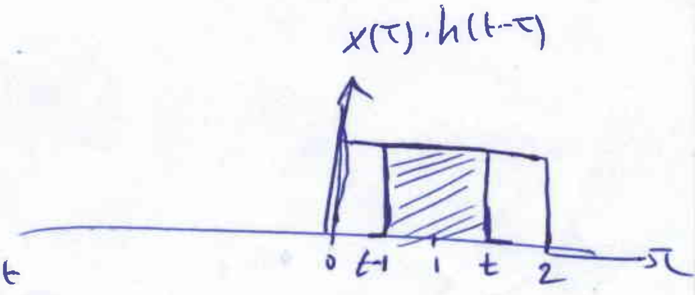
Case 2 : $0 < t < 1$



$$y(t) = \int_0^t 1 \times 1\, d\tau = \tau \Big|_0^t$$

$\boxed{y(t) = t}$

Case 3 : $1 < t < 2$

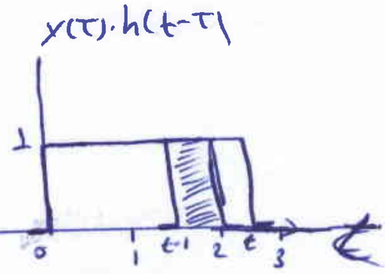


$$y(t) = \int_{t-1}^{t} 1 \times 1\, d\tau = \tau \Big|_{t-1}^{t}$$

$y(t) = t - (t-1) = \cancel{t} - \cancel{t} + 1$

$\boxed{y(t) = +1}$

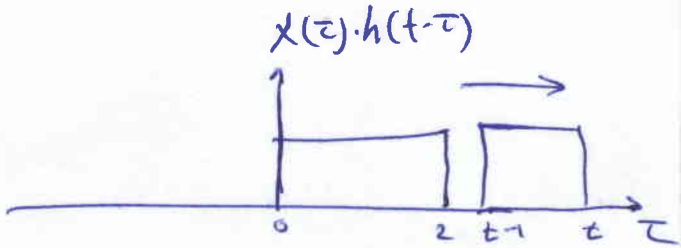
Case 4: $2 < t < 3$



$$y(t) = \int_{t-1}^{2} 1 \, d\tau = \tau \Big|_{t-1}^{2}$$

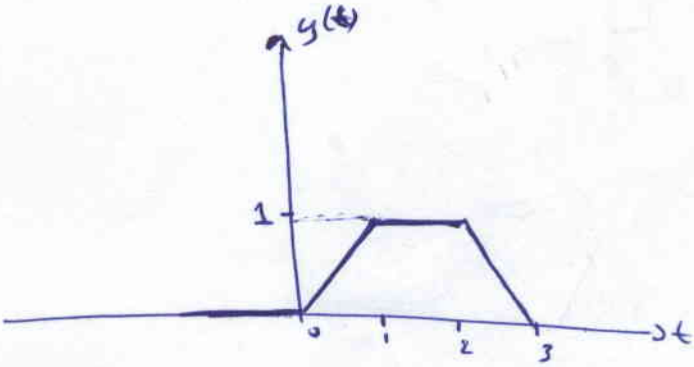
$$\boxed{y(t) = 3 - t}$$

Case 5: $t > 3$

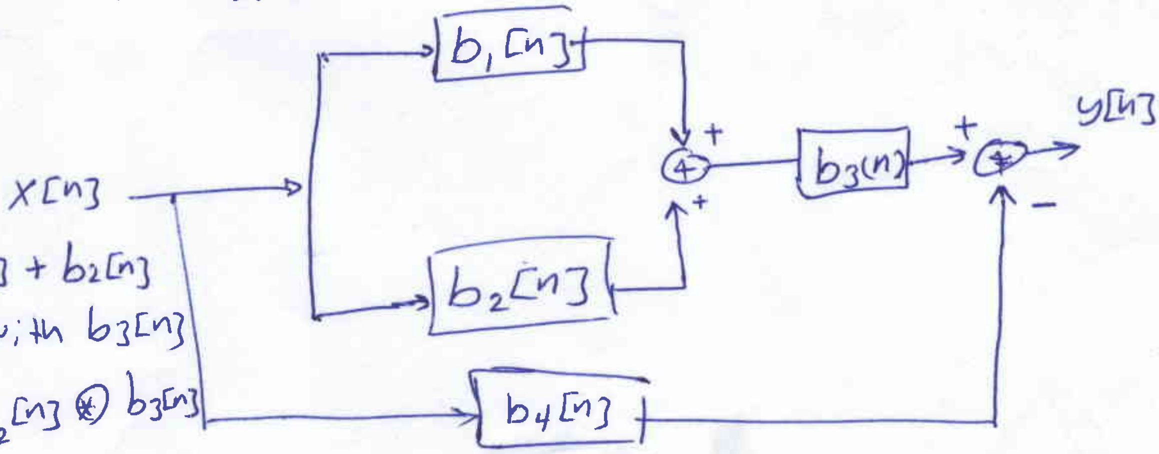


$$\boxed{y(t) = 0}$$

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & t > 3 \end{cases}$$



EX: consider the inter connection of LTI Systems depicted in the figure below. Find the overall impulse response for the System.



solu

* $b_{12}[n] = b_1[n] + b_2[n]$

this is in Series with $b_3[n]$

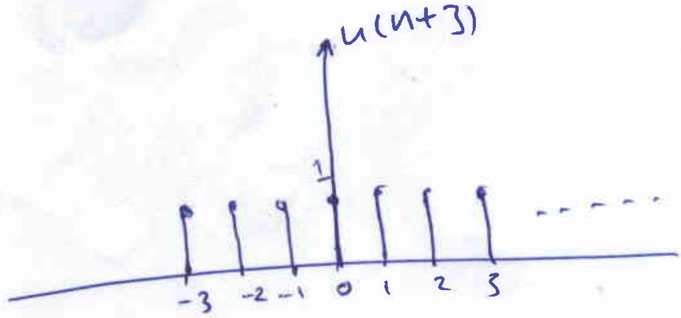
So, $b_{123}[n] = b_{12}[n] \circledast b_3[n]$

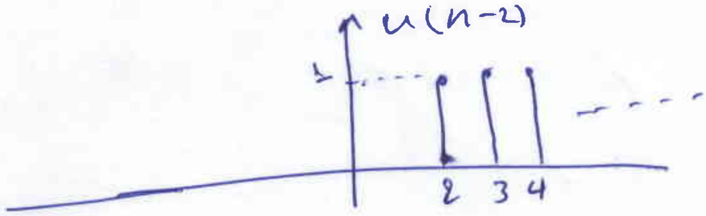
$b_{123}[n] = \left(b_1[n] + b_2[n]\right) \circledast b_3[n]$

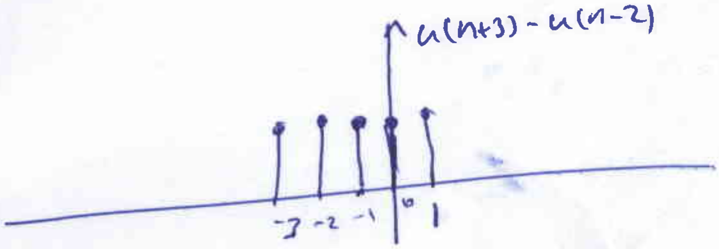
The upper branch is in parallel with the lower branch, $(b_4[n])$; hence the overall System impulse response is $b(n) = b_{123}(n) - b_4(n)$

$\therefore \; b[n] = \left(b_1[n] + b_2[n]\right) * b_3[n] - b_4[n]$

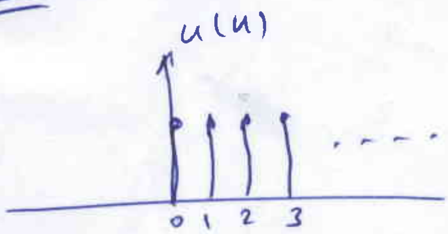
EX: sketch $u(n+3) - u(n-2)$

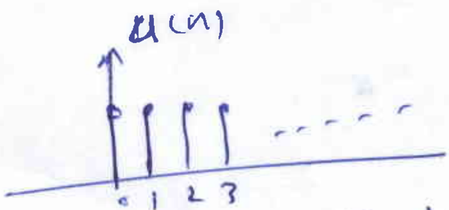


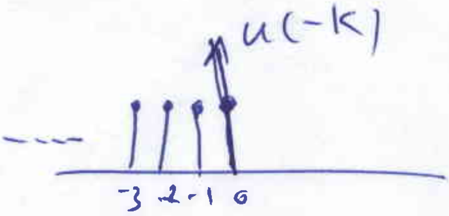


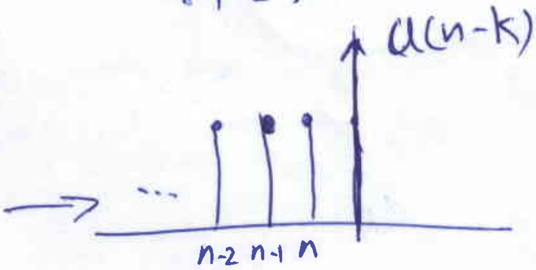


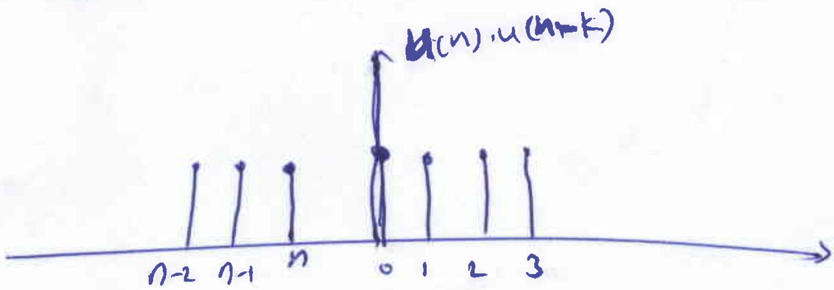
EX: find the convolution of $u(n) \circledast u(n)$











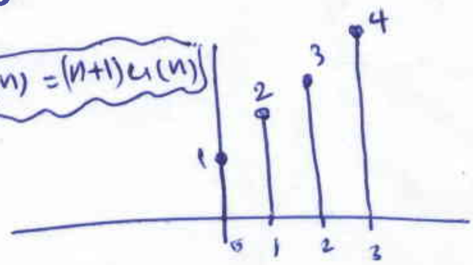
$n < 0 \longrightarrow y(n) = 0$

$n = 0 \longrightarrow y(n) = 1 \times 1 = 1$

$n = 1 \longrightarrow y(n) = 1 \times 1 + 1 \times 1 = 2$

$n = 2 \longrightarrow y(n) = 1 \times 1 + 1 \times 1 + 1 \times 1 = 3$

$\Rightarrow \boxed{y(n) = (n+1)u(n)}$



* Fourier Series (FS):

A continuous periodic signal, $x(t)$ with period T is expressed as a sum of an infinite set of harmonically related sinusoids and a dc component in the FS.

$x(t)$ = dc (avarage value of $x(t)$) + Cosine terms + Sine terms

$$x(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots$$
$$+ b_1 \sin\omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \cdots$$

The frequency of the fundamental or first harmonic is the frequency of the waveform under analysis.

That is $\omega_0 = 2\pi f_0$

$f_0 = \frac{1}{T} \Rightarrow \omega_0 = 2\pi/T$

f_0: fundamental frequency

The frequency of the second harmonic is $2\omega_0$, that of the third harmonic is $3\omega_0$ and so on.

Hence, Fourier Series Equation has dc & Ac terms, So it can be simplified as:

* Trigonometric Fourier Series:

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}$$

To find a_0 :

$$\int_0^T x(t)\,dt = \int_0^T a_0\,dt + \cancel{\int_0^T a_1 \cos\omega_0 t\,dt}^{\,0} + \cancel{\int_0^T a_2 \cos 2\omega_0 t\,dt}^{\,0} + \cdots$$

$$+ \cancel{\int_0^T b_1 \sin\omega_0 t\,dt}^{\,0} + \cancel{\int_0^T b_2 \sin 2\omega_0 t\,dt}^{\,0} + \cdots$$

$$\int_0^T x(t)\,dt = a_0 T$$

$$\boxed{a_0 = \frac{1}{T}\int_0^T x(t)\,dt}$$ and so on

Generally we can write a_0, a_n and b_n as shown below:

Fourier Coefficient

* $a_0 = \dfrac{1}{T}\displaystyle\int_0^T x(t)\,dt$
* $a_n = \dfrac{2}{T}\displaystyle\int_0^T x(t)\cos n\omega_0 t\,dt$
* $b_n = \dfrac{2}{T}\displaystyle\int_0^T x(t)\sin n\omega_0 t\,dt$

if $x(t) \rightarrow$ Even Signal $\Rightarrow b_n = 0$ (when $x(t)$ is even there will be no sine term as sine is an odd signal and this means b_n is equal to zero)

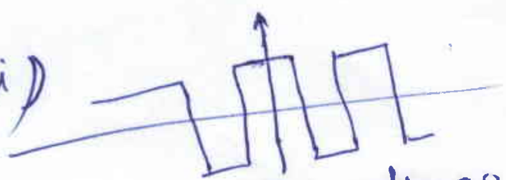
$x(-t) = x(t)$

if $x(t) \rightarrow$ Odd Signal $\Rightarrow a_n = 0$ (when $x(t)$ is odd there will be no cosine term as cosine is an even signal and this mean a_n is equal to zero)

$x(-t) = -x(t)$

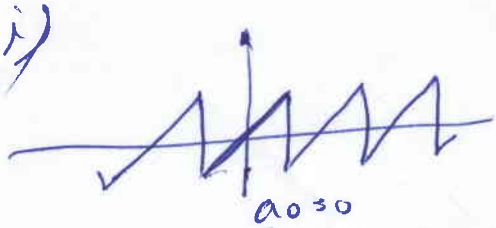
if $x(t) \rightarrow$ Symmetric about t-axis $\rightarrow \underline{\underline{a_0 = 0}}$

Ex: i)



$a_0 = 0$ & $b_n = 0$
$a_n \neq 0$

ii)



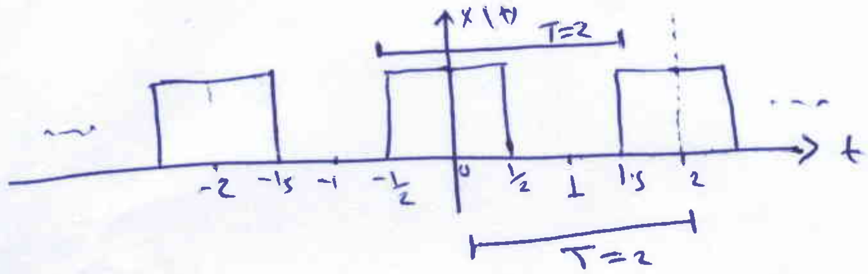
$a_0 = 0$
$a_n = 0$
$b_n \neq 0$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = \frac{2}{T}\int_{0}^{\frac{T}{2}} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos n\omega_0 t\,dt = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\sin n\omega_0 t\,dt = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\sin n\omega_0 t\,dt$$

Ex: For the periodic signal $x(t)$ shown, find Fourier Series.



$T = 2$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = \frac{2}{T}\int_{0}^{\frac{T}{2}} x(t)\,dt$$

$$= \frac{2}{2}\int_{0}^{1} x(t)\,dt = \int_{0}^{\frac{1}{2}} 1\,dt + \cancel{\int_{\frac{1}{2}}^{1} 0}$$

$$= t\Big|_{0}^{\frac{1}{2}} = \boxed{\frac{1}{2}}$$

$b_n = 0$ because $x(t)$ is EVEN

$$a_n = \frac{4}{T}\int_{0}^{\frac{T}{2}} x(t)\cos n\omega_0 t\,dt = \frac{4}{2}\left[\int_{0}^{\frac{1}{2}} 1\cdot\cos n\omega_0 t\,dt + \cancel{\int_{\frac{1}{2}}^{1} 0\cdot\cos n\omega_0 t\,dt}\right]$$

$$= 2\,\frac{\sin n\omega_0 t}{n\omega_0}\Big|_{0}^{\frac{1}{2}}, \quad \boxed{\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi}$$

$$= \frac{2}{n\pi}\left[\sin n\pi\tfrac{1}{2} - \sin 0\right] = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$\therefore\ x(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\underbrace{\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)}_{a_n}\cos n\pi t$$

EX: Find and sketch the spectrum up to 3rd harmonic for the previous example.

* what is the spectrum: It is a domain which shows how the signals change with frequency.

Spectrum has
- Magnitude (Amplitude) $\Rightarrow A_n = \sqrt{a_n^2 + b_n^2}$
- phase $\Rightarrow \phi_n = -\tan^{-1} \dfrac{b_n}{a_n}$

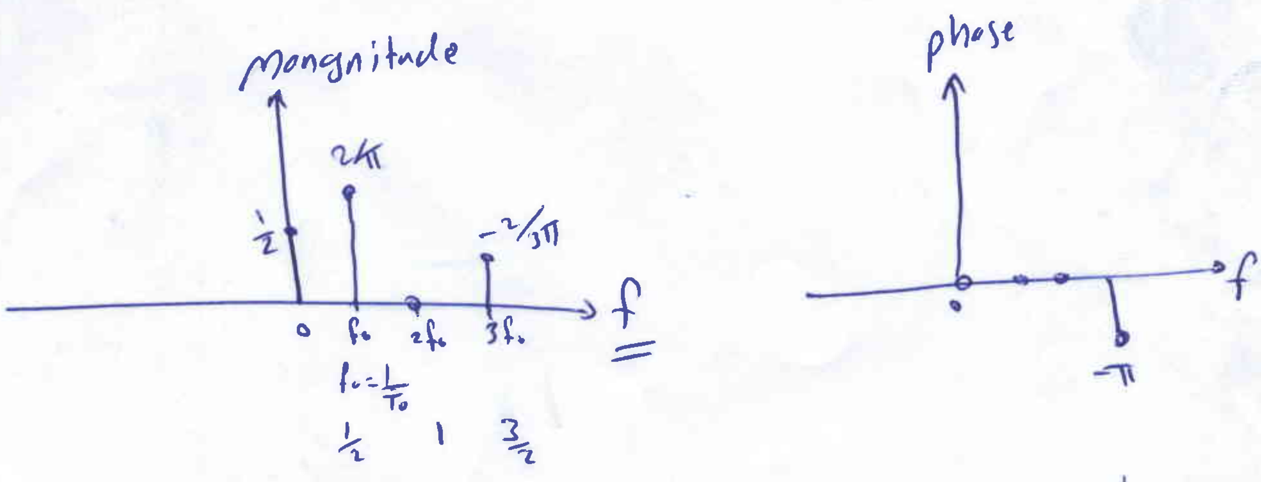
Solution (for previous Ex $a_0 = \frac{1}{2}$, $b_n = 0$, $a_n = \frac{2}{n\pi} \sin \frac{n\pi}{2}$)

$n = 0 \longrightarrow \overset{a_{n=0}}{a_0} = \frac{1}{2}$ & $b_n = 0$ $\rightarrow A_n = \sqrt{(\frac{1}{2})^2 + 0} = \boxed{\frac{1}{2}}$

$\phi_n = -\tan^{-1} \dfrac{0}{0.5} = \boxed{0}$

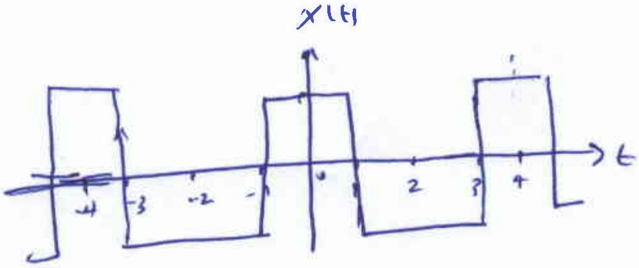
	n	a_n	b_n	A_n	ϕ_n
fundamental	1	$\frac{2}{\pi}$	0	$2/\pi$	0
2nd harmonic	2	0	0	0	0
3rd harmonic	3	$-\frac{2}{3\pi}$	0	$2/3\pi$	$-\pi$ (-180)

[illegible] ϕ, a_n [illegible]



Spectrum

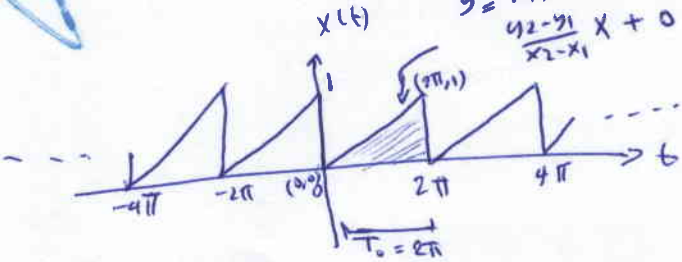
H.W : find Fourier Series of the following Signal



EX 2: find Fourier Series for the following Signal

$y = mx + c$

$\frac{y_2 - y_1}{x_2 - x_1} x + 0$



$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$y = \frac{1-0}{2\pi - 0} \cdot x \Rightarrow y = \frac{x}{2\pi}$$

$$\boxed{x(t) = \frac{t}{2\pi}}$$

$$a_0 = \frac{1}{T} \int_0^{2\pi} x(t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{t}{2\pi}\, dt = \frac{1}{4\pi^2} \left[\frac{t^2}{2} \right|_0^{2\pi}$$

$$= \frac{1}{4\pi^2} \left(\frac{4\pi^2}{2} - 0 \right) = \boxed{\frac{1}{2}}$$

$$a_n = \frac{2}{T_0} \int_0^{T} x(t) \cos n\omega_0 t\, dt$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1 \text{ rad/sec}$$

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} \frac{t}{2\pi} \cos nt\, dt$$

$$a_n = \frac{1}{2\pi^2} \int_0^{2\pi} \underbrace{t}_{u} \cdot \underbrace{\cos nt\, dt}_{dv}$$

$$u\,dv = uv - \int v\,du$$

$$a_n = \frac{1}{2\pi^2}\left[t \cdot \frac{\sin nt}{n} - \int \frac{\sin nt}{n} \cdot 1\, dt\right]$$

$$a_n = \frac{1}{2\pi^2}\left[t \cdot \frac{\sin t}{n} - \left[-\frac{\cos nt}{n^2}\right]\right]_0^{2\pi}$$

$$a_n = \frac{1}{2\pi^2}\left[\frac{t \cdot \sin t}{n} + \frac{\cos nt}{n^2}\right|_0^{2\pi}$$

$$a_n = \frac{1}{2\pi^2}\left[\cancel{\frac{2\pi \sin n 2\pi}{n}}^{0} - \cancel{\frac{0 \sin n(0)}{n}}^{0} + \cancel{\frac{\cos n 2\pi}{n^2}}^{1} - \cancel{\frac{\cos n(0)}{n^2}}^{1}\right]$$

$$a_n = \frac{1}{2\pi^2}\left[0 - 0 + \cancel{\frac{1}{n^2} - \frac{1}{n^2}}^{0}\right]$$

$$a_n = \frac{1}{2\pi^2}[0]$$

$$\boxed{a_n = 0}$$

$$\boxed{b_n = \frac{-1}{n\pi}}$$

So, $x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{n\pi} \sin n \overset{\omega_0}{1} t$

$$\boxed{x(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nt}$$

Parseval's theorem :

If the fourier Series function is :

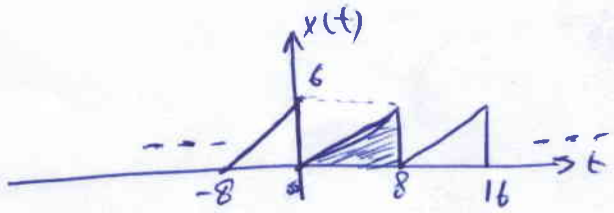
$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

The parseval's theorem is

$$\boxed{P_T = \frac{1}{T}\int_0^T |x(t)|^2 \, dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

Ex : For the periodic Signal $x(t)$ is shown below find the total power & d.c power & percentage of power contained in harmonics up to 3rd.

① The total power



$T = 8$

$$x(t) = \frac{y_2 - y_1}{x_2 - x_1} t + 0$$

$$x(t) = \frac{6-0}{8-0} t$$

$$\boxed{x(t) = \frac{6}{8} t}$$

$$P_T = \frac{1}{T}\int_0^T |x(t)|^2 \, dt$$

$$P_T = \frac{1}{8}\int_0^8 \left|\frac{6}{8} t\right|^2 dt$$

$$= \frac{6^2}{8^3}\left[\frac{t^3}{3}\right|_0^8$$

$$= \frac{6^2 \cdot \cancel{8^3}}{\cancel{8^3} \cdot 3} = \frac{36}{3} = \boxed{12}$$

$* P_{dc} = a_0^2$

$$a_0 = \frac{1}{8}\int_0^8 \frac{6}{8}t \, dt$$

by: Abdulbasit Subaawi

$$a_0 = \frac{6}{8^2}\left[\frac{t^2}{2}\right|_0^8$$

$$a_0 = \frac{6 * \cancel{8^2}}{\cancel{8^2} * 2} = \frac{6}{\cancel{2}} = \boxed{3}$$

$$\boxed{a_0^2 = 9 = P_{d.c}}$$

$*$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t \, dt$$

$$= \frac{1}{4}\int_0^8 \frac{\cancel{6}\,3}{\cancel{8}\,4} \underbrace{t}_{u} \underbrace{\cos n\omega_0 t}_{dv}$$

$$\boxed{u\,dv = uv - \int v\,du}$$

$$= \frac{3}{4*4}\left[t \cdot \frac{\sin n\omega_0 t}{n\omega_0} - \int \frac{\sin n\omega_0 t}{n\omega_0} \cdot 1 \, dt\right]_0^8$$

$$= \frac{3}{16}\left[t \cdot \frac{\sin n\frac{\pi}{4}t}{\frac{n\pi}{4}} + \frac{\cos n\frac{\pi}{4}t}{\left(\frac{n\pi}{4}\right)^2}\right|_0^8\Bigg]$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$= \frac{3 \times 4}{16 * n\pi}\left[\cancel{t \sin n\frac{\pi}{4}t} + \cancel{\frac{\cos n\frac{\pi}{4}t}{\frac{n\pi}{4}}}\right]_0^8$$

$\sin(2\pi) = 0$

$\sin(0) = 0$

$\cos(2\pi) = 1$

$\cos(0) = 1$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1\times 3}{4\times 4}\int_0^8 \underbrace{t}_{u}\cdot \underbrace{\sin n\omega_0 t\, dt}_{dv}$$

$$= \frac{3}{16}\left[-t\cdot\frac{\cos n\omega_0 t}{n\omega_0} + \int \frac{\cos n\omega_0 t}{n\omega_0}\,dt\right]$$

$$= \frac{3}{16}\left[-t\cdot\frac{\cos n\omega_0 t}{n\omega_0} + \frac{\sin n\omega_0 t}{(n\omega_0)^2}\right|_0^8$$

$$= \frac{3}{16\times n\omega_0}\left[-t\cdot\cos n\omega_0 t + \frac{\sin n\omega_0 t}{n\omega_0}\right|_0^8$$

$$= \frac{3}{16\times n\times\frac{\pi}{4}}\left[-t\cdot\cos n\frac{\pi}{4}t + \frac{\sin n\frac{\pi}{4}t}{\frac{n\pi}{4}}\right|_0^8$$

$$= \frac{4\times 3}{16\times n\times\pi}\left[-8\right] = \boxed{\frac{-6}{n\pi}}$$

$$P_n = \frac{1}{2}\left(a_n^2 + b_n^2\right)$$

$$= \frac{1}{2}b_n^2 \qquad , \boxed{b_n = \frac{-6}{n\pi}}$$

$$P_1 = \frac{1}{2}\left(\frac{-6}{\pi}\right)^2$$

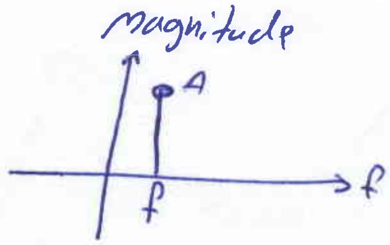
$$P_2 = \frac{1}{2}\left(\frac{-3}{\pi}\right)^2$$

$$P_3 = \frac{1}{2}\left(\frac{-2}{\pi}\right)^2$$

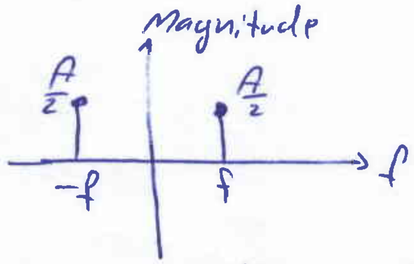
$$P_{\text{up to 3rd}} = P_0 + P_1 + P_2 + P_3 = 11.48$$ (3-7)

$$\% = \frac{P_{\text{up to 3rd}}}{P_T} = \frac{11.48}{12}\times 100 = \boxed{95.6\,\%}$$

Ex: Sketch the single-side and double-side spectrum of the signal $x(t) = A\cos(2\pi f t)$



Single-side spectrum



double-side spectrum.

* Complex Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right]$$

* $\boxed{\frac{1}{j} = -j}$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(\frac{1}{2}e^{jn\omega_0 t}(a_n - jb_n) + \frac{1}{2}e^{-jn\omega_0 t}(a_n + jb_n)\right)$$

$$x(t) = a_0 + \sum_{n=1}\left(C_n e^{jn\omega_0 t} + C_{-n} e^{-jn\omega_0 t}\right)$$

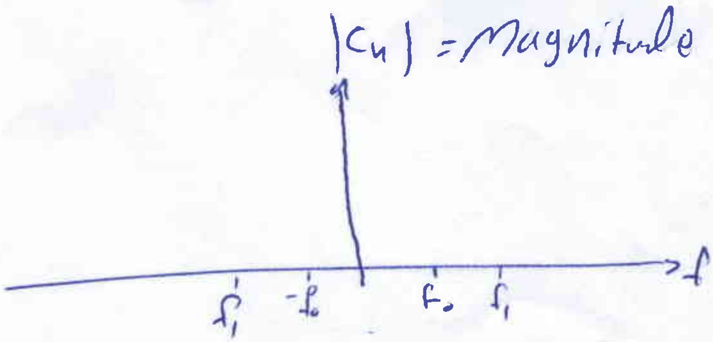
$$C_n = \frac{a_n - jb_n}{2}$$

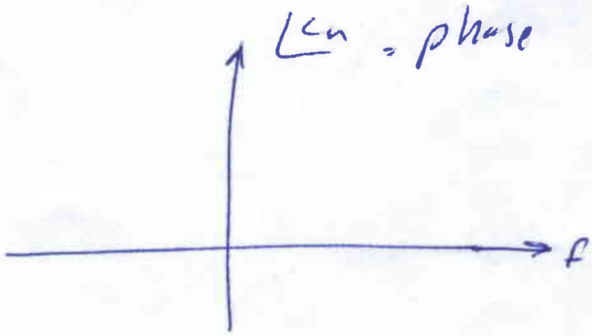
$$C_{-n} = \frac{a_n + jb_n}{2}$$

* $$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

* $$C_0 = a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

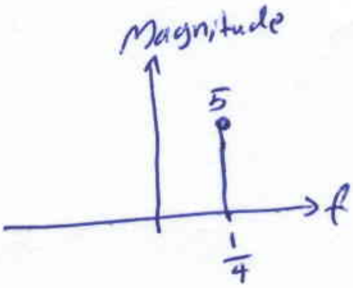
* $$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt$$





EX: Sketch Single side and double side Spectrum of $x(t) = 5\cos\left(\frac{\pi t}{2} + \frac{\pi}{3}\right)$

* Single-side



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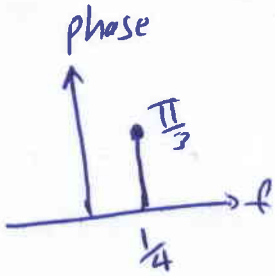


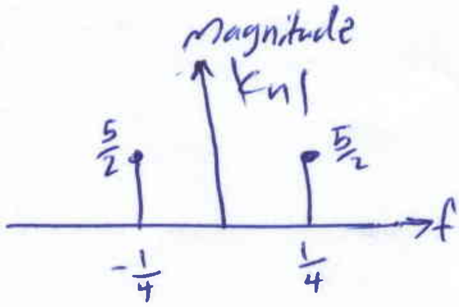
* double-side

$C_n = \frac{1}{2}(a_n - jb_n)$

$x(t)$ is Even So $\underline{\underline{b_n = 0}}$

$C_n = \frac{1}{2} a_n$

$C_1 = \frac{a_1}{2} = \frac{5}{2}$



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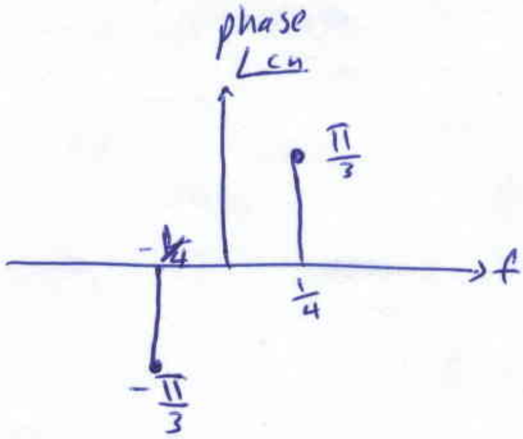
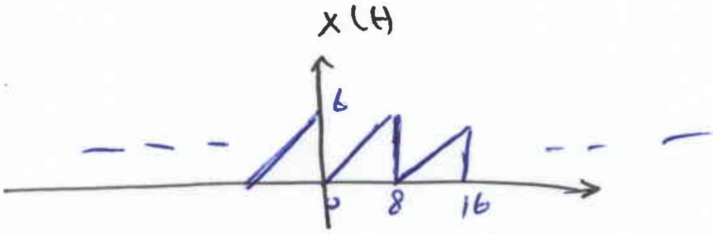


EX: Sketch double-side spectrum of the periodic signal shown:-



$$C_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$T = 8 \,, \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$C_n = \frac{1}{8}\int_0^8 \frac{6}{8}t\, e^{-jn\frac{\pi}{4}t}\, dt \qquad \boxed{udv = uv - \int v du}$$

$$= \frac{6}{8*8}\left[t\, \frac{e^{-jn\frac{\pi}{4}t}}{-j\frac{n\pi}{4}} - \int_0^8 \frac{e^{-j\frac{n\pi}{4}t}}{-j\frac{n\pi}{4}}\, dt \right]$$

$$= \frac{j6*4}{8*8*n\pi}\left[t\, e^{-jn\frac{\pi}{4}t} - \frac{e^{-j\frac{n\pi}{4}t}}{-j\frac{n\pi}{4}} \Big|_0^8 \right]$$

$$= \frac{j6*4}{8*8*n\pi}\left[8\, \underline{\underline{e^{-jn2\pi}}} + \frac{e^{-j2n\pi}}{j\frac{n\pi}{4}} - \frac{1}{j\frac{n\pi}{4}} \right]$$

$$e^{-j2\pi n} = \cos 2\pi n - j\cancel{\sin 2\pi n}^{\,0} = \underline{\underline{1}}$$

$$C_n = \frac{8*j6*4}{8*8*n\pi} = \boxed{\frac{j3}{n\pi}}$$

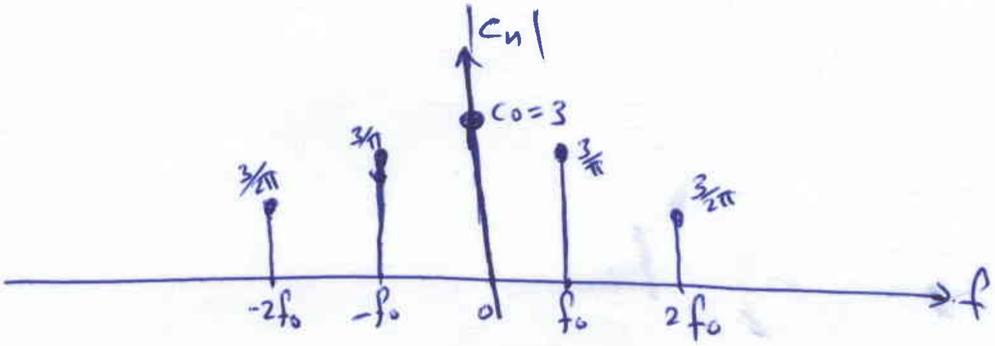
$$C_0 = \frac{1}{T}\int_0^8 x(t)\, dt = \boxed{3}$$

$C_n = \dfrac{j3}{n\pi}$ So, the Magnitude of $|C_n| = \sqrt{(Re)^2 + (Im)^2}$

$|C_n| = \sqrt{(0)^2 + \left(\dfrac{3}{n\pi}\right)^2} \Rightarrow |C_n| = \dfrac{3}{n\pi}$

n = integer , 1, 2, 3 - - - - -

$C_1 = \frac{3}{\pi}$, $C_2 = 3/2\pi$, and so on



We can noticed that the Magnitude → Even Symmetry

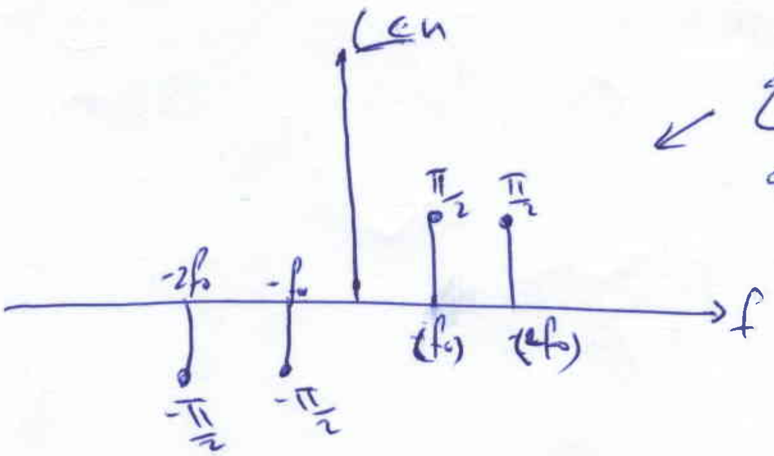
and the phase $\angle C_n$ is odd Symmetry

* notice that

$C_n = j\dfrac{3}{n\pi}$

$\boxed{j = e^{j\frac{\pi}{2}}}$

$\angle C_n = \dfrac{\pi}{2}$



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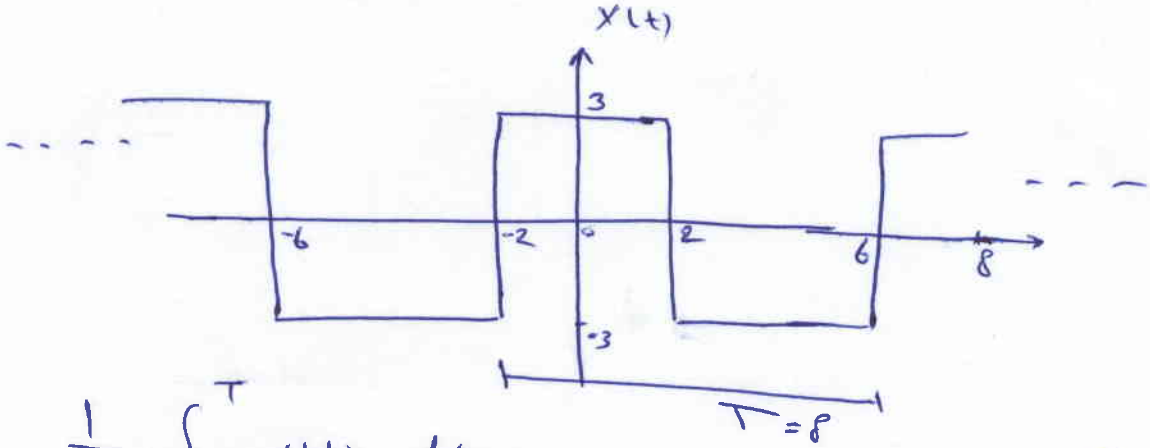
odd Sym.

EX: Use complex fourier Series to find the Spectrum of the following Signal.



$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$T = 8 \;,\; \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$C_0 = \frac{1}{8}\left[\int_{-2}^{2} 3\,dt + \int_{2}^{6} -3\,dt\right]$$

$$C_0 = \frac{3}{8}\left[t\Big|_{-2}^{2} + -t\Big|_{2}^{6}\right]$$

$$= \frac{3}{8}\left[2+2-6+2\right] = \boxed{0}$$

$$C_n = \frac{1}{T}\int_0^T x(t)\,e^{-jn\omega_0 t}\,dt$$

$$C_n = \frac{1}{8}\left[\int_{-2}^{2} 3\,e^{-jn\omega_0 t}\,dt + \int_{2}^{6} -3\,e^{-jn\omega_0 t}\,dt\right]$$

$$\boxed{\omega_0 = \frac{\pi}{4}}$$

$$C_n = \frac{3}{8}\left\{\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\Big|_{-2}^{2} + \frac{-e^{-jn\omega_0 t}}{-jn\omega_0}\Big|_{2}^{6}\right]$$

$$= \frac{3}{-jn\omega_0 \times 8}\left[e^{-j2n\omega_0} - e^{j2n\omega_0} - e^{-j6n\omega_0} + e^{-j2n\omega_0}\right]$$

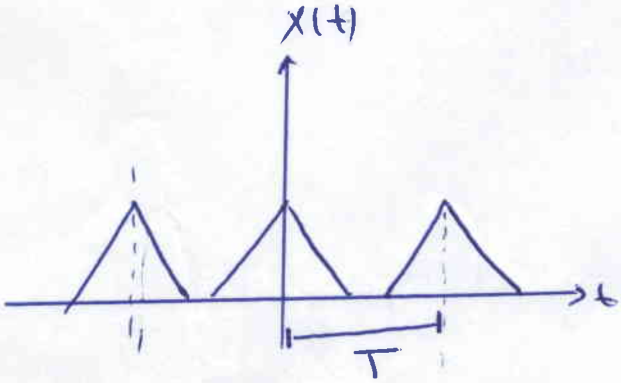
$$= \frac{3}{-j2n\pi}\left[\underline{\underline{e^{-jn\frac{\pi}{2}}}} - e^{jn\frac{\pi}{2}} - e^{-jn\frac{3}{2}\pi} + \underline{\underline{e^{-jn\frac{\pi}{2}}}}\right]$$

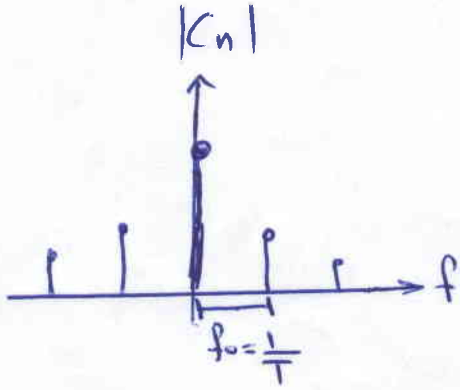
* Fourier Transforms :

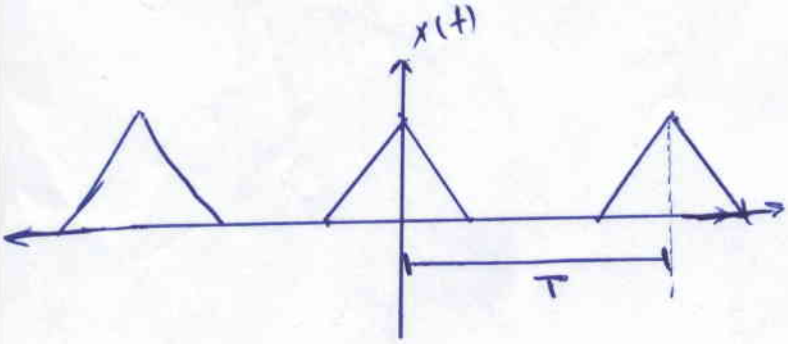
Let $x(t)$ be periodic Signal of period T

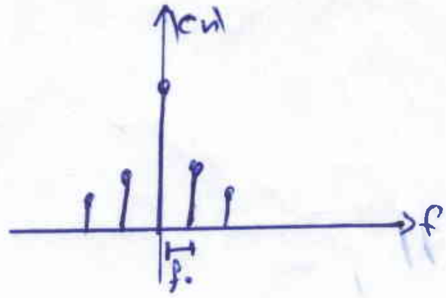
$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$$

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j2\pi n f_0 t} \, dt$$



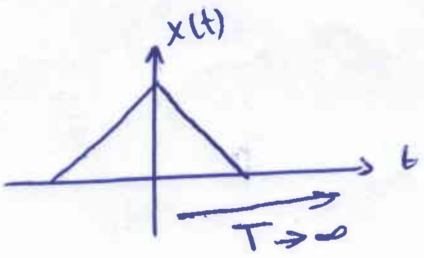


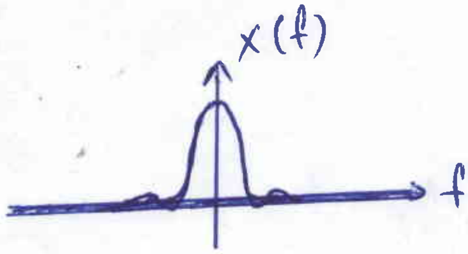




We can noticed that if (T) becomes larger, the frequency (f) becomes closer

as $T \to \infty$ (non-periodic Signal) $nf_0 \to f$





as $T \rightarrow \infty$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, e^{-j2\pi f t}\, dt$$

$$T \cdot c_n = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$\boxed{X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt}$$ Fourier Transform

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df}$$ Inverse Fourier Transform

Q/ why do we care about fourier Transform (FT) ?

Sol/ physically, it will tell us the frequency components of any non-periodic Signal and many Signals are composed of many different frequencies. In other words, Fourier Transform is mathematical tool used for frequency analysis of Signals.

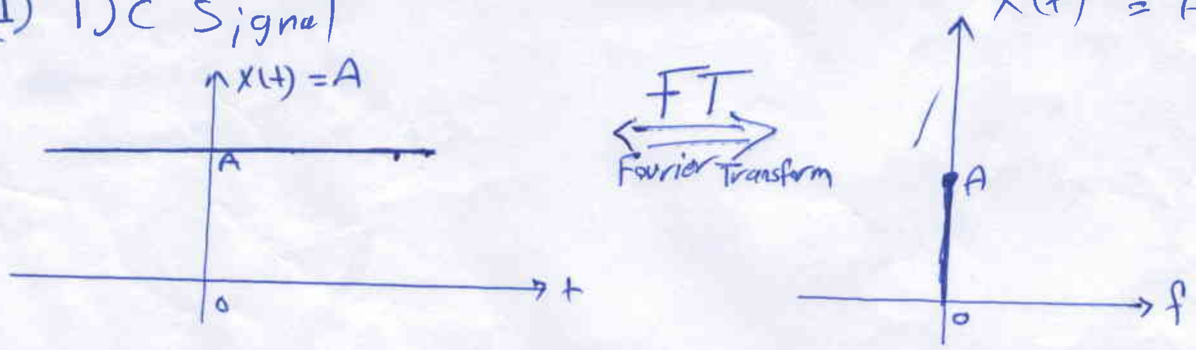
* Determine the Fourier Transform of the following:

1. DC Signal
2. Cosine Signal
3. Rectangular pluse
4. Multiplication
5. convolution
6. Periodic Signal (e.g: rectangular pulses)
7. Sampled Signal
8. periodic and Sampled Signal

" " " " " " " "

(1) DC Signal

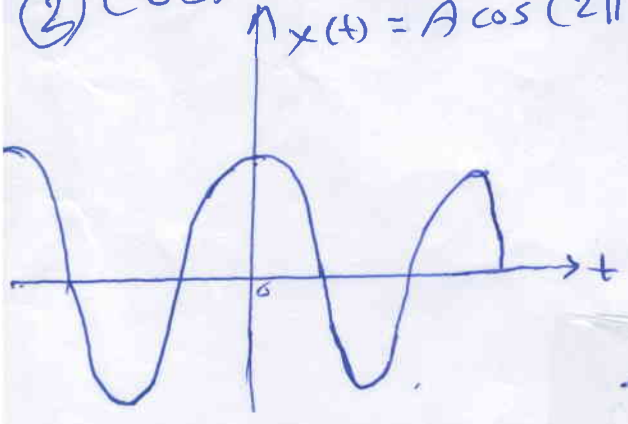
$X(f) = A\,\delta(f)$

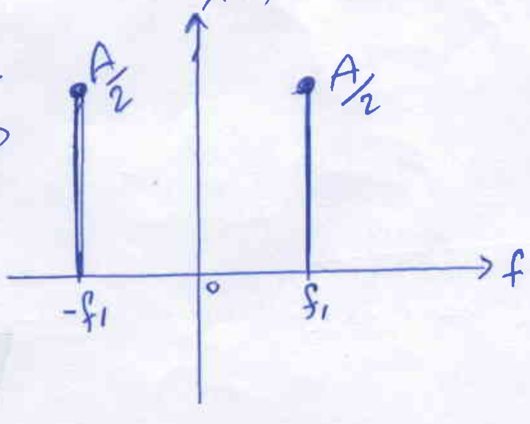


(2) Cosine Signal

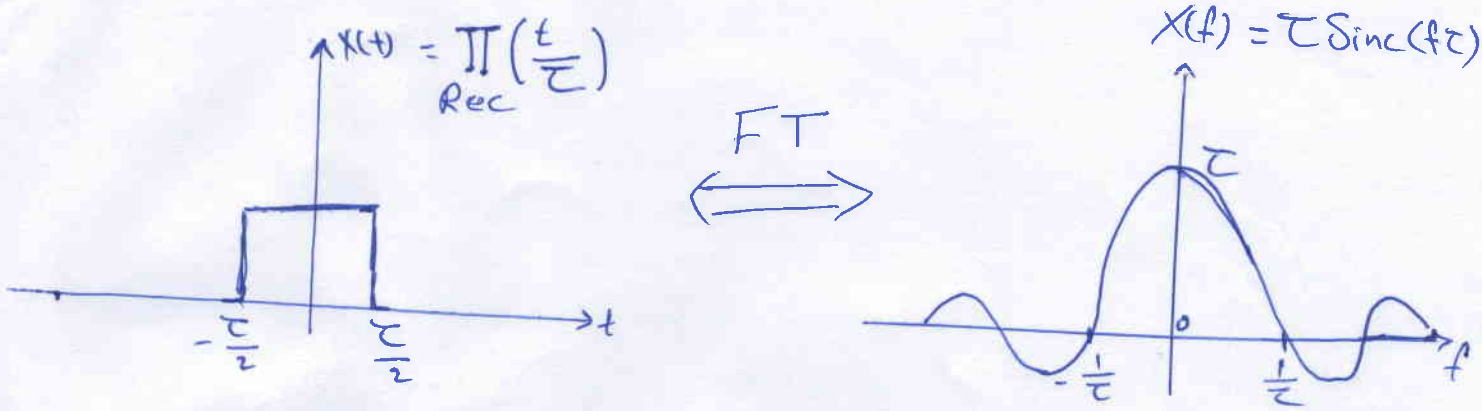
$x(t) = A\cos(2\pi f_1 t)$

$X(f) = \frac{A}{2}[\delta(f+f_1)+\delta(f-f_1)]$





3)- Rectangular plus̲e



4)- Multiplication

Let $X(f) = FT[x(t)]$, $Y(f) = FT[y(t)]$

$$FT[x(t) \cdot y(t)] = X(f) \otimes Y(f)$$

Multiplication in Time domain = convolution in Freq. domain

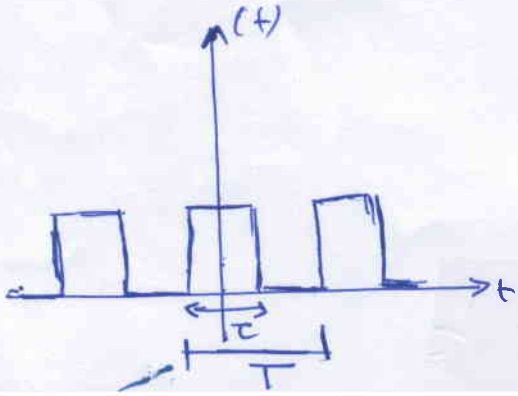
5)- Convolution

Let $X(f) = FT[x(t)]$, $Y(f) = FT[y(t)]$

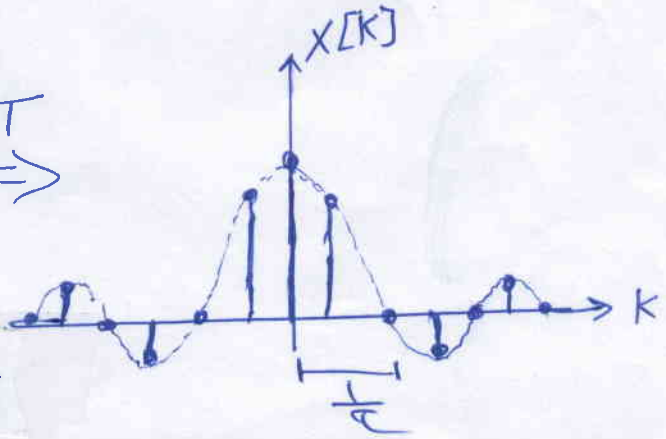
$$FT[x(t) \otimes y(t)] = X(f) \cdot Y(f)$$

convolution in Time domain = Multiplication in Freq. domain

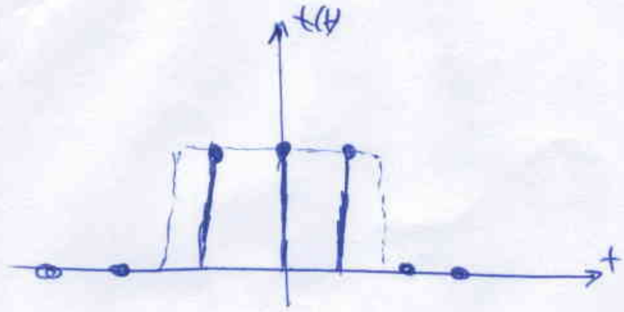
6) Periodic Signal

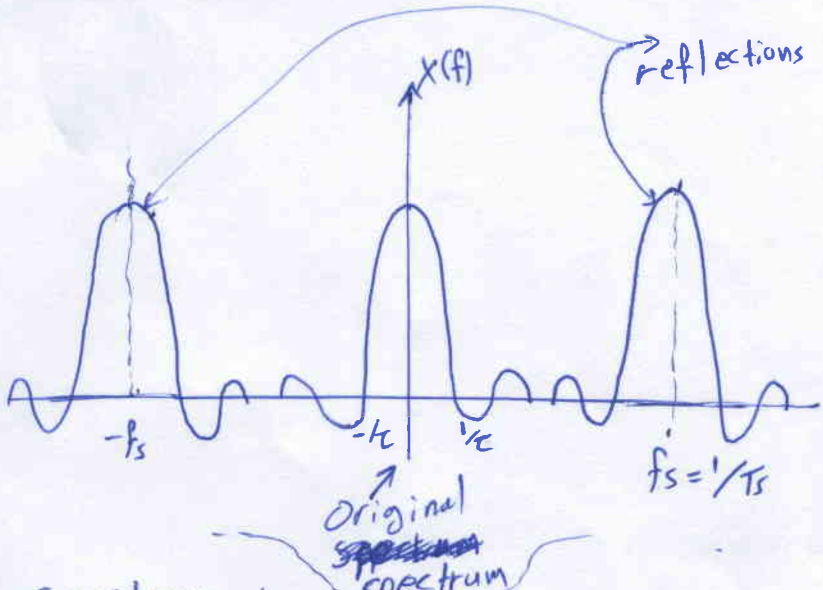


FT



7)- Sampled Signal





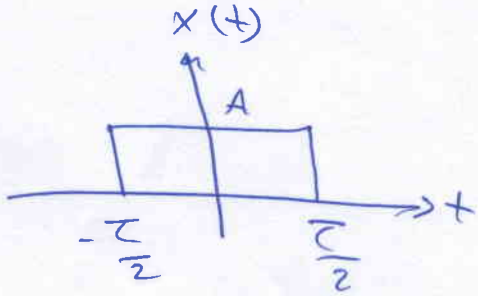
X(f) is periodic i.e. original spectrum is repeated every f_s and these are known as "Reflections".

8) H.w

OBSERVATIONS

Time domain	Freq. Domain
1. Periodic	Discrete
2. Discrete	Periodic
3. Periodic + Discrete	Discrete + Periodic
4. Non-Periodic	Non-discrete (continuous)
5. Non-discrete (continuous)	Non-Periodic

Ex 1: Find Fourier transform of the signal below:



$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\, e^{-j2\pi f t}\, dt$$

$$= A\left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

$$= A\left[\frac{e^{-j2\pi f\frac{\tau}{2}} - e^{+j2\pi f\frac{\tau}{2}}}{-j2\pi f}\right]$$

$$= \frac{A}{-\pi f}\left[\frac{e^{-j\pi f\tau} - e^{j\pi f\tau}}{2j}\right]$$

$$= \frac{A}{\pi f}\left[\frac{e^{j\pi f\tau} - e^{-j\pi f\tau}}{2j}\right]$$

$$= \frac{A}{\pi f}\sin \pi f\tau \times \frac{\tau}{\tau}$$

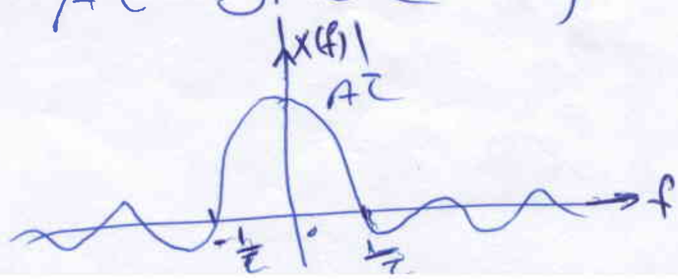
$$= A\tau\, \frac{\sin(\pi f\tau)}{\pi f\tau}$$

$$= A\tau\, \mathrm{Sinc}(\pi f\tau)$$

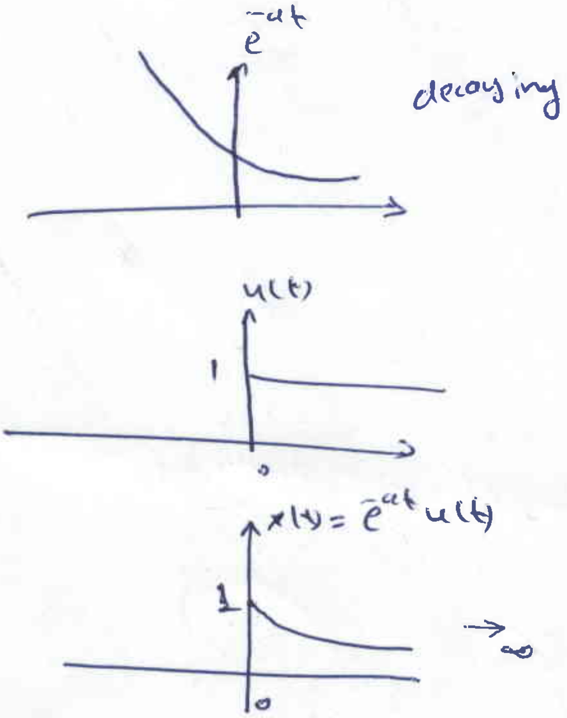
hint:

$$\mathrm{Sinc}(x) = \frac{\sin(x)}{x}$$

$$\mathrm{Sinc}(\pi x) = \frac{\sin(\pi x)}{\pi x}$$



EX: Find Fourier transform of $x(t) = e^{-at}u(t)$, $a>0$ then sketch the spectrum.



$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-at} * \underset{u(t)}{1} * e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt$$

$$= \left. \frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)} \right|_{0}^{\infty}$$

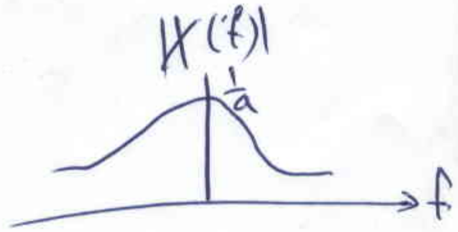
hint: $e^{-\infty} = 0$

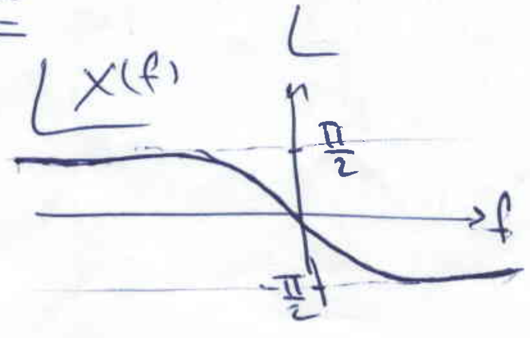
$e^{0} = 1$

$$X(f) = \frac{e^{-\infty} \rightarrow 0}{-(a+j2\pi f)} - \frac{e^{0}}{-(a+j2\pi f)}$$

$$\boxed{X(f) = \frac{1}{a+j2\pi f} = \frac{1}{a+j\omega}}$$

$$\underline{|X(f)|}_{\text{Magnitude}} = \frac{1}{\sqrt{a^2+4\pi^2 f^2}}, \quad \underline{\angle X(f)}_{\text{phase}} = -\tan^{-1}\frac{2\pi f}{a}$$





EX: Find Fourier Transform of $x(t) = \delta(t)$, then sketch its spectrum.

$$x(t) = \delta(t)$$

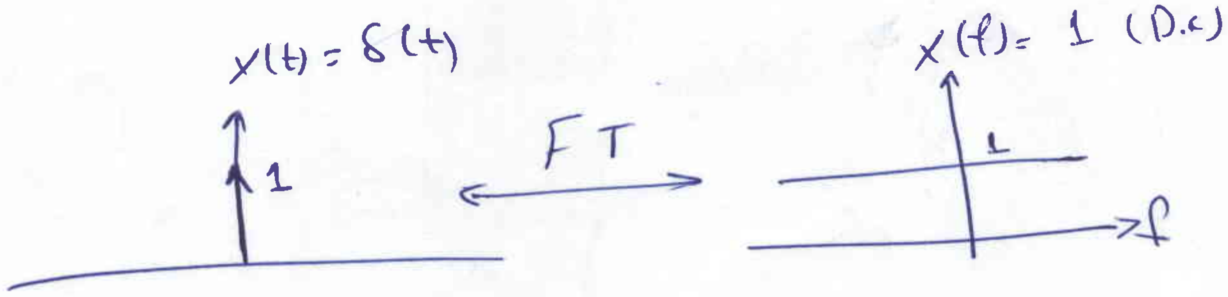
$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) \, e^{-j\omega t} \, dt$$

$$X(f) = \int_{0} \delta(t) \, \underbrace{e^{-j\omega(0)}}_{1} \, dt$$

$$X(f) = 1 \int_{0} \delta(t) \, dt$$

$$= \underline{1} \quad (\text{D.C})$$



EX: Find Fourier Transform of $x(t) = \delta(t-a)$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} \, dt$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t-a) \, e^{-j2\pi ft} \, dt$$

$$\boxed{X(f) = e^{-j2\pi fa}}$$

So, the Fourier transform of shifted impluse is a complex exponential.

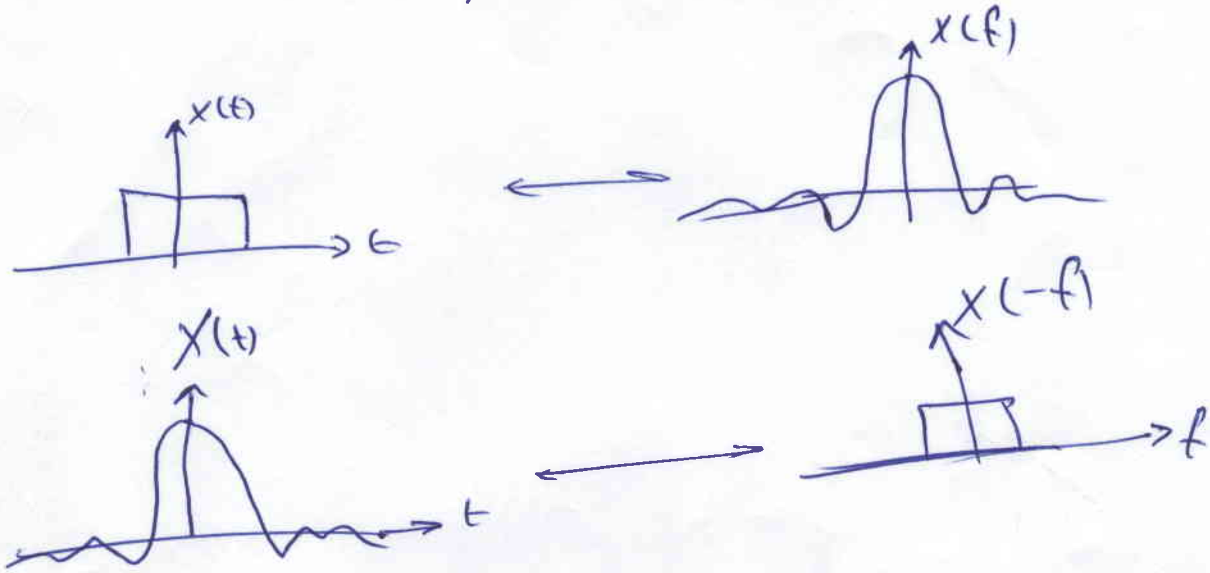
* Properties of Fourier Transform :

1- Linearity : if $x(t) \longrightarrow X(f)$
~~then~~ and $y(t) \longrightarrow Y(f)$
then $ax(t) + by(t) \longleftrightarrow aX(f) + bY(f)$

2- Scaling : if $x(t) \longleftrightarrow X(f)$
then $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

3- Duality :- if $x(t) \longleftrightarrow X(f)$

then $X(t) \longleftrightarrow x(-f)$



4- Time shift :- if $x(t) \longleftrightarrow X(f)$

then $x(t-t_0) \longleftrightarrow X(f)\, e^{-j2\pi f t_0}$

5- Frequency shift :- if $x(t) \longleftrightarrow X(f)$

$x(t)\, e^{j2\pi f_0 t} \longleftrightarrow X(f-f_0)$

6- Time differentiation : if $x(t) \longleftrightarrow X(f)$

then $\dfrac{d^n x(t)}{dt^n} \Longleftrightarrow (j2\pi f)^n X(f)$

7. Multiplication and Convolution properties :

if $f_1(t) \xleftrightarrow{FT} F_1(f)$

$f_2(t) \xrightarrow{FT} F_2(f)$

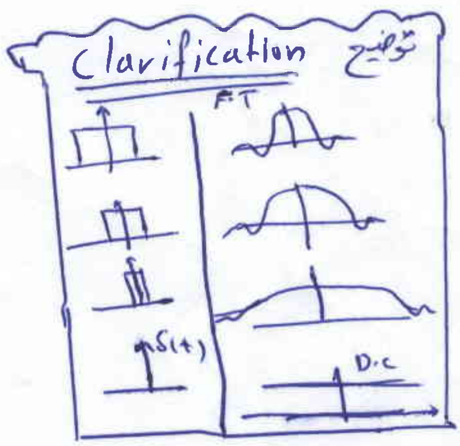
* Then multiplication property states that :

$$x_1(t)\cdot f_2(t) \xrightarrow{F.T} X_1(f) \circledast X_2(f)$$

* Convolution property states that :

$$x_1(t) \overset{conv}{\circledast} x_2(t) \longleftrightarrow X_1(f)\cdot X_2(f)$$

Examples: find fourier transform of :



① $x(t) = 1$

$\delta(t) \longleftrightarrow 1$

Duality

$1 \longleftrightarrow \delta(-f) = \delta(f)$

$X(t) \longrightarrow x(-f)$

② - $3\delta(t-2)$

$\delta(t) \longleftrightarrow 1$

$3\delta(t) \longleftrightarrow 3 * 1 = 3$ (Linearity)

t_0

$3\delta(t-2) \longleftrightarrow 3\, e^{-j2\pi f 2}$

$x(t-t_0) \longrightarrow X(f)\, e^{-j2\pi f t_0}$ time-shift.

$3\delta(t-2) \longrightarrow 3\, e^{-j4\pi f}$

③ - $e^{j\omega_0 t}$

$e^{j\omega_0 t} = 1 . e^{j\omega_0 t} = 1 . e^{j2\pi f_0 t}$

$\omega_0 = 2\pi f_0$

$f_0 = \frac{\omega_0}{2\pi}$

$1 \longleftrightarrow \delta(t)$

$x(t)\, e^{j2\pi f_0 t} \longleftrightarrow X(f-f_0)$ freq. shift

$1 . e^{j2\pi f_0 t} \longleftrightarrow \delta(f - \frac{\omega_0}{2\pi})$

So if we have e^{j6t} ?? Same above ↑ ∴

$\omega = 6$

$\omega = 2\pi f_0$

$f_0 = \frac{6}{2\pi}$

$1 . e^{j6t} \longleftrightarrow \delta(f - f_0)$

$\delta(f - \frac{6}{2\pi})$

$\delta(f - \frac{3}{\pi})$

4) - $5\,\mathrm{rect}\left(\frac{3t}{2}\right)$

$\because\; f(t) = 5\,\mathrm{rect}\left(\frac{3t}{2}\right)$

$$\boxed{A\,\mathrm{rect}\,\frac{t}{T} \longleftrightarrow AT\,\mathrm{Sinc}\, f T}$$

Let $A=5$ $T=1$ and $\boxed{\frac{3}{2}}$ is time Scaling

$$\therefore\; 5\,\mathrm{rect}\,\frac{3}{2}t \longleftrightarrow \frac{5}{3/2}\,\mathrm{Sinc}\left(\frac{f}{3/2}\right)$$

$$x(at) \longleftrightarrow \frac{1}{|a|}\,X\left(\frac{f}{a}\right)$$ Scaling property of FT.

5) $\cos\omega_0 t$

$$\cos\omega_0 t = \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

$$= \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}$$

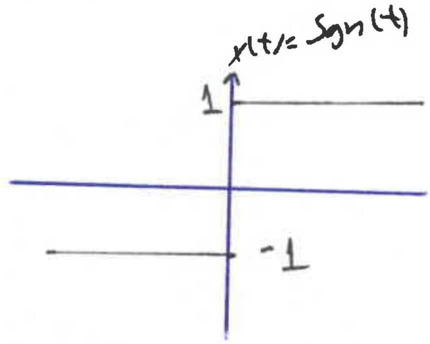
$$= \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$\left\{\begin{array}{l} 1 \longleftrightarrow \delta(t) \\ \frac{1}{2}\times 1 \longleftrightarrow \frac{1}{2}\delta(t) \\ \frac{1}{2}\delta(t)\,e^{j2\pi f_0 t} \longleftrightarrow \frac{1}{2}\delta(f-f_0) \\ \frac{1}{2}\delta(t)\,e^{-j2\pi f_0 t} \longleftrightarrow \frac{1}{2}\delta(f+f_0) \end{array}\right.$$

$$\boxed{H.\omega = F.T[\sin\omega_0 t]}$$

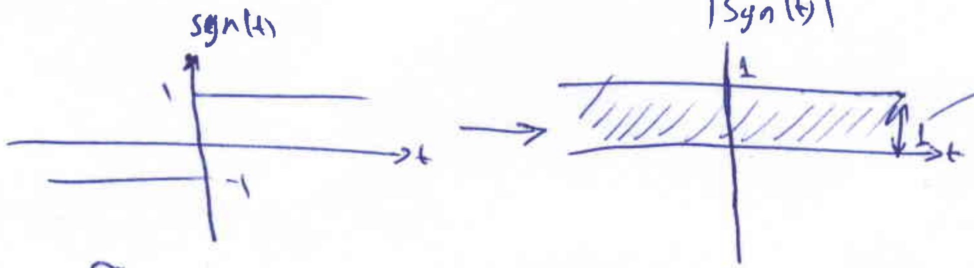
6) Signum function Sgn(t) :

$$Sgn(t) = \begin{cases} -1 & , t < 0 \\ 0 & , t = 0 \\ 1 & , t > 0 \end{cases}$$



لايجاد تكامل اي اشارة (هل قابلة للتكامل ام لا) يجب ان تخضع للشرط

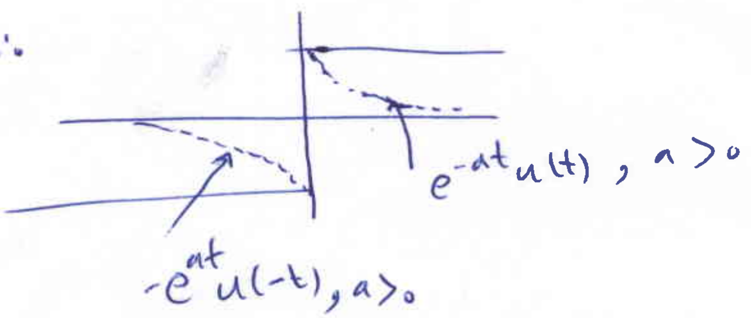
$$\int_{-\infty}^{\infty} |Sgn(t)|\, d < \infty \quad (\text{finite})$$



area under curve = ∞

$$\therefore \int_{-\infty}^{\infty} |Sgn(t)|\, dt = 1 * \infty = \infty$$

$\therefore$



$$u(t) = \lim_{a \to 0} e^{-at} u(t)$$

$$u(-t) = \lim_{a \to 0} e^{at} u(-t)$$

$$Sgn(t) = u(t) - u(-t)$$

$$x(t) = \lim_{a \to 0} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

$$X(f) = \lim_{a \to 0} \left[\frac{1}{a + jw} - \frac{1}{a - jw} \right]$$

$$= \lim_{a \to 0} \frac{\cancel{a} - jw - \cancel{a} - jw}{a^2 - j^2 w^2}$$

$\boxed{j^2 = -1}$

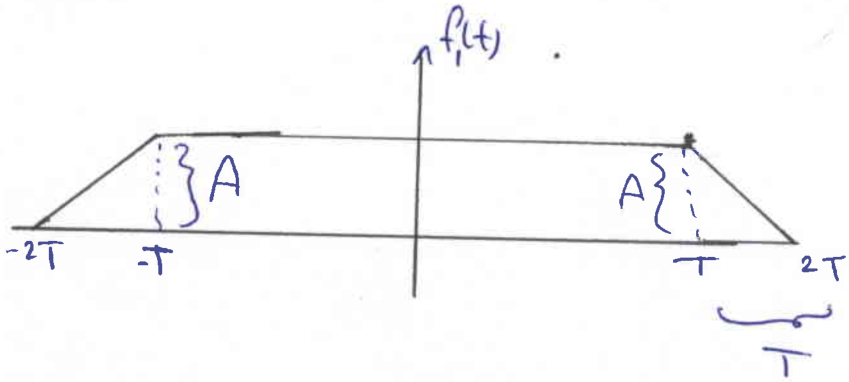
$$= \lim_{a \to 0} \frac{-2jw}{a^2 + w^2} = \frac{-2jw}{w^2} = \frac{-2j}{w} * \frac{j}{j} = \boxed{\frac{2}{jw}}$$

Summary

$$\boxed{Sgn(t) \Longleftrightarrow \frac{2}{jw}}$$

8) Hw

find $f_1'(t)$ then $F_1(f)$



9) $z(t) = \underbrace{e^{-t}u(t)}_{z_1(t)} \overset{conv}{\circledast} \underbrace{e^{-2t}u(t)}_{z_2(t)}$

$Z_1(f) \longleftrightarrow \dfrac{1}{1+j2\pi f}$

$Z_2(f) \longrightarrow \dfrac{1}{2+j2\pi f}$

multiplication

$F.T\left[z_1(t) \circledast z_2(t)\right] = Z_1(f) \cdot Z_2(f)$

$= \dfrac{1}{1+j2\pi f} \cdot \dfrac{1}{2+j2\pi f}$

10) $F.T\left[\cos^2 \omega_0 t\right]$

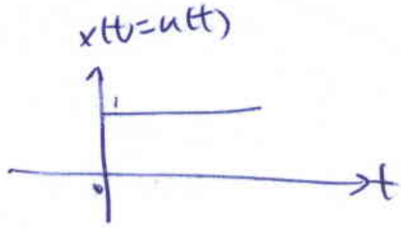
$\cos^2 \omega_0 t = FT\left[\dfrac{1}{2} + \dfrac{\cos 2\omega_0 t}{2}\right]$

$= \dfrac{1}{2}\delta(f) + k$ where $k = \dfrac{\cos 2\omega_0 t}{2}$

$F.T\left[\dfrac{\cos 2\omega_0 t}{2}\right] = F.T\left[\dfrac{e^{+j4\pi f_0 t} + e^{-j4\pi f_0 t}}{4}\right]$

$= \dfrac{1}{2}\delta(f) + \dfrac{1}{4}\delta(f-2f_0) + \dfrac{1}{4}\delta(f+2f_0)$

EX: Find Fourier Transform of $u(t)$
(7)



$$x(t) = u(t)$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$X(f) = \left. \frac{e^{-j2\pi f t}}{-j2\pi f} \right|_{0}^{\infty}$$

$$= \frac{1}{-j2\pi f}\left(e^{-j\infty} - e^{-j0} \right)$$

$$e^{-j\infty} = \infty$$

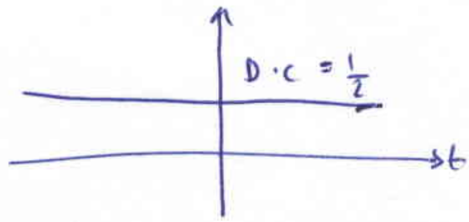
So it cannot be continue in FT Integration way.

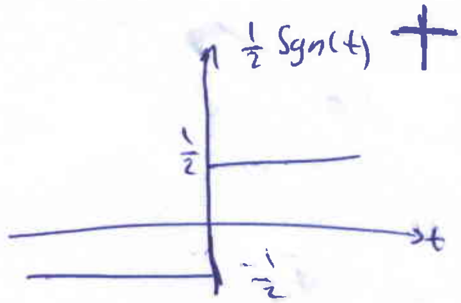
$$x(t) = u(t)$$

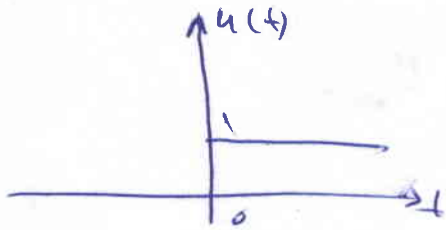
$$x(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{Sgn}(t)$$

$$\updownarrow$$

$$X(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$







$$\boxed{u(t) \longleftrightarrow \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}}$$

Laplace Transforms :

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$x(t) \downarrow$ (FT) fourier Transform

$$y(t) = x(t) \circledast h(t)$$

$\downarrow$ (FT) fourier Transform

$$Y(f) = X(f) \cdot H(f)$$

$$Y(s) = X(s) \cdot H(s)$$

The process of transforming the signal from time_domain $x(t)$ to s-domain is called Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad \text{(Bilateral Laplace Transform)}$$

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt \quad \text{(unilateral Laplace Transform)}$$

Laplace Transform is general case of fourier Transform

where $s = \sigma + j\omega$ (complex number)

when $\sigma = 0 \longrightarrow \boxed{s = j\omega}$

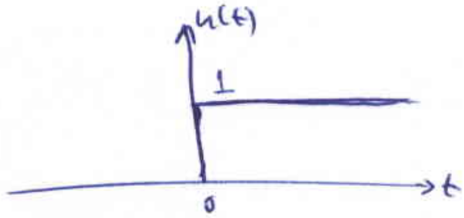
* Symbols

$\mathcal{L}\{x(t)\} \longrightarrow$ Laplace $x(t)$, $x(t) \longrightarrow X(s)$

(Laplace Inverse) $\mathcal{L}^{-1}\{X(s)\}$

EX1: Find Laplace Transform of $x(t) = u(t)$

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$



$$= \int_0^\infty 1 \cdot e^{-st}\, dt$$

$$= \left. \frac{e^{-st}}{-s} \right|_0^\infty = \cancel{\frac{e^{-\infty}}{-s}}^{\,0} - \frac{e^0}{-s}$$

$$= -\frac{1}{-s} = \frac{1}{s}$$

$$\mathcal{L}[u(t)] \longleftrightarrow \frac{1}{s}$$

EX2: Find laplace Transform of $x(t) = e^{at} u(t)$ a: constant

$$x(t) = e^{at} u(t)$$

$$X(s) = \int_0^\infty e^{at} e^{-st}\, dt$$

$$X(s) = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty$$

$$= 0 - \frac{1}{-(s-a)} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at} u(t)\} = \frac{1}{s-a}$$

$\sin at\, u(t) \longleftrightarrow \dfrac{a}{s^2+a^2}$

$\cos at\, u(t) \longleftrightarrow \dfrac{s}{s^2+a^2}$

EX3: Find Laplace inverse of $X(s) = \dfrac{1}{(s-a)(s-b)}$

$$\frac{1}{(s-a)(s-b)} = \frac{A}{(s-a)} + \frac{B}{(s-b)}$$

$$1 = A(s-b) + B(s-a)$$

when $s = b \rightarrow A(\cancel{s-b})$ $\;a\;0$

$$1 = B(b-a) \longrightarrow B = \frac{1}{b-a}$$

when $s = a \rightarrow B(\cancel{s-a})$ $\;a\;0$

$$1 = A(a-b) \longrightarrow A = \frac{1}{a-b}$$

$$= \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

$$= \frac{1}{a-b}\left[e^{at}u(t) - e^{bt}u(t)\right]$$

* Laplace Transform Properties :

1- Linearity

if $x(t) \longleftrightarrow X(s)$ & $y(t) \longleftrightarrow Y(s)$

then $a\,x(t) + b\,y(t) \longleftrightarrow a\,X(s) + b\,Y(s)$

2- Scaling

if $x(t) \longleftrightarrow X(s)$

then $x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)$

3- Time delay

if $x(t) \longleftrightarrow X(s)$

then $x(t-t_0) \longleftrightarrow X(s)\, e^{-st_0}$

4- Shift (shifting)

if $x(t) \longleftrightarrow X(s)$

$e^{-at} x(t) \longleftrightarrow X(s+a)$

5- Multiplication by t :

$t\,x(t) \longleftrightarrow -\frac{dX(s)}{ds}$

$t^n x(t) \longleftrightarrow (-1)^n \frac{d^n X(s)}{ds}$

6- Differentiation

if $x(t) \longleftrightarrow X(s)$

$$\frac{dx(t)}{dt} \longleftrightarrow s\,X(s) - x(0)$$

$$\frac{d^2x(t)}{dt^2} \longleftrightarrow s^2 X(s) - s\,x(0) - x'(0)$$

$$\frac{d^3x(t)}{dt^3} \longleftrightarrow s^3 X(s) - s^2 x(0) - s\,x'(0) - x''(0)$$

7- Convolution & Multiplication

$$x(t) \circledast y(t) \longleftrightarrow X(s) \cdot Y(s)$$

$$x(t) \cdot y(t) \longleftrightarrow X(s) \circledast Y(s)$$

8- Initial value

$$\lim_{s \to \infty} s\,X(s) = \lim_{t \to 0^+} x(t)$$

by: Abdelhousit Sabaan

9- Final Value

$$\lim_{s \to 0} s\,X(s) = \lim_{t \to \infty} x(t)$$

Examples: find Laplace Transform of the following signals:

① $t\,u(t)$

$$u(t) \longleftrightarrow \frac{1}{s}$$

$$t\,u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

② $t^2\,u(t)$

$$t\,u(t) \longleftrightarrow \frac{1}{s^2}$$

$$t^2 u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$$\boxed{\text{In general} \quad t^n u(t) \longleftrightarrow \frac{n!}{s^{n+1}}}$$

③ $t^2 e^{-2t} u(t)$

$$t^2 u(t) \longleftrightarrow \frac{2}{s^3}$$

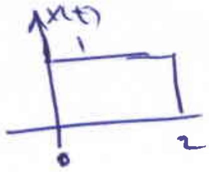
$$e^{-2t} t^2 u(t) \longleftrightarrow \frac{2}{(s+2)^3}$$

④ $t \cos 2t\, u(t)$

$$\cos 2t\, u(t) \longleftrightarrow \frac{s}{s^2+4}$$

$$t \cos 2t\, u(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

5) $x(t) =$ 

$$x(t) = u(t) - u(t-2)$$

$$\downarrow \qquad \downarrow$$

$$\frac{1}{s} - \frac{1}{s} e^{-2s}$$

6) $e^{-at} \sin \omega_0 t \, u(t)$

$$\sin \omega_0 t \, u(t) \longleftrightarrow \frac{\omega_0}{s^2 + \omega_0^2}$$

$$e^{-at} \sin \omega_0 t \, u(t) \longleftrightarrow \frac{\omega_0}{(s+a)^2 + \omega_0^2}$$

EX: Find $\mathcal{L}^{-1} \dfrac{e^{-3s}}{s^3}$ $\qquad$ ($\mathcal{L}^{-1}$ = laplace inverse)

$$t^2 u(t) \longleftrightarrow \frac{2}{s^3}$$

$$\frac{1}{s^3} \longleftrightarrow \frac{1}{2} t^2 u(t)$$

$$\frac{1}{s^3} e^{-3s} \longleftrightarrow \frac{1}{2} (t-3)^2 u(t-3)$$

Example : Find the initial and Final values for

$$X(s) = \frac{1}{s(s+1)} .$$

* Initial value $= \lim_{s\to\infty} s\,X(s)$

$$= \lim_{s\to\infty} \frac{s}{s(s+1)}$$

$$= \frac{1}{\infty+1} = \boxed{0}$$

OR :

$$x(t) = \mathcal{L}^{-1} X(s)$$

$$= \mathcal{L}^{-1} \frac{1}{s(s+1)}$$

$$= \mathcal{L}^{-1} \left(\frac{A}{s} + \frac{B}{s+1} \right)$$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \right.$$

$$= \mathcal{L}^{-1} \left\{ \left[\frac{1}{s(s+1)} = \frac{A(s+1) + Bs}{s(s+1)} \right\} \right.$$

$$A(s+1) + Bs = 1$$

when $s = 0 \longrightarrow A = 1$

$s = -1 \longrightarrow B = -1$

$$\mathcal{L}^{-1} \left(\frac{1}{s} - \frac{1}{s+1} \right) = u(t) - e^{-t} u(t)$$

initial value $= \lim_{t\to 0^+} x(t) = u(0^+) - e^{-0} u(+0)$

$$= 1 - 1 = \boxed{0}$$

* Final value

$$\text{final value} = \lim_{s\to 0} s\,X(s)$$

$$= \frac{\cancel{s}}{\cancel{s}(s+1)} = \frac{1}{0+1} = \boxed{1}$$

OR :

$$\text{final value} = \lim_{t\to\infty} x(t)$$

$$= u(\infty) - e^{-\infty} u(\infty)$$

$$= 1 - 0 = \boxed{1}$$

Example: Solve $\left(\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 0 \right)$ $y(0)=3$, $y'(0)=1$

$$s^2 Y(s) - s\,y(0) - y'(0) + 4s\,Y(s) - 4y(0) + 3Y(s) = 0$$

$$Y(s)\left[s^2 + 4s + 3\right] - 3s - 1 - 12 = 0$$

$$Y(s) = \frac{3s+13}{s^2+4s+3} = \frac{3s+13}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$3s+13 = A(s+3) + B(s+1)$$

when $s=-1 \longrightarrow 2A = 10 \longrightarrow \boxed{A=5}$

$s=-3 \longrightarrow -2B = 4 \longrightarrow \boxed{B=-2}$

$$\mathcal{L}^{-1} Y(s) = \frac{5}{s+1} - \frac{2}{s+3}$$

$$y(t) = \left(5e^{-t} - 2e^{-3t}\right)u(t).$$

* Transfer Function (H(S))

Definition :- Transfer function is defined as the ratio of Laplace transform of o/p to the laplace transform of i/p. when all the initial conditions are assumed to be zero.

$$h(t) \underset{\text{inverse}}{\overset{\text{laplace}}{\rightleftarrows}} H(S)$$

$$\begin{array}{ccccc} X(t) & \longrightarrow & \overset{\text{Sys.}}{\boxed{LTI}} & \longrightarrow & y(t) \\ \downarrow & & & & \downarrow \\ X(S) & & & & Y(S) \end{array}$$

$$H(S) = \left.\frac{Y(S)}{X(S)}\right|_{\text{initial conditions} = 0}$$

LTI (linear + TIV)

when the input of Linear System (i/p) = 0 $\Longrightarrow$ the output (o/p) of the Linear System also will become (o/p = 0). when don't assume intial conditions to be zero, this property i/p = 0 → o/p = 0 becomes violated, Then the system will become not-Linear and so the system become not LTI, hence we cannot define the Transfer function H(S).

Example: A continuous time LTI system is described by

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dx(t)}{dt} + 4x(t)$$

The response $\underset{o/p}{\underline{y(t)}}$ of the system for the input $\underset{i/p}{\underline{x(t) = e^{-2t}u(t)}}$ is ____

Sol:

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t) = ?$$

$\hookrightarrow$ H(S) or Diff. eq.

$$y(t) \rightleftarrows Y(S) = H(S) \cdot X(S)$$

$$H(S) = \frac{Y(S)}{X(S)}$$

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2\frac{dX(t)}{dt} + 4X(t)$$

$$S^2 Y(S) + 4S\,Y(S) + 3Y(S) = 2S\,X(S) + 4X(S)$$

$$Y(S)[S^2 + 4S + 3] = X(S)[2S + 4]$$

$$H(S) = \frac{Y(S)}{X(S)} = \frac{2S+4}{S^2+4S+3} = \boxed{\overset{H(S)}{\frac{2(S+2)}{(S+1)(S+3)}}}$$

$$x(t) = e^{-2t}u(t) \xrightarrow{\text{laplace}} X(S) = \frac{1}{S+2}$$

70-

$$Y(S) = H(S) \cdot X(S) = \frac{2\cancel{(S+2)}}{(S+1)(S+3)} \cdot \frac{1}{\cancel{S+2}} = \frac{2}{(S+1)(S+3)}$$

$$\mathcal{L}^{-1}\frac{2}{(S+1)(S+3)} = \frac{A}{S+1} + \frac{B}{S+3} \qquad \begin{pmatrix} A = 1 \\ B = -1 \end{pmatrix}$$

$$\mathcal{L}^{-1}Y(S) = \frac{1}{S+1} - \frac{1}{S+3} \longrightarrow \underline{\underline{y(t) = e^{-t}u(t) - e^{-3t}u(t)}}$$

H.W: for the following equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 6y(t) = x(t) \quad , \; \underline{x(t) = e^{-t}u(t)}$$

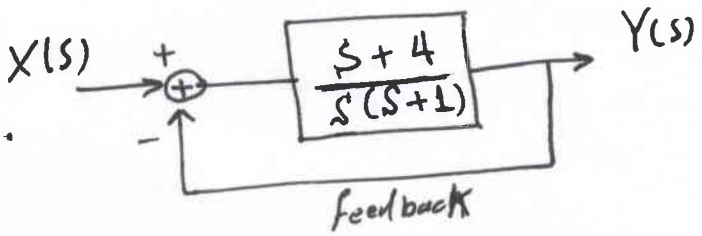
① Find the Transfer function

② Find the output if $y(0) = 0$ & $y'(0) = 5$.

EX:- For the System shown :

① find $H(S)$?

② for $x(t) = e^{-4t}u(t)$, find the $\underline{\underline{\text{output}}}$.



Sol:

①
$$\left\{ \begin{array}{l} X(S) + (-Y(S)) = \text{input} \\ (\text{input}) \cdot H(S) = \underset{\text{output}}{Y(S)} \end{array} \right\} \;\; \underline{\underline{\text{hint}}}$$

$$\therefore \left[X(S) - Y(S)\right] \cdot \left[\frac{S+4}{S(S+1)}\right] = Y(S)$$

$$X(S) \frac{S+4}{S(S+1)} = Y(S)\left[1 + \frac{S+4}{S(S+1)}\right]$$

$$\boxed{\frac{Y(S)}{X(S)} = \frac{S+4}{S^2+2S+4} = H(S)}$$

② - $X(S) = \frac{1}{S+4}$ $\quad (x(t) = e^{-4t} u(t))$

$Y(S) = X(S) \cdot H(S) = \frac{1}{(\cancel{S+4})} \cdot \frac{(\cancel{S+4})}{S^2+2S+4)}$

$Y(S) = \frac{1}{S^2+2S+4}$

$y(t) = \mathcal{L}^{-1} Y(S)$

$= \mathcal{L}^{-1} \frac{1}{S^2+2S+4}$

hint$_1$:

$\left\{ \begin{array}{c} \text{shift property} \\ e^{-at} x(t) = X(S+a) \end{array} \right\}$

$= \mathcal{L}^{-1} \frac{1}{(S\underline{\underline{+1}})^2+3}$

change this shift in S-domain to exp in time domain.

$= e^{-t} \mathcal{L}^{-1} \frac{1}{S^2+3}$

hint$_2$:

$\left\{ \sin at = \frac{a}{S^2+a^2} \right\}$

$= e^{-t} \mathcal{L}^{-1} \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{S^2+(\sqrt{3})^2}$

$$\boxed{y(t) = \frac{1}{\sqrt{3}} e^{-t} \sin\sqrt{3}\,t \; u(t)}$$

Example: The Transfer function of a linear System is:

$$H(S) = \frac{1}{S+a}$$

① Find the differential Equation.

② For input $x(t) = t\,u(t)$, find the output assuming zero initial conditions.

Sol: ①

$$H(S) = \frac{Y(S)}{X(S)} = \frac{1}{S+a}$$

$$S\,Y(S) + a\,Y(S) = X(S)$$

$$\frac{dy(t)}{dt} + a\,y(t) = x(t)$$

② $x(t) = t\,u(t)$

$$X(S) = \frac{1}{S^2} \quad , \quad H(S) = \frac{1}{S+a}$$

$$Y(S) = X(S) \cdot H(S) = \frac{1}{S^2(S+a)}$$

$$y(t) = \mathcal{L}^{-1} \frac{1}{S^2(S+a)} = \mathcal{L}^{-1} \frac{A}{S+a} + \frac{B}{S} + \frac{C}{S^2}$$

$$y(t) = \mathcal{L}^{-1} \frac{1}{a^2}\left[\frac{a}{S^2} - \frac{1}{S} + \frac{1}{S+a}\right]$$

$$y(t) = \frac{1}{a^2}\left[at - 1 + e^{-at}\right]u(t)$$

* Z - Transform

z-transform is a convenient tool in designing analyzing and representing discrete signals & Systems.

It plays in discrete signals, the same role as Laplace transform plays in continuous signals.

z-transform has wide applications such as stability and frequency response.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \qquad \text{Bilateral z-transform}$$

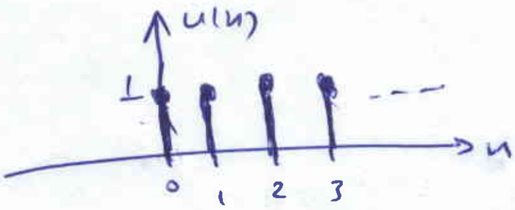
$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \qquad \text{unilateral z-transform}$$

EX: Find z-transform of the following signals:

① - $x(n) = \delta(n)$

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \delta(n)\, z^{-n} = \delta(0)\, z^{0} = \underline{\underline{1}}$$

② - $x(n) = u(n)$



$$X(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= 1 * z^{0} + 1 * z^{-1} + 1 * z^{-2} + 1 * z^{-3} \cdots$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$ حسب قانون مج متتاليه هندسيه.

$$= \frac{\text{الحد الاول}}{1-\text{الاساس}} = \boxed{\frac{1}{1-z^{-1}}} \xleftarrow{X(z)} u(n)$$

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* Properties of Z-transform

1- Linearty

$$a f_1(n) + b f_2(n) \longleftrightarrow a F_1(Z) + b F_2(Z)$$

2- Shifting (delay)

$$f(n-m) \longleftrightarrow Z^{-m} F(Z)$$

3- Shifting (Advance)

$$f(n+1) \longleftrightarrow Z[F(Z) - f(0)]$$

$$f(n+2) \longleftrightarrow Z[F(Z) - f(0)] - Zf(1)$$

$$f(n+3) \longleftrightarrow Z[F(Z) - f(0)] - Zf(1) - Zf(2)$$

4- Convolution & Multiplication

$$x(n) \circledast y(n) \longleftrightarrow X(Z) \cdot Y(Z)$$

$$x(n) \cdot y(n) \longleftrightarrow X(Z) \circledast Y(Z)$$

5- Multiplication by n

$$n f(n) \longleftrightarrow -Z \frac{dF(Z)}{dZ}$$

6- Multiplcation by a^n

$$a^n f(n) \xleftarrow{\text{Example:}} \left[a^n u(n) \longleftrightarrow \frac{Z}{Z-a} \right]$$

$$a^n f(n) \longleftrightarrow F\left(\frac{Z}{a}\right)$$

Ex Find Z-transform of:

① - $u(n-m)$

$$u(n) \longleftrightarrow \frac{z}{z-1} \quad \left(\frac{1}{1-z^{-1}} * \frac{z}{z} \right)$$

$$u(n-m) \longleftrightarrow z^{-m} \frac{z}{z-1}$$

② $n\,u(n)$

$$u(n) \longleftrightarrow \frac{z}{z-1} \leftarrow f(z)$$

$$n\,u(n) \longleftrightarrow -z \frac{df(z)}{dz} \leftarrow \frac{z}{z-1}$$

$$-z \frac{d}{dz} \frac{z}{z-1} = -z \; \frac{(z-1)*1 - z*1}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

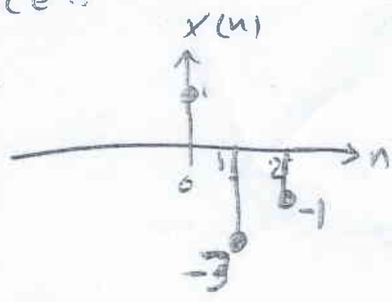
3) $a^n u(n) \longleftrightarrow F\left(\frac{z}{a}\right)$

$$u(n) \longleftrightarrow \frac{z}{z-1}$$

$$a^n u(n) \longleftrightarrow \frac{z/a}{z/a - 1} = \frac{z}{z-a}$$

Ex2: Find Z-transform of $x(n)$ sequence:

$$x(n) = \{ \underset{\substack{\uparrow \\ n=0}}{1}, -3, -1 \}$$



$$X(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$= 1 + (-3) z^{-1} + (-1) z^{-2}$$

$$= 1 - 3z^{-1} - z^{-2} \underset{\text{equation}}{\overset{\text{n-domain}}{\longleftrightarrow}} = \delta(n) - 3\delta(n-1) - \delta(n-2)$$

Ex 3: Find the inverse of Z-transform

$$X(Z) = 2 + 3Z^{-4}$$

$$x(n) = 2\delta(n) + 3\delta(n-4)$$

Ex 4: Find the inverse of $X(Z) = \dfrac{Z}{Z^2 - 6Z + 8}$

$$\frac{X(Z)}{Z} = \frac{1}{Z^2 - 6Z + 8} \Rightarrow$$

$$\frac{X(Z)}{Z} = \frac{1}{(Z-2)(Z-4)} = \frac{A}{Z-2} + \frac{B}{Z-4}$$

$$A = (Z-2)\frac{1}{(Z-2)(Z-4)}\Bigg|_{Z=2} = \frac{1}{2-4} = \boxed{-\frac{1}{2}}$$

$$B = (Z-4)\frac{1}{(Z-2)(Z-4)}\Bigg|_{Z=4} = \frac{1}{4-2} = \boxed{\frac{1}{2}}$$

$$\frac{X(Z)}{Z} = \frac{-\frac{1}{2}}{Z-2} + \frac{\frac{1}{2}}{Z-4}$$

$$X(Z) = -\frac{1}{2}\frac{Z}{Z-2} + \frac{1}{2}\frac{Z}{Z-4}$$

$$u(n) \longrightarrow \frac{Z}{Z-1}$$

$$a^n u(n) \longrightarrow \frac{Z}{Z-a}$$

$$= -\frac{1}{2}(2)^n u(n) + \frac{1}{2}(4)^n u(n)$$

Ex: Find $y(n)$ as function of n when $x(n) = \delta(n)$ using Z-transform of the following equation.

$$y(n) - 4y(n-1) + 3y(n-2) = x(n)$$

$$Y(z) - 4Y(z)z^{-1} + 3Y(z)z^{-2} = X(z)$$

$$Y(z)\left[1 - 4z^{-1} + 3z^{-2}\right] = 1$$

$$Y(z) = \frac{1}{1-4z^{-1}+3z^{-2}} \times \frac{z^2}{z^2}$$

$$Y(z) = \frac{z^2}{z^2-4z+3}$$

$$\frac{Y(z)}{z} = \frac{z}{z^2-4z+3} \Rightarrow \frac{Y(z)}{z} = \frac{z}{(z-3)(z-1)}$$

$A = \frac{3}{2}$
$B = -\frac{1}{2}$

$$\frac{Y(z)}{z} = \frac{A}{z-3} + \frac{B}{z-1}$$

$$A = (z-3)\left.\frac{z}{(z-3)(z-1)}\right|_{z=3} = \frac{3}{3-1} = \frac{3}{2}$$

$$B = (z-1)\left.\frac{z}{(z-1)(z-3)}\right|_{z=1} = \frac{1}{1-3} = \frac{-1}{2}$$

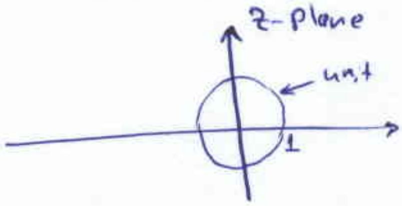
$$\frac{Y(z)}{z} = \frac{3/2}{z-3} + \frac{-\frac{1}{2}}{z-1}$$

$$Y(z) = \frac{3}{2}\frac{z}{z-3} - \frac{1}{2}\frac{z}{z-1}$$

$$\Rightarrow y(n) = \frac{3}{2}(3)^n u(n) - \frac{1}{2}u(n)$$

* Region of Convergence (ROC) of Z-transform.

1- The ROC in Z-transform is indicated as a unit circle



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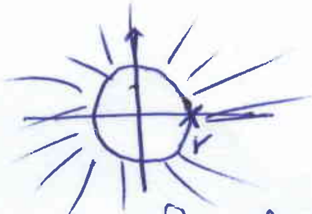
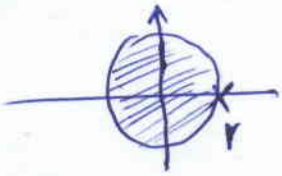
2 If x(n) is a right-sided sequence & if $|Z| = r$ in ROC, then all finite values of Z for which $\boxed{|Z| > r}$ will also be in ROC.

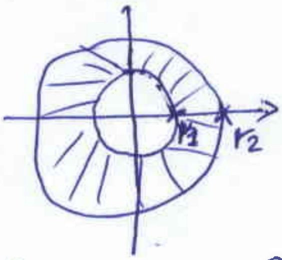


3) If x(n) is left-sided sequence & if $|Z| = r$ in ROC, then all finite values of Z for which $|Z| < r$ will also be in ROC.

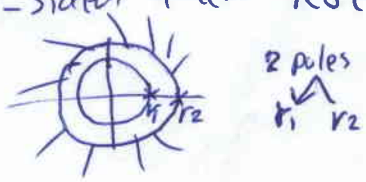


4) If x(n) is two sided signal & if $|Z| = r$, circle is in ROC, then the ROC will contain Ring in Z-plane that include $|Z| = r$

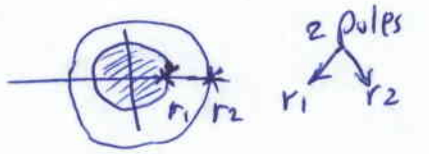
$$r_1 < |Z| < r_2$$



5) If the Z-transform of x(n) is rational & Right-sided then ROC is Z-plane <u>outside</u> the outermost pole.



6) If the Z-transform of x(n) is rational & Left-sided then ROC is Z-plane innermost pole



note: if RoC includes $|Z| = 1$ $(r=1)$, then

DTFT Exists $\Longrightarrow$ converges

* pole - Zero Diagram

$$X(Z) = \frac{N(Z)}{D(Z)}$$

polynomials in Z

$N(Z) = 0 \longrightarrow X(Z) = 0$ "Zeros" : $\xrightarrow{\text{يرسم}}$

$D(Z) = 0 \longrightarrow X(Z) = \infty$ "poles" : $\xrightarrow{\text{يرسم}}$

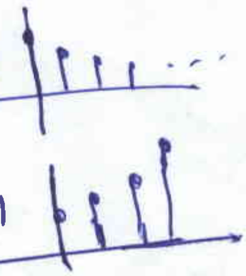
Ex1: (Right-sided Exponential), $x[n] = a^n u(n)$

$$u(n) \xleftrightarrow{Z.T} \frac{Z}{Z-1}$$

$$a^n u(n) \xleftrightarrow{Z.T} \frac{Z}{Z-a}$$

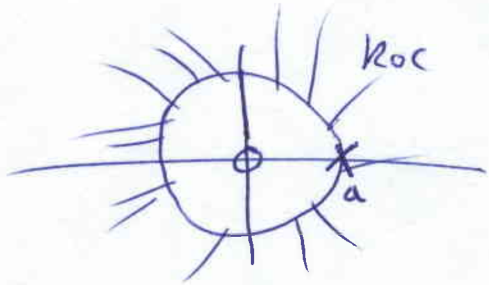
if $|a|<1$

if $|a|>1$



$$X(Z) = \frac{Z}{Z-a} \rightarrow \begin{cases} \text{Zero} = 0 \\ \text{pole} = a \end{cases}$$

RoC, $|Z| > a$



Ex2: Left-sided Exponential, $x(n) = -a^n u(-n-1)$ ($-\infty \to -1$)

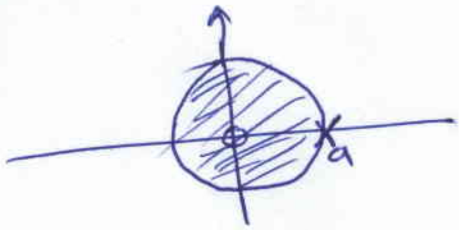
$$X(Z) = \sum_{-\infty}^{\infty} x(n) Z^{-n} = \sum_{-\infty}^{-1} -a^n Z^{-n}$$

$$X(Z) = \sum_{-\infty}^{-1} -\left(\frac{a}{Z}\right)^n = \sum_{n=1}^{\infty} -\left(\frac{Z}{a}\right)^n$$

$$= \frac{-1}{1-\frac{Z}{a}} + 1 \Rightarrow \frac{-a}{a-Z} + \frac{a-Z}{a-Z} = \frac{-Z}{a-Z} = \boxed{\frac{Z}{Z-a}}$$

notice that the Z-transform of Right-sided Exponenetiol looks similar to the one of left sided Exp. But actually they are <u>not</u> the same, Check the RoC of 1st case it is opposite of RoC in 2nd case.

$$|z| < a$$
left-sided



EX3: $x[n] = \left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n)$

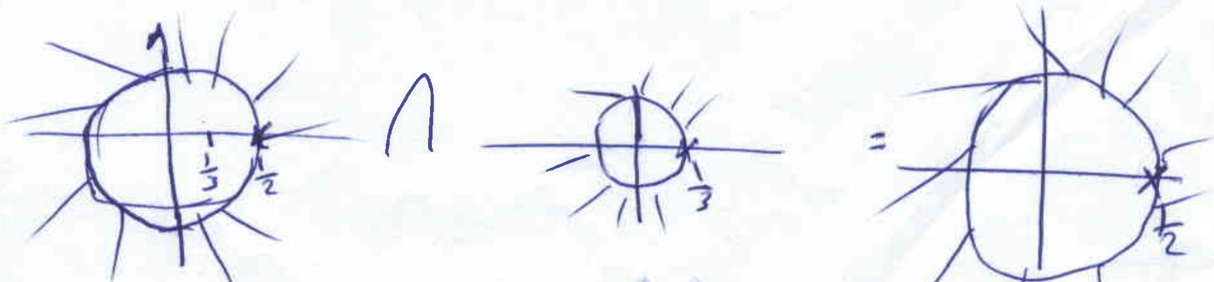
$$\frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z^2 - \frac{1}{6}z}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

$$X(z) = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

RoC: $\dfrac{z}{z-\frac{1}{2}} + \dfrac{z}{z+\frac{1}{3}}$

$$\underset{\text{Roc}}{\downarrow} \qquad \underset{\text{Roc}}{\downarrow}$$

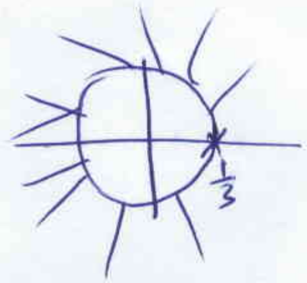
$$|z| > \frac{1}{2} \ \cap \ |z| > \frac{1}{3}$$



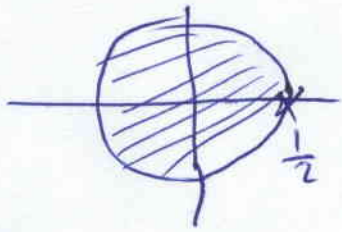
Ex 4: $x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$

$\downarrow$ $\downarrow$

ROC: $|z| > \frac{1}{3}$ $\cap$ ROC: $|z| < \frac{1}{2}$

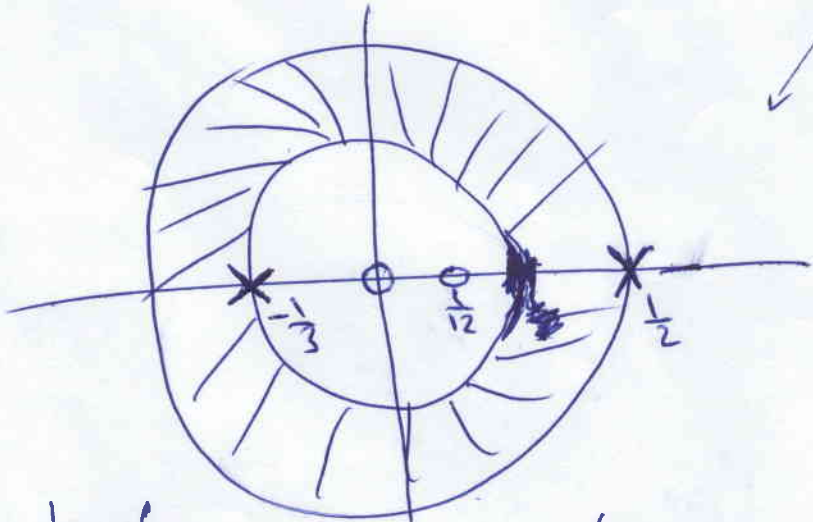


$\cap$



$\parallel$

$$X(z) = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$



* Inverse Z-transform can be taken at least by Four different method:

✓ ① by Inspection

✓ ② partial fraction Expansion

③ power Series Expansion

④ Contour Integration.