Ninevah University
College Of Electronics Engineering
Communication Engineering Department

Class	**Second**			**Theory :**	**3 Hrs/wk**
Subject	**Engineering Analysis**			**Tutorial**	**1 Hrs/wk**
Code	**CE2201**	**Unit**	**6**	**Practical**	**Hrs/wk**

Article	Hrs
Multiple Integrals: i) Double Integral. ii) Area and volumes. iii) Double Integral in Polar Coordinates iv) Evaluation of volume and triple Integrals. v) Evaluation of surface & surface Integrals.	**8**
Sequences And Series: i) Sequences: convergence; Test of monotone ii) series : geometric series; nth partial sum; test of convergence; alternating series. iii) Power and Taylor's series.	**8**
Vectors Functions: i)Equations of lines and planes. ii) Product of three or more vectors. iii) Vector function & motion : velocity and acceleration. iv)Tangential vectors. v) Curvature and normal vector.	**10**
Ordinary Differential Equations: i) First order (variables separable; homogeneous; linear – Bernoulli and exact). ii) Second order (Homogeneous and non homogeneous). iii) Higher order differential equations.	**10**
Solution Of Differential Equations By Power Series: Legendre s equation; Legendre s polynomials; Bessel function of the first and second kinds; Bessel function properties.	**10**
Partial Differentiation Equation: Wave equation; laplace equation; solution of boundary condition problems; general solution; solution by separation of variables.	**10**
Numerical Analysis: i)Solution of non-linear equations (Iteration; bisection and Newton-Raphson). ii) Finite differences. iii) Numerical differentiation and Integration. iv) Numerical solution of 1st order ordinary differential equations.	**10**
Matrix Analysis: Review of matrix theory; Linear transformation; Eigen values & eigen vectors; lace Lap transform of matrices; Application of matrices to electric circuits.	**10**
Statistics: Definition; Frequency distribution (relative & commutative; Mean; Standard deviation).	**10**

Article	Hrs
Probability: Definition; mutually exclusive & conditional probability; permutations & combinations; Probability distribution: Binomial; Normal & Poisson distributions.	**10**
Complex Variable Theory: Function of complex variable; complex differentiation; Analytic function & its properties; Integration in the complex plane; Cauchy s theorem; Cauchy s integral formula for simply & multiply connected regions; Complex variable theory: Taylor's theorem; Laurent series; The residue theorem.	**10**
Applications of Matlab	**14**
Total	**120**

Text book:
1: "Advanced Engineering Mathematics" By KREYSIK
2: "Calculus" By Finney& Thomas

CHAPTER ONE

Multiple Integrals

The integral of functions of several variables called multiple integral (double or triple).

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy. \quad \text{Volume}$$

$$\iint_R dA = \iint_R dx\,dy. \quad \text{.......Area}$$

Ex1: solve the following integral:

$$\int_{-1}^{1}\int_{0}^{2}(1 - 6x^2y)\,dx\,dy$$

Sol:

$$= \int_{-1}^{1} x - 3x^3y\Big]_0^2 dy$$

$$= \int_{-1}^{1} 2 - 16y\,dy$$

$$= 2y - 8y^2\Big]_{-1}^{1}$$

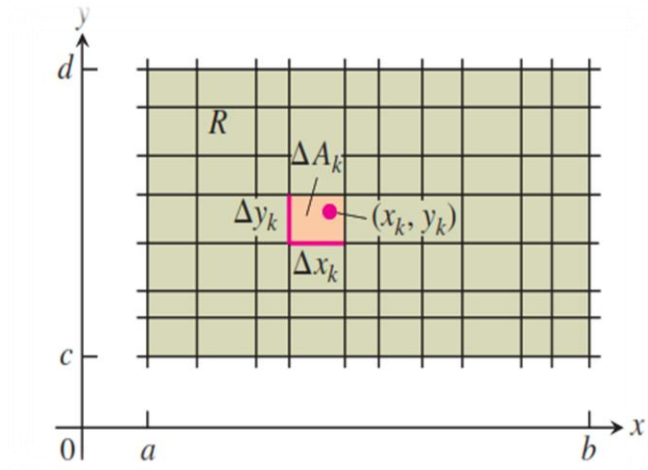
$$= 2 - 8 - (-2 - 8) = 4$$

1-Double Integral over Bounded Rectangular Region

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function f(x, y) defined on a rectangular region R,

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

This region is represent the base region of the solid volume with height is f(x,y). This region is divided into group of small rectangular pieces with area:



$$\Delta A_k = \Delta x_k \Delta y_k$$

Then the volume of small rectangular element is given by:
$\Delta V_k = \Delta A_k f(x_k, y_k)$
Then the sum of the volumes is:

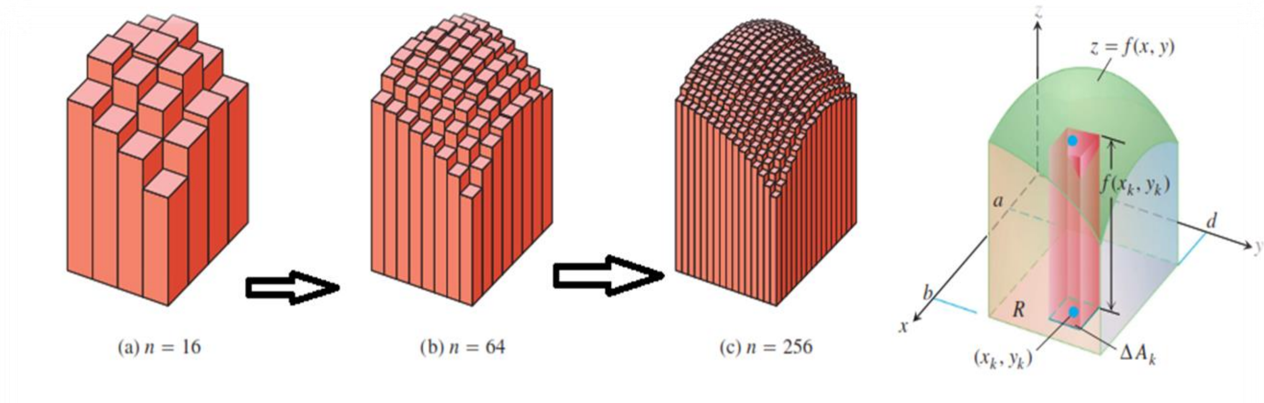
$$S_n = \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} \Delta A_k f(x_k, y_k)$$

Where: *n* represents the No. of rectangular pieces.
To find the total volume of the solid we take limit for both sides of the above equation (as $n \to \infty$). then:

$$Volume = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta A_k f(x_k, y_k)$$

Then :

$$V = \iint_R f(x,y)dA = \iint_R f(x,y)dxdy$$



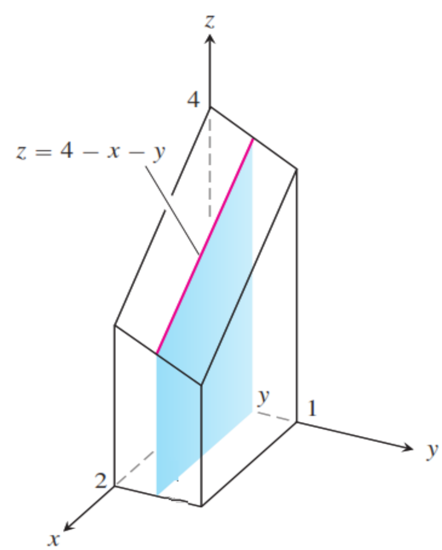
As n increases, the Riemann sum approximations approach the total volume of the solid

In general:

THEOREM 1 Fubini's Theorem (First Form)
If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Ex1: If $z = f(x, y) = 4 - x - y$ over the region R, R: $0 \leq x \leq 2$ & $0 \leq y \leq 1$, find the volume bounded by R and the surface Z.



5
4
3
2
1
-5
-4
-3
-2
-1
1
2
3
4
5
-1
-2
-3
-4
-5

Ex2: If $z = f(x, y) = y \cos xy$ over the region *R*, R: $0 \leq x \leq \pi$ & $0 \leq y \leq 1$, find the volume bounded by *R* and the surface *Z*.

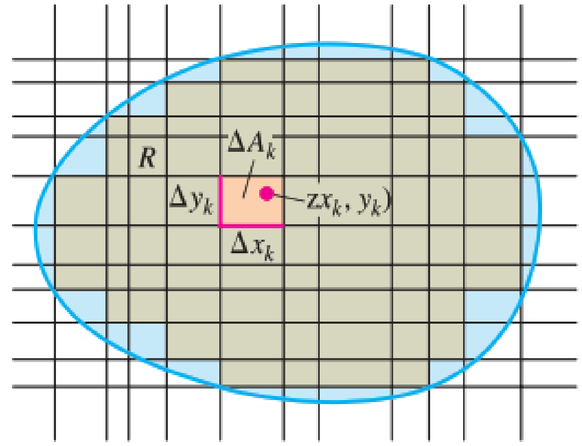
Ex3: If $z = f(x, y) = \frac{1}{xy}$ over the region *R*, R: $1 \le x \le 2$ & $1 \le y \le 2$, find the volume bounded by *R* and the surface *Z*.

HW1: Find the volume of the region bounded above by elliptical paraboloid $z = f(x, y) = 10 + x^2 + 3y^2$ and below the rectangle R: $0 \le x \le 1$ & $0 \le y \le 2$.

HW2: If $z = f(x, y) = \sin x + \cos y$ over the region *R*, R: $0 \le x \le \pi$ & $\pi \le y \le 2\pi$ find the volume bounded by *R* and the surface *Z*.

2-*Dobule Integrals over bounded non-rectangular region*

We considered that the function *z=f(x,y)* is defined over non-rectangular region.



$x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$

Where: either

1) x_1, x_2 are constants and

$$y_1 = g_1(x), y_2 = g_2(x).$$

2) Or y_1, y_2 are constants and

$$x_1 = h_1(y), x_2 = h_2(y).$$

By the same way mentioned in sec.1 then the volume:

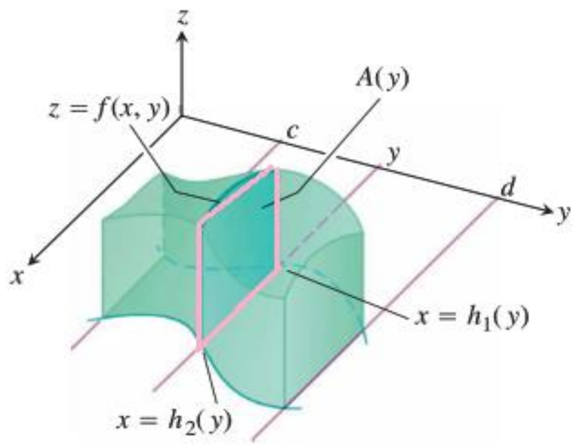
THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

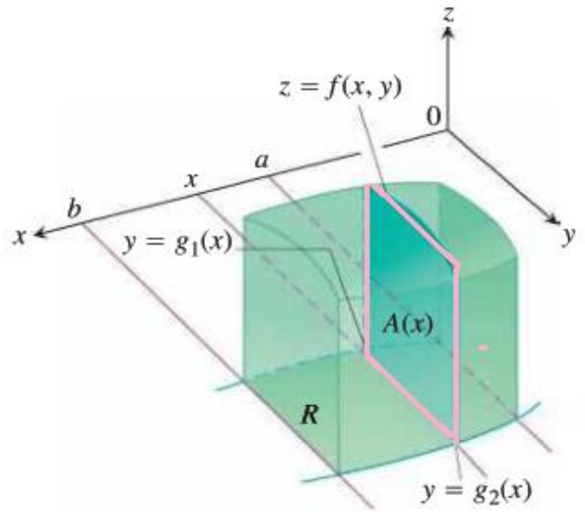
1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$





The volume of the solid shown here is

$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

The volume of the solid shown here is

$$\int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

Finding the limits of integration:

A) Using vertical cross sections (Line in y-axis direction)

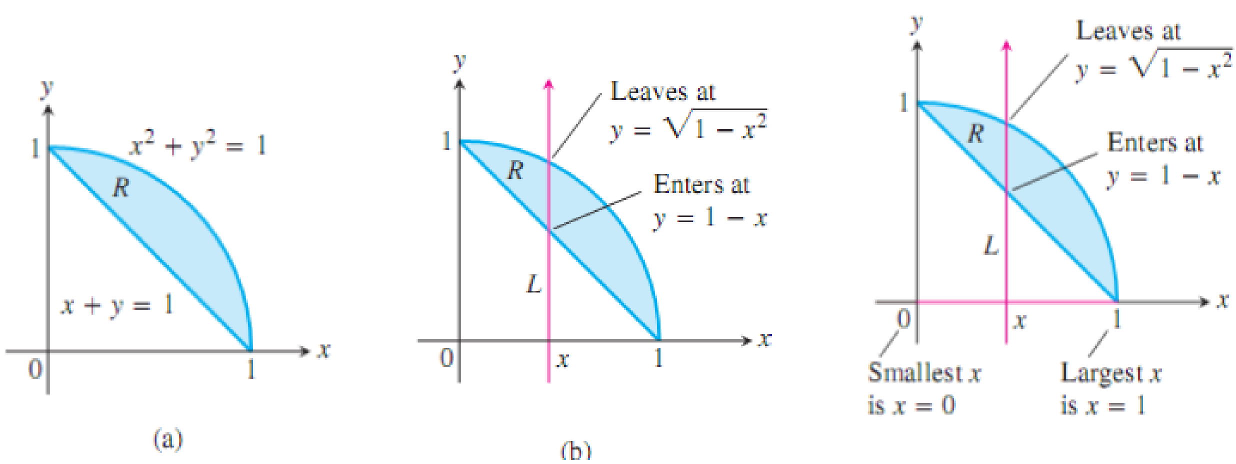


FIGURE Finding the limits of integration when integrating first with respect to y and then with respect to x.

1. *Sketch*. Sketch the region of integration and label the bounding curves.
2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).
3. *Find the x-limits of integration*. Choose x-limits which represents the minimum and maximum values of x.

Then apply:

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$

B) Using Horizontal cross sections(Line in x-axis direction)

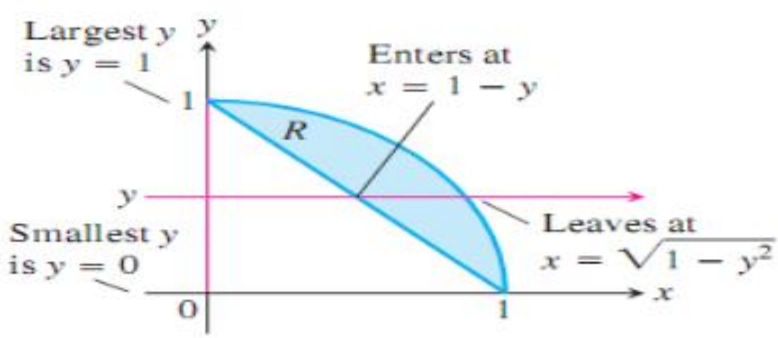


FIGURE Finding the limits of integration when integrating first with respect to x and then with respect to y.

The same procedure can be applied here but the line in step 2 will be in the x-axis direction. Then apply the:

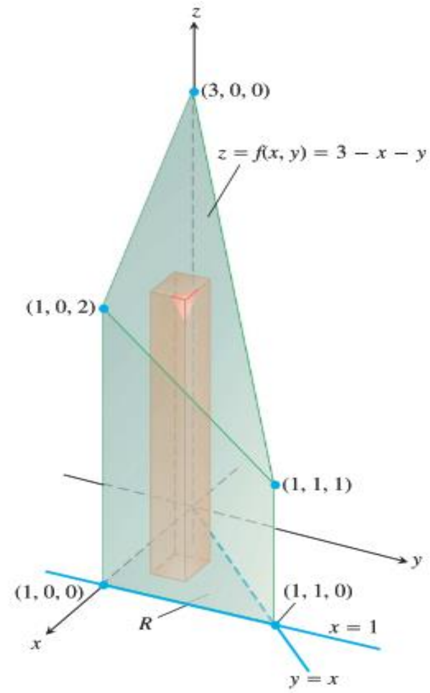
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

$$\int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

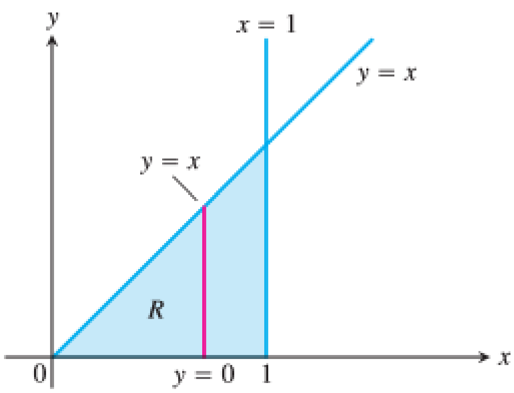
Order of integration: It is mean the arrangement of integral, *dydx* or *dxdy*.

Ex1: Find the volume of the solid whose base is the triangular in the xy-plane bounded by the x-axis and the lines y=x and x=1 and whose top lies in the plane:

$$z = f(x, y) = 3 - x - y.$$



Sol;



$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

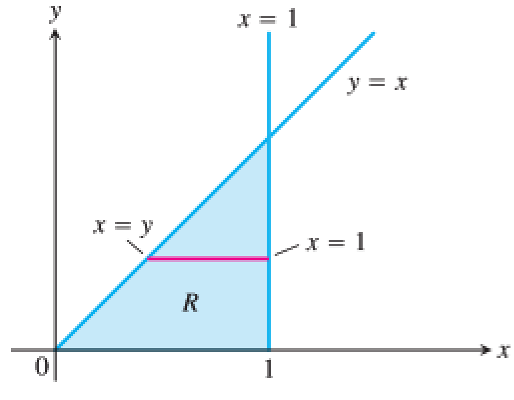
$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed the integral for the volume is

$$V=\int_0^1\int_y^1(3-x-y)\,dx\,dy=\int_0^1\left[3x-\frac{x^2}{2}-xy\right]_{x=y}^{x=1}dy$$

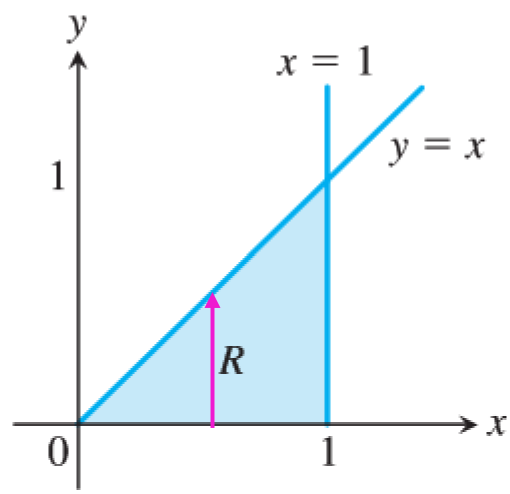
$$=\int_0^1\left(3-\frac{1}{2}-y-3y+\frac{y^2}{2}+y^2\right)dy$$

$$=\int_0^1\left(\frac{5}{2}-4y+\frac{3}{2}y^2\right)dy=\left[\frac{5}{2}y-2y^2+\frac{y^3}{2}\right]_{y=0}^{y=1}=1.$$



Ex2: Find the volume of the solid whose base is the triangular in the xy-plane bounded by the x-axis and the lines y=x and x=1 and whose top lies in the plane:

$$z=f(x,y)=\frac{\sin x}{x}$$

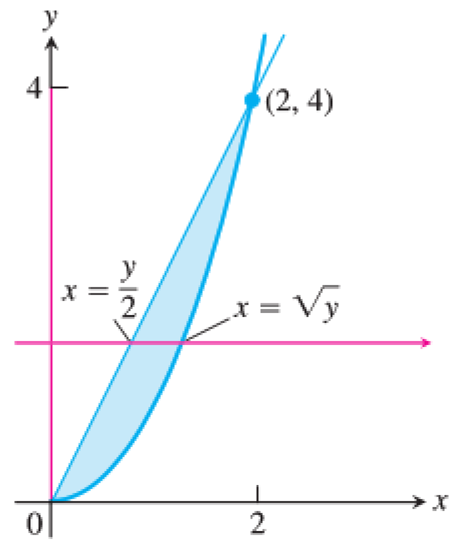


$$\int_0^1\left(\int_0^x\frac{\sin x}{x}\,dy\right)dx=\int_0^1\left(y\frac{\sin x}{x}\right]_{y=0}^{y=x}\right)dx=\int_0^1\sin x\,dx$$

$$=-\cos(1)+1\approx 0.46.$$

Ex3: sketch the region of integration and write the equivalent integral with reversed order of the given integration.

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$



Sol:

x	Y=2x
0	0
1	2
2	4

x	$Y=x^2$
0	0
1	1
2	4

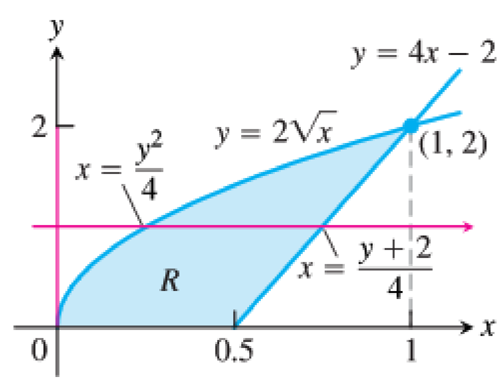
$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

Ex4: Find the volume of the solid whose base in the xy-plane is bounded by the x-axis the line *y=4x-2* and the curve $y = 2\sqrt{x}$ and whose top lies in the plane:

$z = f(x, y) = 16 - x^2 - y^2.$

x	Y=4x-2
0	-2
0.5	0
2	4
1	2
4	14

x	$Y=2\sqrt{x}$
-4	يهمل
-1	يهمل
0	0
1	2
4	4



$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

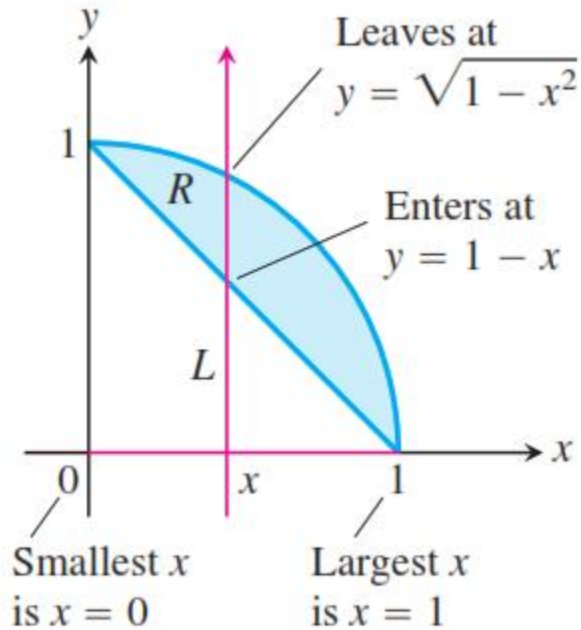
$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2 \right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y + 2) - \frac{(y + 2)^3}{3 \cdot 64} - \frac{(y + 2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4} \right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344} \right]_0^2 = \frac{20803}{1680} \approx 12.4.$$

Ex5: Find the volume of the solid whose base in xy-plane bounded by the circle $x^2 + y^2 = 1$ and the line y=1-x in the 1st quadrant and whose top lies in the plane:

$$z = f(x, y) = 3 - x - y.$$



Sol:

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

H.W1: sketch the region of integration and write the equivalent integral with reversed order of the given integration.

$$V = \int_{0}^{2} \int_{0}^{4-2x} dydx$$

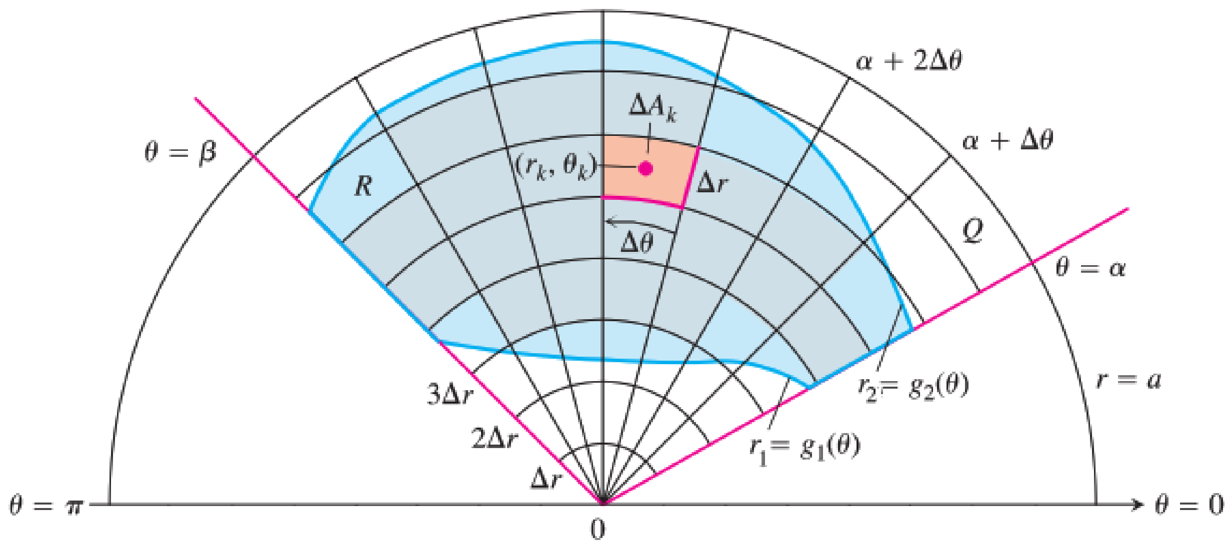
H.W2: Find the volume of the solid whose base in the xy-plane is bounded by the x-axis and the lines *y=4x* and $y = 2 - x$ and whose top lies in the plane:

$$z = f(x,y) = 3 - x - y$$

3-Dobule Integral in Polar form

Integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Suppose that we have two curves $r_1 = g_1(\theta)$ and $r_2 = g_2(\theta)$ as shown in figure below. To find the area bounded between them, divided the bounded area into small rectangular elements.



$Arc\ length\ (dr) = rd\theta$

The area of the small rectangular element is:

$dA = rdrd\theta$

Then the total area in a polar form is;

$$\boxed{A = \int_{\theta_1}^{\theta_2} \int_{r_1=g_1(\theta)}^{r_2=g_2(\theta)} rdrd\theta}$$

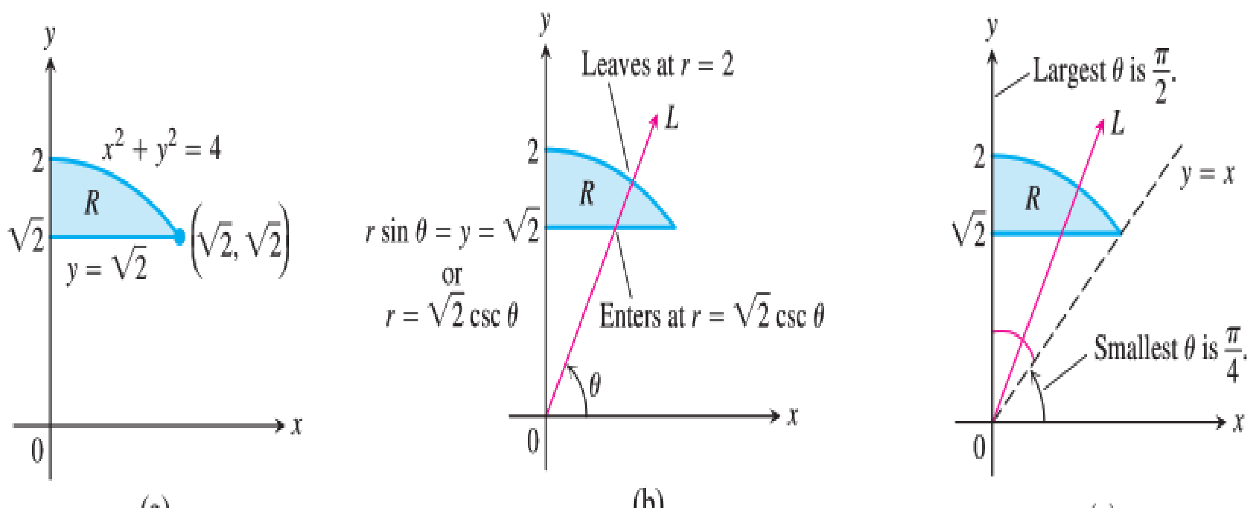
And the volume under the surface *f(r,θ)* is :

$$\boxed{V = \int_{\theta_1}^{\theta_2} \int_{r_1=g_1(\theta)}^{r_2=g_2(\theta)} f(r,\theta)\, r dr d\theta}$$

Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta)\, dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

1. *Sketch.* Sketch the region and label the bounding curves (Figure *a*).
2. *Find the r-limits of integration.* Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis (Figure b).
3. *Find the θ-limits of integration.* Find the smallest and largest θ-values that bound R. These are the θ-limits of integration (Figure c). The polar iterated integral is



(a) (b) (c)

Ex1: Evaluate

$$\iint_R e^{x^2+y^2}\,dy\,dx,$$

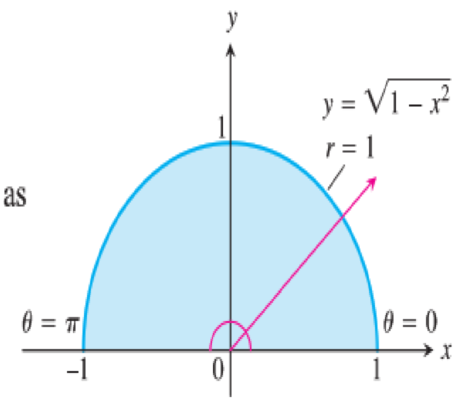
where R is the semicircular region bounded by the x-axis and the curve $x^2 + y^2 = 1$.
in the 1st and 2nd quadrant.

Sol:

Substituting $x = r\cos\theta$, $y = r\sin\theta$
and replacing $dy\,dx$ by $r\,dr\,d\theta$ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1).$$



The semicircular region

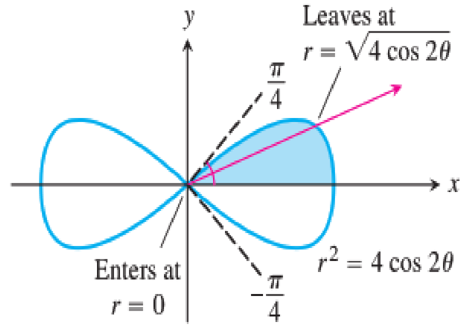
$0 \le r \le 1, \quad 0 \le \theta \le \pi.$

Ex2: Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration (Figure) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$



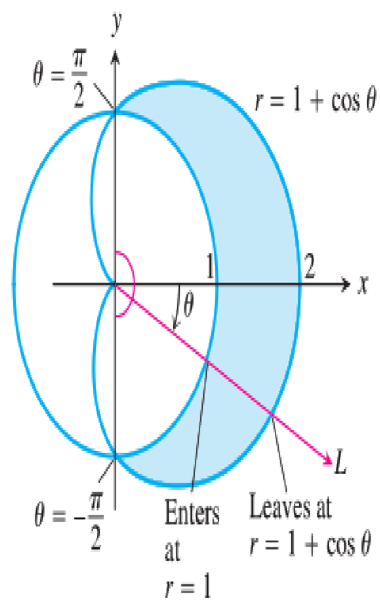
To integrate over the shaded region, we run r from 0 to $\sqrt{4\cos 2\theta}$ and θ from 0 to $\pi/4$

Ex3; Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta.$$



Finding the limits of integration in polar coordinates

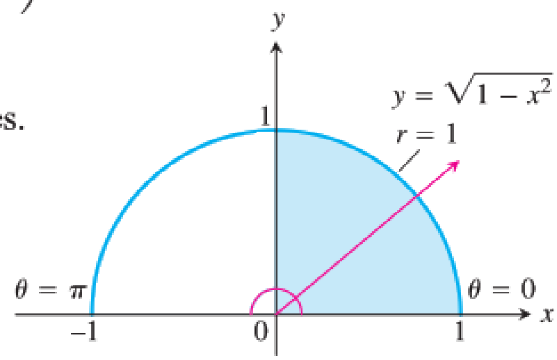
Ex4: Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Solution Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \right) dx,$$

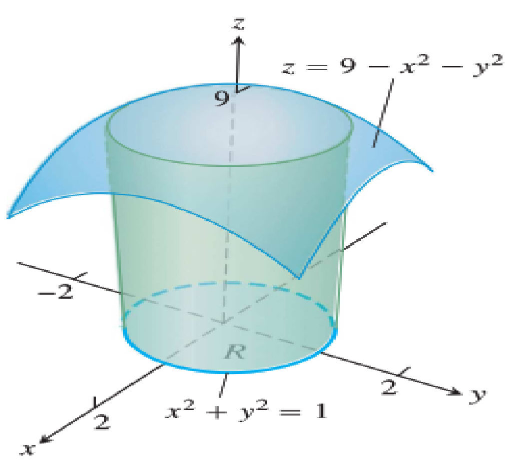
an integral difficult to evaluate without tables.



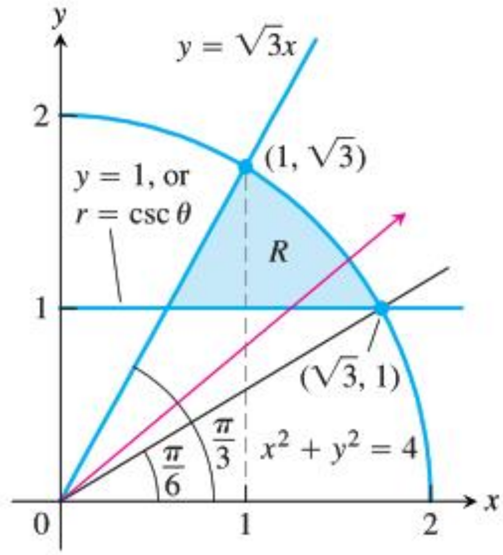
$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

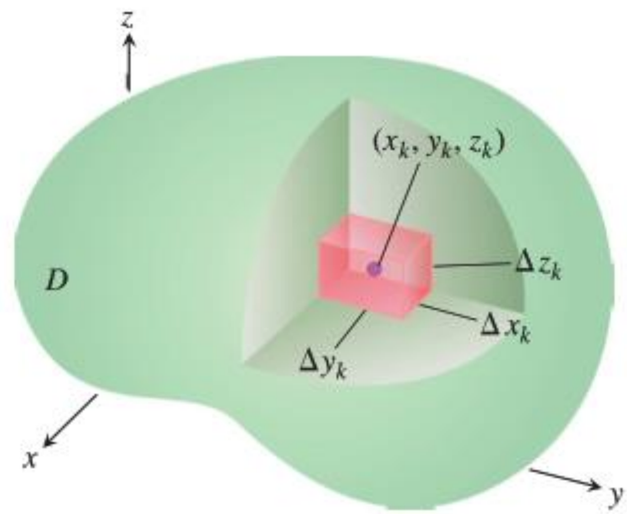
HW1 : Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.



HW2 : Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.



4-Triple Integral in Rectangular Coordinate



Suppose the surface $z = f(x, y)$ to find the volume under the surface divide the volume into small rectangular solids with volume:

$\Delta V = \Delta z \Delta y \Delta x$

Then the total volume

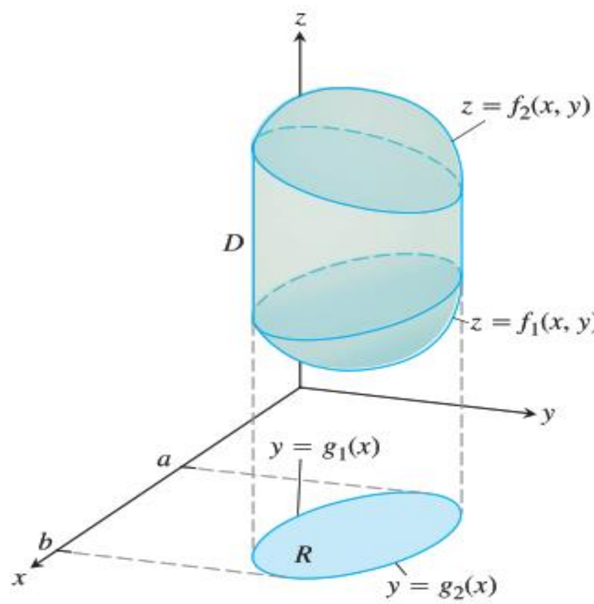
$$V = \sum \Delta V$$

$$\boxed{V = \int_{x_1}^{x_2} \int_{y_1=g_1(x)}^{y_2=g_2(x)} \int_{z_2=f_2(x,y)}^{z_2=f_2(x,y)} dz\, dy\, dx}$$

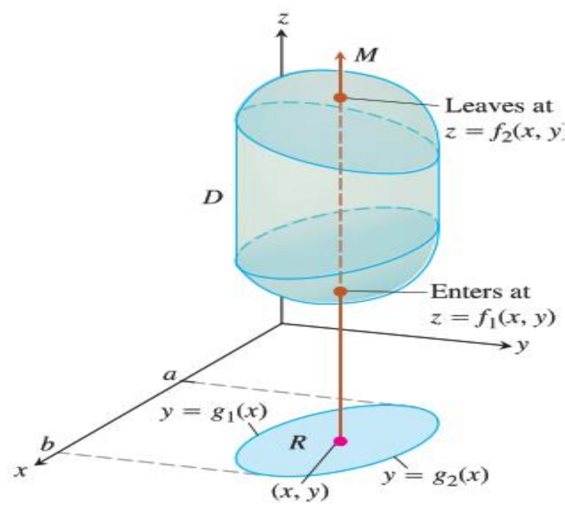
Finding Limits of integration in the order dz dy dx

To evaluate $\iiint F(x, y, z)dV$ over the region D, integrate first with respect to z, then with respect to y, and finally with respect to x. to find the limits of integration:

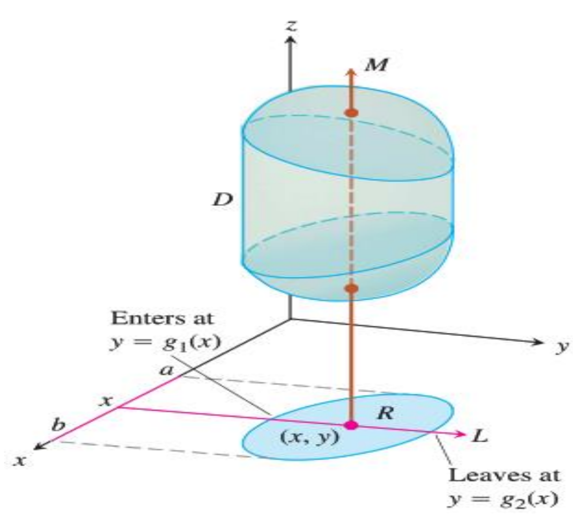
1- Sketch. Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2- Find the z-limits of integration. Draw a line M passing through a typical point (x,y) in R parallel to the z-axis. As z increases, M enters D at z = $f_1(x, y)$ and leaves at z = $f_2(x,y)$. These are the z-limits of integration.



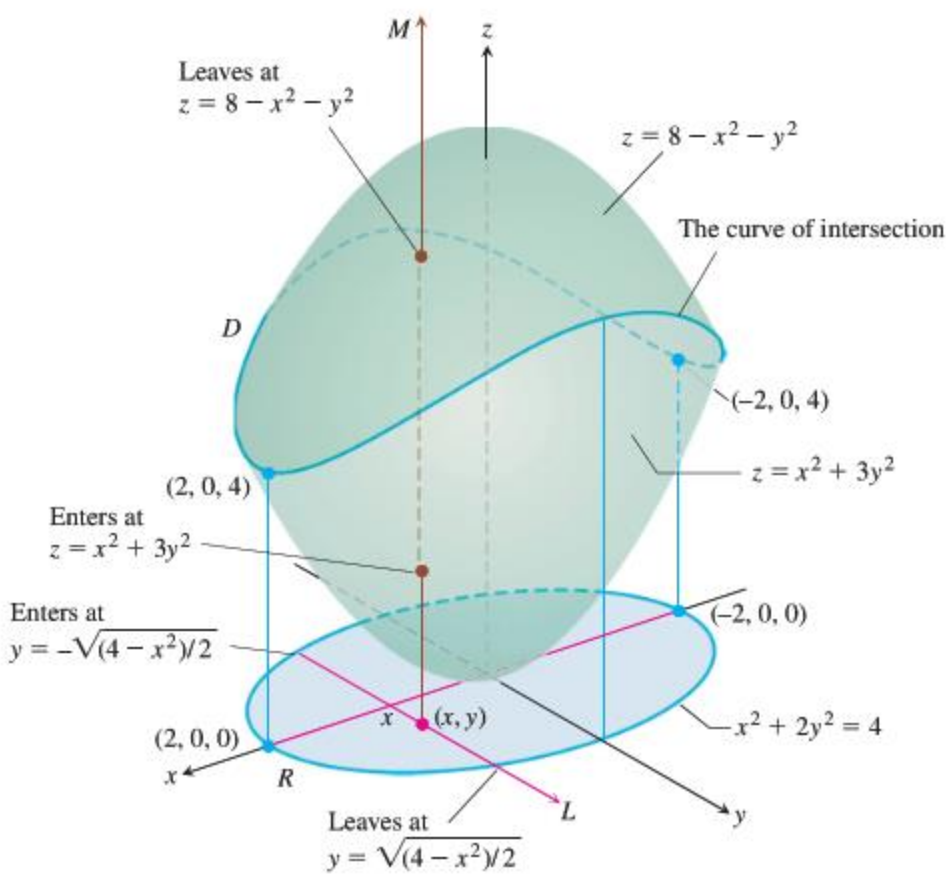
3- Find the y-limits of integration. Draw a line L through (x,y) parallel to the y-axis. As y increases, L enters R at y = $g_1(x)$ and leaves at y = $g_2(x)$. These are the y-limits of integration.



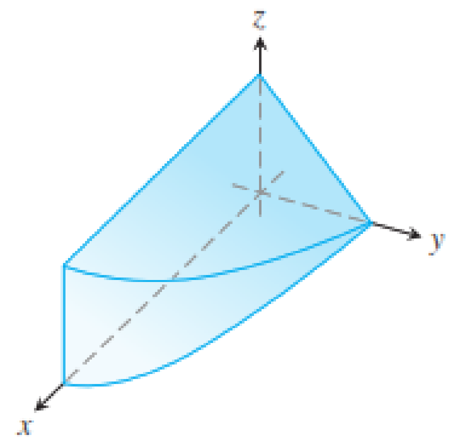
4- Find the x-limits of integration. Choose x-limits that include all lines through R parallel to the y-axis (x = a and x = b in the preceding figure). These are the x-limits of integration. The integral is:

$$V = \int_{x_1}^{x_2} \int_{y_2=g_2(x)}^{y_2=g_2(x)} \int_{z_2=f_2(x,y)}^{z_2=f_2(x,y)} dz\, dy\, dx$$

Ex1 : Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



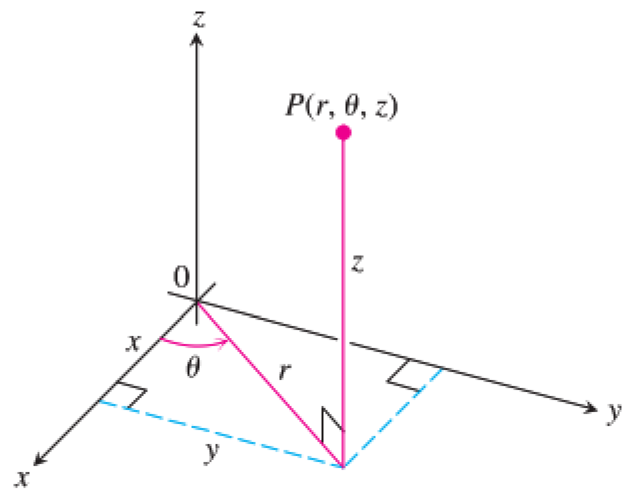
Ex2: Find the volume of the 3D-region in the 1st octant bounded plane and $z + y = 2$ and the cylinder $x = 4 - y^2$.



Ex3: Find the volume bounded by the surface $z = 3 - x - y$ and the area in x-y plane enclosed by the circle $x^2 + y^2 = 1$ and the line y=x and y-axis.

5- Triple Integral in cylindrical coordinate

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ z), as shown in Figure.



The cylindrical coordinates of a point in space are r, θ, and z.

DEFINTION Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane.
2. z is the rectangular vertical coordinate.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

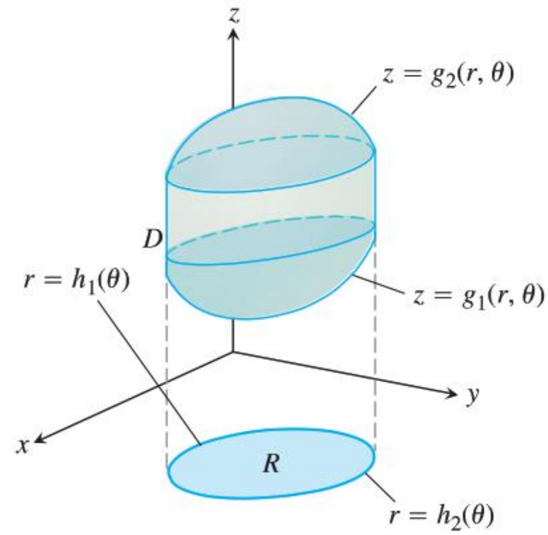
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

How to Integrate in Cylindrical Coordinates

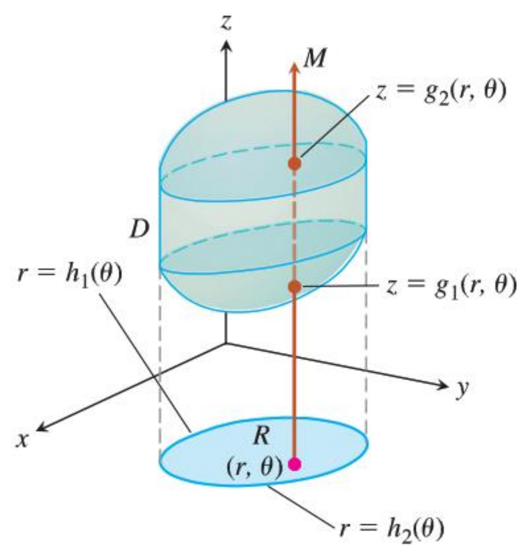
To evaluate:

$$\iiint_D f(r, \theta, z)\, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

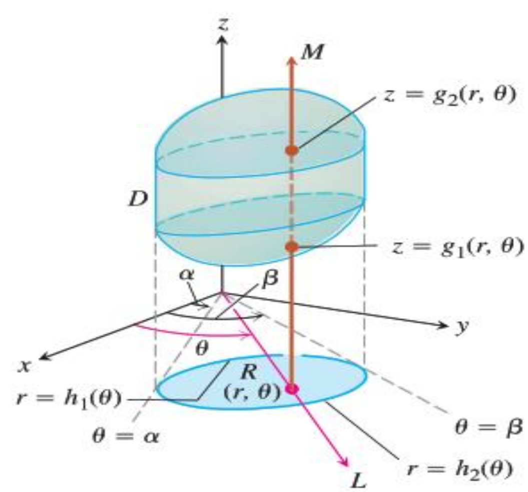


1-**Sketch the region D** along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



2- **Find the z-limits** of integration. Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z_1 = g_1(r, \theta)$ and leaves at $z_2 = g_2(r, \theta)$. These are the z-limits of integration.

3- **Find the r-limits** of integration. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



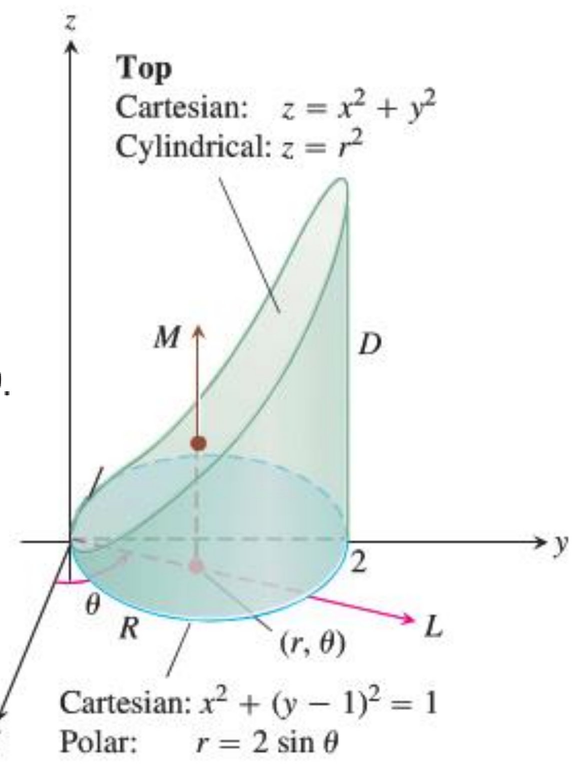
4- **Find the θ-limits** of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from θ = α to θ = β. These are the θ-limits of integration. The integral is

$$\iiint_D f(r,\theta,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for integrating a function $f(r,\theta,z)$ over the region D bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$, and above by the paraboloid $z=x^2+y^2$.

$$\begin{aligned} x^2+(y-1)^2 &= 1 \\ x^2+y^2-2y+1 &= 1 \\ r^2-2r\sin\theta &= 0 \\ r &= 2\sin\theta. \end{aligned}$$

$$\iiint_D f(r,\theta,z)\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$

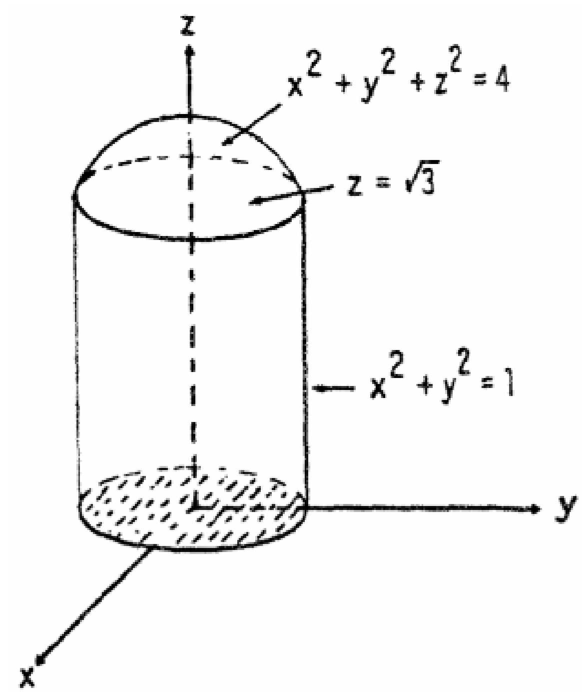


Ex2: Let D be the region bounded below by the plane z = 0, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

a) dz dr dθ .

b) dr dz dθ .

c) dθ dz dr.



$$r^2 + z^2 = 4$$

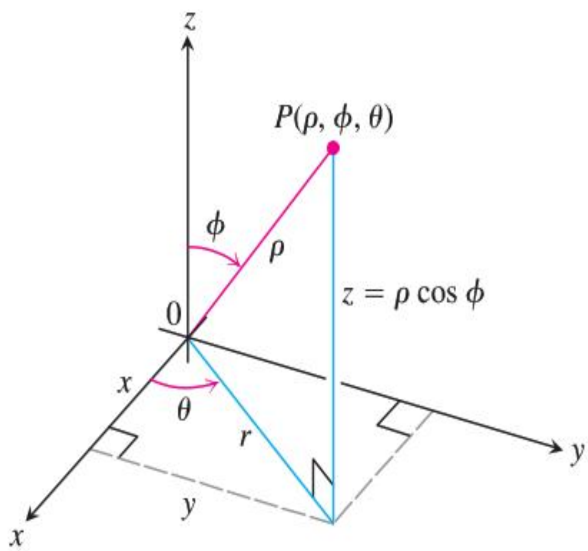
$$\text{(a)} \quad \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$$

$$\text{(b)} \quad \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^1 r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^2\int_0^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta$$

$$\text{(c)} \quad \int_0^1\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta\, dz\, dr$$

6- Triple Integral in spherical coordinate

Spherical coordinates locate points in space with two angles and one distance, as shown in Figure. The first coordinate $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin. Unlike r, the variable ρ is never negative. The second coordinate, $\emptyset$, is the angle $|\overrightarrow{OP}|$ makes with the positive z-axis. It is required to lie in the interval [0, π]. The third coordinate is the angle θ as measured in cylindrical coordinates.



The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

DEFINITION: Spherical coordinates represent a point P in space by ordered triples (ρ, Φ, θ) in which:

1. ρ is the distance from P to the origin.

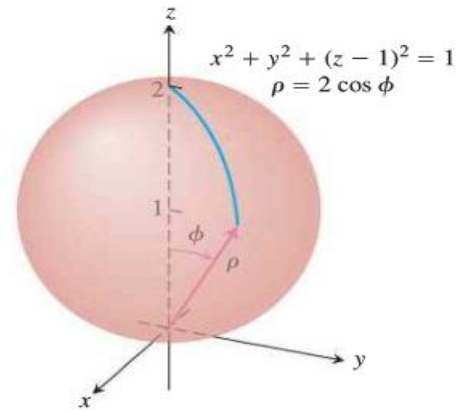
2. Φ is the angle $|\overrightarrow{OP}|$ makes with the positive z-axis $(0 \le \emptyset \le \pi)$.

3. θ is the angle from cylindrical coordinates $(0 \le \theta \le 2\pi)$.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin\phi, \qquad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta,$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

Ex1: Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.



$$x^2 + y^2 + (z - 1)^2 = 1$$

$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

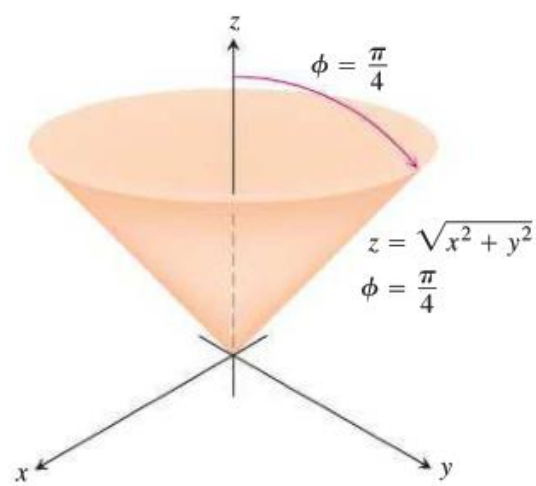
$$\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 2\rho\cos\phi$$

$$\rho^2 = 2\rho\cos\phi$$

$$\rho = 2\cos\phi. \qquad \rho > 0$$

Ex2: Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$



$$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi}$$

$$\rho\cos\phi = \rho\sin\phi$$

$$\cos\phi = \sin\phi$$

$$\phi = \frac{\pi}{4}.$$

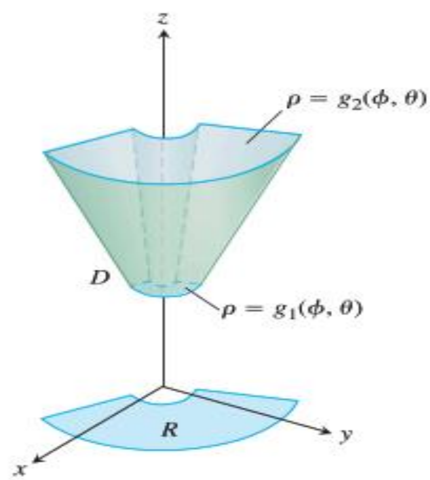
How to Integrate in Spherical Coordinates

To evaluate:

$$\iiint_D f(\rho, \phi, \theta)\, dV$$

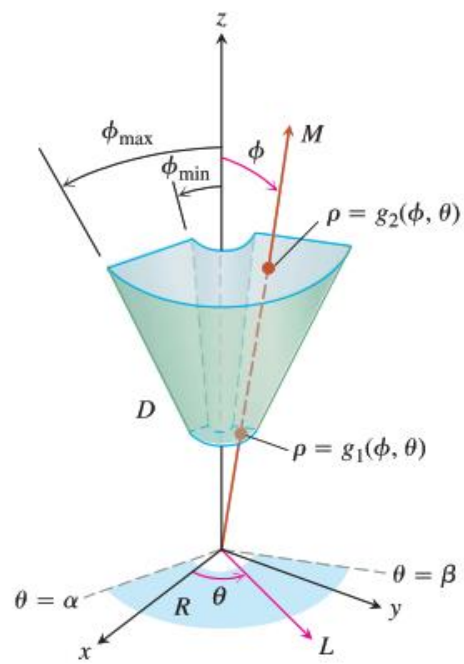
over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to Φ, and finally with respect to θ, take the following steps.

1- Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.



2- Find the ρ-limits of integration. Draw a ray M from the origin through D making an angle Φ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\emptyset, \theta)$ and leaves at $\rho = g_1(\emptyset, \theta)$. These are the ρ-limits of integration.

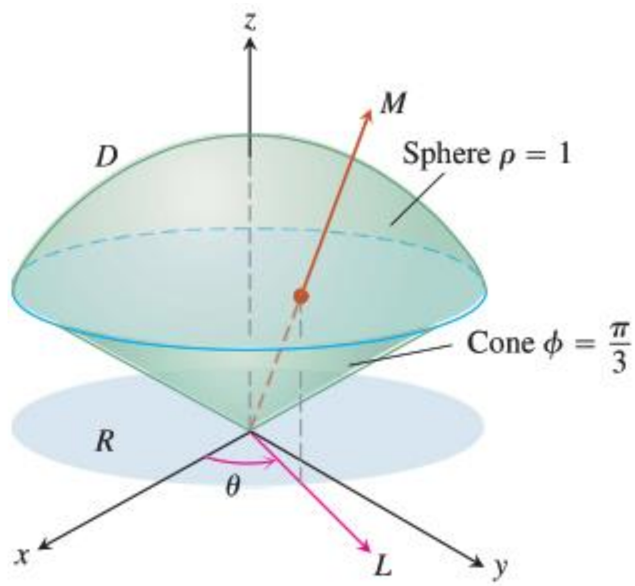
3- Find the Φ-limits of integration. For any given θ, the angle Φ that M makes with the z-axis runs from Φ = Φ min to Φ = Φ max. These are the Φ -limits of integration.



4- Find the θ-limits of integration. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is:

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

Ex1: Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\emptyset = \frac{\pi}{3}$.



$$V = \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi}\left(-\frac{1}{6}+\frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

CHAPTER TWO

Vectors

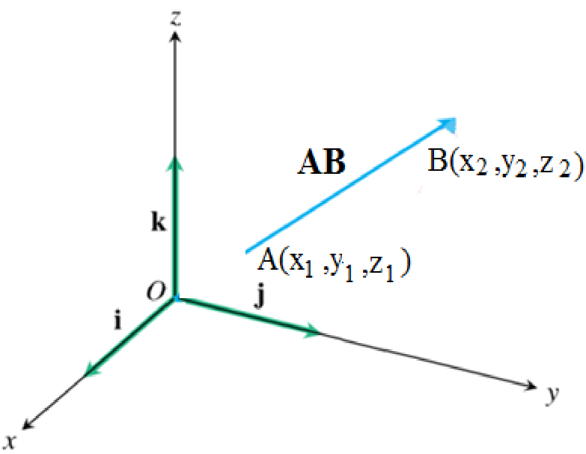
Some of the things we measure are determined simply by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is.

1-Vector in Space:

$$\overrightarrow{AB} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$
$$\overrightarrow{AB} = xi + yj + zk$$

Then the length (magnitude) of the vector is:

$$\boxed{\left|\overrightarrow{AB}\right| = \sqrt{x^2 + y^2 + z^2}}$$



Where as :

i : is a unit vector in the direction of x.

j : is a unit vector in the direction of y.

k : is a unit vector in the direction of z.

Note: Two vectors are equal if they have the same length and direction.

2-Unit Vector:

It is a vector whose length is equal to the one unit of length along the coordinate axis.

$$\boxed{U_{AB} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}}$$

Ex: let *A(-3,4,1)* and *B(-5,2,2)* two points in the space, find:

1- The vector $\overrightarrow{AB}$.

2- Length of $\overrightarrow{AB}$.

3- Unit vector of $\overrightarrow{AB}$.

Sol:

$$\overrightarrow{AB} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k = (-5+3)i + (2-4)j + (2-1)k$$
$$\overrightarrow{AB} = -2i - 2j + k$$

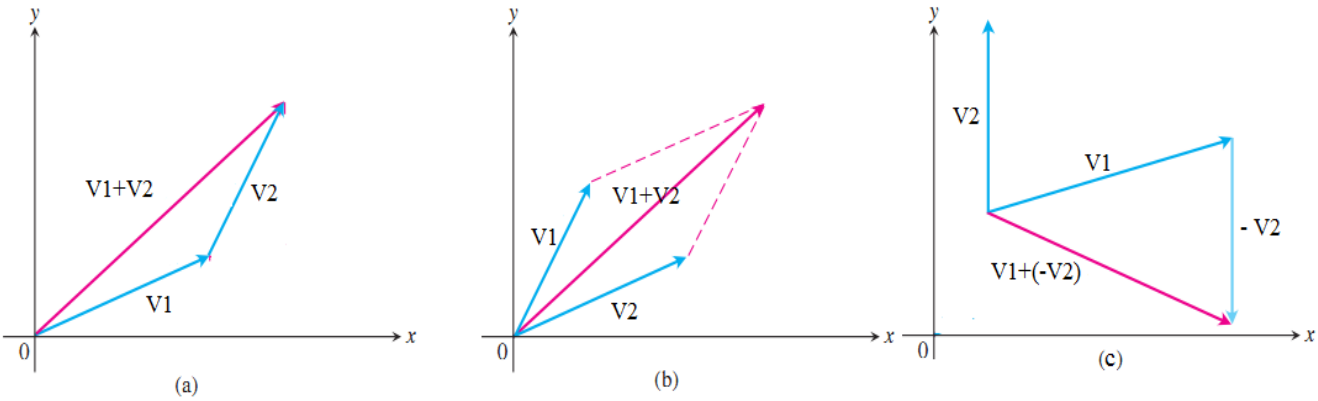
$$\left|\overrightarrow{AB}\right| = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$U_{AB} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{-2i - 2j + k}{3} = \frac{-2}{3}i - \frac{2}{3}j + \frac{1}{3}k$$

....

3-<u>Addition and Subtraction of Vectors</u>: let

$$\vec{V}_1 = x_1 i + y_1 j + z_1 k$$
$$\vec{V}_2 = x_2 i + y_2 j + z_2 k$$
$$\vec{V}_1 + \vec{V}_2 = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$
$$\overrightarrow{V_1} - \overrightarrow{V_2} = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k$$



4-Multiply Vectors with constant:

When we multiply a vector with a constant, that is mean changing in the length of the vector (scaling vector length).

Let C is a constant and the vector $\vec{V}$ is:

$$\vec{V} = xi + yj + zk$$

Then

$$C * \vec{V} = (C * x)i + (C * y)j + (C * z)k$$

Ex: let $\vec{V_1} = -i + 3j + k$ and $\vec{V_2} = 4i + 7j$ find:

1- $\vec{V_1} + \vec{V_2}$. 2- $\vec{V_1} - \vec{V_2}$. 3- $\left|\frac{1}{2}\vec{V_1}\right|$.

$$\vec{V_1} + \vec{V_2} = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k = 3i + 10j + k$$
$$\vec{V_1} - \vec{V_2} = (x_1 - x_2)i + (y_1 - y_2)j + (z_1 - z_2)k = -5i - 4j + k$$

$$\left|\frac{1}{2}\vec{V_1}\right| = \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{11}{4}} = \frac{1}{2}\sqrt{11}$$

5-Slope, tangents(V_T), Normal (V_n) in the plane:

If $y=f(x)$ represent a curve in x-y plane then the 1st derivative

$$\frac{dy}{dx} = y' \text{ slop of tanget at p(x, y)} = S_T = \frac{b}{a}$$

The tangent vector is:

$$\vec{V_T} = ai + bj$$

Slope of tangent $S_n = -\frac{1}{S_T} = -\frac{a}{b}$

$V_n = -bi + aj$ or $V_n = bi - aj$

Ex1 : find the vector tangent and normal to the curve $y = x + \frac{1}{2}$ at p(1,1).

Sol:

$$y' = \frac{dy}{dx} = 1 = \frac{b}{a} = slop$$

$$V_T = i + j$$

$$S_n = -\frac{1}{S_T} = -1 = \frac{a}{b}$$

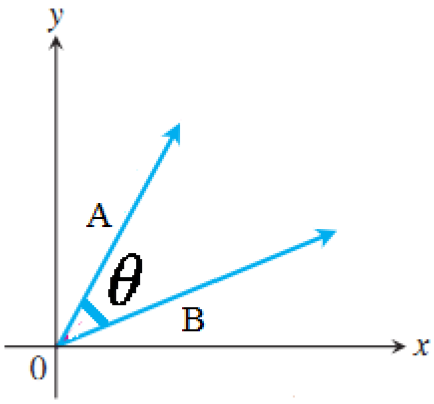
$$V_n = -i + j \quad \text{or} \quad V_n = i - j$$

unit normal

$$\mathrm{U_n} = \frac{N}{|N|} = \frac{-i + j}{\sqrt{2}}$$

6-<u>Dot (Scalar) Product</u>:

If we want to measure the angle between two vectors we apply the dot product. Also we apply it to find the projection of one vector onto another. Then let we have two vectors:



$$\vec{A} = a_1 i + a_2 j + a_3 k$$

$$\vec{B} = b_1 i + b_2 j + b_3 k$$

The $\vec{A} \bullet \vec{B}$ is called the dot (scalar) product of $\vec{A}$ & $\vec{B}$ and given by:

$$\vec{A} \bullet \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{A}||\vec{B}|\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\vec{A} \bullet \vec{B}}{|\vec{A}||\vec{B}|}\right), \quad \text{Where } \theta \text{ is the angle between two vectors}$$

Note:

1) $i \bullet i = j \bullet j = k \bullet k = 1$

2) $i \bullet j = j \bullet k = k \bullet i = 0$

3) $\vec{\mathrm{A}} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

if $\vec{A} \bullet \vec{B} = 0 \quad \therefore \quad \vec{A} \perp \vec{B} \quad$ because $\cos 90 = 0$

if $\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}| \quad \therefore \quad \vec{A} // \vec{B} \quad$ because $\cos 0 = 1$

Ex: Find the angle between $\vec{A} = i - 2j - 2k$ and $\vec{B} = 6i + 3j + 2k$.

Sol:

$$\theta = \cos^{-1}\left(\frac{\vec{A}\bullet\vec{B}}{|\vec{A}||\vec{B}|}\right)$$

$$\vec{A}\bullet\vec{B} = (1*6) + (-2*3) + (-2*2) = -4$$

$$|\vec{A}| = \sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2} = \sqrt{1+4+4} = 3$$

$$|\vec{B}| = \sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2} = \sqrt{36+9+4} = 7$$

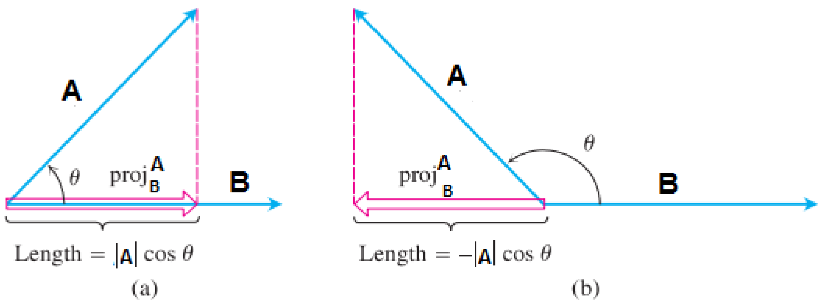
$$\theta = \cos^{-1}\left(\frac{\vec{A}\bullet\vec{B}}{|\vec{A}||\vec{B}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) = 100.98^0$$

Vector Projection:

$$\vec{A}\bullet\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

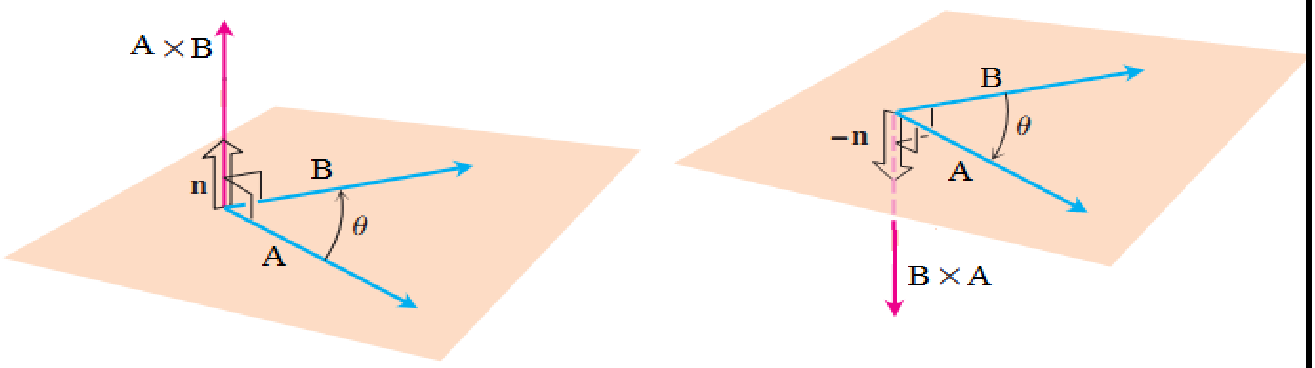
$$|\vec{A}|\cos\theta = \frac{\vec{A}\bullet\vec{B}}{|\vec{B}|}$$

$$\mathrm{Proj}_B^A = \left(\frac{\vec{A}\bullet\vec{B}}{|\vec{B}|}\right)$$



7- Cross Product:

When we apply the cross product onto two vectors we will get a new vector normal to these vectors. Also it gives us information about the area of the parallelogram which contains the vectors.



If we have two vectors:

$\vec{A} = a_1 i + a_2 j + a_3 k$

$\vec{B} = b_1 i + b_2 j + b_3 k$

$$\boxed{\begin{array}{l} \vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k \\ \vec{A}\times\vec{B} = n|\vec{A}||\vec{B}|\sin\theta \end{array}}$$

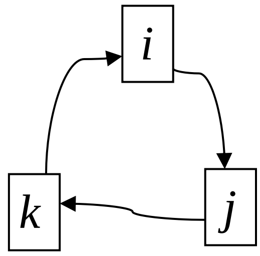
Note:

$1-\ \ if\ \ \vec{A}//\vec{B}\quad \text{then}\quad \sin\theta = 0\quad \vec{A}\times\vec{B} = 0$

$2-\ \ i\times i = j\times j = k\times k = 0$

$3\text{-}\ \ \vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$

$4-\ \ (\vec{A}\times\vec{B})\bullet\vec{C} = (\vec{B}\times\vec{C})\bullet\vec{A} = (\vec{C}\times\vec{A})\bullet\vec{B}$



Ex: Find $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

Ex: 1-Find the normal vector to the plane which contains points A (1,-1,0), B(2,1,-1), C(-1,1,2).

1-Find the normal unit vector to the plane.

Solution

$\overrightarrow{AB} = (2-1)i + (1-(-1))j + (-1-0)k = i + 2j - k$

$\overrightarrow{AC} = (-1-1)i + (1-(-1))j + (2-0)k = -2i + 2j + 2k$

Then the normal vector to the plane is:

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}=6i+6k$$

$$n=\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}=\frac{6}{6\sqrt{2}}i+\frac{6}{6\sqrt{2}}k=\frac{1}{\sqrt{2}}i+\frac{1}{\sqrt{2}}k$$ normal unit vector to the plane

---

H.w: Determine if the two vectors are orthogonal or parallel or not?

1- $\vec{A}=6i+6k$ and $\vec{B}=-2i+2j+2k$

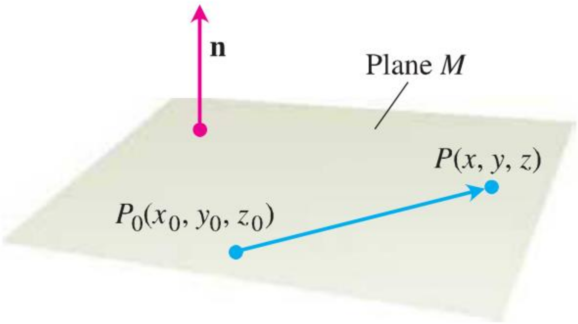
2- $\vec{A}=3i-2j+\mathrm{k}$ and $\vec{B}=2j+4k$

3- $\vec{A}=6i+3j+2\mathrm{k}$ and $\vec{B}=12i+6j+4k$

Lines and Planes in Space

1) Plane Equation in space

Suppose M is a plane passes through the point P_0 (x_0, y_0, z_0). Also M plane is a set of points $P(x, y, z)$. And $\vec{N}$ is a vector normal to the M plane. Then:



$$\overrightarrow{P_0P} = (x - x_0)i + (y - y_0)j + (z - z_0)k$$

$$\vec{N} = Ai + Bj + Ck$$

$$\overrightarrow{P_0P} \perp \vec{N}$$

$$\overrightarrow{P_0P} \cdot \vec{N} = |\overrightarrow{P_0P}||\vec{N}| \cos 90 = 0$$

$$\overrightarrow{P_0P} \cdot \vec{N} = (x - x_0)A + (y - y_0)B + (z - z_0)C = 0$$

$$\boxed{Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D} \quad \text{.........(1)}$$

Equation (1) is called a plane equation, where D is a constant.

Note: to find a plane eq. we must have a normal vector and a point within the plane.

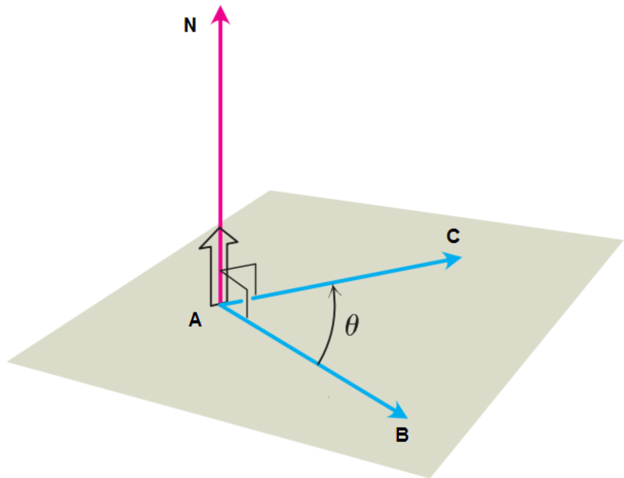
Ex₁: Find the equation for the plane passes through P_0 (-3, 0, 7) and perpendicular to $\vec{N} = 5i + 2j - k$.

Sol:

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0$$
$$5x + 2y - z = -15 + 0 - 7$$
$$5x + 2y - z = -22$$
$$z = f(x, y) = 22 + 5x + 2y$$

Ex₂: Find the Eq. for the plane through *A (0, 0, 1), B (2, 0, 0)* and *C (0, 3, 0).*

Sol: $\vec{V}=(x-x_0)i+(y-y_0)j+(z-z_0)k$



$$\overrightarrow{AB}=(2-0)i+(0-0)j+(0-1)k$$
$$\overrightarrow{AB}=2i-k$$
$$\overrightarrow{AC}=3j-k$$
$$\vec{N}=\overrightarrow{AB}\times\overrightarrow{AC}$$
$$\vec{N}=\begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix}=3i+2j+6k$$

Now we have a vector normal ($\vec{N}$) to the plane and point *A* (0, 0, 1) we can find the plane equation

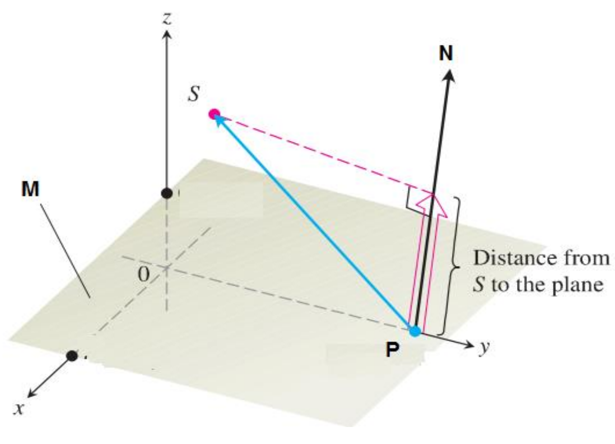
$$Ax+By+Cz=Ax_0+By_0+Cz_0$$
$$3x+2y+6z=6$$

2-<u>The Distance from the Point to a Plane</u>:

If *P* is a point on a plane with normal $\vec{N}$, then the distance from any point S to the plane is the length of the vector projection onto $\vec{N}$. Then the distance from *S* to the plane is:

$$\boxed{d=\left|\frac{\overrightarrow{PS}\cdot\vec{N}}{|\vec{N}|}\right|=\left|\overrightarrow{PS}\cdot\frac{\vec{N}}{|\vec{N}|}\right|} \qquad \dots (2)$$

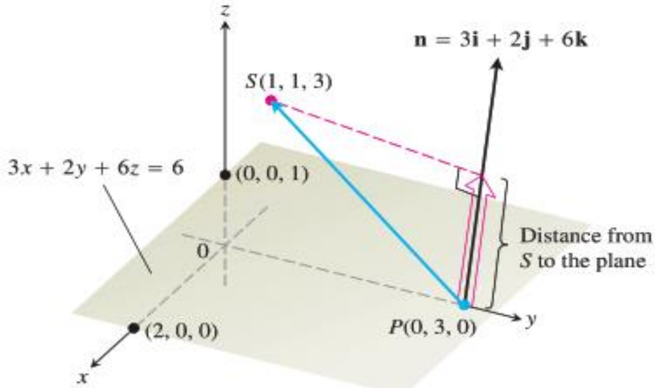
Where $\vec{N}=Ai+Bj+Ck$ is normal to the plane.



Ex₁: Find the distance from *S* (1, 1, 3) to the plane 3x + 2y + 6z = 6.

Sol: $\vec{N} = 3i + 2j + 6k$

P is a point may be a point of intersection between the plane and the y-axis. Then x=z=0 then the point P is (0,3,0)



$$\overrightarrow{SP} = (1-0)i + (1-3)j + (3-0)k = i - 2j + 3k$$

$$d = \overrightarrow{SP}.\frac{\vec{N}}{|\vec{N}|} = i - 2j + 3k.\frac{3i + 2j + 6k}{\sqrt{9+4+36}}$$

$$d = (i - 2j + 3k).(\frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k)$$

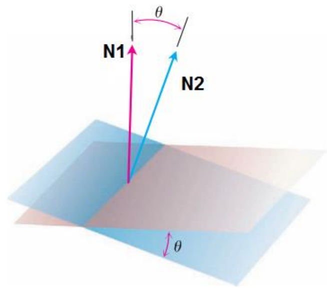
$$d = \frac{3}{7} - \frac{4}{7} + \frac{18}{7} = \frac{17}{7} \quad length\ unit$$

H.W: (Exercisers 12.5 *p694*)

1- Find the distance from the point s(2, -3, 4) to the plane $x + 2y + 2z = 13$

3- <u>Angles Between Two Planes</u>

Two planes that are not parallel will intersect in a line. The angle between two intersecting planes is defined to be the angle between their normal vectors.



If the equations of planes are:

$$Ax + By + Cz = D$$
$$A'x + B'y + C'z = D'$$

Then the corresponding normal vectors are:

$$\vec{N}_1 = Ai + Bj + Ck$$
$$\vec{N}_2 = A'i + B'j + C'k$$
$$\vec{N}_1.\vec{N}_2 = |\vec{N}_1||\vec{N}_2|\cos\theta$$

$$\boxed{\theta = \cos^{-1}\frac{\vec{N}_1.\vec{N}_2}{|\vec{N}_1||\vec{N}_2|}} \qquad \text{………(3)}$$

Ex₁: Find the angle between the planes $3x-6y-2z=0$ and $2x+y-2z=5$.

<u>*Sol*</u>:

$$\left.\begin{aligned}\vec{N}_1 &= 3i-6j-2k\\ \vec{N}_2 &= 2i+j-2k\end{aligned}\right\}$$ The vectors are normal to the planes.

$$\theta=\cos^{-1}\frac{\vec{N}_1.\vec{N}_2}{\left|\vec{N}_1\right|\left|\vec{N}_2\right|}$$

$$\left|\vec{N}_1\right|=\sqrt{3^2+6^2+2^4}=\sqrt{49}=7$$

$$\left|\vec{N}_2\right|=\sqrt{2^2+1^2+2^2}=\sqrt{9}=3$$

$$\theta=\cos^{-1}\frac{(3i-6j-2k).(2i+j-2k)}{7*3}$$

$$\theta=\cos^{-1}\frac{6-6+4}{21}=79^0$$

Ex₂: Find a vector parallel to the line of intersection of the plane3x-6y - 2z =15 and 2x + y - 2z = 5.

<u>*Sol*</u>:

The vector parallel to the line of intersection is the vector results from the cross product between the two normal vectors $\vec{N}_1,\vec{N}_2$.

$$\left.\begin{aligned}\vec{N}_1 &= 3i-6j-2k\\ \vec{N}_2 &= 2i+j-2k\end{aligned}\right\}$$ the vectors normal to the planes.

$$\vec{V}=\vec{N}_1\times\vec{N}_2=\begin{vmatrix} i & j & k\\ 3 & -6 & -2\\ 2 & 1 & -2\end{vmatrix}=14i+2j+15k$$

$$\vec{V}=14i+2j+15k$$

H.W

_H.W_$_1$: find vector parallel to the line of intersection between two planes.

1) $x+y+z=1$ and $x+y=2$.

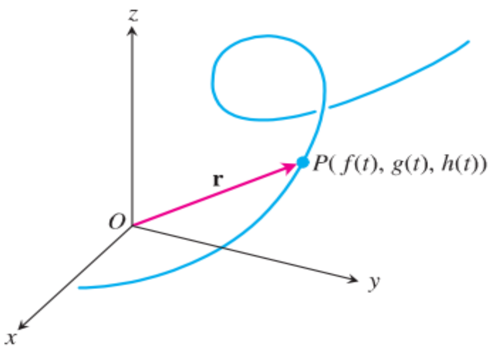
2) $x-2y+4z=2$ and $x+y-2z=5$

***H.W*$_2$**: Find the angle between the planes $x+y=1$ and $2x+y-2z=2$

Vector Valued Functions and Motion in Space

1- Vector Functions:

Now that, we have learned about vectors and the planes in space. Calculus of vector-valued functions will be studied to describe the paths and motions of objects moving in a plane or in space. When a point *P(x,y,z)* (particle) moves through space during a time interval *t*, then the point coordinates will be a functions of *t*(real number):



$x = f(t)\,,\;\; y = g(t)\,,\;\; z = h(t)$

The motion of this points *P(f(t), g(t), h(t))* will make up the curve in space that we call the particle's path (see the figure). A curve in space can also be represented in vector form which is called the vector valued function (also called position vector):

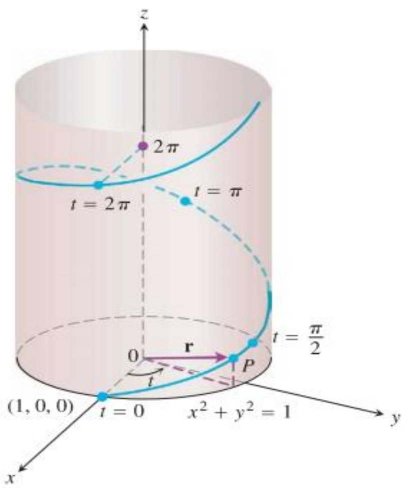
$$\boxed{r(t) = \overline{O}\vec{P} = f(t)i + g(t)j + h(t)k} \qquad ...(4)$$

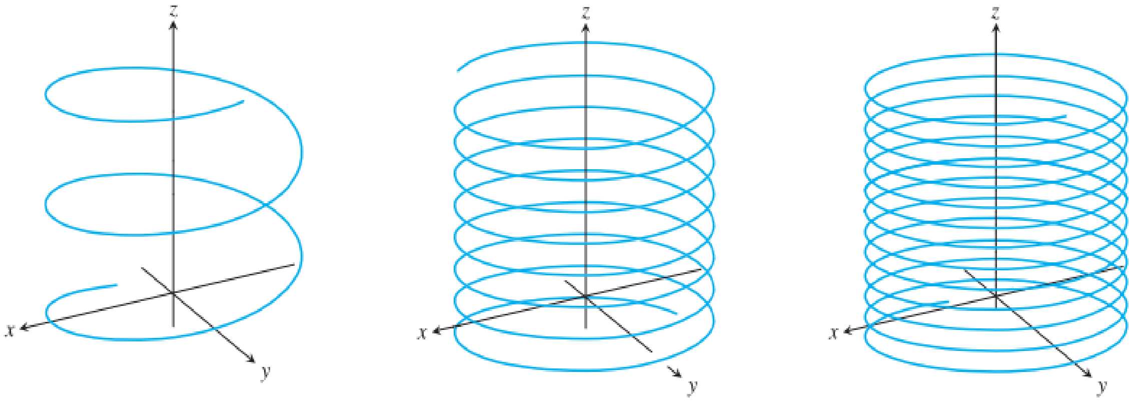
Ex1: Graph the vector function $r(t) = (\cos t)\,i + (\sin t)\; j + t\,k$.

Sol: At *t=0* the curve will be in x-y plane, then the curve Eq. satisfy the circle equation.

$x^2 + y^2 = sin^2 t + cos^2 t = 1$

The curve rises as the k-component z = t increases. Each time t increases by 2π, the curve completes one turn around the cylinder. The curve is called a helix.

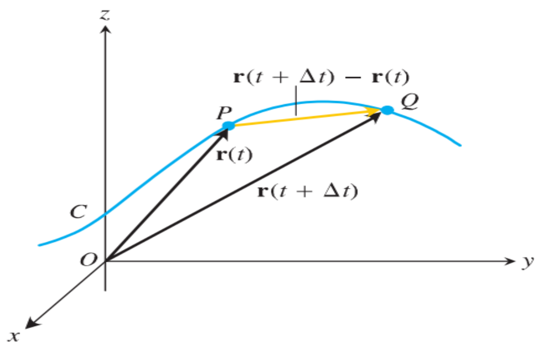




$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$ $\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

2-<u>Derivatives and Motion of Position Vector</u>:

Suppose that $r(t) = f(t)i + g(t)j + h(t)k$ is the position vector of a particle moving along a curve in space and that f, g, and h are differentiable functions of t. Then the difference between the particle's positions at time t and time is $t+\Delta t$:



$\frac{dr}{dt} = Velocity\,Vector\,(\vec{V})$ Then the velocity vector is:

$$\boxed{\begin{aligned}\vec{V} &= \frac{dr(t)}{dt} = \frac{df(t)}{dt}i + \frac{dg(t)}{dt}j + \frac{dh(t)}{dt}k \\ \vec{V} &= v_x i + v_y j + v_z k\end{aligned}} \qquad \text{.........(5)}$$

Then the speed of the particles is the absolute of the velocity vector:

$$\boxed{|\vec{V}| = \sqrt{v_x^2 + v_y^2 + v_z^2}} \qquad \text{.........(6)}$$

$$U_v = \frac{\vec{V}}{|\vec{V}|}$$

$$\vec{V} = |\vec{V}| \left(\frac{\vec{V}}{|\vec{V}|} \right) = speed\,.\,direction$$

Then the acceleration vector of moving particles is second derivatives of the vector function:

$$\frac{d^2 r(t)}{dt^2} = \frac{dV(t)}{dt} = \vec{a}$$

$$\vec{a} = \frac{dv_x(t)}{dt} i + \frac{dv_y(t)}{dt} j + \frac{dv_z(t)}{dt} k \qquad \text{.........(7)}$$

Ex1:Find the velocity and acceleration of a particle whose motion in space is given by the position vector $\overrightarrow{r(t)} = 2\cos t \;\; i + 2\sin t \;\; j + 5\cos^2 t\, k$.

Sol: the velocity vector is:

$$\vec{V} = \frac{dr}{dt} = (-2\sin t) \;\; i + (2\cos t) \; j + (-10\cos t \sin t) \;\; k$$

$$\vec{V} = (-2\sin t) \;\; i + (2\cos t) \; j - (5\sin 2t) \;\; k$$

The accelration vector

$$\vec{a} = \frac{dv}{dt} = (-2\cos t) \;\; i - (2\sin t) \;\; j - (10\cos 2t) \;\; k$$

Ex2: The vector $\overrightarrow{r(t)} = 3\cos t \;\; i + 3\sin t \;\; j + t^2 \, k$ gives the position of moving body at time t find:

1- $\vec{V}$ *and* $\vec{a}$,

2- Speed when *t=2*,

3- Direction of $\vec{V}$ when *t=2*,

4- At what time are the velocity and acceleration orthogonal?

Sol:

1-

$$\vec{V} = \frac{dr}{dt} = (-3\sin t) \;\; i + (3\cos t) \; j + (2t) \;\; k$$

$$\vec{a} = \frac{dv}{dt} = (-3\cos t) \;\; i - (3\sin t) \;\; j + (2) \;\; k$$

2-

$speed = |V| = \sqrt{9sin^2t + 9cos^2t + 4} = \sqrt{9 + 4t^2}$

When t=2

$|V| = \sqrt{9 + (4 * 4)} = \sqrt{25} = 5$

3- Direction of $\vec{V} = U_v = \dfrac{\vec{V}}{|\vec{V}|}$

$$U_v = \frac{(-3\sin t)\ i + (3\cos t)\ j + (2t)\ k}{5}$$

$when\ t = 2\ then$

$$U_v = (\frac{-3\sin 2}{5})i + (\frac{3\cos 2}{5})j + (2*2)k$$

4- The two vectors are orthogonal that mean:

$\vec{V}\ .\ \vec{a} = 0$

$\vec{V}\ .\ \vec{a} = \{(-3\sin t)\ i + (3\cos t)\ j + (2t)\ k\}\ .\{(-3\cos t)\ i - (3\sin t)\ j + (2)\ k\} = 0$

$9\sin t\cos t - 9\cos t\ \sin t + 4t = 0$

$4t = 0$

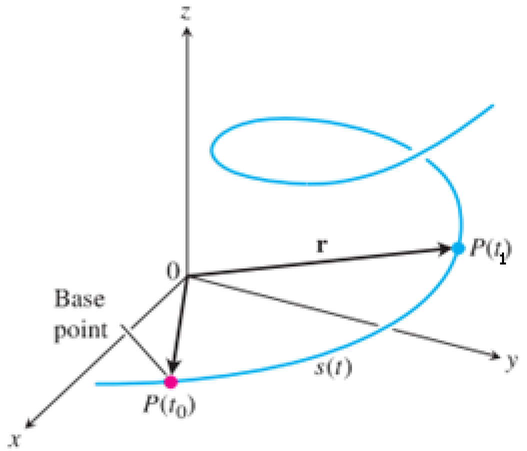
$t = 0$

<u>H.W</u>: (Exercisers 13.1)

1- Find the angle between the velocity and acceleration vectors at time *t=0* of the position vector $\overrightarrow{r(t)} = (3t + 1)i + \sqrt{3}tj + t^2k$.

3-Direct Distance along a curve from (t_0) to (t) (Arc length in space):

The integral of $|\vec{V}|$ from (t_0) to (t_1) gives the direct distance along the curve from $P(t_0)$ to $P(t_1)$ the distance is function of t and we denoted it by $s(t)$:



$$r(t) = f(t)i + g(t)j + h(t)k$$

$$\vec{V} = \frac{dr(t)}{dt} = \frac{df}{dt}i + \frac{dg}{dt}j + \frac{dh}{dt}k$$

$$\vec{V} = V_x i + V_y j + V_z k$$

$$speed = |\vec{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$|\vec{V}| = \frac{ds}{dt}$$

$$ds = |\vec{V}|dt$$

$$\boxed{s(t) = \int_{t_0}^{t_1} |\vec{V}|dt} \quad .....(8)$$

Note: the value of (s) is +ve if $t_1 > t_0$ and -ve if $t_1 < t_0$

Ex1: Find the length of one turn of the helix $r(t) = (\cos t)i + (\sin t)j + tk$.

Sol:

$$\vec{V} = \frac{dr(t)}{dt} = (-\sin t)i + (\cos t)j + k$$

$$|\vec{V}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$s(t) = \int_{t_0}^{t} |\vec{V}|d\tau$$

$$s(t) = \int_{0}^{2\pi} \sqrt{2}\ d\tau = \sqrt{2}\tau\Big|_{0}^{2\pi}$$

$$s(t) = 2\sqrt{2}\pi$$

Ex2: Find the length of the curve $r(t)=(t)i+(t^{\frac{3}{2}})j$ from (0,0,0) to (4,8,0).

Sol:

$$\vec{V}=\frac{dr(t)}{dt}=i+(\frac{3}{2}t^{\frac{1}{2}})j$$

$$\left|\vec{V}\right|=\sqrt{1+(\frac{3}{2}t^{\frac{1}{2}})^2}=\sqrt{1+\frac{9}{4}t}$$

$$s(t)=\int_{t_0}^{t}\left|\vec{V}\right|d\tau$$

To find the limits of integration p(0,0,0), Q(4,8,0)

$$x(t_0)=x_1\to t_0=0$$
$$x(t_1)=x_2\to t_1=4$$

or

$$y(t_0)=y_1\to t_0^{\frac{3}{2}}=0\to t_0=0$$
$$y(t_1)=y_2\to t_1^{\frac{3}{2}}=8\to\sqrt{t_1^3}=8\to t_1^3=64\to t_1=4$$

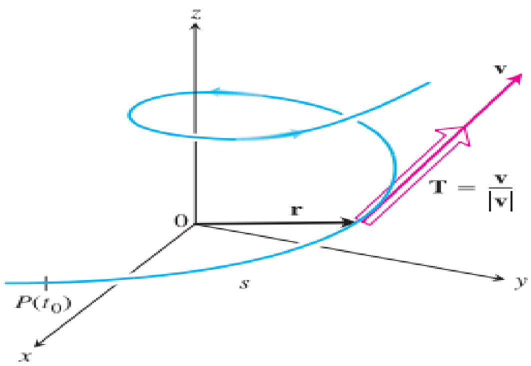
$$s(t)=\int_0^4\sqrt{1+\frac{9}{4}t}\,d\tau=\frac{4}{9}\left.\frac{\left(1+(9/4)t\right)^{\frac{3}{2}}}{3/2}\right|_0^4$$

$$s(t)=\frac{8}{27}\left.\left(1+\frac{9}{4}t\right)^{\frac{3}{2}}\right|_0^4=\frac{8}{27}\left[(10)^{3/2}-1\right]$$

$$s(t)=\frac{8}{27}\left[10\sqrt{10}-1\right]$$

4-Unit Tangent Vector ($\vec{T}$) of the curve:

We can define the unit tangent vector $\vec{T}$ as:



$$\boxed{\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \frac{dr}{ds}} \qquad \text{......(9)}$$

Ex1: Find $\vec{T}$ if $r(t) = (\cos t)i + (\sin t)j + tk$.

Sol:

$$\vec{T} = \frac{\vec{V}}{|\vec{V}|}$$

$$\vec{V} = (-\sin t)i + (\cos t)j + k$$

$$|\vec{V}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

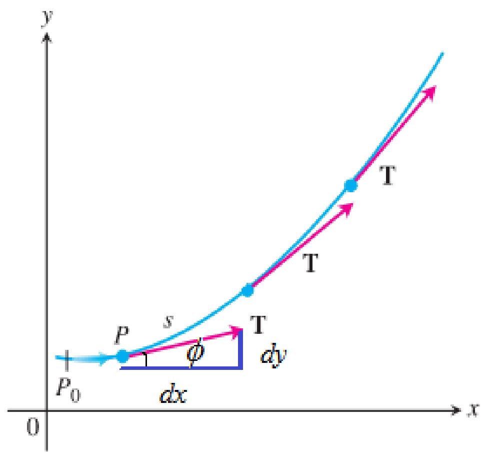
$$\vec{T} = \frac{\vec{V}}{|\vec{V}|} = \frac{(-\sin t)}{\sqrt{2}}i + \frac{(\cos t)}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k$$

5-The Curvature of the Vector Function:

As we move along the differential curve, the unit tangent vector $\vec{T}$ as the curve bend. We measure the rate at which $\vec{T}$ turns by measuring the change in the angle ϕ that $\vec{T}$ makes with i.

The value of the $\left|\frac{d\phi}{ds}\right|$ at point P is called the curvature of the curve.

Then the curvature k is:



$$\boxed{k = \frac{|\vec{V} \times \vec{a}|}{|V|^3} = \frac{1}{|V|}\left|\frac{dT}{dt}\right|} \qquad \text{.......(10)}$$

And the radius of the curvature is:

$$\boxed{\rho = \frac{1}{k}} \qquad \text{.......(11)}$$

Ex1: Find the curvature of the circle of radius a. $r(t) = (a\cos t)i + (a\sin t)j$.

Sol:

$$k = \frac{|\vec{V} \times \vec{a}|}{|V|^3}$$

$$\vec{V} = \frac{dr}{dt} = (-a\sin t)i + (a\cos t)j$$

$$\vec{a} = \frac{d\vec{V}}{dt} = (-a\cos t)i - (a\sin t)j$$

$$\vec{V} \times \vec{a} = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = a^2 k$$

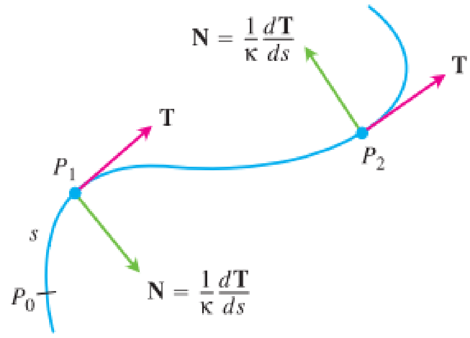
$$|\vec{V} \times \vec{a}| = a^2$$

$$|\vec{V}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = a$$

$$k = \frac{a^2}{a^3} = \frac{1}{a}$$

6- Unit Normal Vector to the Curvature:

Among the vectors orthogonal to the unit tangent vector T is one of particular significance because it points in the direction in which the curve is turning. And it is given by:



$$\boxed{N = \frac{1}{k}\frac{dT}{ds} = \frac{dT/dt}{|dT/dt|}} \qquad \text{.........(12)}$$

Ex1: Find T and N for the circular motion $r(t) = (\cos 2t)i + (\sin 2t)j$.

Sol:

$$T = \frac{\vec{V}}{|V|}$$

$$\vec{V} = \frac{dr(t)}{dt} = (-2\sin 2t)i + (2\cos 2t)j$$

$$|V| = \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2} = 2$$

$$T = (-\sin 2t)i + (\cos 2t)j$$

$$N = \frac{dT/dt}{|dT/dt|}$$

$$\frac{dT}{dt} = (-2\cos 2t)i - (2\sin 2t)j$$

$$\left|\frac{dT}{dt}\right| = \sqrt{(-2\cos 2t)^2 + (-2\sin 2t)^2} = 2$$

$$N = (-\cos 2t)i - (\sin 2t)j$$

Ex2: Find T,k and N of the curve vector $r(t) = (a\cos t)i + (a\sin t)j + btk$, a &b>0

$$a^2 + b^2 \neq 0$$

Sol: we calculate T from the velocity vector:

$$V = \frac{dr(t)}{dt} = (-a\sin t)i + (a\cos t)j + bk$$

$$|V| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$T = \frac{\vec{V}}{|V|} = \frac{(-a\sin t)i + (a\cos t)j + bk}{\sqrt{a^2 + b^2}} = \frac{1}{\sqrt{a^2 + b^2}}\left[(-a\sin t)i + (a\cos t)j + bk\right]$$

$$k = \frac{1}{|V|}\left|\frac{dT}{dt}\right|$$

$$k = \frac{1}{\sqrt{a^2 + b^2}}\left|\left(\frac{1}{\sqrt{a^2 + b^2}}\left[(-a\cos t)i - (a\sin t)j\right]\right)\right|$$

$$k = \frac{1}{\sqrt{a^2 + b^2}}\left(\frac{a}{\sqrt{a^2 + b^2}}\right) = \frac{a}{a^2 + b^2}$$

$$N=\frac{dT/dt}{\left|dT/dt\right|}$$

$$\frac{dT}{dt}=\left(\frac{1}{\sqrt{a^2+b^2}}\left[(-a\cos t)i-(a\sin t)j\right]\right)$$

$$\left|dT/dt\right|=\left(\frac{a}{\sqrt{a^2+b^2}}\right)$$

$$N=\left(\frac{\sqrt{a^2+b^2}}{a}.\frac{1}{\sqrt{a^2+b^2}}\left[(-a\cos t)i-(a\sin t)j\right]\right)$$

$$N=(-\cos t)i-(\sin t)j$$

Ex3: The velocity of a particle moving in space is: $\frac{dr}{dt}=(\cos t)i-(\sin t)j+k$ find a particles position as function of t, if $r=2i+k$ when t=0.

Sol:

$$r(t)=\int V dt$$

$$r(t)=\int (\cos t)i-(\sin t)j+k \ \ dt$$

$$r(t)=(\sin t)i+(\cos t)j+tk+c$$

$$r(0)=j+c$$

$$j+c=2i+k$$

$$c=2i-j+k$$

$$r(t)=(\sin t+2)i+(\cos t-1)j+(t+1)k$$

H.W: (Exercisers 13.4)

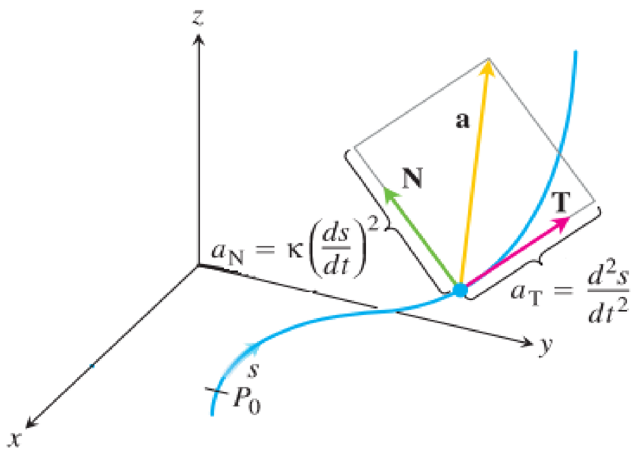
Find T,N and k for the curve:

1- $r(t)=(6\sin 2t)i+(6\cos 2t)j+5tk$.

2- $r(t)=(2t+3)i+(5-t^2)j$.

3- $r(t)=(\cos^3 t)i+(\sin^3 t)j \ \ , \quad 0<\mathrm{t}<\frac{\pi}{2}$

7-Tangential and Normal Components of Acceleration:



If you are traveling along a space curve, the Cartesian i, j, and k coordinate system is used for representing the vectors describing your motion. Another way to describe the motion is to used the vectors, the unit tangent vector $\vec{T}$ and the unit normal vector $\vec{N}$. The acceleration $\vec{a}$ always lies in the plane created by $\vec{T}$ and $\vec{N}$ as shown in figure.

$$\vec{V} = \frac{dr}{dt} = \frac{dr}{ds}.\frac{ds}{dt} = T.\frac{ds}{dt}$$

$$\vec{a} = \frac{dV}{dt} = \frac{d}{dt}\left(T.\frac{ds}{dt}\right)$$

$$\vec{a} = T.\frac{d^2s}{dt^2} + \frac{ds}{dt}\frac{dT}{dt}$$

$$\vec{a} = \frac{d^2s}{dt^2}T + \frac{ds}{dt}\left(\frac{dT}{ds}.\frac{ds}{dt}\right)$$

$$\vec{a} = \frac{d^2s}{dt^2}T + \frac{ds}{dt}\left(kN.\frac{ds}{dt}\right)$$

$$\boxed{\vec{a} = \frac{d^2s}{dt^2}\,T + k\left(\frac{ds}{dt}\right)^2 N} \qquad \text{..........(13)}$$

The tangential and normal scalar components of acceleration are:

$$\boxed{a_T = \frac{d^2s}{dt^2} = \frac{d\,|V|}{dt} \qquad \text{Tangential component}} \qquad \text{..........(14)}$$

$$\boxed{a_N = k\left(\frac{ds}{dt}\right)^2 = k\,|V|^2 \quad \text{Normal component}} \qquad \text{..........(15)}$$

$$\boxed{\begin{array}{l} |\,a\,| = \sqrt{(a_T)^2 + (a_N)^2} \\ a_N = \sqrt{|a|^2 - a_T^2} \end{array}} \qquad \text{..........(16)}$$

By definition, acceleration $\vec{a}$ is the rate of change of velocity $\vec{V}$, and in general, both the length and direction of $\vec{V}$ change as an object moves along its path. The tangential component of acceleration a_T measures the rate of change of the length of $\vec{V}$ (that is, the change in the speed). The normal component of acceleration a_N measures the rate of change of the direction of $\vec{V}$.

***Ex₁*:** Write the acceleration of the motion

$r(t) = (\cos t + t\sin t)i + (\sin t - t\cos t)j$ in the form of $a = a_T T + a_N N$.

Sol:

$$\vec{V} = \frac{dr}{dt} = \left(-\sin t + t\cos t + \sin t\right)i + \left(\cos t + t\sin t - \cos t\right)j$$

$$\vec{V} = (t\cos t)i + (t\sin t)j$$

$$\left|\vec{V}\right| = \sqrt{(t\cos t)^2 + (t\sin t)^2} = \sqrt{t^2} = t$$

$$a_T = \frac{d\left|\vec{V}\right|}{dt} = \frac{d(t)}{dt} = 1$$

$$a = \frac{d\vec{V}}{dt} = \left(-t\sin t + \cos t\right)i + \left(t\cos t + \sin t\right)j$$

$$|a| = \sqrt{\left(-t\sin t + \cos t\right)^2 + \left(t\cos t + \sin t\right)^2} = \sqrt{\left(t^2\sin^2 t - 2t\sin t\cos t + \cos^2 t\right) + \left(t^2\cos^2 t + 2t\sin t\cos t + \sin^2 t\right)}$$

$$|a| = \sqrt{(t^2\sin^2 t + t^2\cos^2 t) + (\cos^2 t + \sin^2 t)} = \sqrt{t^2 + 1}$$

$$a_N = \sqrt{|a|^2 - a_T^2}$$

$$a_N = \sqrt{t^2 + 1 - 1}$$

$$a_N = t$$

$$a = a_T T + a_N N = 1T + t\ N$$

Ex2: Find *T,N* and *k* for the curve $r(t) = ti + \ln(\cos t)\, j, \quad -\pi/2 < t < \pi/2$ Then write $a = a_T T + a_N N$, Find *dT/ds* at *t=π/3*.

Sol: to find T:

$$T = \frac{\vec{V}}{\left|\vec{V}\right|}$$

$$\vec{V} = \frac{dr}{dt} = i - \frac{\sin t}{\cos t}\, j$$

$$\vec{V} = i - \tan t \;\; j$$

$$\left|\vec{V}\right| = \sqrt{1 + \tan^2 t} = \sec t$$

$$T = \frac{1}{\sec t} i - \frac{\tan t}{\sec t}\, j$$

$$T = \cos t\, i - \sin t \;\; j$$

Hint :

$$\frac{d(\ln u)}{du} = \frac{1}{u} du$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\sec^2\theta = 1 + \tan^2\theta$$

Then to find N:

$$N = \frac{dT/dt}{\left|dT/dt\right|}$$

$$\frac{dT}{dt} = (-\sin t)\;\; i - (\cos t\,)\; j$$

$$\left|\frac{dT}{dt}\right| = \sqrt{(-\sin t)^2 - (\cos t\,)^2} = 1$$

$$N = \frac{dT/dt}{\left|dT/dt\right|} = (-\sin t)\, i - (\cos t)\; j$$

$$k = \frac{1}{|V|}\left|\frac{dT}{dt}\right|$$

$$k = \frac{1}{\sec t} = \cos t$$

To find the normal and tangent components:

$$a_T = \frac{d^2 s}{dt^2} \qquad , \qquad \mathrm{a_N} = k * \left(\frac{ds}{dt}\right)^2$$

$$\left|\vec{V}\right| = \frac{ds}{dt} = \sec t$$

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt}(\sec t) = \sec t \tan t$$

$$a_N = k * \left(\frac{ds}{dt}\right)^2 = \cos t \ \sec^2 t$$

$$a = (\sec t \tan t\,)\,T + (\cos t \ \sec^2 t\,)\,N$$

$$a = (\sec t \ \tan t\,)\,T + (\frac{1}{\cos t})\,N$$

$$a = (\sec t \ \tan t) \ T + (\sec t\,)\,N$$

To find *dT/ds*:

$$N = \frac{1}{k}\frac{dT}{ds}$$

$$\frac{dT}{ds} = N.k$$

$$k = \cos t \ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \ at\ t = \pi/3 \ \rightarrow \quad k = 0.5$$

$$N = -\sin t \ i - \cos t \ \ j \quad \cdots\cdots\cdots\cdots\cdots at\ t = \pi/3 \ \rightarrow \quad N = -\frac{\sqrt{3}}{2} i - \frac{1}{2} j$$

$$\frac{dT}{ds} = \left(-\frac{\sqrt{3}}{2} i - \frac{1}{2} j\right).\frac{1}{2}$$

$$\frac{dT}{ds} = -\frac{\sqrt{3}}{4} i - \frac{1}{4} j$$

<u>**H.W**</u>: (Exercisers 13.5)

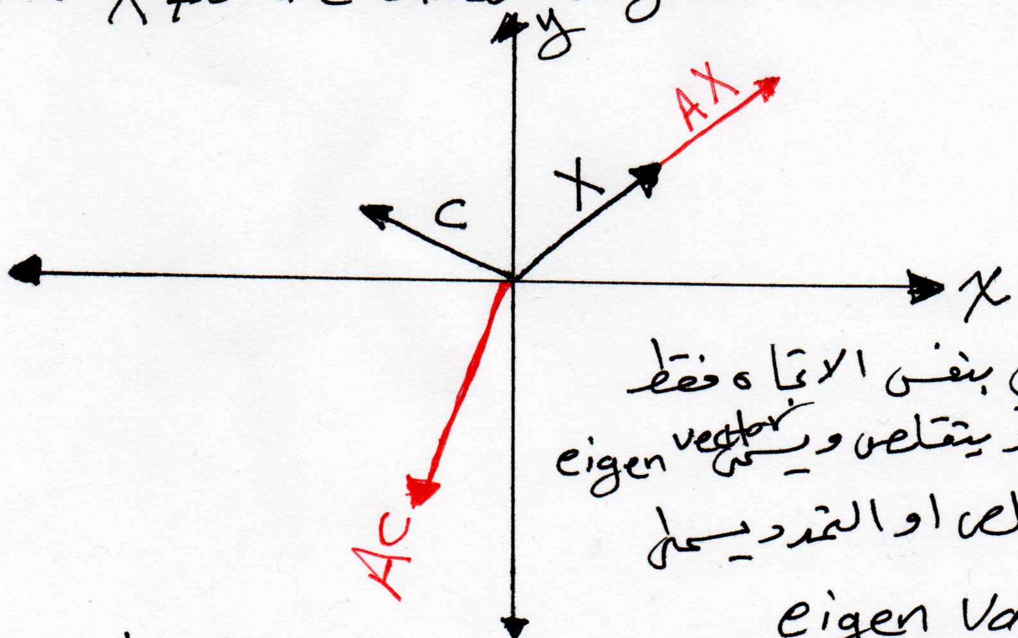
1- Find *T*,*N* and *k* for the curve $r(t) = (e^t \cos t)i + (e^t \sin t)\,j + 2k,$ Then write $a = a_T T + a_N N$, Find *dT/ds* at *t=π/2.*

Eigen Values and Eigen Vectors

Let A is an $(n*n)$ matrix and consider the vector eq. is:

$$AX = \lambda X \quad (1)$$

where X: is an unknown vector and λ an unknown scaler. Then the value of λ for which eq. (1) has a solution $X \neq 0$ is called Eigen value and the crossponding solutions $X \neq 0$ are called Eigen vector.



* المتجه X بقي بنفس الاتجاه فقط اما يتمدد او يتقلص ويسمى eigen vector ومقدار التقلص او التمدد يسمى eigen value

* تغير اتجاه وطول المتجه C لهذا لا يعتبر eigen vector

Ex: For the given matrix A, Prove if u, v are eigen vectors or not?

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}, \quad u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Sol:

نستخرج -4 كعامل مشترك وهي تمثل قيمة λ

$$Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

equal

Then u: is eigen vector.

$$Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} = -3 \begin{bmatrix} 3 \\ \frac{-11}{3} \end{bmatrix}$$

not equal

Then v: is not eigen vector

Calculations of Eigen Values and Eigen Vectors:

To find eigen values and eigen vectors:

$$AX = \lambda X$$

$$AX - \lambda X = 0$$ where I, is identity matrix

$$[A - \lambda I]X = 0$$

There is a solution other than $X = 0$ if and only if:

$$\det.(A - \lambda I) = 0$$

$$\boxed{|A - \lambda I| = 0}$$ ----- (charactristic eq.)

The roots of the charactristic eq. $|A - \lambda I| = 0$ are called eigen values. from the eigen values, eigen vectors can be founded.

Note: An $(n*n)$ matrix has n eigen values.

Ex1: Find the eigen values and eigen Vectors of the following matrix.

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Sol: To find the eigen values:

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-2-\lambda) - 4 = 10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda + 6)(\lambda + 1) = 0$$

$$\left.\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -6 \end{matrix}\right\} \text{eigen values}$$

To find Eigen vectors of A corresponding to

$[A-\lambda I]X=0$

for $\lambda_1=-1$

$[A-\lambda_1 I]X=0$

$$\left[\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right] X = 0$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x_1+2x_2=0$ ------ ①

$2x_1-x_2=0$ ------ ②

from ① and ②

$x_2=2x_1$

If we choose $x_1=1$, then $x_2=2$. then the eigen vector when $\lambda_1=-1$ is:

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find eigen vector of A when $\lambda_2 = -6$

$$[A - \lambda_2 I]X = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 + 2x_2 = 0$ ----- (3)

$2x_1 + 4x_2 = 0$ ----- (4)

from (3) and (4)

$x_1 = -2x_2$

Let $x_1 = 2 \Longrightarrow x_2 = -1$

Then the eigen vector when $\lambda_2 = -6$, is

$$X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Ex3: Determine the eigen values and eigen vectors for the following matrix if one of the roots equal 5.

$$A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$$

Sol: To find eigen values we find the roots of the chrat. eq.

$$|A-\lambda I|=0$$

$$|A-\lambda I|=\left|\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}-\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right|=0$$

$$\begin{array}{c} \begin{matrix}+ & \quad - \quad & +\end{matrix}\\ =\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix}=0\end{array}$$

$$-2-\lambda\left[-\lambda+\lambda^2-12\right]-2\left[-2\lambda-6\right]-3\left[-4+1-\lambda\right]=0$$

$$2\lambda-2\lambda^2+24+\lambda^2-\lambda^3+12\lambda+4\lambda+12+12-3+3\lambda=0$$

$$21\lambda-\lambda^2+45-\lambda^3=0$$

$$\lambda^3+\lambda^2-21\lambda-45=0$$

$\lambda_1 = 5$ (given)

$$(\lambda - 5)(\lambda^2 + 6\lambda + 9) = 0$$

$$(\lambda - 5)(\lambda + 3)(\lambda + 3) = 0$$

$$\begin{array}{r|l} & \lambda^2 + 6\lambda + 9 \\ \hline \lambda - 5 & \lambda^3 + \lambda^2 - 21\lambda - 45 \\ & \underline{-\lambda^3 \mp 5\lambda^2} \\ & \quad 6\lambda^2 - 21\lambda - 45 \\ & \quad \underline{-6\lambda^2 \mp 30\lambda} \\ & \qquad 9\lambda - 45 \\ & \qquad \underline{9\lambda - 45} \\ & \qquad 0 \quad 0 \end{array}$$

then the eigen values:

$\lambda_1 = 5$, $\lambda_2 = \lambda_3 = -3$

To find eigen vectors:

when $\lambda_1 = 5$ $\quad [A - \lambda I]X = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using Gauss elemination:

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 0 & \frac{-24}{7} & \frac{-48}{7} & 0 \\ 0 & \frac{-16}{7} & \frac{-32}{7} & 0 \end{array}\right] \begin{array}{l} \\ R_2 + \frac{2}{7}R_1 \\ R_3 - \frac{1}{7}R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 0 & \frac{-24}{7} & \frac{-48}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 - \frac{2}{3}R_2$$

$$\left.\frac{-24}{7}x_2 - \frac{48}{7}x_3 = 0\ \right] * \frac{7}{24}$$

$$-x_2 - 2x_3 = 0$$

$$x_2 = -2x_3$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$

Let $x_3 = 1 \Longrightarrow \left.\begin{array}{l} x_2 = -2 \\ x_1 = -1 \end{array}\right\}$ Then the eigen vector

$$X = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

when $\lambda_2 = \lambda_3 = -3$ $\quad [A + 3I]X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$$x_1 = -2x_2 + 3x_3$$

Let $x_3 = 0 \Rightarrow x_1 = -2x_2$

Let $x_1 = 2 \Rightarrow x_2 = -1$

then the eigen vector

$$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

to find ~~third~~ third eigen vector :

Let $x_2 = 0 \Rightarrow x_1 = 3x_3$

Let $x_3 = 1 \Rightarrow x_1 = 3$

$$X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Ex4: Find the eigen Values of the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$1-\lambda\left[(-5-\lambda)(4-\lambda)+18\right] + 3\left[12-3\lambda-18\right] + 3\left[-18+36+6\lambda\right] = 0$$

$$1-\lambda\left[\lambda^2+\lambda-2\right] + 3\left[-3\lambda-6\right] + 3\left[12+6\lambda\right] = 0$$

$$\lambda^2+\lambda-2-\lambda^3-\lambda^2+2\lambda-9\lambda-18+36+18\lambda = 0$$

$$-\lambda^3+12\lambda-16 = 0$$

$$\lambda^3 - 12\lambda - 16 = 0$$

ملاحظة: لحل المعادلة اعلاه، نفرض قيمة لـ λ ونعوض عنها في المعادلة اعلاه الى ان نحصل على قيمة λ التي تحقق المعادلة اعلاه.

let $\lambda = 0 \longrightarrow$ ~~eq~~ this value do not satisfy the eq.

$\lambda = 1 \longrightarrow$ " " " " " "

$\lambda = 2 \longrightarrow$ " " " " " "

$\lambda = 3 \longrightarrow$ " " " " " "

$\lambda = 4 \longrightarrow$ satisfy the eq.

then the first root is $(\lambda - 4)$

$$\lambda^3 - 12\lambda - 16 = 0$$

	1	0	−12	−16
4	↓	4	16	16
	1	4	4	0

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = 0$$

$$(\lambda - 4)(\lambda + 2)(\lambda + 2) = 0$$

then

$\lambda_1 = -4 \;,\; \lambda_2 = \lambda_3 = -2$

ملاحظات اضافية :

① مجموع الـ eigen values = مجموع عناصر القطر الرئيسي .

$$a_{11} + a_{22} + a_{nn} = \lambda_1 + \lambda_2 + \lambda_n$$

② لو كان هنالك احدى قيم الـ λ مساوية للصفر ، فهنا $|A|$ للمصفوفة الاصلية مساوي للصفر .

③ $$\lambda_1 * \lambda_2 * \lambda_3 = |A|$$

④ الـ (λ) للمصفوفة الاصلية = $\frac{1}{\lambda}$ لمعكوس (inverse) المصفوفة الاصلية.

$$AX = \lambda X$$

$$A^{-1}X = \frac{1}{\lambda}X$$

⑤

$$A^nX = \lambda^nX$$

⑥ الـ (eigen vector) للمصفوفة الاصلية A لا يساوي الـ eigen vector للمصفوفة A^T .

$$AX = \lambda X \neq A^TX = \lambda X$$

CHAPTER FOUR

1. Sequences

A sequence is a list of numbers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the terms of the sequence. For example, the sequence:

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$, and nth term $a_n = 2n$. The integer n is called the index of a_n, and indicates where a_n occurs in the list. Order is important. The sequence 2, 4, 6, 8 ... is not the same as the sequence 4, 2, 6, 8 .... .Sequences are fundamental to the study of infinite series and many applications of mathematics.

....

Ex1: Writ down the first few terms of the following sequences.

a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

$$\left\{\frac{\boldsymbol{n+1}}{\boldsymbol{n^2}}\right\}_{\boldsymbol{n=1}}^{\infty} = \left\{2, \frac{\mathbf{3}}{\mathbf{4}}, \frac{\mathbf{4}}{\mathbf{9}}, \frac{\mathbf{5}}{\mathbf{16}}, \frac{\mathbf{6}}{\mathbf{25}}, \frac{\mathbf{7}}{\mathbf{36}}, \ldots\ldots\ldots\right\}$$

b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \frac{1}{32} \ldots\ldots.\right\}$$

Convergence and Divergence of the Sequences:

1) The sequence is convergence if:

$$\lim_{n\to\infty} a_n = L$$ Where L is constant.

2) The sequence is Divergence if:

$$\lim_{n\to\infty} a_n = \infty$$

Some important Rules:

THEOREM Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

$$\frac{0}{0} = Undefine$$
$$\frac{\infty}{0} = Undefined$$
$$\frac{\infty}{\infty} = Undefined$$

$$\frac{0}{\infty} = 0$$

$$\frac{Number}{\infty} = 0$$

$$(0)^{Number} = 0$$

$$(0)^{\infty} = 0$$

$$\frac{\infty}{Number} = \infty$$

$$(Number)^{\infty} = \infty$$

$$(Number)^{0} = 1$$

$$(\infty)^{0} = 1$$

....

Ex1: Determine if the following sequences convergence or divergence?

1) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

2) $\{2\}_{n=0}^{\infty}$

3) $\{n^2\}_{n=1}^{\infty}$

4) $\left\{(-1)^{n+1}\frac{n-1}{n}\right\}_{n=1}^{\infty}$

Test of monotone for the sequences:

In the previous section, we introduced the concept of a sequence and talked about limits of sequences and the idea of convergence and divergence for a sequence. In this section, we want to take a quick look at some ideas involving sequences.

Let's start with some definitions.

Given any sequence $\{a_n\}$ we have the following.

1. We call the sequence **increasing** if $a_n < a_{n+1}$ for every n.
2. We call the sequence **decreasing** if $a_n > a_{n+1}$ for every n.
3. If $\{a_n\}$ is an increasing sequence or $\{a_n\}$ is a decreasing sequence we call it **monotonic**.
4. If there exists a number m such that $m \le a_n$ for every n we say the sequence is **bounded below**. The number m is sometimes called a **lower bound** for the sequence.
5. If there exists a number M such that $a_n \le M$ for every n we say the sequence is **bounded above**. The number M is sometimes called an **upper bound** for the sequence.
6. If the sequence is both bounded below and bounded above we call the sequence **bounded**.

Note: that in order for a sequence to be increasing or decreasing it must be increasing /decreasing for every n. In other words, a sequence that increases for three terms and then decreases for the rest of the terms is NOT a decreasing sequence. Also note that a monotonic sequence must always increase or it must always decrease.

Ex: determine if the following sequences are monotonic and / bounded.

a) $\left\{-n^2 \right\}_{n=0}^{\infty}$

b) $\left\{(-1)^{n+1} \right\}_{n=1}^{\infty}$

2. Infinite Series:

An infinite series is the sum of an infinite sequence of numbers:

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The number a. is the **nth term** of the series. The sequence $\{s_n\}$ defined by:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

$\{s_n\}$ is the sequence of partial sums of the series, the number s_n being the nth partial sum. If the sequence of partial sums converges to a limit L, we say that the series converges and that its sum is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series diverges.

2.1 Geometric Series

Geometric series are series of the form:

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $r \neq 0$. And r is:

$$r = \frac{a_{n+1}}{a_n}$$

The value of r can be positive or negative. The sum of the geometric series is depending on the value of r:

1) $|r| < 1 \rightarrow \text{Converges} \rightarrow s_n = \frac{a}{1-r}$

2) $|r| \geq 1 \rightarrow \text{Diverges} \rightarrow s_n = \frac{a(1-r^n)}{1-r}$

Ex1: Find the sum of the following series:

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{5}\right)^{n-1}$$

Sol: from the series a=2 and $r = \frac{1}{5}$, r < 1, then the series converges

$$s = \frac{a}{1-r} = \frac{2}{1-\frac{1}{5}} = \frac{2}{\frac{4}{5}} = 2 * \frac{5}{4} = 2.5$$

Ex2: Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{2^{n+3}}{3^n}$$

Sol:

$$\sum_{n=0}^{\infty} \frac{2^n * 2^3}{3^n} = \sum_{n=0}^{\infty} 2^3 \left(\frac{2}{3}\right)^n$$

from the series a=8 and $r = \frac{2}{3}$, r < 1, then the series converges

$$s = \frac{a}{1-r} = \frac{8}{1-\frac{2}{3}} = \frac{8}{\frac{1}{3}} = 8 * \frac{3}{1} = 24$$

2.2 Power Series

We begin with the formal definition, which specifies the notation and terms used for power series.

DEFINITIONS A **power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \quad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \quad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

2.3 Taylor Series

This section shows how functions that are infinitely differentiable generate power series called Taylor series. In many cases, these series can provide useful polynomial approximations of the generating functions. Assume that f(x) is the sum of a power series:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Where:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Where $f^{(n)}(a)$ is the nth derivatives of the function at x=a. Then the Tylor series is a representation of the function f(x) in form of power series about x=a:

Tylor series:

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \ldots.}$$

If we use a=0, so we are talking about the Tylor series about x=0, then we call it Maclurian series:

Maclurian series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

Ex1: Find the Tylor series for the $f(x) = e^x$ about x=0.

The Tylor series at a=0

$$f(x)= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \cdots + \cdots$$

$f(x) = e^x \quad -----\rightarrow \quad f(0) = 1$

$f'(x) = e^x \quad -----\rightarrow \quad f'(0) = 1$

$f''(x) = e^x \quad -----\rightarrow \quad f''(0) = 1$

$f'''(x) = e^x \quad -----\rightarrow \quad f'''(0) = 1$

$f^4(x) = e^x \quad -----\rightarrow \quad f^4(0) = 1$

$$e^x = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \cdots + \cdots$$

$$e^x = \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \cdots$$

Ex2: Find the Tylor series for the $f(x) = \cos(x)$ about x=0.

The Tylor series at a=0

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \cdots + \cdots$$

$f(x) = \cos x \quad ----\rightarrow \quad f(0) = 1$

$f'(x) = -\sin x \quad ----\rightarrow \quad f'(0) = 0$

$f''(x) = -\cos x \quad ----\rightarrow \quad f''(0) = -1$

$f'''(x) = \sin x \quad ----\rightarrow \quad f'''(0) = 0$

$f^4(x) = \cos x \quad ----\rightarrow \quad f^4(0) = 1$

$$\cos x = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \cdots +$$

$$\cos x = \frac{x^0}{0!} + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \cdots + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n}}{2n!}$$

H.W: Find the Tylor series about x=0 for the:

1) $f(x) = e^{-x}$.

2) $f(x) = \sin(x)$.

Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

CHAPTER FIVE

Differential Equations

The equations contain one or more derivatives. These equations obtained from modelling of the physical systems. They can be classified by:

1) **Type:** Ordinary or Partial.
2) **Order**: The order of D.E. is the highest order of derivatives in the equation.
3) **Degree**: The exponent of the highest derivative.

Ordinary D.E.: is a differential equation that unknown function depends on only **one** independent variable.

Partial D.E.: is a differential equation that unknown function depends on **two or more** independent variable.

Ex: Find the order and the degree of the following differential equations?

1) $\frac{dy}{dx} = 5x + 3$

2) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 = 2x + 3$

3) $4\frac{d^3y}{dx^3} + \sin x\,\frac{d^2y}{dx^2} + 5xy = 0$

4) $y''' + 2(y'')^2 = xy$

1- **General Solution:** of an n-th order, D.E. is the solution that contains an arbitrary constant.

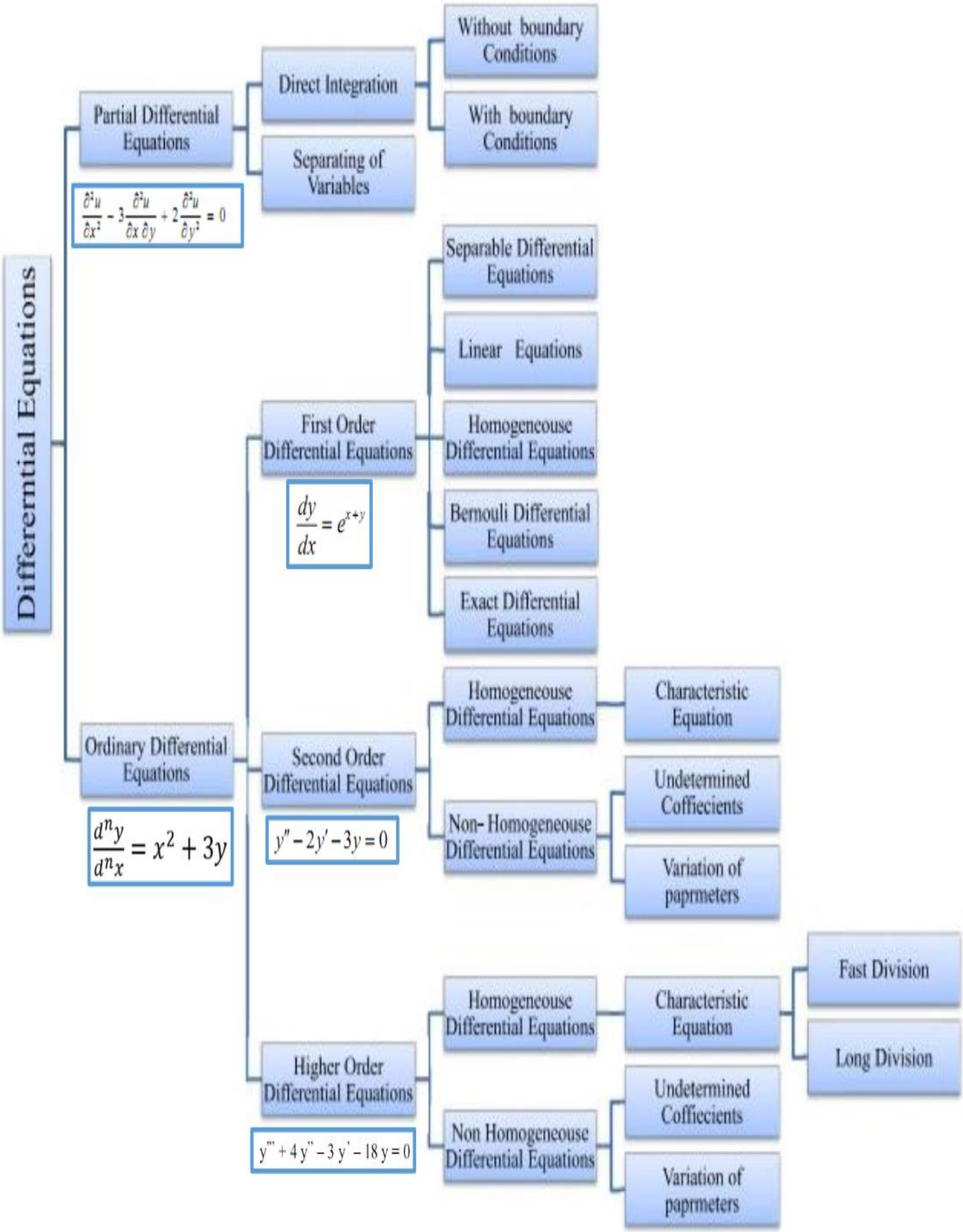
2- **A Particular Solution:** is a solution obtained from the general solution by assigning specific value to the arbitrary constant.

3- **Linear D.E.** : Any D.E. that can be written in the following form:

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=g(t)$$

1) $4x^2y''+12xy'+3y=0$ ………….. Linear D.E.

2) $\sin y\frac{d^2y}{dx^2}=(1-y)\frac{dy}{dx}+y^2e^{-5y}$ ………. Non-Linear D.E.


Differerntial Equations
Partial Differential Equations
$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$
Direct Integration
Without boundary Conditions
With boundary Conditions
Separating of Variables
Ordinary Differential Equations
$\frac{d^n y}{d^n x} = x^2 + 3y$
First Order Differential Equations
$\frac{dy}{dx} = e^{x+y}$
Separable Differential Equations
Linear Equations
Homogeneouse Differential Equations
Bernouli Differential Equations
Exact Differential Equations
Second Order Differential Equations
$y'' - 2y' - 3y = 0$
Homogeneouse Differential Equations
Characteristic Equation
Non- Homogeneouse Differential Equations
Undetermined Coffiecients
Variation of paprmeters
Higher Order Differential Equations
$y''' + 4y'' - 3y' - 18y = 0$
Homogeneouse Differential Equations
Characteristic Equation
Fast Division
Long Division
Non Homogeneouse Differential Equations
Undetermined Coffiecients
Variation of paprmeters

1)First Order Differential Equations

1-1 Separable D.Es.

Any D.E. can be written in the form below is called separable.

$$\boxed{\int N(y)dy = \int M(x)dx}$$

Ex1: Find the general solution of the $y' = 1 + y^2$

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1+y^2} = dx.$$

By integration,

$$\tan^{-1} y = x + c$$

or

$$y = \tan(x + c).$$

Ex2: Find the particular solution of the $y' = -2xy$, knowing that $y(0) = 1.8$.

$$\frac{dy}{y} = -2x\,dx, \qquad \ln y = -x^2 + \tilde{c}, \qquad y = ce^{-x^2}.$$

This is the general solution. From it and the initial condition, $y(0) = ce^0 = c = 1.8$.

$y = 1.8e^{-x^2}$. This is a particular solution.

1-2 Exact D.Es.

The conditions for this method are:

1) The D.E. must be in the following form:

$$M(x,y)dx + N(x,y)dy = 0$$

2) The equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then the general solution is:

$$\boxed{c = \int M(x,y)dx + \int (Terms\ in\ N\ do\ not\ contain\ x)\ dy}$$

Where c is the integration constant.

Ex1: Solve the following D.E.

$$(x^2 + y^2)dx + (2xy + \cos y)dy = 0$$

Sol:

Test the equation if it exact or not.

$\frac{\partial M}{\partial y} = 2y \qquad , \frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the D.E. is exact.

$$c = \int (x^2 + y^2)dx + \int (\cos y)dy$$

$$c = \frac{x^3}{3} + xy^2 + siny$$

Ex2: Solve the following D.E.

$$\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0.$$

Sol:

$$M = \cos(x+y),$$

$$N = 3y^2 + 2y + \cos(x+y).$$

$$\frac{\partial M}{\partial y} = -\sin(x+y),$$

$$\frac{\partial N}{\partial x} = -\sin(x+y).$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the D.E. is exact.

$$c = \int M(x,y)dx + \int (Terms\ in\ N\ do\ not\ contains\ x)\ dy$$

$$c = \int \cos(x+y)\,dx + \int 3y^2 + 2y\,dy$$

$$c = \sin(x+y) + y^3 + y^2$$

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c$$

To check the solution apply the following equation:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = \cos(x+y)\,dx + (\cos(x+y) + 3y^2 + 2y)\,dy = 0.$$

1-3 Linear First Order D.Es.

A first order D.E. is said to be linear if it can be written in the form of:

$\frac{dy}{dx} + P(x)y = Q(x)$

Steps for solution:

1) Find the integrating factor

$$\mu(x) = e^{\int P(x)dx}$$

2) Then the general solution is:

$$\boxed{y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx}$$

………

Ex1: Solve the following equation:

$xy' - 3y = x^2$

Sol:

$$\frac{dy}{dx} - \frac{3}{x}y = x$$

$P(x) = -\frac{3}{x}$ and $Q(x) = x$

$$\mu(x) = e^{\int P(x)dx}$$

$$\mu(x) = e^{\int -\frac{3}{x}dx} = e^{-3ln|x|} = x^{-3} = \frac{1}{x^3}$$

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx = x^3\int \frac{1}{x^3}xdx = x^3\int x^{-2}dx$$

$$y = x^3\left[-\frac{1}{x} + c\right] = -x^2 + cx^3$$

Ex2: Find the particular solution of following equation:

$y' + y\tan x = \sin 2x$ whereas $y(0) = 1$

Sol:

$$\frac{dy}{dx} + y\tan x = 2\sin x\cos x$$

$P(x) = \tan x$ and $Q(x) = 2\sin x\cos x$

$$\mu(x) = e^{\int P(x)dx}$$

$$\mu(x) = e^{\int \tan x dx} = e^{ln|\sec x|} = \sec x$$

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx = \frac{1}{\sec x}\int \sec x * 2\sin x\cos x\, dx$$

$$y = \frac{1}{\sec x}\int 2\sin x\, dx$$

$$y = \frac{1}{\sec x}[-2\cos x + c] = -2cos^2 x + c\cos x$$

At x=0 $\rightarrow y(0) = 1$

$1 = -2cos^2(0) + c\cos(0)$

$1 = -2 + c \rightarrow c = 3$

Then the particular solution is:

$y = -2cos^2 x + 3\cos x$

H.W: Solve the following equation:

1) $y' - y = e^{2x}$

2) $y' + y\sin x = e^{\cos x}$

1-4 First Order Homogenous D.Es.

A first order D.E. is said to be homogenous if it can be written in the form of:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Put $\frac{y}{x} = u$ then $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

Ex1: Solve the following equation

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

$$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{y}{x}} \Rightarrow \text{homo.} \quad \text{Put} \quad \frac{y}{x} = u \quad \Rightarrow \frac{dy}{dx} = x\cdot\frac{du}{dx} + u$$

$$x\cdot\frac{du}{dx} + u = \frac{1+u^2}{u} \quad \Rightarrow \quad x\cdot\frac{du}{dx} = \frac{1+u^2-u^2}{u}$$

$$x\cdot\frac{du}{dx} = \frac{1}{u} \quad , \quad \int u\cdot du = \int\frac{dx}{x}$$

$$\frac{u^2}{2} = \ln x + c \quad \Rightarrow \quad \frac{y^2}{2x^2} = \ln x + c$$

Ex2: Solve the following D.E.

$$x\frac{dy}{dx} = y(lnx - \ln y)$$

Sol:

$$x\frac{dy}{dx} = y(lnx - \ln y) \qquad \div x$$

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\frac{x}{y}\right)$$

$$\frac{dy}{dx} = -\frac{y}{x}\left(\ln\frac{y}{x}\right)$$

Put $\frac{y}{x} = u$ then $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = -u(\ln u)$$

$$x\frac{du}{dx} = -u(\ln u) - u$$

$$xdu = (-u(\ln u) - u)dx$$

$$\frac{1}{(-u(\ln u) - u)}du = \frac{1}{x}dx$$

$$\int \frac{1}{-u\big((\ln u) + 1\big)}du = \int \frac{1}{x}dx$$

$$-\ln|(\ln u) + 1| = \ln|x| + c$$

$$\ln|(\ln u) + 1| \quad = -\ln|cx|$$

$$\ln u + 1 = cx^{-1}$$

$$\ln u = cx^{-1} - 1$$

$$u = e^{cx^{-1}-1}$$

$$y = xe^{\frac{1}{cx}-1}$$

1-5 Bernoulli D.Es.

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful is the Bernoulli equation.

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$$

To convert it to linear form:

1) Find the value of n.
2) Let $V = y^{(1-n)}$.
3) Then the new equation in linear form is:

$$\boxed{\left(\frac{1}{1-n}\right)V' + P(x)V = Q(x)}$$

………………………………………………………………………………………………

Ex1: solve the following D.E. $y' + \frac{4}{x}y = x^3y^2$

Sol:

1) n=2
2) $V = y^{(1-2)} = y^{-1}$
3) $\left(\frac{1}{1-n}\right)V' + P(x)V = Q(x)$

$$-V' + \frac{4}{x}V = x^3 \longrightarrow V' - \frac{4}{x}V = -x^3$$

$$P(x) = -\frac{4}{x} \quad and\ Q(x) = -x^3$$

$$\mu(x) = e^{\int P(x)dx}$$

$$\mu(x) = e^{\int -\frac{4}{x}dx}$$

$$\mu(x) = e^{-4\int \frac{1}{x}dx} = e^{-4lnx} = x^{-4}$$

$$V = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx$$

$$V = \frac{1}{x^{-4}}\int x^{-4} * (-x^3)dx$$

$$V = x^4\int -\frac{1}{x}dx = x^4(-\ln x + c)$$

$$V = -x^4\ln x + x^4c$$

$$y^{-1} = -x^4\ln x + x^4c$$

……

Ex2: Solve the following D.E. $y' + (x+1)y = e^{x^2}y^3$

Sol:

1) $n = 3$
2) Let $V = y^{1-3} = y^{-2}$
3) $\left(\frac{1}{1-n}\right)V' + P(x)V = Q(x)$

$$-\frac{1}{2}V' + (x+1)V = e^{x^2} \longrightarrow V' - 2(x+1)V = -2e^{x^2}$$

$$P(x) = -2(x+1) \quad , \quad Q(x) = -2e^{x^2}$$

$$\mu(x) = e^{\int P(x)dx}$$

$$\mu(x) = e^{\int -2(x+1)dx}$$

$$\mu(x) = e^{-x^2-2x}$$

$$V = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx$$

$$V = \frac{1}{e^{-x^2-2x}} \int e^{-x^2-2x} * -2e^{x^2} dx$$

$$V = \frac{1}{e^{-x^2-2x}} \int -2e^{-2x} dx$$

$$V = \frac{1}{e^{-x^2-2x}} (e^{-2x} + c)$$

$$V = e^{x^2+2x}(e^{-2x} + c)$$

$$V = \left(e^{x^2} + e^{x^2+2x}c\right)$$

$$y^{-2} = e^{x^2}(1 + e^{2x}c)$$

2) Second Order Differential Equations

A second-order ODE is called linear if it can be written:

$$y'' + P(x)y' + q(x)y = F(x)$$

Where P(x) and q(x) are called the coefficient of the ODEs. These coefficients may be variables (functions) or constants. In the case where we assume constant coefficients, we will use the following differential equation form:

$$\boxed{ay'' + by' + cy = F(x)} \qquad \text{………….(1)}$$

Where a,b and c are constant coefficients.

1) If $F(x) = 0$ then Eq.1 is called Homogenous D.E.
2) If $F(x) \neq 0$ then Eq.1 is called Non-Homogenous D.E.

Ex: What is the type of the following D.E.?

1) $y''-x^2y' + \sin x\, y = 0$ is linear, 2^{nd} order, homo.
2) $y''-(y')^2 + y = \sin x$ is nonlinear, 2^{nd} order, non homo.

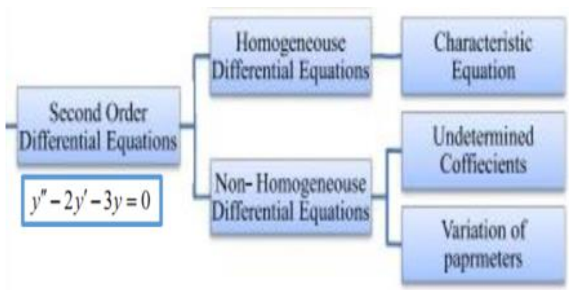
Then we have two type of 2^{nd} order linear D.Es.

1) The second order, constants coefficients, linear, ***Homogeneous*** D.E.s, we use *Characteristic Equation* method to solve these equations.

2) The second order, constants coefficients, linear, ***Non-Homogeneous*** D.E.s, There are two method to solve these equations:

   a) Undetermined coefficients.

   b) Variation of parameters.



……………………………………………………………………………………………………

2-1 The Second order Linear Homogenous D.Es. With constant coefficients

The general form of these equations is:

$$ay'' + by' + cy = 0 \qquad \text{……(2)}$$

Where a,b and c are constants. We use ***characteristic equation*** method to solve them. Then the general solution is:

1) Put $y' = Dy$ and $y'' = D^2y$ in eq.2. (where D is an operator).

$$aD^2y + bDy + cy = 0$$

$$(aD^2 + bD + c)y = 0$$

2) Replace D by r and delete y, then:

$$\boxed{ar^2 + br + c = 0}$$

This equation is called characteristic equation of the differential equation and the solution of this equation (the roots r) give the solution of the differential equation where:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three cases for the roots value r:

Case one: If $b^2 - 4ac > 0$ then r_1 and r_2 are distinct $(r_1 \neq r_2)$ and real roots and the general solution is:

$$\boxed{y = C_1 e^{r_1 x} + C_2 e^{r_2 x}}$$

Case Two: If $b^2 - 4ac = 0$ then r_1 and r_2 are equal $(r_1 = r_2 = r)$ and real roots and the general solution is:

$$\boxed{y = (C_1 + C_2 x)\, e^{rx}}$$

Case Three: If $b^2 - 4ac < 0$ then r_1 and r_2 are two complex conjugate roots. $r = \alpha \pm j\beta$, and the general solution is:

$$\boxed{y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)}$$

Ex.1: Solve $y'' - 2y' - 3y = 0$

Solution:

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = 0 \qquad , \quad y = 1 \ , \ y' = r \quad , \ y'' = r^2$$

$$(r+1)(r-3) = 0$$

(Not equal roots)

$$r + 1 = 0 \quad \Rightarrow \quad r1 = -1$$

$$r - 3 = 0 \quad \Rightarrow \quad r2 = 3$$

the general solution is

$$y = c_1 e^{-x} + c_2 e^{3x}$$

Ex.2: Solve $y'' - 6y' + 9y = 0$

Solution:

$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0 \qquad \Rightarrow \quad r_1 = r_2 = 3$$

(Equal roots)

$$\therefore \quad y = (c_1 + c_2 x) e^{3x}$$

Ex.3: Solve $y'' + y' + y = 0$

Solution:

$$y'' + y' + y = 0$$

$$r^2 + r + 1 = 0 \qquad a = 1, b = 1, c = 1$$

$$r = \frac{-b \pm \sqrt{1 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} \qquad = \frac{-1 \pm \sqrt{3} i}{2}$$

$$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i \qquad \alpha = \frac{-1}{2} \ , \ \beta = \frac{\sqrt{3}}{2}$$

, (Complex roots)

$$\therefore \ y = e^{\frac{-1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right)$$

2-2 The Second order Linear Non-Homogenous D.Es. With constant coefficients

The general form of these equations is:

$ay'' + by' + cy = F(x)$ ……(3)

Where a,b and c are constants.

The general solution is:

$$\boxed{y(x) = y_h(x) + y_p(x)}$$

Whereas $y_h(x)$ is the solution of the homogenous D.E. (3): $ay'' + by' + cy = 0$

$y_p(x)$: is the particular solution of the non-homogenous D.E. and can find it using two methods.

Methods of Finding $y_p(x)$:

1) Undetermined Coefficients
2) Variations of Parameters

2-2-1 Undetermined Coefficients

In this method y_p depends on the roots r_1 and r_2 of the characteristic equation and on the form of $F(x)$ in eq. (3) as follows:

$F(x)$	Choice of y_p
kx^n nth degree polynomial	$k_n x^n + k_{n-1}x^{n-1} + k_{n-2}x^{n-2} + \cdots + k_0$
ke^{px}	ce^{px}
$(k\sin\beta x)$ *or* $(k\cos\beta x)$	$c_1\cos\beta x + c_2\sin\beta x$

Note: For the repeated roots, multiply by x.

Ex: Use the table to find y_p

1) $F(x) = 3x^2$

   $y_p = k_2x^2 + k_1x + k_0$

2) $F(x) = -\frac{1}{2}e^{-3x}$

   $y_p = ce^{-3x}$

3) $F(x) = 2\cos 3x$

   $y_p = c_1\cos 3x + c_2\sin 3x$

4) $F(x) = 3x^2 - 3x + 5 - 2e^{3x}$

$$y_p = k_2x^2 + k_1x + k_0 + ce^{3x}$$

5) $F(x) = 2\cos x - \frac{1}{2}\sin x$

$$y_p = c_1 \cos x + c_2 \sin x$$

6) $F(x) = \sin x - \cos 2x$

$$y_p = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$$

....

Ex1: Solve the Following D.E. $y'' - y' - 2y = 4x^2$

Sol:

$$y = y_h + y_p$$

First we will find y_h :

$$y'' - y - 2y = 0$$

the char. Eq. $r^2 - r - 2 = 0$

$$(r+1)(r-2) = 0$$

$$r_1 = -1,\ r_2 = 2$$

r1 and r2 are not equal roots. then :

$$y_h = c_1e^{-x} + c_2e^{2x}$$

Second we will find y_p :

$F(x) = 4x^2$, is polynomial of second degree then

$$y_p = k_2x^2 + k_1x + k_0$$

Now, we are going to find k_2, k_1, k_0.

$$y_p = k_2x^2 + k_1x + k_0$$
$$\Rightarrow \quad y_p' = 2k_2x + k_1 \quad , \quad y_p'' = 2k_2$$

differentiate the $\mathbf{y_p}$ first and second derivative.

Substitution y_p, y_p', y_p'' in (1)

$$2k_2 - (2k_2x + k_1) - 2(k_2x^2 + k_1x + k_0) = 4x^2$$

Then, find k_2, k_1, k_0.

$$coeff.of\ x^2 : -2k_2 = 4 \quad \Rightarrow \quad k_2 = -2$$
$$coeff.of\ x : -2k_2 - 2k_1 = 0 \quad \Rightarrow \mathrm{k}_1 = 2$$
$$const : 2k_2 - k_1 - 2k_0 = 0 \quad \Rightarrow \mathrm{k}_0 = -3$$

Now, the y_p is,

$$y_p = -2x^2 + 2\mathrm{x} - 3$$

Then the solution of the equation **(1)** is :

$$\mathrm{y} = \mathrm{y_h} + y_p = (c_1e^{-x} + c_2e^{2x}) - 2x^2 + 2x - 3$$

……………………………………………………………………………………………….

Ex2: Solve the following D.E. $y'' - y' - 2y = e^{3x}$

Sol:

$$\mathrm{y}'' - \mathrm{y}' - 2\mathrm{y} = \mathrm{e}^{3\mathrm{x}} \qquad \text{....(1)}$$

$$\mathrm{y}'' - \mathrm{y}' - 2\mathrm{y} = 0$$

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0 \Rightarrow r_1 = 2, r_2 = -1$$

$$y_h = (c_1 e^{2x} + c_2 e^{-x}) \text{, Put}$$

$$y_p = ce^{3x} \qquad \qquad ....(2)$$
$$y_p' = 3ce^{3x} \quad , \quad y_p'' = 9ce^{3x}$$

Substitute In (1)

$$9\,c\,e^{3x} - 3\,c\,e^{3x} - 2\,c\,e^{3x} = e^{3x}$$

$$9c-3c-2c=1 \Rightarrow 4c = 1 \quad \Rightarrow \quad c = \frac{1}{4}$$

In (2)$\Rightarrow$ $y_p = \frac{1}{4}e^{3x}$

$$y = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4}e^{3x}$$

....

Ex3: Solve the following D.E. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 5e^x$

Sol:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0$$

$r_1 = 2 \quad , r_2 = 1$

$y_h = c_1 e^{2x} + c_2 e^{x}$

$y_p = cxe^{x}$

$y_p{}' = cxe^{x} + ce^{x}$

$y_p{}'' = cxe^{x} + ce^{x} + ce^{x} = cxe^{x} + 2ce^{x}$

substitute y_p, $y_p{}'$ and $y_p{}''$ in the main Eq.

$cxe^{x} + 2ce^{x} - 3(cxe^{x} + ce^{x}) + 2cxe^{x} = 5e^{x}$

$cxe^{x} + 2ce^{x} - 3cxe^{x} - 3ce^{x} + 2cxe^{x} = 5e^{x}$

$-ce^{x} = 5e^{x}$ then, $c = -5$

$y_p = -5xe^{x}$

$y = y_h + y_p = c_1 e^{2x} + c_2 e^{x} - 5xe^{x}$

Ex4: Solve the following D.E. $y'' - 6y' + 9y = e^{3x}$

Sol:

$y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r_1 = r_2 = 3$ then:

$y_h = (C_1 + C_2x)\, e^{3x} = C_1e^{3x} + C_2xe^{3x}$

$y_p = cx^2e^{3x}$

$y_p{}' = 3cx^2e^{3x} + 2cxe^{3x}$

$y_p{}'' = 9cx^2e^{3x} + 6cxe^{3x} + 6cxe^{3x} + 2ce^{3x} = 9cx^2e^{3x} + 12cxe^{3x} + 2ce^{3x}$

substitute y_p, $y_p{}'$ and $y_p{}''$ in the main Eq.

$9cx^2e^{3x} + 12cxe^{3x} + 2ce^{3x} - 6(3cx^2e^{3x} + 2cxe^{3x}) + 9cx^2e^{3x} = e^{3x}$

$c = \frac{1}{2}$

$y_p = \frac{1}{2}x^2e^{3x}$

$y = y_h + y_p = C_1e^{3x} + C_2xe^{3x} + \frac{1}{2}x^2e^{3x}$

....

Ex5: Solve the following D.E. $y'' + y = \sin x$

Sol:

$$y'' + y = 0$$

$r^2+1=0,\ r^2=-1 \Rightarrow r = \pm i,\ \boldsymbol{\alpha=0},\ \beta=1$

$y_h = c_1 \cos x + c_2 \sin x$

$y_p = x(c_3\cos x + c_4\sin x)$,

$y'_p = x(-c_3\sin x + c_4\cos x) + (c_3\cos x + c_4\sin x)$

$y''_p = x(-c_3\cos x - c_4\sin x) + (-c_3\sin x + c_4\cos x) + (-c_3\sin x + c_4\cos x)$

Substitution y_p, y'_p, y''_p .

$-2c_3\sin x + 2c_4\cos x = \sin x$

$-2c_3 = 1 \Rightarrow c_3 = -1/2$,

$2c_4 = 0 \Rightarrow c_4 = 0$

$$y_g = c_1\cos x + c_2\sin x - \frac{x}{2}\cos x$$

2-2-2 Variation of Parameters

To solve $ay'' + by' + cy = F(x)$ using Variation of parameters method, let the homogenous solution of $ay'' + by' + cy = F(x)$ is:

$$\boxed{y_h = c_1u_1 + c_2u_2}$$

and the particular solution is:

$$\boxed{y_p = u_1v_1 + u_2v_2}$$

$u_1 \& u_2$ are functions of x should be founded from the homogenous solution, whereas $v_1 \& v_2$ are unknown functions of x must be determined. Firstly, solve the following linear equations for v'1 and v'2:

$$\boldsymbol{v'_1u_1 + v'_2u_2 = 0}$$

$$\boldsymbol{v'_1u'_1 + v'_2u'_2 = F(x)}$$

Which can be solved with respect to v'1 and v'2 by Grammar rule as follows:

$$D = \begin{vmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} 0 & u_2 \\ F(x) & u'_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} u_1 & 0 \\ u'_1 & F(x) \end{vmatrix}$$

$$\text{and } v'_1 = \frac{D_1}{D}, \quad v'_2 = \frac{D_2}{D}$$

By integration of v'_1 and v'_2 with respect to x we can find v_1 and v_2.

$$\boxed{\begin{aligned} v_1 &= \int \frac{-F(x)u_2}{D}\,dx \\ v_2 &= \int \frac{F(x)u_1}{D}\,dx \end{aligned}}$$

Then the general solution is:

$$y = y_h + y_p$$

………

Ex1: Solve the following D.E. $y'' - y' - 2y = e^{3x}$

Sol: To find y_h

$$y'' - y' - 2y = e^{3x}$$
$$r^2 - r - 2 = 0$$
$$(r + 1)(r - 2) = 0$$

Then $r_1 = -1$ and $r_2 = 2$, then:

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

Then, $u_1 = e^{-x}$ and , $u_2 = e^{2x}$

Now, to solve with variation of parameters method,

$$u_1 = e^{-x} \quad \Rightarrow \quad u_1' = -e^{-x}$$
$$u_2 = e^{2x} \quad \Rightarrow \quad u_2' = 2e^{2x}$$

$$y_p = u_1 v_1 + u_2 v_2$$

$$D = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^{x}$$

$$v_1 = \int \frac{-F(x)u_2}{D} dx = \int -\frac{e^{3x} * e^{2x}}{3e^{x}} dx = \int -\frac{e^{4x}}{3} dx = -\frac{e^{4x}}{12}$$

$$v_2 = \int \frac{F(x)u_1}{D} dx = \int \frac{e^{3x} * e^{-x}}{3e^{x}} dx = \int \frac{e^{x}}{3} dx = \frac{e^{x}}{3}$$

$$y_p = u_1 v_1 + u_2 v_2 = e^{-x} * -\frac{e^{4x}}{12} + e^{2x}\frac{e^{x}}{3} = -\frac{e^{3x}}{12} + \frac{e^{3x}}{3}$$

$$y_p = \frac{-e^{3x} + 4e^{3x}}{12} = \frac{3e^{3x}}{12} = \frac{1}{4}e^{3x}$$

$$y = y_h + y_p$$

$$y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4}e^{3x}$$

Ex2: Solve the following D.E. $y'' + y = \sec x$

Sol: To find y_h

y'' + y=0

$r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r = \pm i \qquad \alpha=0, \beta=1$

$y_h = c_1\cos x + c_2\sin x, \quad u_1 = \cos x, u_2 = \sin x, \quad f(x)=\sec x$

$$y_p = u_1v_1 + u_2v_2$$

$$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_1' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = cos^2\theta + sin^2\theta = 1$$

$$v_1 = \int \frac{-F(x)u_2}{D}dx = \int -\frac{\sec x * \sin x}{1}dx = \int -\frac{\sin x}{\cos x}dx = \ln|\cos x|$$

$$v_2 = \int \frac{F(x)u_1}{D}dx = \int \frac{\sec x * \cos x}{1}dx = \int 1dx = x$$

$$y_p = u_1v_1 + u_2v_2 = \ln|\cos x|\cos x + x\sin x$$

$$y = y_h + y_p = c_1\cos x + c_2\sin x + \ln|\cos x|\cos x + x\sin x$$

2-3 Euler–Cauchy Equations

It is homogenous differential equation (f(x) = 0) with non-constant coefficients as the following form:

$$a\boldsymbol{x}^2 y'' + b\boldsymbol{x}y' + cy = 0$$

With given constants a, b and c. We substitute y(x):

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

Then the auxiliary equation is:

$$ar^2 + (b-a)r + c = 0$$

Case I: Real different roots r_1 and r_2 ($r_1 \neq r_2$) and the general solution is:

$$\boxed{y = C_1 x^{r_1} + C_2 x^{r_2}}$$

Case II: A real double root ($r_1 = r_2 = r$) and the general solution is:

$$\boxed{y = (C_1 + C_2 \ln x)x^r}$$

Case III: Complex conjugate roots $r = \alpha \pm j\beta$

$$\boxed{y = x^\alpha(C_1\cos(\beta\ln|x|) + C_2\sin(\beta\ln|x|)}$$

Ex1: solve the following D.E. $x^2y'' + 2xy' - 2y = 0$

$a = 1 \quad b = 2 \quad c = -2$

$r^2 + (2-1)r - 2 = 0$

$r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$

$r_1 = -2 \qquad , \qquad r_2 = 1$

$y = C_1x^{-2} + C_2x^1$

....

Ex2: solve the following D.E. $x^2y'' - 5xy' + 9y = 0$

$a = 1 \quad b = -5 \quad c = 9$

$r^2 + (-5-1)r + 9 = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$r_{1,2} = 3$

$y = (C_1 + C_2 \ln|x|)x^3$

....

Ex3: Solve the following D.E. $x^2y'' - 3xy' + 68y = 0$

Sol:

$a = 1 \quad b = -3 \quad c = 68$

$r^2 + (-3-1)r + 68 = 0$

$$r^2 - 4r + 68 = 0$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a = 1 \quad b = -4 \quad c = 68$$

$$r_{1,2} = \frac{-(-4) \pm \sqrt{(-4)^2 - (4 \times 68)}}{2} = \frac{4 \pm \sqrt{16 - 272}}{2} = \frac{4 \pm \sqrt{-1} \times \sqrt{256}}{2} = \frac{4 \pm j1}{2}$$

$$r_{1,2} = 2 \pm j8$$

$$\alpha = 2 \quad , \quad \beta = 8$$

$$y = x^2 (C_1 \cos(8 \ln|x|) + C_2 \sin(8 \ln|x|)$$

3) Higher Order Differential Equations

The general form with constant coefficient is:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y + a_n y = F(x) \qquad \ldots\ldots(1)$$

If $F(x) = 0$ then (1) is called homogenous, otherwise (1) is called nonhomogeneous. The methods of solving second order homogenous D.Eqs. with constant coefficients can be extended to solve higher order homogenous and nonhomogeneous D.Eq. with constant coefficients.

3-1 Higher Order Differential Equations Homogenous

The characteristic equation of nth order homogenous D. E.:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y + a_n y = 0$$
$$r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0$$

Let $r_1, r_2, r_3 \ldots \ldots r_n$ be the roots of characteristic equation then:

1) If $r_1, r_2, r_3 \ldots \ldots r_n$ are all distinct then the solution is:

$$\boxed{y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}}$$

2) If r_1 repeated m times, then y_h will contain the terms:

$$\boxed{c_1 e^{r_1 x} + c_2 x e^{r_1 x} + \ldots + c_m x^{m-1} e^{r_1 x}}$$

3) If some of roots are complex ($r = \alpha \mp j\beta$) then y_h will contain:

$$\boxed{(c_1 \cos \beta x + c_2 \sin \beta x) e^{\alpha x}}$$

Now, we are going to find the roots and solve D.E. of higher order. There are two methods help us to find the roots: long division & fast division.

1- Fast Division

2- Long Division

Ex1: Find all roots of the given differential equation and solve it. $y''' + 4y'' - 3y' - 18y = 0$ using fast and long division.

Sol:

$r^3 + 4r^2 - 3r - 18 = 0$

First method: Fast division

Find all roots of $r^3 + 4r^2 - 3r - 18 = 0$,

$r: \mp 1, \mp 2,$

$f(2) = 8 + 16 - 6 - 18 = 0,$

r= 2 is the root that make the equation above is zero, then:

	1	4	-3	-18
2	↓	2	12	18
	1	6	9	0

$$r^2 + 6r + 9 = 0$$

$$(r-2)(r^2 + 6r + 9) = 0$$

$$(r-2)(r+3)(r+3) = 0$$

Then the roots that we got it using fast division are:

$r_1 = 2, r_2 = -3, r_3 = -3$

Then, the solution of the given Differential Equation is:

$y = c_1 e^{2x} + c_2 e^{-3x} + c_3 x e^{-3x}$

Second method: long division

$r^3 + 4r^2 - 3r - 18 = 0$,

$r_1 = 2$, is the root make the Eq. is equal to zero.

$$\begin{array}{r|l} & r^2 + 6r + 9 \\ \hline (r-2) & r^3 + 4r^2 - 3r - 18 \\ & \underline{\mp r^3 \pm 2r^2} \\ & 6r^2 - 3r \\ & \underline{\mp 6r^2 \pm 12r} \\ & 9r - 18 \\ & \underline{\mp 9r \pm 18} \\ & 0 \end{array}$$

$(r-2)(r^2 + 6r + 9) = 0$

$(r-2)(r+3)(r+3) = 0$

$r_1 = 2, r_2 = -3, r_3 = -3$

$y = c_1 e^{2x} + c_2 e^{-3x} + c_3 x e^{-3x}$

Ex2: Solve the following D.E.

$$y'''' - 3y''' - 2y'' + 2y' + 12y = 0$$

$$r^4 - 3r^3 - 2r^2 + 2r + 12 = 0$$

$r = 2$ is a root $\Rightarrow$ $(r - 2)$ is a factor

	1	-3	-2	2	12
2	↓	2	-2	-8	-12
	1	-1	-4	-6	0

$\Rightarrow \; r^3 - r^2 - 4r - 6 = 0$

$\Rightarrow (r-2)(r^3 - r^2 - 4r - 6) = 0$, $r = 3$ root $\Rightarrow$ $(r - 3)$ is a factor

	1	-1	-4	-6
3	↓	3	6	6
	1	2	2	0

$\Rightarrow \; r^2 + 2r + 2 = 0$

$$(r-2)(r-3)(r^2 + 2r + 2) = 0$$

$r_1 = 2, \quad r_2 = 3, \quad r = -1 \mp i \quad \alpha = -1 \;, \; \beta = 1$

$$\Rightarrow \quad y_h = c_1 e^{2x} + c_2 e^{3x} + (c_3 \cos x + c_4 \sin x) e^{-x}$$

3-2 Higher Order Differential Equations Non-Homogenous

The general form of nth order nonhomogeneous differential equation is:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y + a_n y = F(x)$$

Then the general solution is: $y = y_h + y_p$

Methods of finding y_p:

1) Undetermined coefficients

We can extend the methods of solving second order non-homogenous D.Es. with constant coefficients to solve higher order non-homogenous D.E. with constant coefficients.

Ex1: Solve the following D.E. $y^4 - 8y'' + 16y = -18\sin x$

Sol:

$$y_g = y_h + y_p$$

$$y^{(4)} - 8y'' + 16y = 0$$

$$r^4 - 8r^2 + 16 = 0 \Rightarrow (r^2-4)^2 = 0 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$$

$$\mathbf{y_h = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 x e^{-2x}}$$

Now, we will find y_p,

let $y_p = A\cos x + B\sin x$, $y'_p = -A\sin x + B\cos x$, $y''_p = -A\cos x - B\sin x$

$y'''_p = A\sin x - B\cos x$, $y^{(4)}_p = A\cos x + B\sin x$

$A\cos x + B\sin x + 8A\cos x + 8B\sin x + 16A\cos x + 16B\sin x = -18\sin x$

$25A\cos x + 25B\sin x = -18\sin x$

$25A = 0 \Rightarrow A = 0$

$25B = -18 \Rightarrow B = -18/25$

$$y = c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x} - \frac{18}{25}\sin x$$

……

2) Variation of parameters

In this method, the particular solution yp has the form $y_p = v_1u_1 + v_2u_2 + \ldots + v_nu_n$

Where $u_1, u_2, \ldots, u_n$ are taken from $y_h = c_1u_1 + c_2u_2 + \ldots + c_nu_n$.

Ex1: Solve the following D.E. $y''' + y' = \sec x$

Let $y''' + y' = 0$

$r^3 + r = 0 \Rightarrow r(r^2+1) = 0 \Rightarrow r = 0, r^2 = -1 \Rightarrow r1 = 0, r = \pm i$

$y_h = c_1 + c_2\cos x + c_3\sin x$

$u_1 = 1$, $u_2 = \cos x$, $u_3 = \sin x$, $f(x) = \sec x$

$$v'_1 + v'_2 cosx + v'_3 sinx = 0$$

$$v'_1(0) + v'_2(-\sin x) + v_3(\cos x) = 0$$

$$v'_1(0) + v'_2(-\cos x) - v'_3(\sin x) = \sec x$$

$$D=\begin{vmatrix}1 & \cos x & \sin x\\ 0 & -\sin x & \cos x\\ 0 & -\cos x & -\sin x\end{vmatrix}=\sin^2 x+\cos^2 x=1$$

$$D_1=\begin{vmatrix}0 & \cos x & \sin x\\ 0 & -\sin x & \cos x\\ \sec x & -\cos x & -\sin x\end{vmatrix}=\sec x(\sin^2 x+\cos^2 x)=\sec x$$

$$D_2=\begin{vmatrix}1 & 0 & \sin x\\ 0 & 0 & \cos x\\ 0 & \sec x & -\sin x\end{vmatrix}=\begin{vmatrix}0 & \cos x\\ \sec x & -\sin x\end{vmatrix}=-\cos x\sec x=-1$$

$$D_3=\begin{vmatrix}1 & \cos x & 0\\ 0 & -\sin x & 0\\ 0 & -\cos x & \sec x\end{vmatrix}=\begin{vmatrix}-\sin x & 0\\ -\cos x & \sec x\end{vmatrix}=-\sin x\sec x=-\tan x$$

$$v_1{}'=\frac{D_1}{D}=\sec x\Rightarrow v_1=\int\sec x dx=\ln(\sec x+\tan x)$$

$$v_2{}'=\frac{D_2}{D}=-1\Rightarrow v_2=\int -1\,dx=-x$$

$$v_3{}'=\frac{D_3}{D}=-\tan x\Rightarrow v_3=-\int\tan x\,dx=\ln\cos x$$

$y_p = \ln(\sec x+\tan x)-x\cos x-\ln\cos x\sin x$

$y_g = c_1+c_2\cos x+c_3\sin x+\ln(\sec x+\tan x)-x\cos x-\ln\cos x\sin x$

CHAPTER SIX

Solution of D.Eqs by Power Series

In the previous chapter, we have seen that linear ODEs with constant coefficients can be solved by algebraic methods. This chapter will study some of ODEs with variable coefficients, which can be solved using power series.

1.Power Series Method:

The power series method is the standard method for solving linear ODEs with variable coefficients. It gives solutions in the form of power series. The power series is an infinite series, if the series is about x=a (see chapter 4) then it is:

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Here, x is a variable. a_0, a_1, a_2, are constants, called the coefficients of the series. We shall assume that all variables and constants are real.

Familiar Power Series are the Maclaurin series

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \qquad (|x| < 1, \textbf{ geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots .$$

Ex1: Solve the following D.E. by power series method: $y' - y = 0$

Sol:

1) Assume the solution in the form of power series:

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

then y':

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

2) Insert the above series in the given D.E.

$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0$

3) Then we collect like powers of x, finding

$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0$

4) Equating the coefficient of each power of x to zero, we have

$(a_1 - a_0) = 0 \qquad (2a_2 - a_1) = 0 \qquad (3a_3 - a_2) = 0$, … ….

5) Solving these equations, we may express a_1, a_2 a_3 in terms of a_0 , which remains arbitrary.

$a_1 = a_0$

$a_2 = \dfrac{a_1}{2} = \dfrac{a_0}{2} = \dfrac{a_0}{2!}$

$a_3 = \dfrac{a_2}{3} = \dfrac{a_0}{3 * 2} = \dfrac{a_0}{3!}$

With these values of the coefficients, the series solution becomes:

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots$$

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$y = a_0 e^x$$

……………………………………………………………………………………………………

Ex2: Solve the following D.E. by power series method: $y'' + y = 0$

Sol:

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots$$

$$(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots) = 0$$

$$(2a_2 + a_0) + (6a_3 + a_1)x + (12a_4 + a_2)x^2 + (20a_5 + a_3)x^3 + \cdots = 0$$

$$(2a_2 + a_0) = 0 \quad \rightarrow \quad a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!}$$

$$(6a_3 + a_1) = 0 \quad \rightarrow \quad a_3 = -\frac{a_1}{6} = -\frac{a_1}{3!}$$

$$(12a_4 + a_2) = 0 \quad \rightarrow \quad a_4 = -\frac{a_2}{12} = \frac{a_0}{24} = \frac{a_0}{4!}$$

$$(20a_5 + a_3) = 0 \quad \rightarrow \quad a_5 = -\frac{a_3}{20} = \frac{a_1}{120} = \frac{a_1}{5!}$$

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + \cdots$$

$$y = \left(a_0 - \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 - \cdots\right) + \left(a_1 x - \frac{a_1}{3!}x^3 + \frac{a_1}{5!}x^5 + \cdots\right)$$

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$y = a_0(\cos x) + a_1(\sin x)$$

………

2. Legendre's Equation

Legendre's differential equation is one of the most important ODEs in physics. The equation involves a parameter n, whose value depends on the physical or engineering problem. The general form of the equation is:

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

The general solution of the Legendre's D.E. is:

$$y=c_1P_n(x)+c_2Q_n(x)$$

Where n=0,1,2,3,……… , $P_n(x)$ is called Legendre polynomial and $Q_n(x)$ is called Legendre function. $Q_n(x)$ series are unbounded. $P_n(x)$ is a series and convergence for $|x|<1$. Where:

$$P_n(x)=\sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!}x^{n-2m}$$

where $M=\frac{n}{2}$ or $M=\frac{n-1}{2}$, whichever is an integer. The first few $P_n(x)$ polynomials of these functions are:

1) $P_0(x)=1$
2) $P_1(x)=x$
3) $P_2(x)=\frac{1}{2}(3x^2-1)$

Whereas the first few terms of $Q_n(x)$ for $|x<1|$ is:

$$Q_0(x)=x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$Q_1(x)=x\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots\right)=\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1$$

Recursion Formulas:

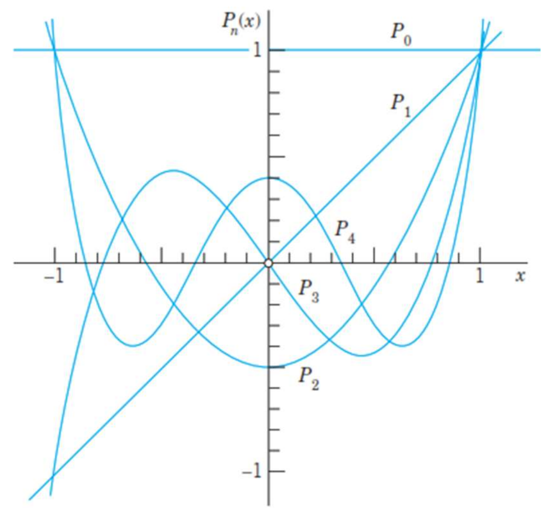
1) $P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$

2) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

3) $Q_{n+1}(x) = \frac{2n+1}{n+1} x Q_n(x) - \frac{n}{n+1} Q_{n-1}(x)$

4) $Q'_{n+1} - Q'_{n-1} = (2n+1)Q_n(x)$

The first few Legendre polynomials are shown in the figure.



Ex1: Write the general solution of $(1-x^2)y'' - 2xy' + 2y = 0$

Sol:

$n(n+1) = 2$

$n = 1$

$y = c_1 P_1(x) + c_2 Q_1(x)$

$P_1(x) = x$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

$$y = c_1 x + c_2 \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

Ex2: Find $P_2'(x)$, $Q_2(x)$, $P_3'(x)$, $P_5(x)$:

Sol:

1) $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_2'(x) = \frac{1}{2}(6x) = 3x$

2) to find $Q_2(x)$ we have:

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1$$

then $n = 1$

$$Q_{n+1}(x) = \frac{2n+1}{n+1}xQ_n(x) - \frac{n}{n+1}Q_{n-1}(x)$$

$$Q_2(x) = \frac{3}{2}xQ_1(x) - \frac{1}{2}Q_0(x) = \frac{3x}{2}\left[\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1\right] - \frac{1}{2}\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right]$$

3) to find $P_3'(x)$:

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$$

$$n + 1 = 3 \qquad \rightarrow \quad n = 2$$

$$P_{n+1}'(x) = (2n+1)P_n(x) + P_{n-1}'(x)$$

$$P_3'(x) = (5)P_2(x) + P_1'(x)$$

$$P_3'(x) = \frac{5}{2}(3x^2 - 1) + 1$$

4) To find $P_5(x)$ use the general formula:

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!\,(n-m)!\,(n-2m)!} x^{n-2m}$$

$$P_5(x) = \sum_{m=0}^{2} (-1)^m \frac{(10-2m)!}{2^5 m!\,(5-m)!\,(5-2m)!} x^{5-2m}$$

$$P_5(x) = \frac{(10!)}{2^5(5!)(5!)} x^5 - \frac{(8!)}{2^5(4!)(3!)} x^3 + \frac{(6!)}{2^5(2!)(3!)(1!)} x$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

HW:

1) Find $P_3\ (x)$, $P_6(x)$

3. Bessel's Equation, Bessel Functions $J_v(x)$:

One of the most important Eq. in applied mathematics is Bessel's D.Eq. It appears in connection with electric field, vibrations, heat conduction. It is given by:

$$\boxed{x^2y'' + xy' + (x^2 - v^2)y = 0} \qquad \text{......(1)}$$

Where v is assumed to be real and positive number.

The solutions of the Bessel's Eq. are called Bessel Function. The general solution of Eq.1 is given by:

$$\boxed{y = c_1J_v(x) + c_2y_v(x)} \qquad \text{......(2)}$$

The solution of $J_v(x)$ is called Bessel function of the **first kind** of order v. The second solution $y_v(x)$ is called Bessel function of the **second kind** of order v. $J_v(x)$ has a finite limit as x approaches to zero, while $y_v(x)$ has no limit (i.e unbounded) as x approaches to zero.

....

3.1 Bessel Functions of the First Kind $J_v(x)$

The Bessel function of the first kind of order v is given as:

$$J_v(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left({}^{x}/_{2}\right)^{v+2m}}{m!\,\Gamma(v+m+1)} \qquad \text{......(3)}$$

Where $\Gamma(v+1)$ is the gamma function and it is given by:

$\Gamma(v+1) = v\Gamma(v)$

There are two cases for v:

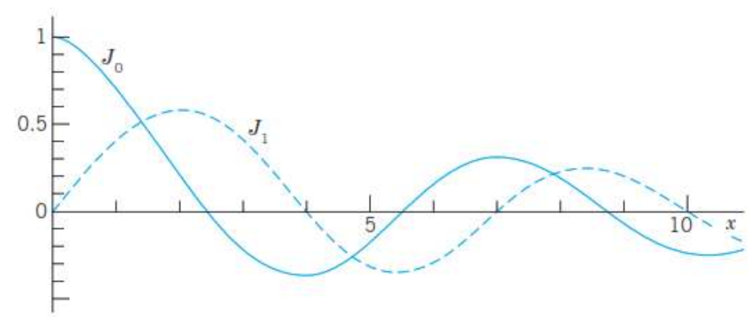
Case One: When v is integer then the Bessel function $J_n(x)$:

The integer values of v are denoted by n. In this case

$\Gamma(n+m+1) = (n+m)!$

$\Gamma(n+1) = n!$.

Eq.(3) can be written by replacing v by n. for example for n=0, then:



Bessel functions of the first kind J_0 and J_1

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m}}{2^{2m} m!\; m!}$$

for n=1

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+1}}{2^{2m+1} m!\,(m+1)!}$$

A function of $J_{-n}(x)$ can be calculated depending on $J_n(x)$ as given below:

$J_{-n}(x) = (-1)^n J_n(x)$

The above equation show linear dependence.

Case Two: When v is non-integer then the Bessel function $J_v(x)$:

If v is not integer i.e $v \neq 1,2,3, ...$ then the general solution is:

$y = c_1 J_v(x) + c_2 J_{-v}(x)$

where $J_v(x)$and $J_{-v}(x)$ are linearly independent.

If v is half odd integer, $J_v(x)$can be expressed in terms of sines and cosines.

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Note:

- $\Gamma(n + m + 1) = (n + m)!$ for n= integer.
- $\Gamma(v + 1) = v\Gamma(v)$ for any value of v
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$
- $\Gamma(v + 2) = (v + 1)\Gamma(v + 1) = (v + 1)v\Gamma(v)$

Some Properties of Bessel Functions:

1) $J_{v+1}(x) = \frac{2v}{x} J_v(x) - J_{v-1}(x)$

2) $J'_v(x) = \frac{1}{2}[J_{v-1}(x) - J_{v+1}(x)]$

3) $\frac{d}{dx}[x^v J_v(x)] = x^v J_{v-1}(x)$

4) $\frac{d}{dx}[x^{-v} J_v(x)] = -x^{-v} J_{v+1}(x)$

5) For large values of x :

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

Ex1: Find $J_{3/2}(x)$?

Sol:

$$J_{v+1}(x) = \frac{2v}{x} J_v(x) - J_{v-1}(x)$$

$$v = \frac{1}{2}$$

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$J_{3/2}(x) = \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

....

Ex2: Evaluate $I = \int_1^2 x^{-3} J_4(x)dx$

Sol: from Properties

$$\frac{d}{dx}[x^{-v}J_v(x)] = -x^{-v}J_{v+1}(x)$$

$v = 3$, by integrating both sides

$$[x^{-v}J_v(x)] = \int -x^{-v}J_{v+1}(x)dx$$

$$I = \int_1^2 x^{-3}J_4(x)dx = -[x^{-3}J_3(x)]\begin{matrix}2\\1\end{matrix}$$

$$I = -\left(\frac{1}{8}J_3(2) - J_3(1)\right)$$

To find J_3, firstly we find J_2 depending on J_0 and J_1. To find J_2 we use the properties:

$$J_{v+1}(x) = \frac{2v}{x}J_v(x) - J_{v-1}(x)$$

$v = 1$

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

Then find $J_3(x)$:

$$J_{v+1}(x) = \frac{2v}{x}J_v(x) - J_{v-1}(x)$$

$$v = 2$$

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$$

$$J_3(x) = \frac{4}{x}\left(\frac{2}{x}J_1(x) - J_0(x)\right) - J_1(x)$$

$$J_3(x) = \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x)$$

$$J_3(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$$

For $x = 2$:

$$J_3(2) = \left(\frac{8}{4} - 1\right)J_1(2) - \frac{4}{2}J_0(2)$$

from the table:

$J_1(2) = 0.5767$

$J_0(2) = 0.2239$

$J_3(2) = 1 * 0.5767 - 2 * 0.2239$

$J_3(2) = 0.1289$

By the same way we find

$J_3(1) = 0.0199$

$$I = -\left(\frac{1}{8}J_3(2) - J_3(1)\right) = -\left(\frac{1}{8} * 0.1289 - 0.0199\right)$$

$$I = 0.0038$$

x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$	x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000	3.0	−0.2601	0.3391	6.0	0.1506	−0.2767
0.1	0.9975	0.0499	3.1	−0.2921	0.3009	6.1	0.1773	−0.2559
0.2	0.9900	0.0995	3.2	−0.3202	0.2613	6.2	0.2017	−0.2329
0.3	0.9776	0.1483	3.3	−0.3443	0.2207	6.3	0.2238	−0.2081
0.4	0.9604	0.1960	3.4	−0.3643	0.1792	6.4	0.2433	−0.1816
0.5	0.9385	0.2423	3.5	−0.3801	0.1374	6.5	0.2601	−0.1538
0.6	0.9120	0.2867	3.6	−0.3918	0.0955	6.6	0.2740	−0.1250
0.7	0.8812	0.3290	3.7	−0.3992	0.0538	6.7	0.2851	−0.0953
0.8	0.8463	0.3688	3.8	−0.4026	0.0128	6.8	0.2931	−0.0652
0.9	0.8075	0.4059	3.9	−0.4018	−0.0272	6.9	0.2981	−0.0349
1.0	0.7652	0.4401	4.0	−0.3971	−0.0660	7.0	0.3001	−0.0047
1.1	0.7196	0.4709	4.1	−0.3887	−0.1033	7.1	0.2991	0.0252
1.2	0.6711	0.4983	4.2	−0.3766	−0.1386	7.2	0.2951	0.0543
1.3	0.6201	0.5220	4.3	−0.3610	−0.1719	7.3	0.2882	0.0826
1.4	0.5669	0.5419	4.4	−0.3423	−0.2028	7.4	0.2786	0.1096
1.5	0.5118	0.5579	4.5	−0.3205	−0.2311	7.5	0.2663	0.1352
1.6	0.4554	0.5699	4.6	−0.2961	−0.2566	7.6	0.2516	0.1592
1.7	0.3980	0.5778	4.7	−0.2693	−0.2791	7.7	0.2346	0.1813
1.8	0.3400	0.5815	4.8	−0.2404	−0.2985	7.8	0.2154	0.2014
1.9	0.2818	0.5812	4.9	−0.2097	−0.3147	7.9	0.1944	0.2192
2.0	0.2239	0.5767	5.0	−0.1776	−0.3276	8.0	0.1717	0.2346
2.1	0.1666	0.5683	5.1	−0.1443	−0.3371	8.1	0.1475	0.2476
2.2	0.1104	0.5560	5.2	−0.1103	−0.3432	8.2	0.1222	0.2580
2.3	0.0555	0.5399	5.3	−0.0758	−0.3460	8.3	0.0960	0.2657
2.4	0.0025	0.5202	5.4	−0.0412	−0.3453	8.4	0.0692	0.2708
2.5	−0.0484	0.4971	5.5	−0.0068	−0.3414	8.5	0.0419	0.2731
2.6	−0.0968	0.4708	5.6	0.0270	−0.3343	8.6	0.0146	0.2728
2.7	−0.1424	0.4416	5.7	0.0599	−0.3241	8.7	−0.0125	0.2697
2.8	−0.1850	0.4097	5.8	0.0917	−0.3110	8.8	−0.0392	0.2641
2.9	−0.2243	0.3754	5.9	0.1220	−0.2951	8.9	−0.0653	0.2559

$J_0(x) = 0$ for $x = 2.40483, 5.52008, 8.65373, 11.7915, 14.9309, 18.0711, 21.2116, 24.3525, 27.4935, 30.6346$

$J_1(x) = 0$ for $x = 3.83171, 7.01559, 10.1735, 13.3237, 16.4706, 19.6159, 22.7601, 25.9037, 29.0468, 32.1897$

x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$	x	$Y_0(x)$	$Y_1(x)$
0.0	$(-\infty)$	$(-\infty)$	2.5	0.498	0.146	5.0	−0.309	0.148
0.5	−0.445	−1.471	3.0	0.377	0.325	5.5	−0.339	−0.024
1.0	0.088	−0.781	3.5	0.189	0.410	6.0	−0.288	−0.175
1.5	0.382	−0.412	4.0	−0.017	0.398	6.5	−0.173	−0.274
2.0	0.510	−0.107	4.5	−0.195	0.301	7.0	−0.026	−0.303

CHAPTER SEVEN

Probabilities and Statistics

Probability and statistics are concerned with events which occur by chance. Examples include occurrence of accidents, errors of measurements. In each case, we may have some knowledge of the likelihood of various possible results, but we cannot predict with any certainty the outcome of any particular trial. Probability and statistics are used throughout engineering. In electrical and communication engineering, signals and noise are analyzed by means of probability theory.

Probability versus Statistics

Probability deals with predicting the likelihood of future events, while **statistics** involves the analysis of the frequency of past events.

Statistics

Describing a set of Data with numerical measures. Graphs can help you describe the basic shape of a data distribution.

After the sampling process, this is the next step in every statistical study and usually consists of:

1. To classify, group and sort the data of the sample.

2. To tabulate and plot data according to their frequencies.

3. To calculate numerical measures that summarize the information contained in the sample (sample statistics).

Sample classification:

It consists of grouping the values that are the same and sorting them if there is an order among them.

Example. X = Height

Frequency count:

It consists of counting the number of times that every value appears in the sample.

Example. X=Height

Sample frequencies

Definition - Sample frequencies. Given a sample of n values of a variable X, for every value x_i of the variable we define

- **Absolute Frequency** n_i: The number of times that value x_i appears in the sample.
- **Relative Frequency** f_i: The proportion of times that value x_i appears in the sample.

$$f_i = n_i / n$$

- **Cumulative Absolute Frequency** N_i: The number of values in the sample less than or equal to x_i.

$$N_i = n_1 + \cdots + n_i = N_{i-1} + n_i$$

- **Cumulative Relative Frequency** F_i: The proportion of values in the sample less than or equal to x_i.

$$F_i = N_i / n$$

Frequency table:

The set of values of a variable with their respective frequencies is called frequency distribution of the variable in the sample, and it is usually represented as a frequency table.

X values	Absolute frequency	Relative frequency	Cumulative absolute frequency	Cumulative relative frequency
x_1	n_1	f_1	N_1	F_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	n_i	f_i	N_i	F_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_k	n_k	f_{kf}	N_k	F_k

Example - Quantitative variable and non-grouped data.

Find (fi, Ni, Fi) for the following of number of children in 25 families are:

1, 2, 4, 2, 2, 2, 3, 2, 1, 1, 0, 2, 2, 0, 2, 2, 1, 2, 2, 3, 1, 2, 2, 1, 2

Solution :The frequency table for the number of children in this sample is

Relative Frequency f_i: $f_i = n_i / n$

Cumulative Absolute Frequency N_i: $N_i = n_1 + \cdots + n_i$

Cumulative Relative Frequency F_i: $F_i = N_i / n$

x_i	n_i	f_i	N_i	F_i
0	2	0.08	2	0.08
1	6	0.24	8	0.32
2	14	0.56	22	0.88
3	2	0.08	24	0.96
4	1	0.04	25	1
$\sum$	25	1		

Example - Quantitative variable and grouped data. The heights (in cm) of 30 students are:

179, 173, 181, 170, 158, 174, 172, 166, 194, 185,
162, 187, 198, 177, 178, 165, 154, 188, 166, 171,
175, 182, 167, 169, 172, 186, 172, 176, 168, 187.

Solution :The frequency table for the height in this sample is :

Relative Frequency f_i: $f_i = n_i / n$

Cumulative Absolute Frequency N_i: $N_i = n_1 + \cdots + n_i$

Cumulative Relative Frequency F_i: $F_i = N_i / n$

x_i	n_i	f_i	N_i	F_i
(150, 160]	2	0.07	2	0.07
(160, 170]	8	0.27	10	0.34
(170, 180]	11	0.36	21	0.70
(180, 190]	7	0.23	28	0.93
(190, 200]	2	0.07	30	1
$\sum$	30	1		

Sample statistics:

According to the aspect of the distribution that they study, there are different types of statistics: Location statistics and Measures of dispersion.

Location statistics:

There are two groups:

1-Central location measures (Measures of center):

They measure the values where data are concentrated, usually at the center of the distribution. The most important are:

1) Arithmetic mean
2) Median
3) Mode

Central location measures (Measures of center):

1- Arithmetic mean

Sample arithmetic mean $\overline{X}$. The *sample arithmetic mean* of a variable X is the sum of observed values in the sample divided by the sample size:

$$\boxed{\bar{x} = \frac{\sum x_i}{n}}$$

Also , it can be calculated from the ***frequency table*** with the formula :

$$\boxed{\bar{x} = \frac{\sum x_i n_i}{n} = \sum x_i f_i}$$

Example – Non-grouped data.

The number of children in 25 families are: 1, 2, 4, 2, 2, 2, 3, 2, 1, 1, 0, 2, 2, 0, 2, 2, 1, 2, 2, 3, 1, 2, 2, 1, 2

Solution:

$$\boxed{\bar{x} = \frac{\sum x_i}{n}}$$

$$\bar{x} = \frac{1+2+4+2+2+2+3+2+1+1+0+2+2}{25} + $$
$$+ \frac{0+2+2+1+2+2+3+1+2+2+1+2}{25} = \frac{44}{25} = 1.76 \text{ children.}$$

or using the frequency table

x_i	n_i	f_i	$x_i n_i$	$x_i f_i$
0	2	0.08	0	0
1	6	0.24	6	0.24
2	14	0.56	28	1.12
3	2	0.08	6	0.24
4	1	0.04	4	0.16
$\sum$	25	1	44	1.76

$$\bar{x} = \frac{\sum x_i n_i}{n} = \frac{44}{25} = 1.76 \text{ children} \qquad \bar{x} = \sum x_i f_i = 1.76 \text{ children.}$$

Example – Grouped data. Using the data of the sample of student heights, the arithmetic mean is

$$\bar{x} = \frac{179 + 173 + \cdots + 187}{30} = 175.07 \text{ cm.}$$

or using the frequency table and taking the class marks as x_i,

X	x_i	n_i	f_i	$x_i n_i$	$x_i f_i$
(150, 160]	155	2	0.07	310	10.33
(160, 170]	165	8	0.27	1320	44.00
(170, 180]	175	11	0.36	1925	64.17
(180, 190]	185	7	0.23	1295	43.17
(190, 200]	195	2	0.07	390	13
$\sum$		30	1	5240	174.67

$$\bar{x} = \frac{\sum x_i n_i}{n} = \frac{5240}{30} = 174.67 \text{ cm} \qquad \bar{x} = \sum x_i f_i = 174.67 \text{ cm.}$$

Observe that when the mean is calculated from the table the result differs a little from the real value, because the values used in the calculations are the class marks instead of the actual values.

Weighted mean

In some cases the values of the sample have different importance. In that case the importance or *weight* of each value of the sample must be taken into account when calculating the mean.

weighted mean of variable X is the sum of the product of each value by its weight, divided by sum of weights

$$\bar{x}_w = \frac{\sum x_i w_i}{\sum w_i}$$

From the frequency table can be calculated with the formula

$$\bar{x}_w = \frac{\sum x_i w_i n_i}{\sum w_i}$$

Example. Assume that a student wants to calculate a representative measure of his/her performance in a course. The grade and the credits of every subjects are

Subject	Credits	Grade
Maths	6	5
Economics	4	3
Chemistry	8	6

The arithmetic mean is

$$\bar{x} = \frac{\sum x_i}{n} = \frac{5+3+6}{3} = 4.67 \text{ points.}$$

However, this measure does not represent well the performance of the student, as not all the subjects have the same importance and require the same effort to pass. Subjects with more credits require more work and must have more weight in the calculation of the mean.

In this case it is better to use the weighted mean, using the credits as the weights of grades, as a representative measure of the student effort

$$\bar{x}_w = \frac{\sum x_i w_i}{\sum w_i} = \frac{5\cdot 6 + 3\cdot 4 + 6\cdot 8}{6+4+8} = \frac{90}{18} = 5 \text{ points.}$$

2- ***Median:***

A second measure of ***central tendency*** is the ***median***, which is the value in the ***middle position*** in the set of measurements ***ordered from smallest to largest***.

Definition : The median ***m*** of a set of n measurements is the value of ***x*** that falls in the ***middle position*** when the measurements are ordered ***from smallest to largest***.

We can know the order and the value of median by using the following :

The value " **.5(n + 1)** " indicates the **position** of the median in the ordered data set. If the position of the median is a number that ends in the value .5, you need to average the two adjacent values.

EXAMPLE Find the median for the set of measurements 2, 9, 11, 5, 6.

Solution $n = 5$ measurements from smallest to largest:

$$\begin{array}{ccccc} 2 & 5 & 6 & 9 & 11 \\ & & \uparrow & & \end{array}$$

The middle observation, marked with an arrow, is in the center of the set, or $m = 6$.

EXAMPLE Find the median for the set of measurements 2, 9, 11, 5, 6, 27.

Solution Rank the measurements from smallest to largest:

$$2 \quad 5 \quad \boxed{6 \quad 9} \quad 11 \quad 27$$
$$\uparrow$$

Now there are two "middle" observations, shown in the box. To find the median, choose a value halfway between the two middle observations:

$$m = \frac{6+9}{2} = 7.5$$

Now if we use the value " **.5(n + 1)** :

For the $n = 5$ ordered measurements from Example , the position of the median is $.5(n + 1) = .5(6) = 3$, and the median is the *3rd ordered observation*, or $m = 6$. For the $n = 6$ ordered measurements from Example , the position of the median is $.5(n + 1) = .5(7) = 3.5$, and the median is the *average of the 3rd and 4th ordered observations*, or $m = (6 + 9)/2 = 7.5$.

3- The Mode :

Another way to ***locate the center of a distribution*** is to look for the value of x that occurs with the ***highest frequency***. This measure of the center is called the **mode**.

Note : The mode is generally used to describe ***large data sets***, whereas the mean and median are used for both ***large*** and ***small*** data sets.

EXAMPLE

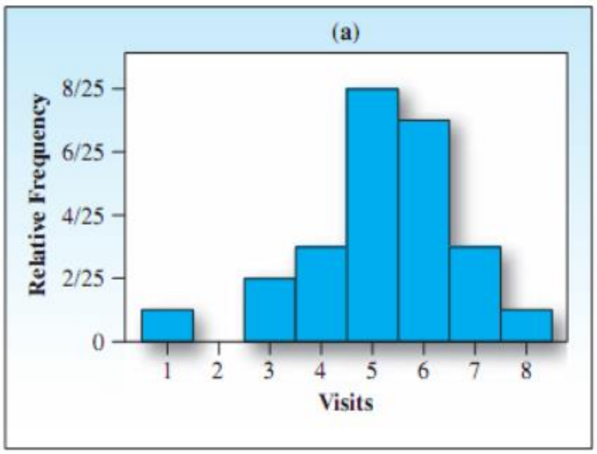
Starbucks and birth weight data

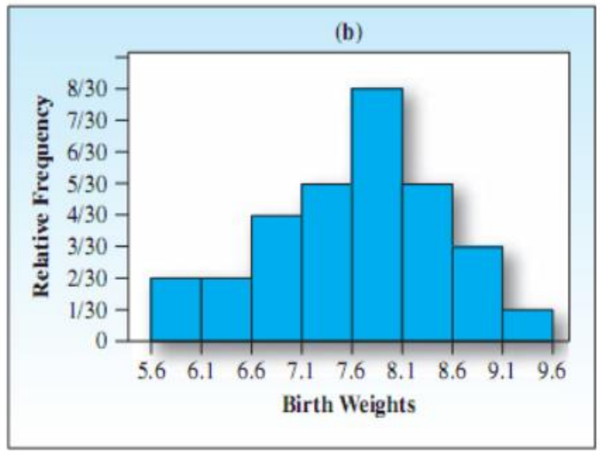
(a) Starbucks data

6	7	1	5	6
4	6	4	6	8
6	5	6	3	4
5	5	5	7	6
3	5	7	5	5

(b) Birth weight data

7.2	7.8	6.8	6.2	8.2
8.0	8.2	5.6	8.6	7.1
8.2	7.7	7.5	7.2	7.7
5.8	6.8	6.8	8.5	7.5
6.1	7.9	9.4	9.0	7.8
8.5	9.0	7.7	6.7	7.7





Solution :

For ***The visits*** :

Table: From the data in Example reproduced in Table (a), the **mode** of the distribution of the number of reported weekly visits to Starbucks for 30 Starbucks customers is 5.

For ***the birth weight*** :

Table: For the birth weight data in Table (b), a birth weight of 7.7 occurs four times, and therefore the **mode** for the distribution of birth weights is 7.7

2-Dispersion statistics (Measure of variability)

Dispersion or spread refers to the variability of data. So, dispersion statistics measure how the data values are scattered in general, or with respect to a central location measure. For quantitative variables, the most important are:

1- Range

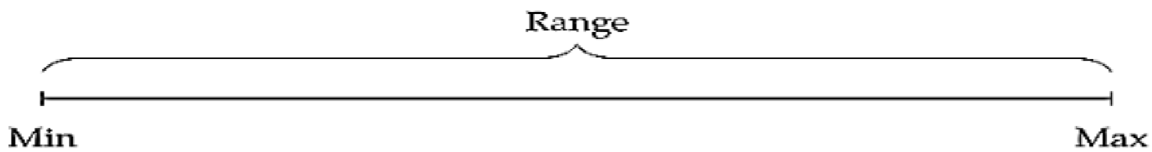
2- Variance

3- Standard deviation

1- Range :

Definition - Sample range. The sample range of a variable X is the difference between the the maximum and the minimum values in the sample.

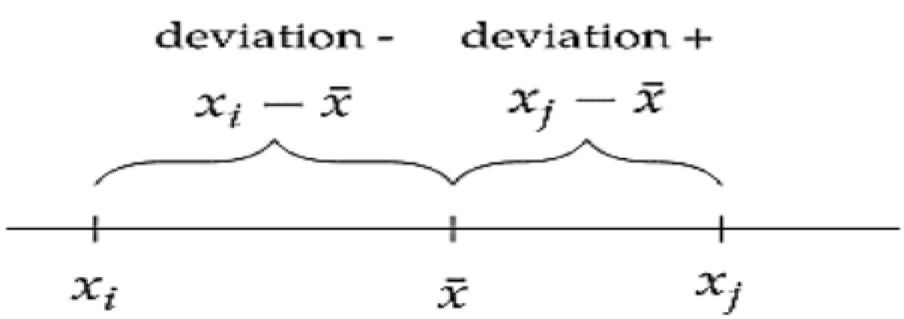
$$\text{Range} = \max_{x_i} - \min_{x_i}$$



The range measures the largest variation among the sample data. However, it is very sensitive to ***outliers***, as they appear at the ends of the distribution, and ***for that reason is rarely used***.

Example:
the measurements " 5, 7, 1, 2, 4 " vary from 1 to 7. Hence, the range is (7 - 1 = 6) .
The range is easy to calculate, easy to interpret, and is an adequate measure of variation for small sets of data. ***For large data sets, the range is not an adequate measure of variability***.

Deviations from the mean:
Another way of measuring spread of data is with respect to a central tendency measure, as for example the mean.
In that case, it is measured the distance from every value in the sample to the mean, that is called **deviation from the mean**.



If deviations are big, the mean is less representative than when they are small.

2- Variance and standard deviation

Variance of a population:

The ***variance of a population*** of N measurements is the average of the squares of the deviations of the measurements about their mean "$\bar{X}$". The population variance is denoted by " σ^2 " and is given by the formula.

$$\sigma^2 = \frac{\Sigma(x_i - \bar{X})^2}{N}$$

Most often, you will not have all the population measurements available but will need to calculate the *variance of a sample* of n measurements.

The variance of a sample :

The ***variance of a sample*** of **"n"** measurements is the sum of the squared deviations of the measurements about their mean " $\bar{x}$ " divided by (n - 1). The sample variance is denoted by s^2 and is given by the formula.

$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1}$$

For the set of $n = 5$ sample measurements presented in Table , the square of the deviation of each measurement is recorded in the third column. Adding, we obtain

$$\Sigma(x_i - \bar{x})^2 = 22.80$$

and the sample variance is

$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n-1} = \frac{22.80}{4} = 5.70$$

TABLE Computation of $\Sigma(x_i - \bar{x})^2$

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
5	1.2	1.44
7	3.2	10.24
1	−2.8	7.84
2	−1.8	3.24
4	.2	.04
19	0.0	22.80

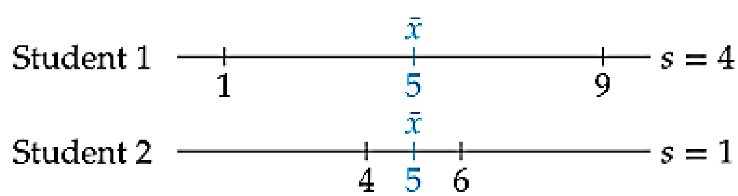
The **variance** (s^2) is measured in terms of the square of the original units of measurement. If the original measurements are in inches, the variance is expressed in square inches. Taking the square root of the variance, we obtain the ***standard deviation***, which returns the measure of variability to the original units of measurement.

Definition : The ***standard deviation(S)*** of a set of measurements is equal to the positive square root of the variance $\boxed{s = +\sqrt{s^2}}$

Both variance and standard deviation measure the spread of data around the mean. When the variance or the standard deviation are small, the sample data are concentrated around the mean, and the mean is a good representative measure. In contrast, when variance or the standard deviation are high, the sample data are far from the mean, and the mean does not represent so well.

Standard deviation small	$\Rightarrow$	Mean is representative
Standard deviation big	$\Rightarrow$	Mean is unrepresentative

Example. The following samples contains the grades of 2 students in 2 subjects



Which mean is more representative?

NOTATION

n: number of measurements in the sample

s^2: sample variance

$s = \sqrt{s^2}$: sample standard deviation

N: number of measurements in the population

σ^2: population variance

$\sigma = \sqrt{\sigma^2}$: population standard deviation

For the set of $n = 5$ sample measurements in Table , the sample variance is $s^2 = 5.70$, so the sample standard deviation is $s = \sqrt{s^2} = \sqrt{5.70} = 2.39$. The more variable the data set is, the larger the value of s.

THE COMPUTING FORMULA FOR CALCULATING s^2

$$s^2 = \frac{\Sigma(x_i - \overline{x})^2}{n-1} \iff s^2 = \frac{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}}{n-1}$$

Σx_i^2 = Sum of the squares of the individual measurements
$(\Sigma x_i)^2$ = Square of the sum of the individual measurements

EXAMPLE Calculate the variance and standard deviation for the five measurements in Table , which are 5, 7, 1, 2, 4. Use the computing formula for s^2 and compare your results with those obtained using the original definition of s^2.

Table for Simplified Calculation of s^2 and s

x_i	x_i^2
5	25
7	49
1	1
2	4
4	16
19	95

Solution The entries in Table are the individual measurements, x_i, and their squares, x_i^2, together with their sums. Using the computing formula for s^2, you have

$$s^2 = \frac{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}}{n-1} = \frac{95 - \frac{(19)^2}{5}}{4} = \frac{22.80}{4} = 5.70$$

and $s = \sqrt{s^2} = \sqrt{5.70} = 2.39$, as before.

Now that you have learned how to compute the variance and standard deviation, remember these points:

- The value of s is always greater than or equal to zero.
- The larger the value of s^2 or s, the greater the variability of the data set.
- If s^2 or s is equal to zero, all the measurements must have the same value.
- In order to measure the variability in the same units as the original observations, we compute the standard deviation $s = \sqrt{s^2}$.

Example - Non-grouped data. Using the data of the sample with the number of children of families, with mean $\bar{x} = 1.76$ children, and adding a new column to the frequency table with the squared values,

$$\boxed{s^2 = \frac{\sum x_i^2 n_i}{n} - \bar{x}^2}$$

$$\boxed{\bar{x} = \frac{\sum x_i n_i}{n} = \frac{44}{25} = 1.76 \text{ children}}$$

x_i	n_i	$x_i^2 n_i$
0	2	0
1	6	6
2	14	56
3	2	18
4	1	16
$\sum$	25	96

$$s^2 = \frac{\sum x_i^2 n_i}{n} - \bar{x}^2 = \frac{96}{25} - 1.76^2 = 0.7424 \text{ children}^2.$$

and the standard deviation is $s = \sqrt{0.7424} = 0.8616$ children.

Compared to the range, that is 4 children, the standard deviation is not very large, so we can conclude that the dispersion of the distribution is small and consequently the mean, $\bar{x} = 1.76$ children, represents quite well the number of children of families of the sample.

Example - Grouped data. Using the data of the sample with the heights of students and grouping heights in classes, we got a mean $\bar{x} = 174.67$ cm. The calculation of variance is the same than for non-grouped data but using the class marks.

$$\boxed{s^2 = \frac{\sum x_i^2 n_i}{n} - \bar{x}^2}$$

$$\boxed{\bar{x} = \frac{\sum x_i n_i}{n} = \frac{5240}{30} = 174.67 \text{ cm}}$$

X	x_i	n_i	$x_i^2 n_i$
(150, 160]	155	2	48050
(160, 170]	165	8	217800
(170, 180]	175	11	336875
(180, 190]	185	7	239575
(190, 200]	195	2	76050
$\sum$		30	918350

$$s^2 = \frac{\sum x_i^2 n_i}{n} - \bar{x}^2 = \frac{918350}{30} - 174.67^2 = 102.06 \text{ cm}^2,$$

and the standard deviation is $s = \sqrt{102.06} = 10.1$ cm.

This value is quite small compared to the range of the variable, that goes from 150 to 200 cm, therefore the distribution of heights has little dispersion and the mean is very representative.

PERMUTATIONS

Permutations:
The total number of ways of **arranging** n objects, taking r at a time is given by

$$\frac{n!}{(n-r)!}$$

Notation: We use the notation ^nP_r (read as "n–p–r') to denote $\frac{n!}{(n-r)!}$.

That is, $^nP_r = \frac{n!}{(n-r)!}$.

example
the total number of arrangements of 8 books on a bookshelf if only 5 are used

solution

$$^8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = 6720 .$$

example In how many ways can 5 boys be arranged in a row
(a) using three boys at a time?
(b) using 5 boys at a time?

We have 5 boys to be arranged in a row with certain constraints.

(a) The constraint is that we can only use 3 boys at a time. In other words, we want the number of arrangements (permutations) of 5 objects taken 3 at a time.
From rule 4: $n = 5, r = 3,$

$$\text{Therefore, number of arrangements} = {}^5P_3 = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$$

(b) This time we want the number of arrangements of 5 boys taking all 5 at a time.
From rule 4: $n = 5, r = 5,$

$$\text{Therefore, number of arrangements} = {}^5P_5 = \frac{5!}{(5-5)!} = \frac{120}{0!} = 120$$

permutations with repetitions:

The number of permutations of n objects of which n_1 are identical, n_2 are identical, . . . , n_k are identical is given by $\frac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$.

Example : How many different arrangements of the letters of the word HIPPOPOTAMUS are there?

Solution : $\frac{12!}{3! \times 2!} = 39916800$ arrangements.

COMBINATIONS

On the otherhand, combinations represent a counting process where the order has no importance. For example, the number of combinations of the letters A, B, C and D, if only two are taken at a time, can be enumerated as:

AB, AC, AD, BC, BD, CD,

That is, the combination of the letters A and B, whether written as AB or BA, is considered as being the same.

Instead of **combination** the term **selection** is often used.

Combinations:

The total number of ways of **selecting** n objects, taking r at a time is given by

$$\boxed{\frac{n!}{(n-r)!\,r!}}$$

Notation: We use the notation $\binom{n}{r}$ (read as "n–c–r') to denote $\frac{n!}{(n-r)!\,r!}$.

That is, $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$. Note:Sometimes ${}^{n}C_{r}$ is used instead of $\binom{n}{r}$.

Example : in how many ways can 5 books be selected from 8 different books?

Solution In this instance, we are talking about selections and therefore, we are looking at combinations. Therefore we have. the selection of 8 books taking 5 at a time is equal to

$$\binom{8}{5} = \frac{8!}{(8-5)!\,5!} = \frac{8!}{3!\,5!} = 56$$

Example : A sports committee at the local hospital consists of 5 members. A new committee is to be elected, of which 3 members must be women and 2 members must be men. How many different committees can be formed if there were originally 5 women and 4 men to select from?

First we look at the number of ways we can select the women members (using Rule 6):

We have to select 3 from a possible 5, therefore, this can be done in ${}^{5}C_{3} = 10$ ways.

Similarly, the men can be selected in ${}^{4}C_{2} = 6$ ways.

Using Rule 2, we have that the total number of possible committees = ${}^{5}C_{3} \times {}^{4}C_{2} = 60$.

CHAPTER EIGHT

Probability

Probability: The likelihood that something will happen.

How can data obtained?

Data are obtained either by observing uncontrolled events in nature or by observing events in controlled situations. We use the term experiment to describe either method of data collection.

Some Important Terms:

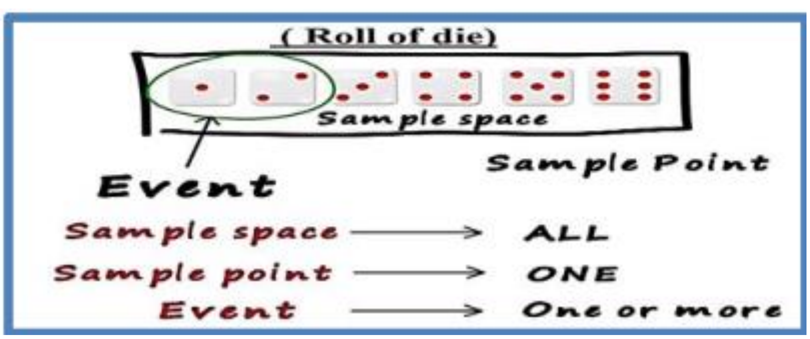
Experiment: is the process by which an observation (or measurement) is obtained.

Outcome: A possible result of one trial of a probability experiment.

Sample Point: is the one of each outcome.

Event: is the outcome that observed on a single repetition of the experiment.

Sample space: is a collection of events. or, the set of all events.



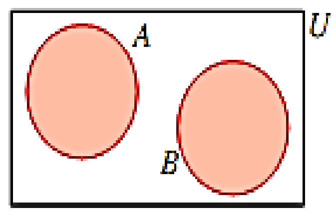
If an experiment has equally likely outcomes and of these the event A is defined, then the **theoretical probability of event A** occuring is given by

$$\mathrm{P}(A) = \frac{n(A)}{n(U)} = \frac{\text{Number of outcomes in which A occurs}}{\text{Total number of outcomes in the sample space}}$$

Where $n(U)$ is the total number of possible outcomes in the sample space, U, (i.e., $n(U) = N$).
As a consequence of this definition we have what are known as the **axioms of probability**:

1. $0 \le \mathrm{P}(A) \le 1$
2. $\mathrm{P}(\varnothing) = 0$ and $\mathrm{P}(\varepsilon) = 1$
   That is, if $A = \varnothing$, then the event A can never occur.
   $A = U$ implies that the event A is a certainty.
3. If A and B are both subsets of U and are mutually exclusive, then
   $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

Note:
Two events A and B are said to be **mutually exclusive** (or disjoint) if they have no elements in common, i.e., if $A \cap B = \varnothing$.

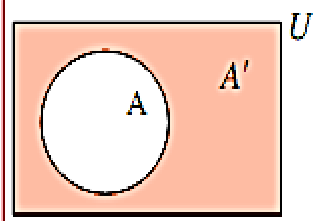
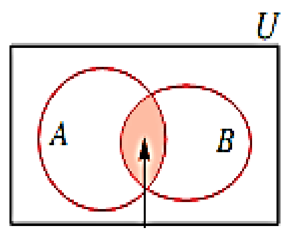
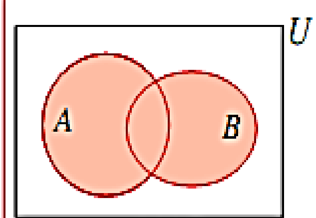
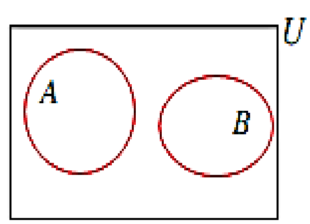


EXAMPLE A fair die is thrown. List the sample space of the experiment and hence find the probability of observing: (a) a multiple of 3
(b) an odd number.
Are these events mutually exclusive?

Solution
(a) The sample space is $U = \{1, 2, 3, 4, 5, 6\}$.
Let A be the event 'obtaining a multiple of 3'.
We then have that $A = \{3, 6\}$. Therefore, $\mathrm{P}(A) = \frac{n(A)}{n(U)} = \frac{2}{6} = \frac{1}{3}$.

(b) Let B be the event 'obtaining an odd number'.
Here $B = \{1, 3, 5\}$ and so $\mathrm{P}(B) = \frac{n(B)}{n(U)} = \frac{3}{6} = \frac{1}{2}$.

In this case, $A = \{3, 6\}$ and $B = \{1, 3, 5\}$, so that $A \cap B = \{3\}$. Therefore, as $A \cap B \ne \varnothing$ A and B are not mutually exclusive.

Event	Set language	Venn diagram	Probability result
The **complement** of A is denoted by A'.	A' is the complement to the set A, i.e., the set of elements that do not belong to the set A.		$P(A') = 1 - P(A)$ $P(A')$ is the probability that event A does not occur.
The **intersection** of A **and** B : $A \cap B$	$A \cap B$ is the intersection of the sets A and B, i.e., the set of elements that belong to **both** the set A **and** the set B.	 $A \cap B$	$P(A \cap B)$ is the probability that both A and B occur.
The **union** of events A and B : $A \cup B$	$A \cup B$ is the union of the sets A and B, i.e., the set of elements that belong to A **or** B or both A **and** B.	 $A \cup B$	$P(A \cup B)$ is the probability that either event A or event B (or both) occur. From this we have what is known as the '**Addition rule**' for probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
If $A \cap B = \varnothing$ the events A and B are said to be **disjoint**. That is, they have no elements in common.	If $A \cap B = \varnothing$ the sets A and B are **mutually exclusive**.	 $A \cap B = \varnothing$	If A and B are mutually exclusive events then event A and event B cannot occur simultaneously, i.e., $n(A \cap B) = 0$ $\Rightarrow P(A \cap B) = 0$ Therefore: $P(A \cup B) = P(A) + P(B)$

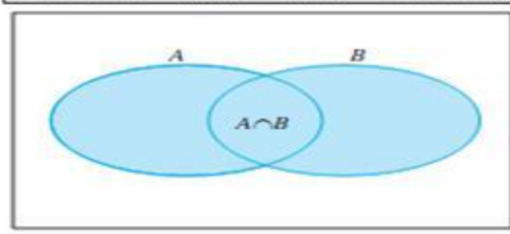
Calculating Probabilities for Unions

When we can write the event of interest in the form of a union, a complement, or an Calculating Probabilities for Unions intersection, there are special probability rules that can simplify our calculations. The first rule deals with unions of events.

General addition rule

Given two events, A and B, the probability of their union, $A \cup B$, is equal to

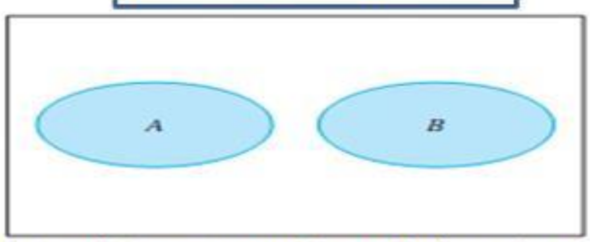
$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



Special case of additon rule (mutually exclusive)

When two events A and B are **mutually exclusive** or **disjoint,** it means that when A occurs, B cannot, and vice versa. This means that the probability that they both occur, $P(A \cap B)$, must be zero. Figure is a Venn diagram representation of two such events with no simple events in common.

$$\boxed{P(A \cup B) = P(A) + P(B)}$$



When two events A and B are **mutually exclusive,** then $P(A \cap B) = 0$ and the Addition Rule simplifies to

Example :

1	2	3	4	5	6	7	8	9	10

A ⟶ Even B ⟶ Greater than 5

P (A ∪ B) = P (A happening) + P (B happening) − P (A & B happening together)

$= \frac{5}{10} + \frac{5}{10} - \frac{3}{10}$ A --> 2 , 4 , 6 , 8 , 10

$= 0.7$ B --> 6 , 7 , 8 , 9 , 10

2 , 4 , 6 , 7 , 8 , 9 , 10

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1	2	3	4	5	6

A ⟶ Odd

B ⟶ Even

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A & B ---> Mutually Exclusive

$P(A \cup B) = P(A) + P(B)$

$= \frac{3}{6} + \frac{3}{6} = 1$

$P(A \cup B) = P(A) + P(B)$ Mutually Exclusive

EXAMPLE

A card is randomly selected from an ordinary pack of 52 playing cards. Find the probability that it is either a 'black card' or a 'King'.

Solution

Let B be the event 'A black card is selected.' and K the event ' A King is selected'.
We first note that event B has as its elements the Jack of spades(J♠), the Jack of clubs (J♣), the Queen of spades(Q♠), the Queen of clubs(Q♣) and so on.
This means that
B ={K♠,K♣,Q♠,Q♣,J♠,J♣,10♠,10♣,9♠,9♣,8♠,8♣,7♠,7♣,6♠,6♣,5♠,5♣,4♠,4♣,3♠, 3♣,2♠,2♣,A♠,A♣} and
K = {K♠, K♦, K♥, K♣}, so that $B \cap K$ = {K♠, K♣}.
Using the addition rule, $P(B \cup K) = P(B) + P(K) - P(B \cap K)$

we have $$P(B \cup K) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{7}{13}.$$

Note the importance of subtracting $\frac{2}{52}$ as this represents the fact that we have included the event {K♠, K♣} twice when finding B and K.

EXAMPLE

A bag has 20 coins numbered from 1 to 20. A coin is drawn at random and its number is noted. What is the probability that the coin has a number that is divisible by 3 or by 5?

Solution

Let T denote the event "The number is divisible by 3" and S, the event "The number is divisible by 5".
Using the addition rule we have $P(T \cup S) = P(T) + P(S) - P(T \cap S)$
Now, $T = \{3, 6, 9, 12, 15, 18\}$ and $S = \{5, 10, 15, 20\}$ so that $T \cap S = \{15\}$.
Therefore, we have $P(T) = \frac{6}{20}$ and $P(S) = \frac{4}{20}$ and $P(T \cap S) = \frac{1}{20}$.

This means that $P(T \cup S) = \frac{6}{20} + \frac{4}{20} - \frac{1}{20} = \frac{9}{20}$.

EXAMPLE

If $p(A) = 0.6$, $p(B) = 0.3$ and $p(A \cap B) = 0.2$, find
(a) $p(A \cup B)$ (b) $p(B')$

Solution

(a) Using the addition formula we have, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
$$\Rightarrow p(A \cup B) = 0.6 + 0.3 - 0.2 = 0.7$$

(b) Using the complementary formula, we have $p(B') = 1 - p(B) = 1 - 0.3 = 0.7$.

Multiplication Rule

Conditional Probability	$P(A \cap B) = P(A\|B) \times P(B)$
Independence	$P(A \cap B) = P(A) \times P(B)$

Conditional probability

If the events are *not independent*, one event affects the probability for the other event. In this case *conditional probability* must be used. The conditional probability of B given that A occurs, or *on condition* that A occurs, is written **Pr [B | A]**. This is read as the probability of B given A, or the probability of B on condition that A occurs.

Independence : Two events, A and B, are said to be independent if and only if the probability of event B is not influenced or changed by the occurrence of event A, or vice versa

***Example '*:** A bag contains green balls and yellow balls. You are going to choose two balls without replacement. If the probability of selecting a green ball and a yellow ball is $\frac{14}{39}$, what is the probability of selecting a yellow ball on the second draw, if you know that the probability of selecting a green ball on the first draw is $\frac{4}{9}$.

Solution:

Step 1: List what you know

$$P(Green) = \frac{4}{9}$$

$$P(Green\ AND\ Yellow) = \frac{14}{39}$$

$$\boxed{P(\textit{first choice and second choice}) = P(\textit{second} | \textit{first}) \times P(\textit{first choice})}$$

Step 2: Calculate the probability of selecting a yellow ball on the second draw with a green ball on the first draw

$$P(Y|G) = \frac{P(Green\ AND\ Yellow)}{P(Green)}$$

$$P(Y|G) = \frac{14/39}{4/9}$$

$$P(Y|G) = \frac{14}{39} \times \frac{9}{4}$$

$$P(Y|G) = \frac{126}{156}$$

$$P(Y|G) = \frac{21}{26}$$

Step 3: Write your conclusion: Therefore the probability of selecting a yellow ball on the second draw after drawing a green ball on the first draw is $\frac{21}{26}$.

Example : Two cards are chosen from a deck of cards. What is the probability that they both will be face cards? (**draw** without replacement.)

Solution

Let A = 1st Face card chosen

Let B = 2nd Face card chosen

Therefore, the total number of face cards in the deck $= \underbrace{4}_{\text{4 suits}} \times \underbrace{3}_{\text{3 face cards per suit}} = 12$

$P(A) = \frac{12}{52}$

$P(B) = \frac{11}{51}$

$P(A\ AND\ B) = \frac{12}{52} \times \frac{11}{51}$ or $P(A \cap B) = \frac{12}{52} \times \frac{11}{51} = 0.049$

CHAPTER Nine

Partial Differential Equations

9.1 Introduction

Partial differential equations arise in connection with various Engineering, physical and geometrical problems, when the functions involved depend on two or more independent variables, usually on time (t) and on one or several space variables.

It is fair to say that only the simplest physical systems can be modeled by ordinary differential equations, whereas most problems in electromagnetic theory, fluid mechanics and other areas of physics lead to partial differential equations.

Order of P.D.E: is the highest derivative of the equation.

Some important engineering partial differential equations:

There are many types of partial differential equations. Some typically found in engineering and science include:

(a) The wave equation, where the equation of motion is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where $c^2 = \frac{T}{\rho}$, with T being the tension in a string and ρ being the mass/unit length of the string.

(b) The heat conduction equation, is of the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

where $c^2 = \frac{h}{\sigma\rho}$, with h being the thermal conductivity of the material, σ the specific heat of the material, and ρ the mass/unit length of material.

(c) Laplace's equation, used extensively with electrostatic fields is of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(d) The transmission equation, where the potential u in a transmission cable is of the form:

$\frac{\partial^2 u}{\partial x^2} = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} + Cu$ where A, B and C are constants.

9.2 Solving Partial Differential Equations

We will study the following methods to solve P.D.E:

1) **Direct Partial Integration**
2) **Separation of variables**

9.2.1 Direct Partial Integration

As explained in the previous class, the integration is the reverse process of differentiation. We can use the integration to find the solution.

Ex1: Integrate the partial differential equation given below with respect to t,

$$\partial u/\partial t = 5\cos x \sin t$$

Sol: The (5 cosx) term is considered as a constant

$$u = \int 5\cos x \sin t \, dt = 5\cos x \int \sin t \, dt$$

$$u = 5\cos x \; (-\cos t) + f(x)$$

....

Ex2: Integrate the partial differential equation given below

$$\frac{\partial^2 u}{\partial x \partial y} = 6x^2 \cos 2y$$

Sol: Integrate with respect to y:

$$\frac{\partial u}{\partial x} = \int 6x^2 \cos 2y \, dy = (6x^2) \int \cos 2y \, dy$$

$$= (6x^2)\left(\frac{1}{2}\sin 2y\right) + f(x)$$

$$= 3x^2 \sin 2y + f(x)$$

Then we integrate with respect to x:

$$u = \int [3x^2 \sin 2y + f(x)] \, dx$$

$$= x^3 \sin 2y + \int f(x) \, dx + g(y)$$

F(x) and g(y) can be calculated if extra information are known like boundary conditions or initial conditions.

Ex3: solve the following P.D.E $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$, if the boundary conditions are:

$x = 0 \rightarrow \frac{\partial u}{\partial x} = \sin 2y \quad , u = \cos y$

Sol:

$$\frac{\partial u}{\partial x} = \int 6x^2(2y-1)\mathrm{d}x = (2y-1)\int 6x^2 \mathrm{d}x$$

$$= (2y-1)\frac{6x^3}{3} + f(y)$$

$$= 2x^3(2y-1) + f(y)$$

where $f(y)$ is an arbitrary function. From the boundary conditions, when $x=0$,

$$\frac{\partial u}{\partial x} = \sin 2y.$$

Hence, $\sin 2y = 2(0)^3(2y-1) + f(y)$, from which, $f(y) = \sin 2y$

Now $\frac{\partial u}{\partial x} = 2x^3(2y-1) + \sin 2y$

Integrating partially with respect to x gives:

$$u = \int [2x^3(2y-1) + \sin 2y]\mathrm{d}x$$

$$= \frac{2x^4}{4}(2y-1) + x(\sin 2y) + g(y)$$

From the boundary conditions, when $x=0$, $u=\cos y$, hence

$$\cos y = \frac{(0)^4}{2}(2y-1) + (0)\sin 2y + g(y)$$

from which, $g(y)=\cos y$

Hence, the solution of $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ for the given boundary conditions is:

$$\boldsymbol{u = \frac{x^4}{2}(2y-1) + x\sin 2y + \cos y}$$