Advanced Control Systems

LECTURE 1

STATE VARIABLE MODELS

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State Variable Models

Advantages of state-space description

- The system is modeled as a set of first-order differential equations.
- The power of modern control has its roots in the fact that the state-space model can represent a MIMO (multi-input multi-output) system due to the use of vectors and matrices.

Why use state-space approach?

- State variable form is a convenient way to work with complex dynamics; it uses matrix format that is easy to use on computers.
- Transfer functions in classical control theory only deal with input/output behavior, while state-space form provides easy access to the internal features and response of the system.
- State-space approach is great for MIMO (multi-input multi-output) system, which are very hard to work with using transfer functions.
- State variables can be also used for feedback.

State Variable Models from differential equation

We consider physical systems described by nth-order ordinary differential equation. Utilizing a set of variables, known as state variables, we can obtain a set of first-order differential equations. We group these first-order equations using a compact matrix notation in a model known as the state variable model.

The time-domain state variable model lends itself readily to computer solution and analysis. The Laplace transform is utilized to transform the differential equations representing the system to an algebraic equation expressed in terms of the complex variable **s**.

Utilizing this algebraic equation, we are able to obtain a transfer function representation of the input-output relation ship. With the ready availability of digital computers, it is convenient to consider the time-domain formulation of the equations representing control system.

The time domain techniques can be utilized for nonlinear, time varying, and multivariable systems

A time-varying control system is a system for which one or more of the parameters of the system may vary as a function of time.

For example, the mass of a missile varies as a function of time as the fuel is expended during flight. (multivariable system is a system with several input and output).

The time-domain analysis and design of control systems utilizes the concept of the state of a system.

The variables used to write these n first-order equations are called state variables. The collection of state variables at any given time is known as the state of the system, and the set of all values that can be taken on by the state is known as the state space.

Example.

The system model used to illustrate state variables, is given in Figure 1.

The differential equation describing this system was already determined in 1.

$$M\frac{d^2y(t)}{dt^2} + B\frac{dy}{dt} + Ky(t) = f(t) \qquad 1$$

and the transfer function given by

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} \qquad 2$$

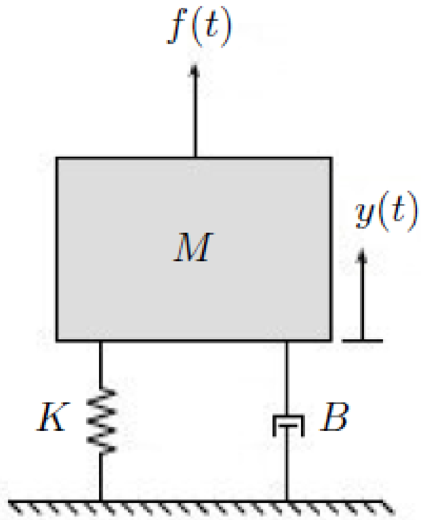

Figure 1: Simple mechanical system.

- This equation gives a description of the position y(t) as a function of the force f(t).
- Suppose that we also want information about the velocity.
- Using the state variable approach, we dene the two state variables x1(t) and x2(t) as

$$x_1(t) = y(t)$$

and

$$x_2(t) = \frac{dy(t)}{dt} = \frac{dx_1(t)}{dt} = \dot{x}_1(t)$$

Thus x1(t) is the position of the mass and x2(t) is its velocity.

$$\frac{d^2y(t)}{dt^2} = \frac{dx_2(t)}{dt} = \dot{x}_2(t) = -\left(\frac{B}{M}\right)x_2(t) - \left(\frac{K}{M}\right)x_1(t) + \left(\frac{1}{M}\right)f(t)$$

The state variable model is usually written in a specific format which is given by rearranging the equations as

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\left(\frac{K}{M}\right)x_1(t) - \left(\frac{B}{M}\right)x_2(t) + \left(\frac{1}{M}\right)f(t)$$

$$y(t) = x_1(t)$$

Usually state equations are written in a vector-matrix format as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} f(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The most general state space representation of a LTI system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (3) \text{ state equation}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad (4) \text{ output equation}$$

where

x(t) = state vector = (n * 1) vector of the states of an nth-order system

A = (n * n) system matrix

B = (n* r) input matrix

u(t) = input vector = (r * 1) vector composed of the system input functions

y(t) = output vector = (p* 1) vector composed of the defined outputs

C = (p* n) output matrix

D = (p* r) matrix to represent direct coupling between input and output

Equation (3) is called the state equation, and Equation (4) is called the output equation, together they are referred to as the state-variable equations.

Equations (3) and (4) are shown in block diagram form in Figure 2.

The heavier lines indicate that the signals are vectors, and the integrator symbol really indicates n scalar integrators.

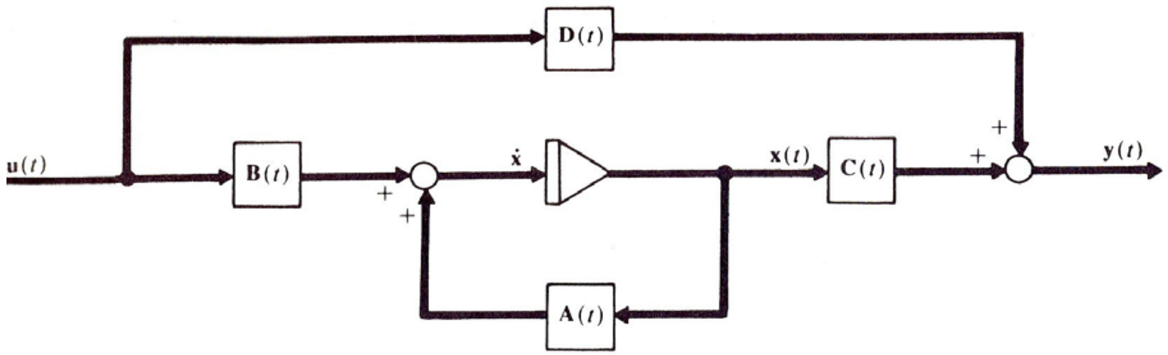

Figure 2 State space representation of CT linear system.

Simulation Diagrams

In the previous section we presented examples of finding the state model of a system directly from the system differential equations.

However, sometimes only a transfer function may be available to describe a system. We obtain state models directly from a transfer function by means of a simulation diagram.

A simulation diagram is a certain type of a block diagram or a flow graph that is constructed to have a given transfer function or to model a set of differential equations

Simulation diagrams are very useful in constructing either digital or analog computer simulations of a system

The basic element of the simulation diagram is the integrator which can be easily constructed using electronic devices. Figure 3 shows the block diagram of an integrating device

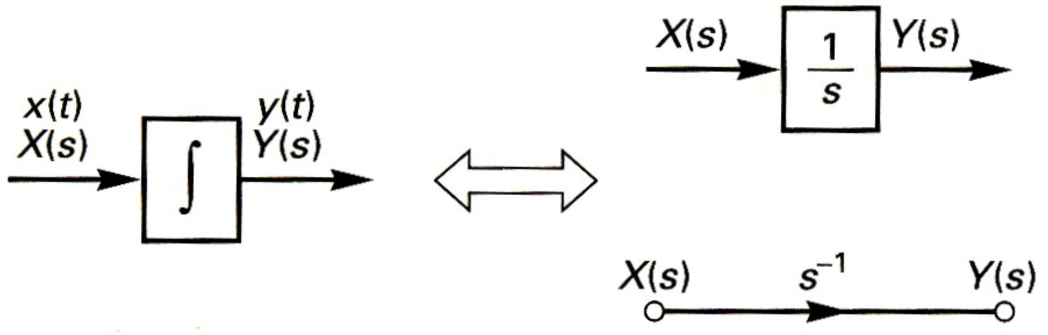

Figure 3 : Integrating device.

The differential equation describing the system in figure 1 is.

$$\ddot{y}(t) = -\frac{B}{M}\dot{y}(t) - \frac{K}{M}y(t) + \frac{1}{M}f(t)$$

A simulation diagrams for the system as shown in figure 4

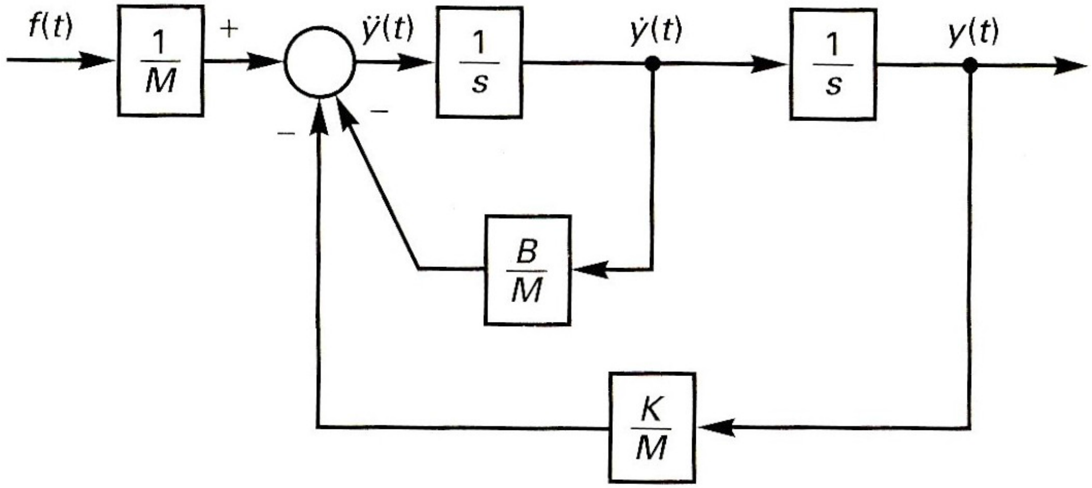

Figure 4. Simulation diagrams.

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LECTURE 2

Control Canonical Form

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State-Variable Models from Transfer Function

A simulation diagram constructed from the system differential equations will usually be unique. However, if the transfer function is used to construct the simulation diagram, the simulation diagram can be of many forms, that is, the simulation diagram is not unique.

Next, we consider two common and useful forms of the simulation diagram, namely, control canonical form and observer canonical form. The two different simulation diagrams are derived from the general transfer functions of the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} = \frac{b_0 + \cdots + b_{m-1}s^{m-1} + b_m s^m}{a_0 + \cdots + a_{n-1}s^{n-1} + s^n} \qquad \text{------1}$$

Where

$m < n$ and $a_n = 1$

Control Canonical Form

Also called the phase variable model, as an example consider m = 2 and n = 3 in (1), therefore,

$$Y(s) = \frac{b_0 + b_1 s + b_2 s^2}{a_0 + a_1 s + a_2 s^2 + s^3} U(s)$$

Divide numerator and denominator by s^n, in this example that is s^3, hence,

$$Y(s) = \frac{b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}}{a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1} U(s)$$

Set

$$W(s) = \frac{U(s)}{a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1} + 1}$$

This gives

$$W(s) = U(s) - [a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1}] W(s)$$

and

$$Y(s) = [b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}] W(s)$$

A simulation diagram, called the control canonical form shown in Figure 1 can be drawn.

Once a simulation diagram of a transfer function is constructed, a state model of the system is easily obtained. The procedure is as follows:

1. Assign a state variable to the output of each integrator starting from right to left. (We could assign state variables from left to right to obtain what we call input feedforward canonical form).
2. Write an equation for the input of each integrator and an equation for each system output.

$$W(s) = U(s) - [a_0 s^{-3} + a_1 s^{-2} + a_2 s^{-1}]W(s)$$

$$Y(s) = [b_0 s^{-3} + b_1 s^{-2} + b_2 s^{-1}]W(s)$$

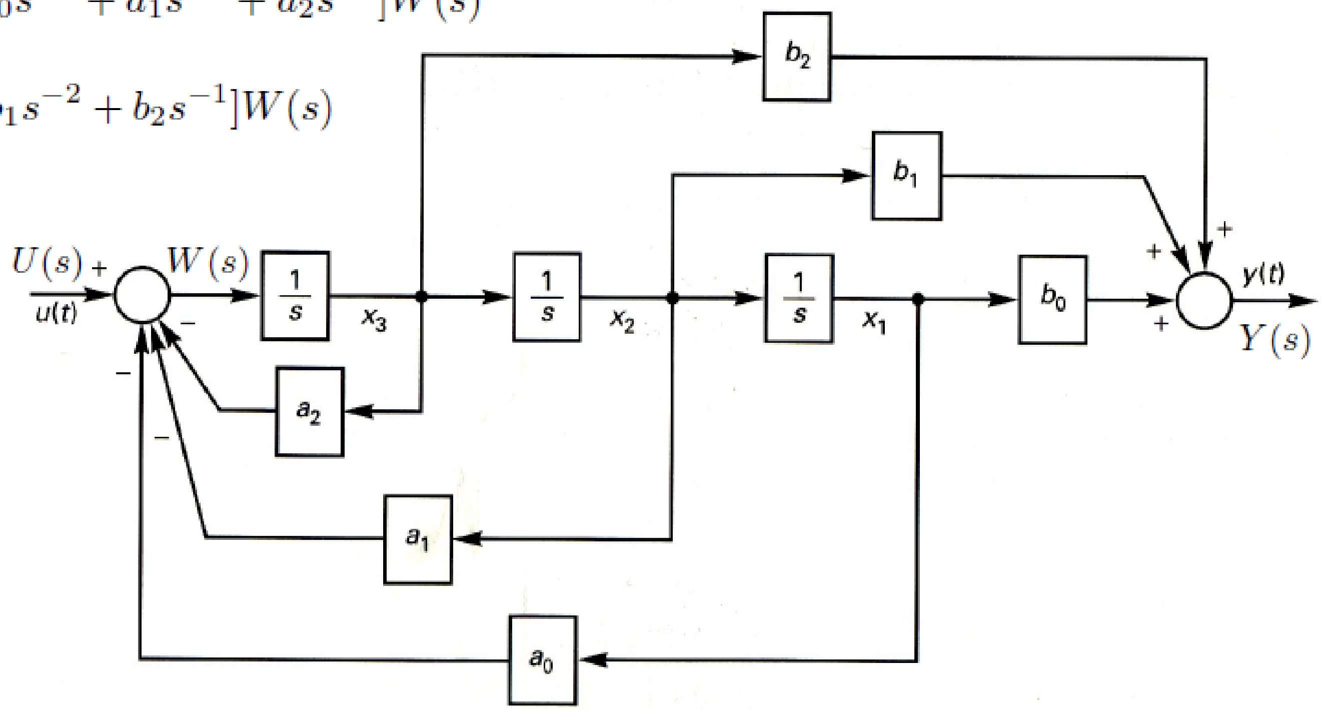


Figure 1: Control canonical form.

Following the procedure above the state variable satisfy:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + u(t)$$

while the output is

$$y(t) = b_0 x_1 + b_1 x_2 + b_2 x_3$$

In matrix form this yields the following state-variable model

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \mathbf{x}$$

Note the direct connection with coefficients of the transfer function.

The bottom row of the A matrix contains the negatives of the coefficients of the characteristic equation (i.e., the denominator of G(s)), starting on the left with -a0 and ending on the right with -a2.

Above the bottom row is a column of zeros on the left and a22 identity matrix on the right.

The B matrix is similarly very simple all the elements are zero except for the bottom element, which is the gain from the original system.

The C matrix contains the positive of the coefficients of the numerator of the transfer function, starting on the left with b0 and ending on the the right with b2.

It is important to note that state matrices are never unique, and each G(s) has infinite number of state models.

Observer Canonical Form

In addition to control canonical form, we can draw a simulation diagram called the observer canonical form. To show how observer canonical form can be derived, consider the transfer function in (1) with m = 2 and n = 3. Equation (1) is written in the form

$$Y(s)[a_0 + a_1 s + a_2 s^2 + s^3] = [b_0 + b_1 s + b_2 s^2]U(s)$$

Divide both sides by s^3 to obtain

$$Y(s)[1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}] = [b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}]U(s)$$

leading to

$$Y(s) = -[a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}]Y(s) + [b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}]U(s)$$

This relationship can be implemented by using a simulation diagram as shown in Figure 2.

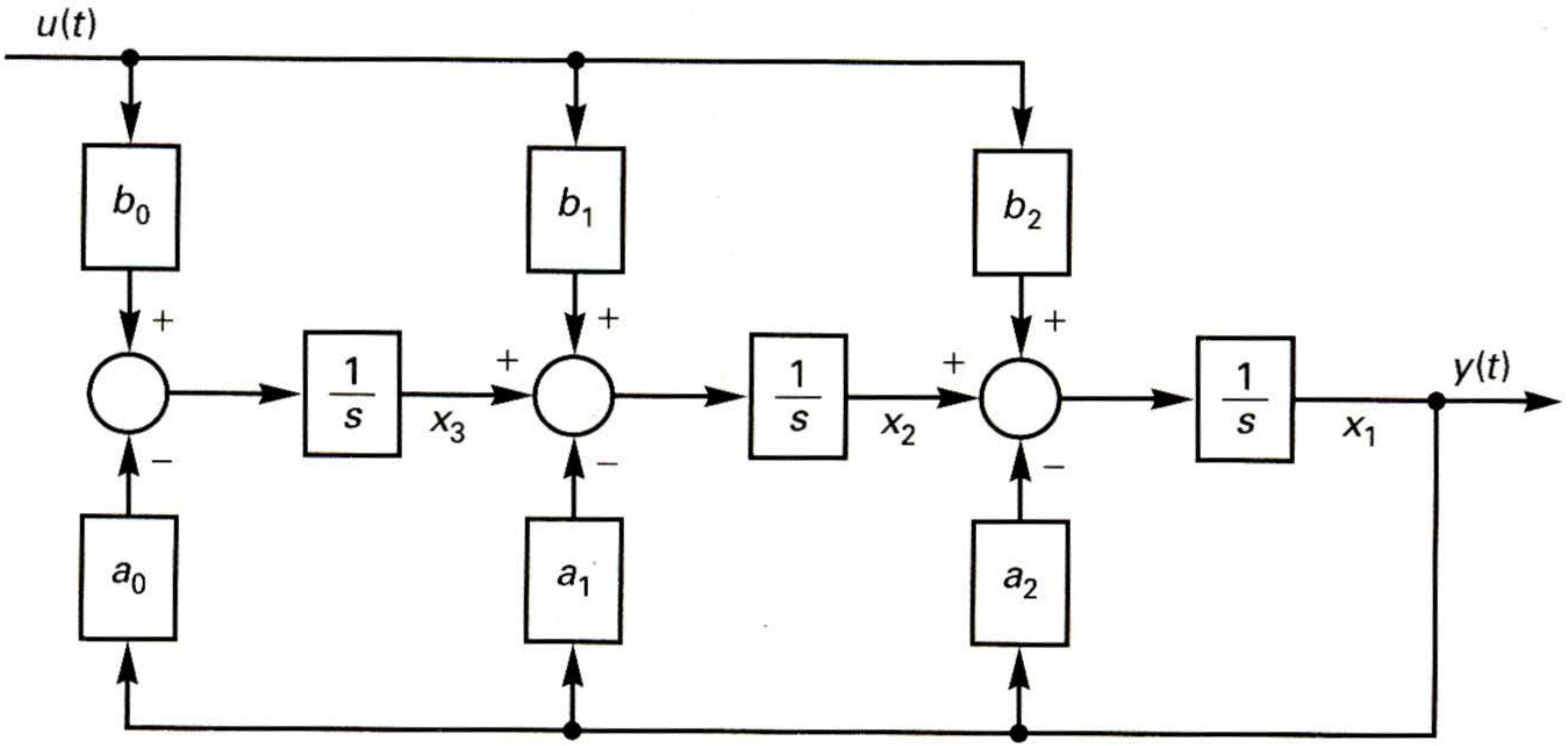


Figure 2: Observer canonical form.

Note

The state equations are written as

$$\dot{\mathrm{x}} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \mathrm{x} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathrm{x}$$

$$x_1 = y(t)$$

$$\dot{x_1} = -a_2 x_1 + x_2 + b_2 u$$

$$\dot{x_2} = -a_1 x_1 + x_3 + b_1 u$$

$$\dot{x_3} = -a_0 x_1 + b_0 u$$

Example 1: Find the state and output equations for $G(s) = \frac{5s^2 + 7s + 4}{s^3 + 3s^2 + 6s + 2}$

in control canonical form.

■ **Solution** State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

The output equation is

$$y = \begin{bmatrix} 4 & 7 & 5 \end{bmatrix} \mathbf{x} \qquad \blacksquare$$

Example 2: Find the state and output equations for $G(s) = \frac{1}{2s^2 - s + 3}$

■ **Solution** State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3/2 & 1/2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The output equation is

$$y = \begin{bmatrix} 1/2 & 0 \end{bmatrix} x \ \blacksquare$$

Example 3: Write a state variable expression for the following differential equation $2\ddot{y} - \dot{y} + 3y = \dot{u} - 2u$

A useful formulation for state variables here is to obtain a transfer function and then using a simulation diagram to obtain the state model. The transfer function of the system is

$$2s^2Y(s) - sY(s) + 3Y(s) = sU(s) - 2U(s)$$

$$Y(s) = \frac{s-2}{2s^2 - s + 3}U(s)$$

The state model in control canonical form is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3/2 & 1/2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 1/2 \end{bmatrix} \mathbf{x}$$

To show that the answer is true let us construct a simulation diagram.
First we have to express the transfer function in standard form:

$$Y(s) = \frac{\frac{1}{2}s^{-1} - s^{-2}}{1 - \frac{1}{2}s^{-1} + \frac{3}{2}s^{-2}} U(s)$$

introduce an auxiliary signal W(s):

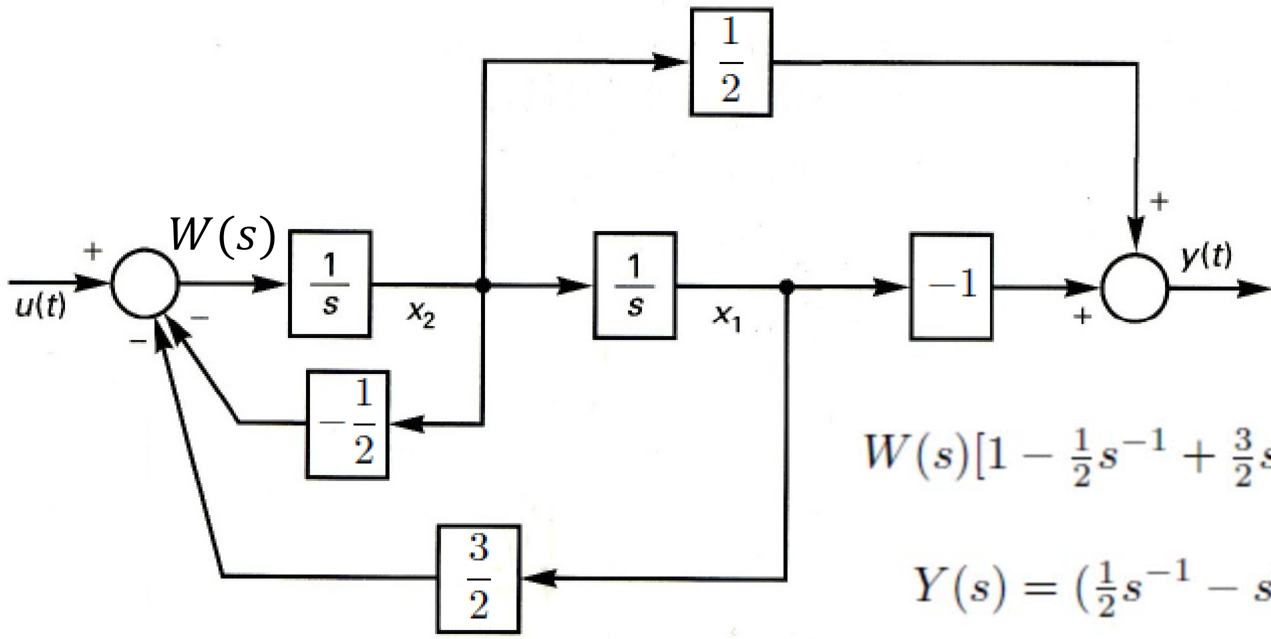
Therefore,

$$Y(s) = (\tfrac{1}{2}s^{-1} - s^{-2}) \underbrace{\frac{1}{1 - \frac{1}{2}s^{-1} + \frac{3}{2}s^{-2}} U(s)}_{=:W(s)}$$

and

$$W(s)[1 - \tfrac{1}{2}s^{-1} + \tfrac{3}{2}s^{-2}] = U(s)$$

$$Y(s) = (\tfrac{1}{2}s^{-1} - s^{-2})W(s)$$



$$W(s)[1 - \tfrac{1}{2}s^{-1} + \tfrac{3}{2}s^{-2}] = U(s)$$

$$Y(s) = (\tfrac{1}{2}s^{-1} - s^{-2})W(s)$$

Figure 3: Simulation diagram for Example 3 in control canonical form.

After we assign a state variable to the output of each integrator from right to left we get,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{3}{2}x_1 + \frac{1}{2}x_2 + u$$

$$y = -x_1 + \frac{1}{2}x_2 \qquad \blacksquare$$

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LECTURE 3

Transfer Functions From State Variable Models

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Transfer Functions from State-Variable Models

Consider a state-variable model initially at rest

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = 0 \qquad \text{------ (1)}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad \text{------ (2)}$$

Taking Laplace transforms yields

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \qquad \text{------ (3)}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \qquad \text{------ (4)}$$

The term sX(s) in 3 must be written as sIX(s), where I is the identity matrix. This additional step is necessary, since the subtraction of the matrix A from the scalar s is not defined. Then,

$$s\mathbf{X}(s) - \mathbf{A}\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \qquad \text{------ (5)}$$

Substituting 5 in 2 the output equation, we get

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$ - - - - - - (6)

We conclude that the transfer function from Y(s) / U (s) is then

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$ - - - - - - (7)

Example 1: The state equations of a system are given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \mathbf{x}$$

Determine the transfer function for the system.

Solution The transfer function is given by

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

First, we calculate $(s\mathbf{I} - \mathbf{A})^{-1}$. Now,

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ 3 & s+1 \end{bmatrix}$$

Therefore,

$$\det(s\mathbf{I} - \mathbf{A}) = (s+2)(s+1) = s^2 + 3s + 2$$

Then, letting $\det(s\mathbf{I} - \mathbf{A}) = \Delta(s)$ for convenience, we have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \begin{bmatrix} \dfrac{s+1}{\Delta(s)} & 0 \\ \dfrac{-3}{\Delta(s)} & \dfrac{s+2}{\Delta(s)} \end{bmatrix}$$

and the transfer function is given by

$$G(s) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+1}{\Delta(s)} & 0 \\ \frac{-3}{\Delta(s)} & \frac{s+2}{\Delta(s)} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+1}{\Delta(s)} \\ \frac{2s+1}{\Delta(s)} \end{bmatrix} = \frac{5s+4}{s^2+3s+2}$$

Example 2:

Find the transfer function of the system with state space representation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 15 & 5 & 0 \end{bmatrix} x$$

Solution

$$sI_3 - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -14 & -7 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 8 & 14 & s+7 \end{bmatrix}$$

The matrix of minors and matrix of cofactors are

$$M = \begin{bmatrix} (s^2+7s+14) & 8 & -8s \\ -(s+7) & s(s+7) & 14s+8 \\ 1 & -s & s^2 \end{bmatrix}, \quad C_{of} = \begin{bmatrix} (s^2+7s+14) & -8 & -8s \\ (s+7) & s(s+7) & -(14s+8) \\ 1 & s & s^2 \end{bmatrix}$$

$$Adj\,(sI_n - A) = \begin{bmatrix} (s^2+7s+14) & (s+7) & 1 \\ -8 & s(s+7) & s \\ -8s & -(14s+8) & s^2 \end{bmatrix}$$

$$|sI_n - A| = s\left(s^2+7s+14\right) + 1\,(8) + 0\,(-8s) = s^3 + 7s^2 + 14s + 8$$

The steps in the calculation of the numerator are

$$Adj\,(sI_n - A)\,B = \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}, \qquad C\,Adj\,(sI_n - A)\,B = 15 + 5s$$

$$G(s) = C\,(sI_n - A)^{-1}\,B + D = \frac{C\,Adj\,(sI_n - A)\,B}{|sI_n - A|} + D$$

so the transfer function is

$$H(s) = \frac{5s+15}{s^3+7s^2+14s+8} = \frac{5\,(s+3)}{(s+1)\,(s+2)\,(s+4)}$$

H. W.:

1-Find the transfer function of the system with state space representation

$$\dot{x} = \begin{bmatrix} -7 & 1 & 0 \\ -14 & 0 & 1 \\ -8 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 20 \\ 125 \\ 185 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x + [5]\, u$$

Solution of State Equations

We have developed procedures for writing the state equations of a system, given the system differential equations, the system transfer function, or the simulation diagram.

In this section we present two methods for finding the solution of the state equations.

The standard form of the state equation is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad \text{------} \ (1)$$

This equation will now be solved using the Laplace transform. Taking Laplace transforms

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad \text{------} \ (2)$$

We wish to solve this equation for X(s); to do this we rearrange the last equation 2

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s) \quad \text{------} \ (3)$$

Pre-multiplying by $(s\mathbf{I}-\mathbf{A})^{-1}$, to eq.3 we obtain

$$\mathbf{X}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)+(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \qquad \text{------ (4)}$$

and the state vector x(t) is the inverse Laplace transform of this equation (4).

Therefore $x(t)=\mathcal{L}^{-1}[(sI-A)^{-1}]x(0)+\mathcal{L}^{-1}[(sI-A)^{-1}BU(s)]$

Note that

$$(sI-A)^{-1}=\frac{I}{s}+\frac{A}{s^2}+\frac{A^2}{s^3}+\cdots \Rightarrow \mathcal{L}^{-1}[(sI-A)^{-1}]=I+At+\frac{A^2t^2}{2!}+\frac{A^3t^3}{3!}+\cdots=e^{At}$$

$$\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)+\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \qquad \text{------ (5)}$$

if the initial time is t_0, then

$$\mathbf{x}(t)=e^{\mathbf{A}(t-t_0)}\mathbf{x}(0)+\int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \qquad \text{------ (6)}$$

The exponential matrix $e^{\mathbf{A}t}$ is called the state transition matrix $\Phi(t)$ and is defined as

$$\Phi(t) = \mathcal{L}^{-1}(s\mathbf{I} - \mathbf{A})^{-1} = e^{\mathbf{A}t} \qquad \text{------ (7)}$$

and

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} \qquad \text{------ (8)}$$

The exponential matrix $e^{\mathbf{A}t}$ represents the following power series of the matrix At, and

$$\Phi(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots \qquad \text{------ (9)}$$

Equation (5) can be written as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)\, d\tau \qquad \text{------ (10)}$$

In Equation (10) the first term represents the response to a set of initial conditions (zero-input response), whilst the integral term represents the response to a forcing function u(t) (zero-state response).

Similarly, the output equation is given by

$$\mathbf{y}(t) = \mathbf{C}\mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{C}\mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t) \qquad \text{------ (11)}$$

Example 1: Use the infinite series in (9) to evaluate the transition matrix $\Phi(t)$ if

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solution This is a good method only if A has a lot of zeros, since this guarantees a quick convergence of the infinite series. Clearly,

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and we stop here, since $\mathbf{A}^2 = 0$ and any higher powers are zero. Therefore,

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Example 2: Use the Laplace transform to find the transition matrix if A is the same as in Example 1.

Solution :We first calculate the matrix $(s\mathbf{I} - \mathbf{A})$,

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

The determinant of this matrix is

$$\det(s\mathbf{I} - \mathbf{A}) = s^2$$

and the ad joint matrix is

$$\operatorname{adj}(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

Next we determine the inverse of the matrix $(s\mathbf{I} - \mathbf{A})$,

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

The state transition matrix is the inverse Laplace transform of this matrix

$$e^{\mathbf{A}t} = \Phi(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Example 3: Consider the system described by the transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2}$$

(a) Write down the state equation in observer canonical form.
(b) Evaluate the state transition matrix $\Phi(t)$
(c) Find the zero-state response if a unit step is applied.

Solution (a) The state equation in observer canonical form is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(b) To find the state transition matrix, we first calculate the matrix $(s\mathbf{I} - \mathbf{A})$,

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

The determinant of this matrix is

$$\det(s\mathbf{I} - \mathbf{A}) = s^2 + 3s + 2 = (s+1)(s+2)$$

and the ad joint matrix is $\operatorname{adj}(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$

Next we determine the inverse of the matrix $(s\mathbf{I} - \mathbf{A})$,

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\operatorname{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \begin{bmatrix} \dfrac{s}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-1}{s+1} + \dfrac{2}{s+2} & \dfrac{1}{s+1} + \dfrac{-1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{2}{s+1} + \dfrac{-1}{s+2} \end{bmatrix}$$

The state transition matrix is the inverse Laplace transform of this matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} -e^{-t} + 2e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

(c) If a unit step is applied as an input. Then U(s) = 1/s, and the second term in (4) becomes

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) = \begin{bmatrix} \dfrac{s}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s+3}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s}$$

$$= \begin{bmatrix} \dfrac{1}{s(s+1)(s+2)} \\ \dfrac{s+3}{s(s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{1}{2}}{s} + \dfrac{-1}{s+1} + \dfrac{\frac{1}{2}}{s+2} \\ \dfrac{\frac{3}{2}}{s} + \dfrac{-2}{s+1} + \dfrac{\frac{1}{2}}{s+2} \end{bmatrix}$$

The inverse Laplace transform of this term is

$$\mathcal{L}^{-1}((s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s)) = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ \frac{3}{2} - 2e^{-t} + \frac{1}{2}e^{-2t} \end{bmatrix}$$

Advanced Control Systems

LECTURE 4

Diagonal Canonical Form

Prepared by: Mr. Abdullah I. Abdullah

Diagonal Canonical Form

The diagonal canonical form is a state space model in which the poles of the transfer function are arranged diagonally in the A matrix.

1-Distinct Real Roots

Given the system transfer function having a denominator polynomial that can be factored into distinct $(p_1 \neq p_2 \neq \ldots \neq p_n)$ roots as follows:

$$\frac{Y(s)}{U(s)} = \frac{P_m(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \cdots + \frac{k_n}{s+p_n}$$

The simulation diagram of such a form is shown in Figure1 .

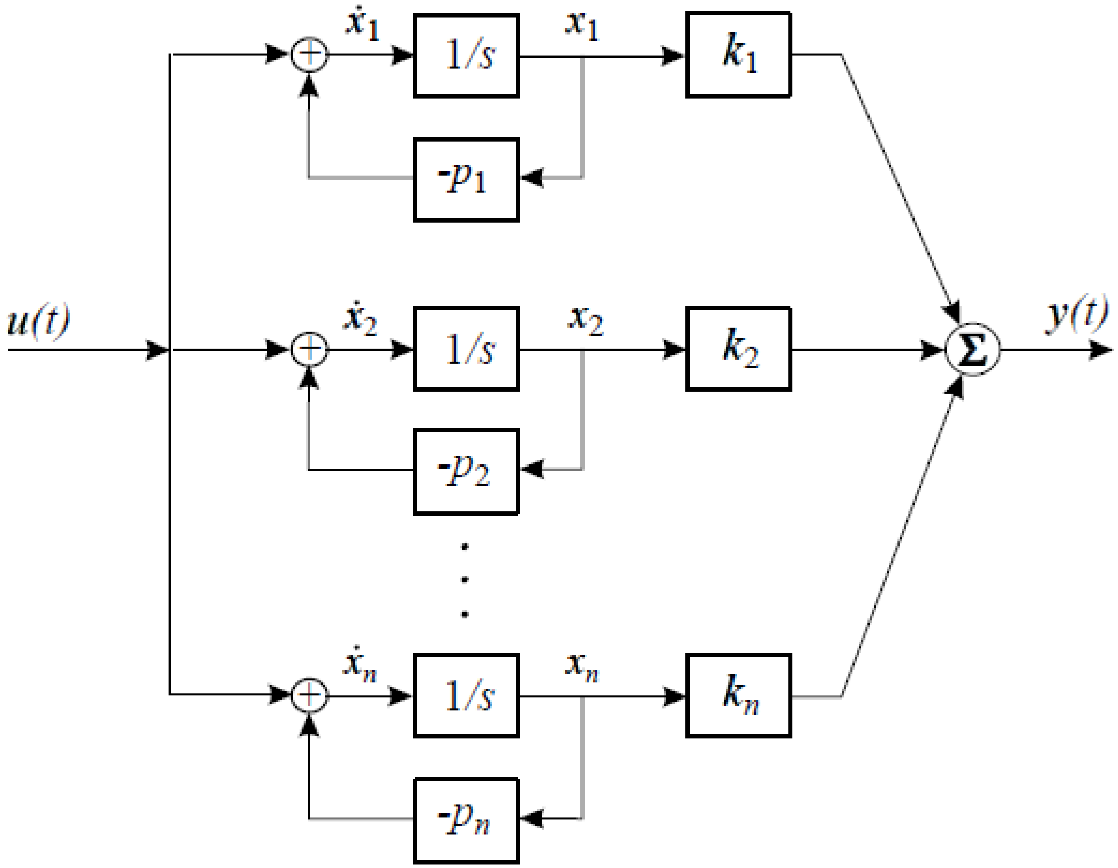


Figure 1: The simulation diagram for the Diagonal Canonical form

Then the diagonal canonical form state space model can be written as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \quad \text{-------(1)}$$

$$y = \begin{bmatrix} k1 & k2 & \ldots & kn \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} . \quad \text{-------(2)}$$

Example 1: Find the state space model of a system described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{(s+5)(s+4)}{(s+1)(s+2)(s+3)}$$

Employing the partial fraction expansion (which can be obtained by the MATLAB function residue), the transfer function is written as

$$\frac{Y(s)}{U(s)} = \frac{(s+5)(s+4)}{(s+1)(s+2)(s+3)} = \frac{6}{s+1} - \frac{6}{s+2} + \frac{1}{s+3}$$

The state space model, directly written using (1 & 2), are

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 6 & -6 & 1 \end{bmatrix} \mathbf{x}$$

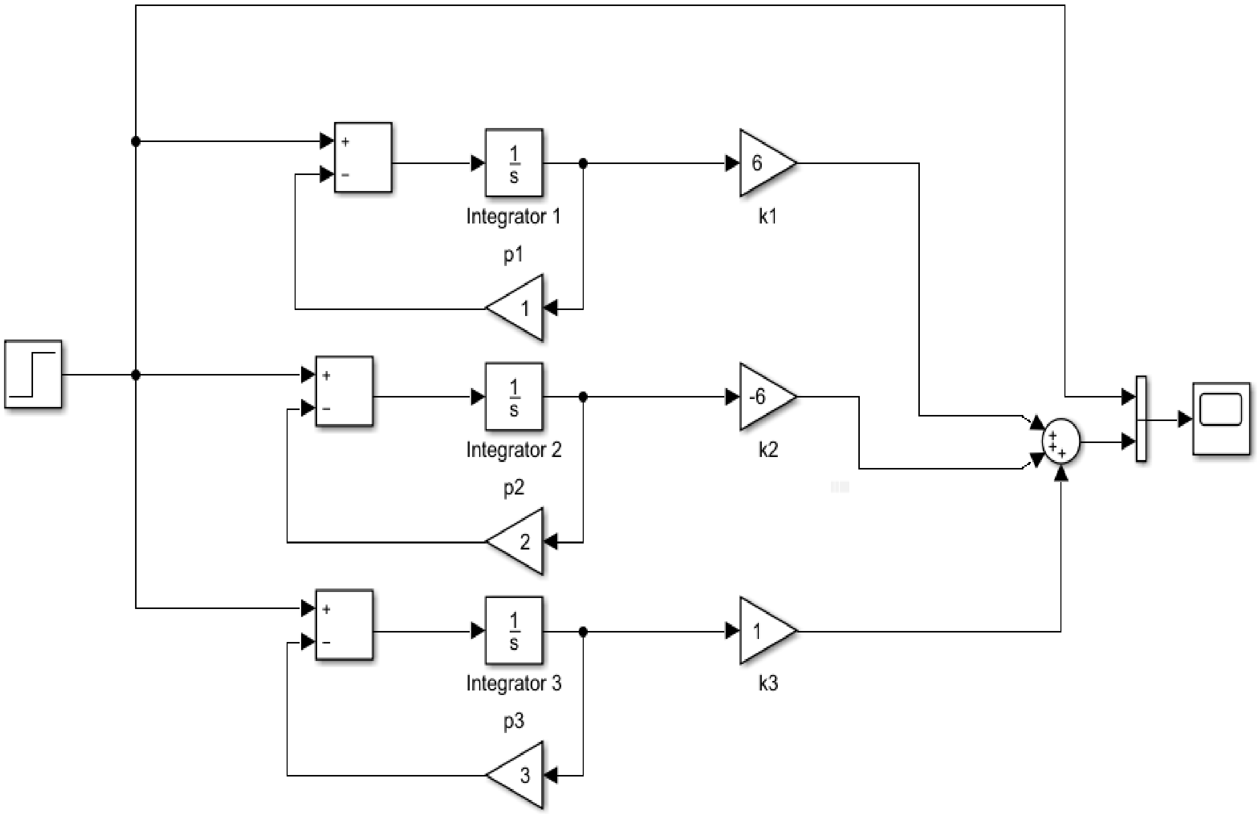


Fig. 2 : Simulation Diagram for the Diagonal Canonical form (DCF)

2-complex conjugate roots

If complex conjugate roots appear they should be combined in pairs corresponding to the second-order transfer functions, which can be independently implemented as demonstrated in the example 2.

Example 2: Let a transfer function containing a pair of complex conjugate roots be given by

$$G(s) = \frac{4}{s+1-j} + \frac{4}{s+1+j} + \frac{2}{s+5} + \frac{3}{s+10}$$

We first group the complex conjugate poles in a second-order transfer function, that is

$$G(s) = \frac{8s+8}{s^2+2s+2} + \frac{2}{s+5} + \frac{3}{s+10}$$

Then, distinct real poles are implemented like in the case of parallel programming. A second-order transfer function, corresponding to the pair of complex conjugate poles, is implemented using direct programming, and added in parallel to the first-order transfer functions corresponding to the real poles

The simulation diagram is given in Figure 3, where the controller canonical form is used to represent a second-order transfer function corresponding to complex conjugate poles

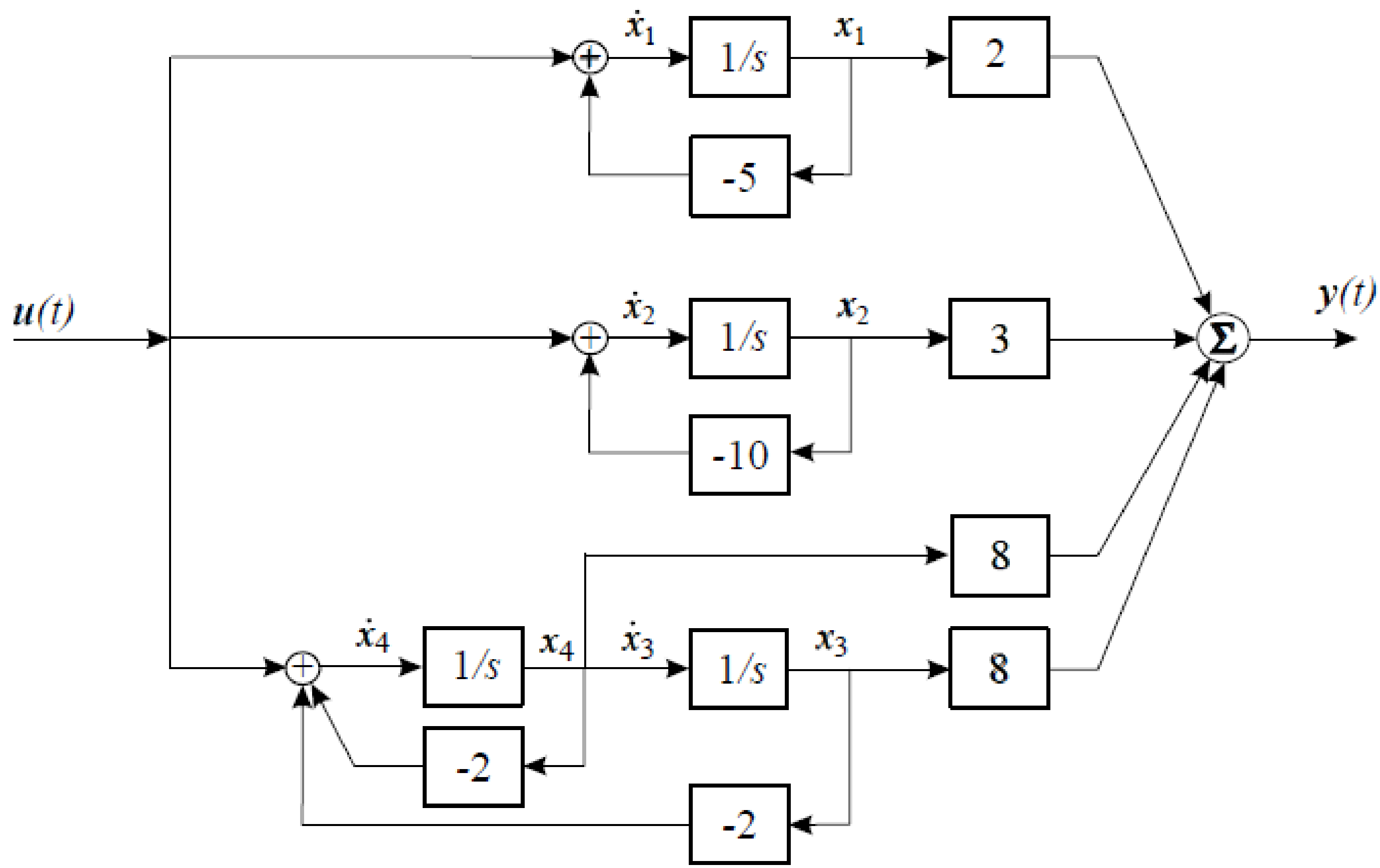


Figure 3: Simulation diagram for a system with complex conjugate poles

From this simulation diagram we have

$$\begin{aligned}
\dot{x}_1 &= -5x_1 + u \\
\dot{x}_2 &= -10x_2 + u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -2x_3 - 2x_4 + u \\
y &= 2x_1 + 3x_2 + 8x_3 + 8x_4
\end{aligned}$$

so that the required state space form is

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 & 8 & 8 \end{bmatrix} \mathbf{x}$$

3-Multiple Real Roots (Jordan canonical form)

When the transfer function has multiple real poles, it is not possible to represent the system in uncoupled form. Assume that a real pole p_1 of the transfer function has multiplicity r and that the other poles are real and distinct, that is

$$\frac{Y(s)}{U(s)} = \frac{N(s)}{(s+p_1)^r (s+p_{r+1}) \cdots (s+p_n)}$$

The partial fraction form of the above expression is

$$\frac{Y(s)}{U(s)} = \frac{k_{11}}{s+p_1} + \frac{k_{12}}{(s+p_1)^2} + \cdots + \frac{k_{1r}}{(s+p_1)^r} + \frac{k_{r+1}}{s+p_{r+1}} + \cdots + \frac{k_n}{s+p_n}$$

The simulation diagram for such a system is shown in Figure 4.

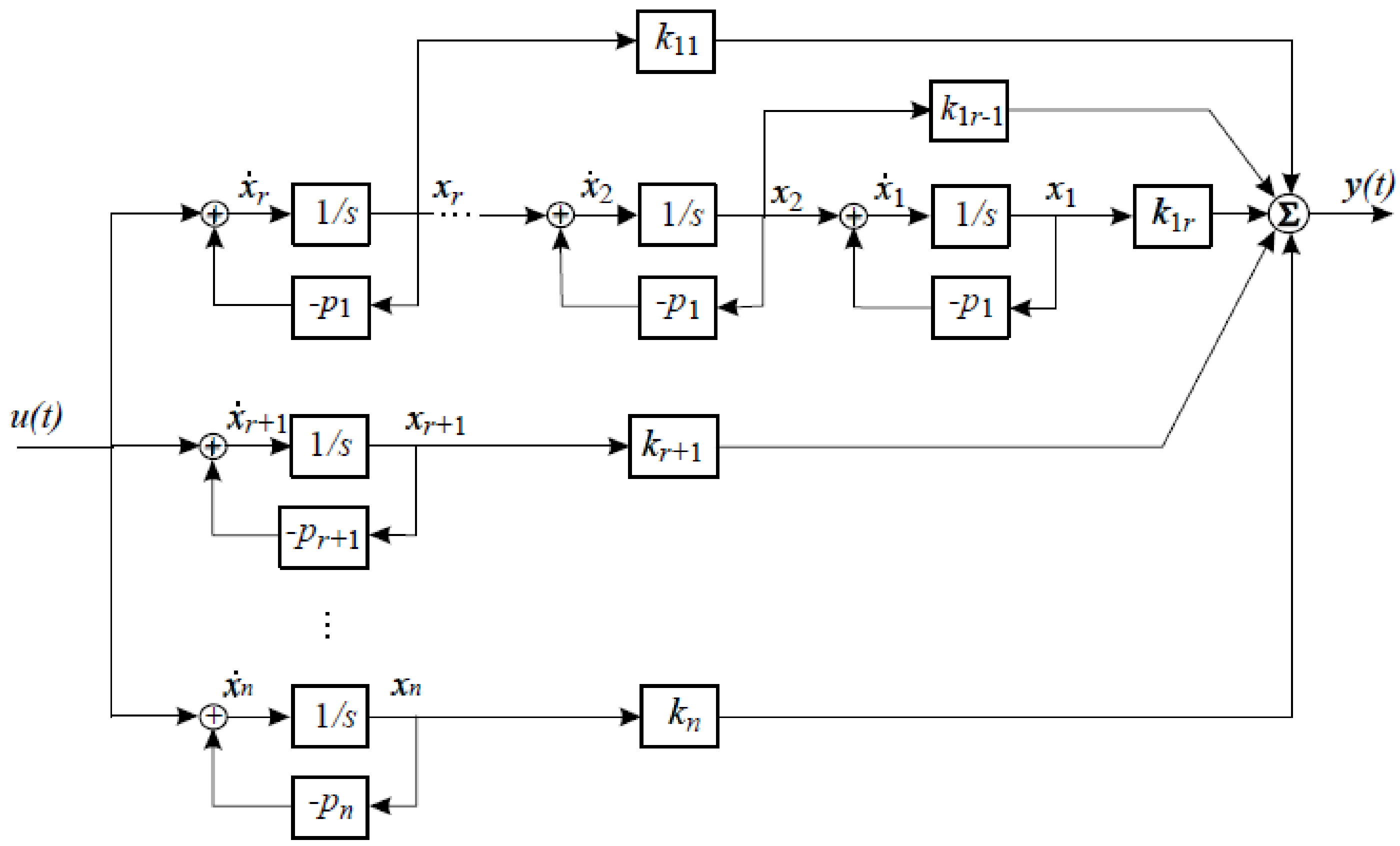


Figure 4: The simulation diagram for the Jordan canonical form

Taking for the state variables the outputs of integrators, the state space model is obtained as follows

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -p_1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -p_1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p_1 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & -p_4 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \dots & 0 & 0 & & -p_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} k1 & k2 & \cdots & kn \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

Example 3: Find the state space model from the transfer function using the Jordan canonical form

$$G(s) = \frac{s^2 + 6s + 8}{(s+1)^2(s+3)} = \frac{k1}{(s+1)} + \frac{k2}{(s+1)^2} + \frac{k3}{(s+3)}$$

This transfer function can be expanded as

$$G(s) = \frac{1.25}{s+1} + \frac{1.5}{(s+1)^2} - \frac{0.25}{s+3}$$

so that the required state space model is

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1.5 & 1.25 & -0.25 \end{bmatrix} \mathbf{x}$$

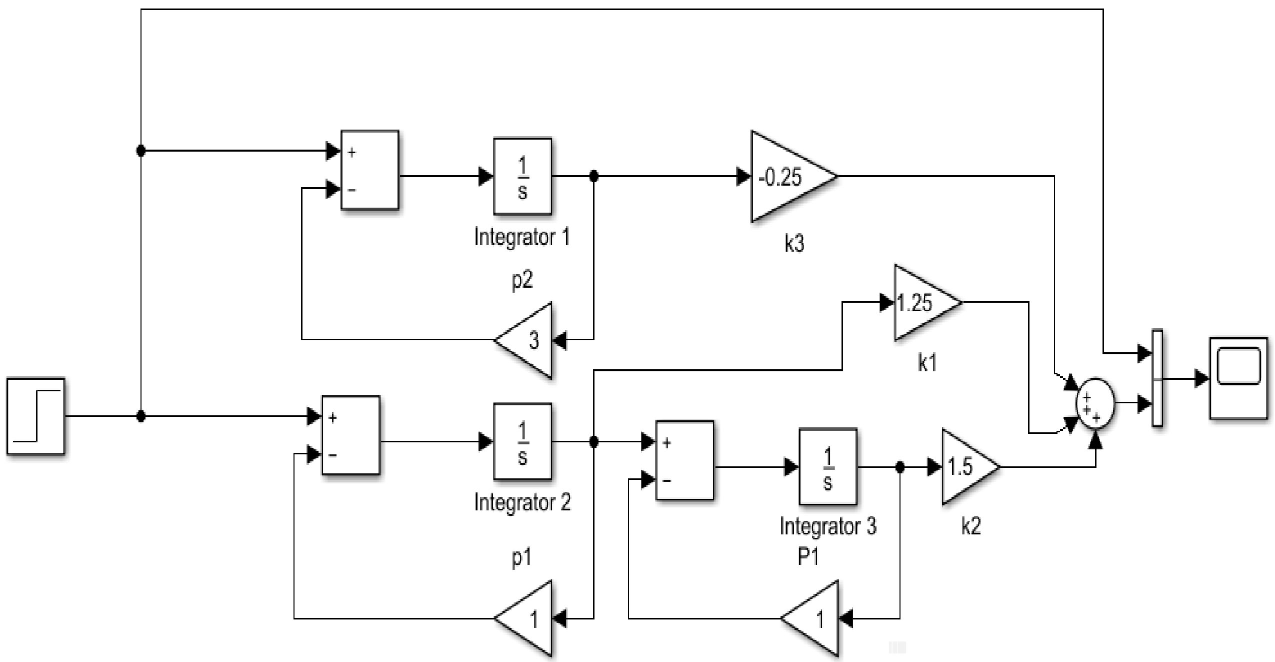


Fig. 3 : Simulation Diagram for the Jordan Canonical form (JCF)

Advanced Control Systems

LECTURE 5

Similarity Transformation

Prepared by: Mr. Abdullah I. Abdullah

Characteristic Equations

Characteristic equations play an important role in the study of linear systems. They can be defined with respect to differential equations, transfer functions, or state equations.

1-Characteristic Equation from a Differential Equation

Consider that a linear time-invariant(LTI) system is described by the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t) \quad \text{- - - (1)}$$

where n > m. By defining the operator S as

$$s^k = \frac{d^k}{dt^k} \qquad k = 1, 2, \cdots, n$$

Equation (1) is written

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)y(t) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)u(t)$$

The Characteristic equation of the system is defined as

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad \text{- - - (2)}$$

2-Characteristic Equation from a Transfer Function

The transfer function of the system described by (1) is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \qquad \text{- - - (3)}$$

The characteristic equation is obtained by equating the denominator polynomial of the transfer function to zero.

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad \text{- - - (4)}$$

3-Characteristic Equation from a State Equation

Recall that

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

which can be written as

$$\begin{aligned} G(s) &= \mathbf{C}\frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{|s\mathbf{I} - \mathbf{A}|}\mathbf{B} + \mathbf{D} \\ &= \frac{\mathbf{C}[\mathrm{adj}(s\mathbf{I} - \mathbf{A})]\mathbf{B} + |s\mathbf{I} - \mathbf{A}|\mathbf{D}}{|s\mathbf{I} - \mathbf{A}|} \end{aligned}$$

Setting the denominator of the transfer-function matrix G(s) to zero, we get the characteristic equation

$$|s\mathbf{I} - \mathbf{A}| = 0 \qquad \text{- - - (5)}$$

which is an alternative form of the characteristic equation, but should lead to the same equation as in (2).

Eigenvalues

The roots of the characteristic equation are often referred to as the eigenvalues of the matrix A.

Example 1: Find the eigenvalues of the matrix A given by $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

Solution The characteristic equation of A is $|s\mathbf{I} - \mathbf{A}| = 0$

$$|s\mathbf{I} - \mathbf{A}| = s^2 - 1$$

Replace s by λ

$$\lambda^2 - 1 = 0$$

$$(\lambda_1 - 1)(\lambda_2 + 1) = 0$$

Therefore, the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1.$

Eigenvectors

Any nonzero vector pi which satisfies the matrix equation

$$(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{p}_i = \mathbf{0} \qquad \text{- - - - - - - - (6)}$$

where $\lambda_i,\ i = 1, 2, \cdots, n,$ denotes the i^{th} eigenvalue of A, p_i is called the Eigen-vector of A associated with the eigenvalue λ_i : The procedure for determining eigenvectors can be divided into two possible cases depending on the results of the eigenvalue calculations.

Case 1: All eigenvalues are distinct.

Case 2: Some eigenvalues are multiple roots of the characteristic equation.

Case 1: Distinct Eigenvalues

If A has distinct eigenvalues, the eigenvectors can be solved directly from (6).

$$(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{p}_i = \mathbf{0}$$

Example 2 : Consider that a state equation has the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$.
Find the eigenvectors.

Solution The eigenvalues were determined in Example 1 as $\lambda_1 = 1$ and $\lambda_2 = -1$
Let the eigenvectors be written as

$$(\lambda_i I - A)p_i = 0$$

$$\mathbf{p}_1 = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

Substituting $\lambda_1 = 1$ and p1 into (6), we get $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus, p21 = 0, and p11 is arbitrary which in this case can be set equal to 1.

Similarly, for $\lambda_2 = -1$ (5) becomes $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

which leads to $-2p_{12} + p_{22} = 0$

The last equation has two unknowns, which means that one can be set arbitrarily.

Let $p_{12} = 1$, then $p_{22} = 2$. The eigenvectors are $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0.4472 \\ 0 & 0.8944 \end{bmatrix}$$

Case 2: Repeated Eigenvalues

Example 3: Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & -8 \\ 2 & -6 \end{bmatrix}$

$$|\lambda I - A| = 0$$

The characteristic equation is $|s\mathbf{I} - \mathbf{A}| = 0 \Rightarrow s^2 + 4s + 4 = 0 \Rightarrow \lambda^2 + 4\lambda + 4 = 0$

$$(\lambda_1 + 2)(\lambda_2 + 2) = 0$$

Thus, A has only one eigenvalue $\lambda = -2$ that is repeated twice.

$$(\lambda_i I - A)p_i = 0 \quad , \quad \begin{bmatrix} \lambda - 2 & 8 \\ -2 & \lambda + 6 \end{bmatrix}_{\lambda=-2} \underbrace{\begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}}_{p_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Making the substitution $\lambda = -2$ in the matrix yields

$$\begin{bmatrix} -4 & 8 \\ -2 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}}_{p_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

resulting in the equations

$$-4p_{11} + 8p_{21} = 0$$
$$-2p_{11} + 4p_{21} = 0$$

Both equations tell us the same thing, that $p_{11} = 2p_{21}$ ➡ let $p_{21} = 1$

Thus, one choice for the eigenvector is $p_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

For the generalized eigenvector that is associated with the second eigenvalue, we use a variation of the equation we used to find the eigenvector. That is, we write

$$\begin{bmatrix} \lambda - 2 & 8 \\ -2 & \lambda + 6 \end{bmatrix}_{\lambda=-2} \underbrace{\begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}}_{p_2} = - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which yields the equations

$$-4p_{12} + 8p_{22} = -2$$
$$-2p_{12} + 4p_{22} = -1$$

Either of these equations yields $p_{22} = \dfrac{2p_{12} - 1}{4}$

Choosing ➡ $p_{12} = 1/2$ ➡ $p_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

$$P = \begin{bmatrix} 2 & 0.5 \\ 1 & 0 \end{bmatrix}$$

Similarity Transformation

In this chapter, procedures have been presented for finding a state-variable model from system differential equations, from system transfer functions, and from system simulation diagrams.

In this section, a procedure is given for finding a deferent state model from a given state model. It will be shown that a system has an unlimited number of state models

The state model of an LTI single-input, single-output system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \qquad \text{- - - - - (1)}$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t) \qquad \text{- - - - - (2)}$$

Let us consider that the state model of Equations (1) and (2) and suppose that the state vector x(t) can be expressed as

$$\mathbf{x}(t) = \mathbf{P}\mathbf{v}(t) \qquad \text{- - - - - (3)}$$

where P is an n×n nonsingular matrix, called a transformation matrix, or simply, a transformation. We can write,

$$\mathbf{v}(t) = \mathbf{P}^{-1}\mathbf{x}(t) \qquad \text{- - - - - (4)}$$

Substituting (3) into the state equation in (1) yields

$$\mathbf{P}\dot{\mathbf{v}}(t) = \mathbf{APv}(t) + \mathbf{B}u(t) \qquad \text{- - - - - (5)}$$

Pre multiplying the above equation (5) by $\mathbf{P}^{-1}$ to solve for $\dot{\mathbf{v}}(t)$ results in the state model for the state vector v(t):

$$\dot{\mathbf{v}}(t) = \mathbf{P}^{-1}\mathbf{APv}(t) + \mathbf{P}^{-1}\mathbf{B}u(t) \qquad \text{- - - - - (6)}$$

Using (3), we find that the output equation in (2) becomes

$$y(t) = \mathbf{CPv}(t) + Du(t) \qquad \text{- - - - - (7)}$$

We have the state equations expressed as a function of the state vector x(t) in (1) and (2) and as a function of the transformed state vector v(t) in (6) and (7).

The state equations as a function of v(t) can be expressed in the standard format as

$$\dot{\mathbf{v}}(t) = \mathbf{A}_v \mathbf{v}(t) + \mathbf{B}_v u(t) \quad \text{- - - - - (8)}$$

$$y(t) = \mathbf{C}_v \mathbf{x}(t) + D_v u(t) \quad \text{- - - - - (9)}$$

Comparing (6) with (8) we get

$$\mathbf{A}_v = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

and

$$\mathbf{B}_v = \mathbf{P}^{-1}\mathbf{B}$$

Similarly, comparing (7) with (9), we see that

$$\mathbf{C}_v = \mathbf{C}\mathbf{P}$$

and

$$D_v = D$$

The transformation just described is called a similarity transformation, since in the transformed system such properties as the characteristic equation, eigenvectors, eigenvalues, and transfer functions are all preserved by the transformation.

Diagonal Canonical From a State Model

This form makes use of the eigenvectors. If A has distinct eigenvalues, there is a nonsingular transformation P that can be formed by use of the eigenvectors of A as its columns; that is

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & \cdots & \mathbf{p}_n \end{bmatrix} \quad \text{- - - - - (10)}$$

where pi, i = 1; 2;   ; n; denotes the eigenvector associated with the eigenvalue λ_i
The Av matrix is a diagonal matrix,

$$\mathbf{A}_v = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the n distinct eigenvalues of A.

Example 4 : Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$, find the diagonal canonical form of A

$|\lambda I - A| = 0$

From Example 2 the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

$(\lambda_i I - A)p_i = 0$

And the eigenvectors where determined $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Thus,

$$\mathbf{P} = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

and the diagonal canonical form of A is written

$$\mathbf{A}_v = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In general, when the matrix A has repeated eigenvalues, it cannot be transformed into a diagonal matrix. However, there exists a similarity transformation such that the Av matrix is almost diagonal. the matrix Av is called the Jordan canonical form. A typical Jordan canonical form is shown below

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

where it is assumed that A has a repeated eigenvalue λ_1 and distinct eigenvalues λ_2 and λ_3 . The Jordan canonical form generally has the following properties:

1. The elements on the main diagonal are the eigenvalues.
2. All elements below the main diagonal are zero.
3. Some of the elements immediately above the repeated eigenvalues on the main diagonal are 1s.

Example 5 : Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & -8 \\ 2 & -6 \end{bmatrix}$, find the Jordan canonical form of A

$|\lambda I - A| = 0$

A has only one eigenvalue $\lambda = -2$ that is repeated twice

The generalized eigenvector were found in example 3 to be

$(\lambda_i I - A)p_i = 0$

$$\mathbf{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

Thus,

$$\mathbf{P} = [\mathbf{p}_1 \quad \mathbf{p}_2] = \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

and the Jordan canonical form of A is written $A_v = p^{-1} A\, p$

$$\mathbf{A}_v = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = -2 \begin{bmatrix} 0 & -\frac{1}{2} \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -8 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

Example 6 : Diagonalize the following system dynamics matrice

Using coordinate transformations.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$|\lambda\, I - A| = 0$$

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0$$

The eigenvalues of **A** are the roots of the characteristic equation, or –1, –2, and –3.

$$(\lambda_i I - A)p_i = 0$$

$$p = \begin{bmatrix} -0.5774 & 0.2182 & -0.1048 \\ 0.5774 & -0.4364 & 0.3145 \\ -0.5774 & 0.8729 & -0.9435 \end{bmatrix}$$

$$p^{-1} = \begin{bmatrix} -5.1962 & -4.3301 & -0.8660 \\ -13.7477 & -18.3303 & -4.5826 \\ -9.5394 & -14.3091 & -4.7697 \end{bmatrix}$$

$$A_v = P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Vander Monde Matrix: The Vander monde matrix of order n is a square matrix specified variously as:

Case 1 distinct Eigen value : $$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ {\lambda_1}^2 & {\lambda_2}^2 & {\lambda_3}^2 \end{bmatrix}$$

Example 7 : Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$ find the diagonal canonical form of A .

$$|\lambda I - A| = 0 \qquad \lambda = -1 , -2 , -3$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ {\lambda_1}^2 & {\lambda_2}^2 & {\lambda_3}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$A_v = P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$B_v = P^1\mathrm{B} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

Note: $\mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\quad \mathrm{C} = [1 \quad 0 \quad 0]$

$$C_v = C\mathrm{P} = [1 \quad 1 \quad 1]$$

Case 2 : Repeated Eigen value

$$P = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ {\lambda_1}^2 & 2\,\lambda_1 & {\lambda_3}^2 \end{bmatrix}$$

Example 8 : Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & -21 & -8 \end{bmatrix}$,find the Jordan canonical form of A

$|\lambda I - A| = 0 \qquad \lambda_1 = \lambda_2 = -3\,, \qquad \& \;\; \lambda_3 = -2$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 1 & -2 \\ 9 & -6 & 4 \end{bmatrix}$$

$$A_v = P^{-1}AP = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Advanced Control Systems

LECTURE 6

Similarity Transformation of the Control Canonical Form

Prepared by: Mr. Abdullah I. Abdullah

Control Canonical Form

The transformation matrix **P** that transforms the state model into control canonical form is computed from the controllability matrix C.

$$C = [B \quad AB \;\; A^2B \;\cdots\cdots\cdots\; A^{n-1}B]$$

If this matrix is nonsingular, it will be invertible. Assume that Av and Bv are in control canonical form. Now

$$P^{-1}AP = A_v \implies \boxed{P^{-1}A = A_v P^{-1}}$$

and

$$\boxed{P^{-1}B = B_v}$$

Let $p_1, p_2, \cdots, p_n$ denote the rows of $\mathbf{P}^{-1}$. Then, $\mathbf{A}_v\mathbf{P}^{-1} = \mathbf{P}^{-1}\mathbf{A}$ is given by

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} p_1\mathbf{A} \\ p_2\mathbf{A} \\ p_3\mathbf{A} \\ \vdots \\ p_n\mathbf{A} \end{bmatrix}$$

Therefore,

$$p_2 = p_1\mathbf{A}$$
$$p_3 = p_2\mathbf{A} = p_1\mathbf{A}^2$$
$$p_4 = p_3\mathbf{A} = p_1\mathbf{A}^3$$
$$\vdots$$
$$p_n = p_{n-1}\mathbf{A} = p_1\mathbf{A}^{n-1}$$

$$p_n = p_1 A^{n-1}$$

Also, $\mathbf{B}_v = \mathbf{P}^{-1}\mathbf{B}$ yields

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} p_1\mathbf{B} \\ p_2\mathbf{B} \\ \vdots \\ p_n\mathbf{B} \end{bmatrix}$$

which implies

Note $p_n = p_1 A^{n-1}$

$$p_1\mathbf{B} = 0$$

$$p_2\mathbf{B} = p_1\mathbf{A}\mathbf{B} = 0$$

$$p_3\mathbf{B} = p_1\mathbf{A}^2\mathbf{B} = 0$$

$$\vdots$$

$$p_n\mathbf{B} = p_1\mathbf{A}^{n-1} = 1$$

therefore, in vector matrix form, we have

$$p_1[B \quad AB \quad A^2B \cdots A^{n-1}B] = [0 \quad 0 \quad 0 \cdots 0 \quad 1]$$

$$p_1 \underbrace{\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}}_{=\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Hence,

$$\boxed{p_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \mathcal{C}^{-1}}$$

Having found p1 we can now go back and construct all the rows of $\mathbf{P}^{-1}$. Note that we are only interested in the last row of C^{-1} to define p1.

$$p_n = p_1 A^{n-1}$$

Example 1 : Transform the following state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

to control canonical form.

Solution: $p_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \mathcal{C}^{-1}$

We first need to construct the C matrix. Therefore,

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 10 \\ 0 & 3 & 9 \\ 1 & 2 & 7 \end{bmatrix}$$

Next we need to find $\mathcal{C}^{-1}$, hence,

NOTE

$$p_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \mathcal{C}^{-1}$$

$$\mathcal{C}^{-1} = \begin{bmatrix} -0.3333 & -0.6667 & 1.3333 \\ -1 & 0.3333 & 1 \\ 0.3333 & 0 & -0.3333 \end{bmatrix}$$

We are only interested in the last row of $\mathcal{C}^{-1}$ to define p1,

$$p_1 = [0.3333 \quad 0 \quad -0.3333]$$

$$p_n = p_1 A^{n-1}$$

Next, we compute p2 and p3 as follows

$$p_2 = p_1 \mathbf{A} = \begin{bmatrix} 0 & 0.3333 & 0 \end{bmatrix}$$

$$p_3 = p_1 \mathbf{A}^2 = \begin{bmatrix} 0 & 0.3333 & 1 \end{bmatrix}$$

Therefore,

$$P^{-1} = \begin{bmatrix} p_1 \\ p_2 \\ p_2 \end{bmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{bmatrix} 0.3333 & 0 & -0.3333 \\ 0 & 0.3333 & 0 \\ 0 & 0.3333 & 1 \end{bmatrix}$$

Hence, the system can be transformed into the control canonical form,

$$\mathbf{A}_v = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0.3333 & 0 & -0.3333 \\ 0 & 0.3333 & 0 \\ 0 & 0.3333 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{B}_v = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 0.3333 & 0 & -0.3333 \\ 0 & 0.3333 & 0 \\ 0 & 0.3333 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the control canonical form model is given by

$$\dot{\mathrm{v}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 3 \end{bmatrix} \mathrm{v} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Example 2

A dynamic system is described by the following set of coupled liner ordinary differential equation

$$\dot{x}_1 + 2x_1 - 4x_2 = 5u \quad ---- -1$$

$$\dot{x}_1 - \dot{x}_2 + 4x_1 + x_2 = 0 ----- -2$$

$$y = \dot{x}_1 - \dot{x}_2 \quad --------- -3$$

1) Represent the system in state space form.
2) Draw the simulation diagram in control canonical form (CCF).

Solution

$$\dot{x}_1 = -2x_1 + 4x_2 + 5u \quad ---- -from\ (1)$$

$$\dot{x}_2 = \dot{x}_1 + 4x_1 + x_2 -------- -from\ (2)$$

$$\dot{x}_2 = -2x_1 + 4x_2 + 5u + 4x_1 + x_2 = 2x_1 + 5x_2 + 5u$$

$$y = \dot{x}_1 - \dot{x}_2 = -4x_1 - x_2 --- -from\ 2\&3$$

$$\dot{x}(t) = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} x(t) + \begin{bmatrix} 5 \\ 5 \end{bmatrix} u(t) \quad ----\ 4$$

$$y = [-4 \quad -1] x(t) \quad ----------\ 5$$

In control canonical form CCF

$$Con = [B \quad AB] = \begin{bmatrix} 5 & 10 \\ 5 & 35 \end{bmatrix} \quad controlable\ matrix$$

$$\det(Con) = 125$$

$$Con^{-1} = \begin{bmatrix} 0.28 & -0.08 \\ -0.04 & 0.04 \end{bmatrix} \quad invers\ of\ \ controlable\ matrix$$

$$p_1 = [-0.04 \quad 0.04] \quad last\ row\ of\ \ Con^{-1}$$

$$p_2 = p_1 A = [0.16 \quad 0.04]$$

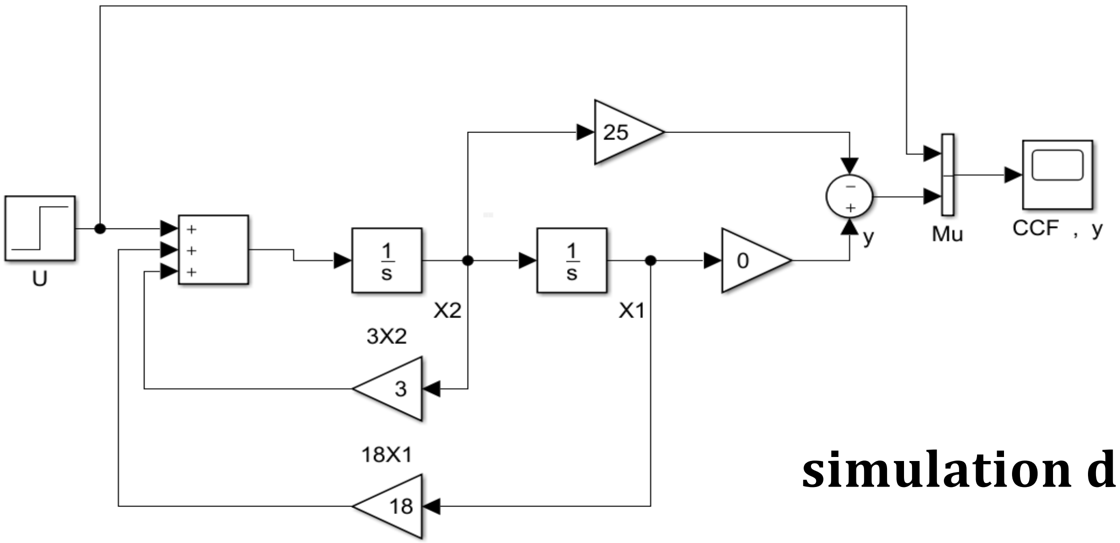
$$p^{-1} = \begin{bmatrix} -0.04 & 0.04 \\ 0.16 & 0.04 \end{bmatrix} \quad invers\ of\ transformation\ matrix$$

$$p = (p^{-1})^{-1} = \begin{bmatrix} -5 & 5 \\ 20 & 5 \end{bmatrix} \qquad \textit{transformation matrix}$$

$$A_v = p^{-1} A\, P = \begin{bmatrix} 0 & 1 \\ 18 & 3 \end{bmatrix}$$

$$B_v = p^{-1} B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_v = Cp = \begin{bmatrix} 0 & -25 \end{bmatrix}$$



simulation diagram in CCF

Advanced Control Systems

LECTURE 7

Similarity Transformation of the Observer Canonical Form

Prepared by: Mr. Abdullah I. Abdullah

Observer Canonical Form

The transformation matrix P that transforms the state model into observer canonical form is computed from the observability matrix O.

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

If this matrix is nonsingular, it will be invertible. Assume that Av and Cv are in observer canonical form. Now

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{A}_v \implies \boxed{\mathbf{AP} = \mathbf{PA}_v \quad \text{- - -(1)}}$$

and

$$\boxed{\mathbf{CP} = \mathbf{C}_v}$$

Let $p_1, p_2, \cdots, p_n$ denote the columns of $\mathbf{P}$. Then, $\mathbf{AP} = \mathbf{PA}_v$ is given by

$$[p_1 \quad p_2 \quad p_3 \quad \cdots \quad p_n] \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & 0 \\ -a_2 & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix} = [\mathbf{A}p_1 \quad \mathbf{A}p_2 \quad \mathbf{A}p_3 \quad \cdots \mathbf{A}p_n]$$

Therefore,

$$p_1 = \mathbf{A}p_2 = \mathbf{A}^{n-1}p_n$$

$$\vdots$$

$$p_{n-3} = \mathbf{A}p_{n-2} = \mathbf{A}^3 p_n$$

$$p_{n-2} = \mathbf{A}p_{n-1} = \mathbf{A}^2 p_n$$

$$p_{n-1} = \mathbf{A}p_n$$

Note : $p_1 = Ap_2$, $p_2 = Ap_3$, $p_1 = A^2 p_3$

$p_1 = A^{n-1} p_n$

In general $p_{n-m} = A^m p_n$, where m=1,…,n-1

Also, $\mathbf{C}_v = \mathbf{CP}$ yields

$$[1 \quad 0 \quad \cdots \quad 0] = [\mathbf{C}p_1 \quad \mathbf{C}p_2 \quad \cdots \quad \mathbf{C}p_n]$$

which implies

$$\mathbf{C}p_1 = \mathbf{C}\mathbf{A}^{n-1}p_n = 1$$

$$\vdots$$

$$\mathbf{C}p_{n-2} = \mathbf{C}\mathbf{A}^2 p_n = 0$$

$$\mathbf{C}p_{n-1} = \mathbf{C}\mathbf{A}p_n = 0$$

$$\mathbf{C}p_n = 0$$

$p_1 = A^{n-1} p_n$

therefore, in vector matrix form, we have

$$\underbrace{\begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}}_{=\mathcal{O}} p_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathcal{O}p_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Hence,

$$p_n = \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Having found p_n we can now go back and construct all the columns of P.

Note that we are only interested in the last column of $\mathcal{O}^{-1}$ to define p_n

Example1 : Transform the following state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}$$

to observer canonical form.

Solution We first need to construct the O matrix. Therefore,

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 4 \\ 5 & 9 & 14 \end{bmatrix}$$

Next we need to find $\mathcal{O}^{-1}$, hence,

$$\mathcal{O}^{-1} = \begin{bmatrix} 0.5 & -1.667 & 0.3333 \\ 0.5 & 1.1667 & -0.3333 \\ -0.5 & -0.3333 & 0.1667 \end{bmatrix}$$

We are only interested in the last column of $\mathcal{O}^{-1}$ to define p3, in this case

$$p_3 = \begin{bmatrix} 0.3333 \\ -0.3333 \\ 0.1667 \end{bmatrix}$$

Next, we compute p1 and p2 as follows

In general $p_{n-m} = A^m \; p_n$, where m=1,…,n-1

n=order of matrix $\mathcal{O}^{-1}$ =3 , m=1 ,2

$$p_2 = \mathbf{A} p_3 = \begin{bmatrix} -0.1667 \\ 0.1667 \\ 0.1667 \end{bmatrix} \qquad p_1 = \mathbf{A}^2 p_3 = \begin{bmatrix} 0.3333 \\ 0.6667 \\ 0.1667 \end{bmatrix}$$

Therefore,

$$\mathbf{P} = \begin{bmatrix} 0.3333 & -0.1667 & 0.3333 \\ 0.6667 & 0.1667 & -0.3333 \\ 0.1667 & 0.1667 & 0.1667 \end{bmatrix}$$

Hence, the system can be transformed into the observer canonical form,

$$P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\mathbf{A}_v = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.3333 & -0.1667 & 0.3333 \\ 0.6667 & 0.1667 & -0.3333 \\ 0.1667 & 0.1667 & 0.1667 \end{bmatrix}$$

and

$$\mathbf{C}_v = \mathbf{CP} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.3333 & -0.1667 & 0.3333 \\ 0.6667 & 0.1667 & -0.3333 \\ 0.1667 & 0.1667 & 0.1667 \end{bmatrix}$$

Thus, the observer canonical form model is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix} \mathbf{v} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Example 2

A dynamic system is described by the following set of coupled liner ordinary differential equation

$$\dot{x}_1 + 2x_1 - 4x_2 = 5u \quad ----1$$

$$\dot{x}_1 - \dot{x}_2 + 4x_1 + x_2 = 0 -----2$$

$$y = \dot{x}_1 - \dot{x}_2 \; ---------3$$

1) Represent the system in state space form.
2) Draw the simulation diagram in observable canonical form.

Solution

$$\dot{x}_1 = -2x_1 + 4x_2 + 5u \quad ----from\ (1)$$

$$\dot{x}_2 = \dot{x}_1 + 4x_1 + x_2 --------from\ (2)$$

$$\dot{x}_2 = -2x_1 + 4x_2 + 5u + 4x_1 + x_2 = 2x_1 + 5x_2 + 5u$$

$$y = \dot{x}_1 - \dot{x}_2 = -4x_1 - x_2 ---from\ 2\&3$$

$$\dot{x}(t) = \begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix} x(t) + \begin{bmatrix} 5 \\ 5 \end{bmatrix} u(t) \quad ---- \; 4$$

$$y = \begin{bmatrix} -4 & -1 \end{bmatrix} x(t) \quad ---------- \; 5$$

In observable canonical form CCF

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 6 & -21 \end{bmatrix} \quad observable\ matrix$$

$$\det(\mathcal{O}) = 90$$

$$\mathcal{O}^{-1} = \begin{bmatrix} -0.2333 & 0.0111 \\ -0.0667 & -0.0444 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} 0.0111 \\ -0.0444 \end{bmatrix} \quad last\ columan\ of\ \ \mathcal{O}^{-1}$$

$$p_1 = Ap_2 = \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix}$$

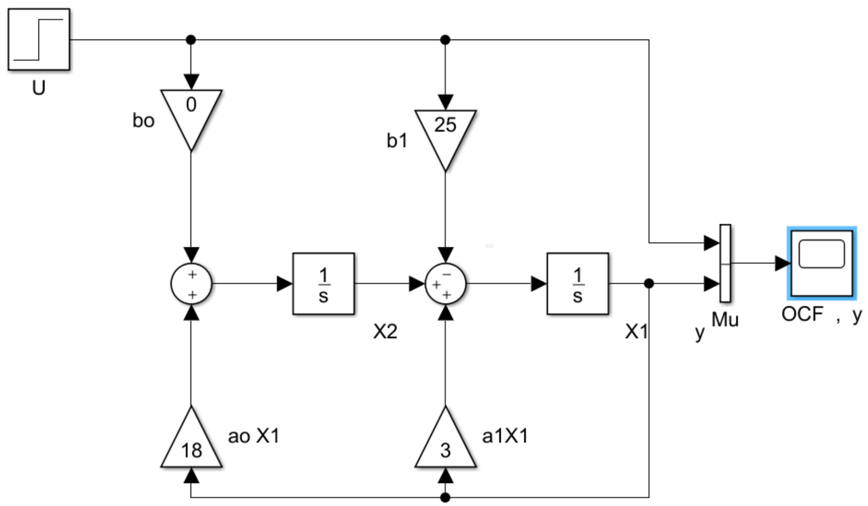
$$P = \begin{bmatrix} -0.2 & 0.0111 \\ -0.2 & -0.0444 \end{bmatrix} \quad invers\ of\ transformation\ matrix$$

$$p^{-1} = \begin{bmatrix} -4 & -1 \\ 18 & -18 \end{bmatrix}$$

$$A_v = p^{-1} A\, P = \begin{bmatrix} 3 & 1 \\ 18 & 0 \end{bmatrix}$$

$$B_v = p^{-1} B = \begin{bmatrix} -25 \\ 0 \end{bmatrix}$$

$$C_v = Cp = \begin{bmatrix} 1 & 0 \end{bmatrix}$$



S. D in OCF

Advanced Control Systems

LECTURE 8

Controllability and Observability

Prepared by: Mr. Abdullah I. Abdullah

Controllability and Observability

Controllability: In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable.

Observability: In order to see what is going on inside the system under observation, the system must be observable.

Motivation Examples

It is often a common practice in control applications to design a control input u(t) that makes the output y(t) behave in a desired manner.

If one has a state space model, then it is possible that while you are making the outputs behave nicely, some of the states x(t) may be misbehaving badly. This is best illustrated with the following examples.

Example 1: Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}$$

We compute the transition matrix $\boldsymbol{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$

$$\emptyset(t) = e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}] = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) & \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t - e^{-t}) & \frac{1}{2}(e^t + e^{-t}) \end{bmatrix}$$

and so, if we use the initial condition x(0) = 0, and the input u(t) is the unit step function we get

$$x(t) = \emptyset(t)x(0) + \int_0^t \emptyset(t-\tau)\, B\, u(\tau)d\tau$$

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} \frac{1}{2}\left(e^{(t-\tau)} + e^{-(t-\tau)}\right) & \frac{1}{2}\left(e^{(t-\tau)} - e^{-(t-\tau)}\right) \\ \frac{1}{2}\left(e^{(t-\tau)} - e^{-(t-\tau)}\right) & \frac{1}{2}\left(e^{(t-\tau)} + e^{-(t-\tau)}\right) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$x(t) = \int_0^t \begin{bmatrix} \frac{1}{2}(e^{(t-\tau)} - e^{-(t-\tau)}) \\ \frac{1}{2}(e^{(t-\tau)} + e^{-(t-\tau)}) \end{bmatrix} d\tau = \begin{bmatrix} -\frac{1}{2}e^{(t-\tau)} - \frac{1}{2}e^{-(t-\tau)} \\ -\frac{1}{2}e^{(t-\tau)} + \frac{1}{2}e^{-(t-\tau)} \end{bmatrix}_0^t$$

$$= \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) - 1 \\ \frac{1}{2}(e^t - e^{-t}) \end{bmatrix}$$

$$y = [1 \quad -1]\ x(t) = [1 \quad -1] \begin{bmatrix} \frac{1}{2}(e^{t} + e^{-t}) - 1 \\ \frac{1}{2}(e^{t} - e^{-t}) \end{bmatrix} = e^{-t} - 1$$

The output is $y(t) = e^{-t} - 1$ which we plot in Figure 1.

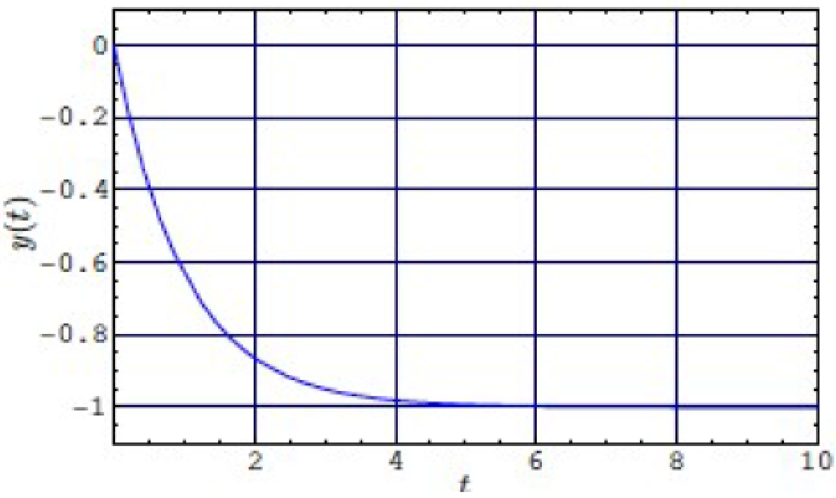


Figure 1: Output response of the system to a step input.

Example 2 : Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}$$

We compute the transition matrix $\boxed{\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}}$

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}(e^t - e^{-t}) & e^{-t} \end{bmatrix}$$

and so, if we use the initial condition x(0) = 0, and the input u(t) is the unit step function we get

$$x(t) = \emptyset(t)x(0) + \int_0^t \emptyset(t-\tau)\, B\, u(\tau) d\tau$$

$$\mathbf{x}(t) = \int_0^t \begin{bmatrix} e^{(t-\tau)} & 0 \\ \frac{1}{2}\left(e^{(t-\tau)} - e^{-(t-\tau)}\right) & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau \;\; = \begin{bmatrix} 0 \\ 1 - e^{-t} \end{bmatrix}$$

The output is $y(t) = 1 - e^{-t}$

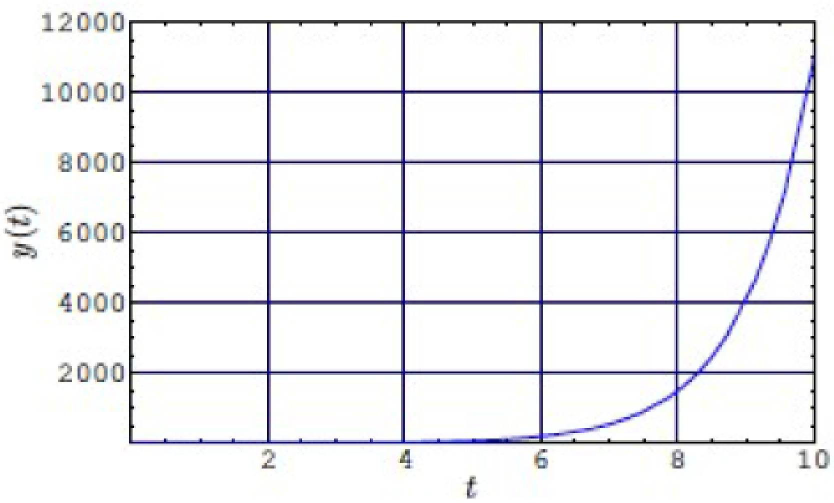
Everything looks okay, the output is behaving nicely, and the states are not blowing up to ∞

Let's change the initial condition to x(0) = (1; 0). We then compute

$$x(t) = \emptyset(t)x(0) + \int_0^t \emptyset(t-\tau)\, B\, u(\tau) d\tau$$

$$x(t) = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}(e^t - e^{-t}) & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - e^{-t} \end{bmatrix} = \begin{bmatrix} e^t \\ 1 + \frac{1}{2}(e^t - 3e^{-t}) \end{bmatrix}$$

and $y(t) = 1 + \frac{1}{2}(e^t - 3e^{-t})$ which we plot in Figure 2.



This system is uncontrollable

Figure 2: Output response of the system to a step input and non-zero initial conditions.

Controllability Tests

Controllability is a property of the coupling between the input and the state, and thus involves the matrices A and B.

A linear system is said to be controllable at t0 if it is possible to find some input function u(t), that when applied to the system will transfer the initial state x(t0) to the origin at some finite time t1, i.e., x(t1) = 0.

The most common test for controllability $n \times n$ is that the controllability matrix C defined as

$$\mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}] \quad \text{- - - - (1)}$$

contains n linearly independent row or column vectors, i.e. is of rank n (that is, the matrix is non-singular, i.e. the determinant is non-zero). Since only the matrices A and B are involved, we sometimes say the pair (A;B) is controllable.

Example 3 : Is the following system completely controllable

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ 3 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}$$

Solution From (1) the controllability matrix is

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix}$$

where

$$\mathbf{AB} = \begin{bmatrix} -2 & 0 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

hence

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

Clearly the matrix is nonsingular since it has a non-zero determinant. Therefore the system is controllable.

Observability Tests

Observability is a property of the coupling between the state and the output, and thus involves the matrices A and C.

A linear system is said to be observable at t0 if for an initial state x(t0), there is a finite time t1 such that knowledge of y(t) for $t_0 < t \leq t_1$ is sufficient to determine x(t0).

Observability is a major requirement in filtering and state estimation problems.

In many feedback control problems, the controller must use output variables y rather than the state vector x in forming the feedback signals.

If the system is observable, then y contains sufficient information about the internal states.

The most common test for observability is that the $n \times n$ observability matrix O defined as

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad \text{- - - - - (2)}$$

is of rank n (that is, the matrix is non-singular, i.e. the determinant is nonzero). Since only the matrices A and C are involved, we sometimes say the pair (A;C) is observable.

Example 4 : Is the following system completely observable

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ 3 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{x}$$

Solution From (2) the observability matrix is $\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix}$

where

$$\mathbf{CA} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 5 \end{bmatrix}$$

hence

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -5 & 5 \end{bmatrix}$$

Clearly the matrix is singular since it has a zero determinant.

The system is unobservable.

Advanced Control Systems

LECTURE 9

Stability in State Space

Prepared by: Mr. Abdullah I. Abdullah

Stability

الاستقرارية هي القضية الأكثر أهمية في تصميم أي نظام تحكم.

Stability is the most crucial issue in designing any control system. One of the most common control problems is the design of a closed loop system such that its output follows its input as closely as possible.

Unstable systems have at least one of the state variables blowing up to infinity as time increases.

Consider the ball which is free to roll on the surface shown in Figure 1. The ball could be made to rest at points A, E, F, and G and anywhere between points B and D, such as at C. Each of these points is an equilibrium point of the system.

نعتبر أن الكرة حرة في التدحرج على السطح الموضح في الشكل 1. يمكن جعل الكرة تستريح عند النقاط A و E و F و G وفي أي مكان بين النقطتين B و D، مثل عند النقطة C .كل من هذه النقاط هي نقطة توازن النظام.

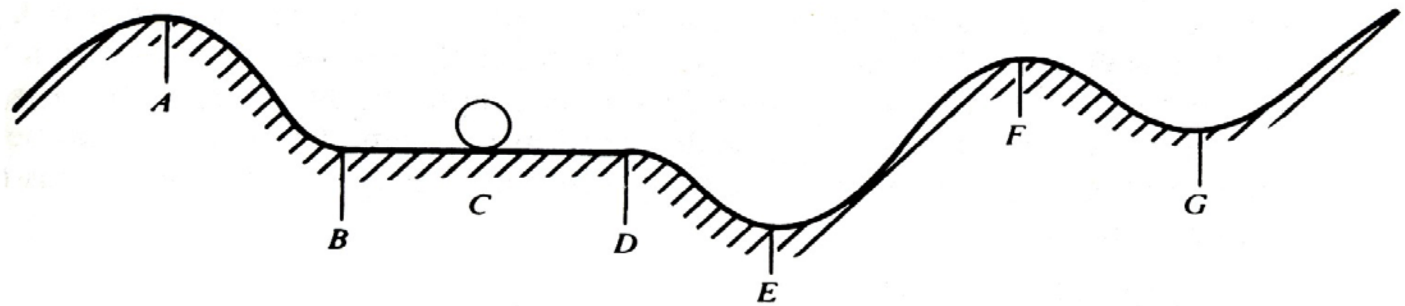


Figure 1: Equilibrium points

In state space, an equilibrium point for a system is a point at which $\dot{x}$ is zero in the absence of all inputs and disruptive disturbances. Thus if the system is placed in that state, it will remain there.

A small perturbation away from points A or F will cause the ball to diverge from these points. This behavior justifies labeling points A and F as unstable equilibrium points.

After small perturbations away from E and G, the ball will eventually return to rest at these points. Thus E and G are labeled as stable equilibrium points.

If the ball is displaced slightly from point C, it will normally stay at the new position. Points like C are sometimes said to be neutrally stable.

We say the system is stable locally. Stability therefore depends on the size of the original perturbation and on the nature of any disturbances.

Stability in State Space

Given a state space description

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du$$

The transfer function is given by

$$H(s) = C(sI - A)^{-1}B + D$$

$$= \frac{C\left[adj(sI - A)\right]B}{|sI - A|} + D$$

$$= \frac{C\left[adj(sI - A)\right]B + D|sI - A|}{|sI - A|}$$

The denominator of this is the characteristic polynomial

$$\Delta(s) = |sI - A| .$$

The system poles are the roots of the characteristic equation

Example 1

Let $\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x.$ Then $\Delta(s) = \begin{vmatrix} s & -1 \\ 1 & s+2 \end{vmatrix} = s^2 + 2s + 1 = (s+1)^2$

so that both poles are at s= -1. Therefore the system is stable.

Example 2

$$\text{Let } \dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = Cx = \begin{bmatrix} -1 & 1 \end{bmatrix} x .$$

The characteristic polynomial is

a. The characteristic polynomial is

$\Delta(s) = \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} = s^2 - 1 = (s+1)(s-1)$. The poles are at s=-1, s=1, so the system is not AS. It is unstable. The natural modes are e^{-t}, e^{t}.

b. The transfer function is

$$H(s) = C(sI - A)^{-1} B = \frac{s-1}{(s-1)(s+1)} = \frac{1}{s+1},$$

which has poles at s=-1. Therefore, the system is BIBOS. Note that the unstable pole at s=1 has cancelled with a zero at s=1.

Note that the eigenvalues of A appear as exponents in the solution of state x(t) (although some of them may not appear at the output due to pole-zero cancellations).

As a result for a given (A;B;C;D) to be stable (internal stability), all eigenvalues of A should be stable.

Example 3 Consider the state space

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{u}(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{aligned}$$

where

- Eigenvalues of A are - 2 (stable) and 3 (unstable).
- Output is equal to the first state, which is decoupled from the second state: y(t) = x1(t).

The transfer function of this system:

$$H(s) = \frac{\overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{\mathbf{C}} \overbrace{\begin{bmatrix} s-3 & 0 \\ 0 & s+2 \end{bmatrix}}^{Adj(s\mathbf{I}-\mathbf{A})} \overbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}^{\mathbf{B}}}{\underbrace{(s-3)(s+2)}_{det(s\mathbf{I}-\mathbf{A})}}$$

$$= \frac{(s-3)}{(s-3)(s+2)} = \frac{1}{s+2}$$

The transfer function has only a stable pole (-2) !(after the pole-zero cancellation).

Advanced Control Systems

LECTURE 10

Modern Control Design

Prepared by: Mr. Abdullah I. Abdullah

Modern Control Design

Classical design techniques are based on either frequency response or the root locus. Over the last decade new design techniques has been developed, which are called modern control methods to differentiate them from classical methods. In this chapter we present a modern control design method known as pole placement, or pole assignment. This method is similar to the root-locus design, in that poles in the closed-loop transfer function may be placed in desired locations. Achievement of suitable pole locations is one of the fundamental design objectives as this will ensure satisfactory transient response. The placing of all poles at desired locations requires that all state variables must be measured.

State feedback

Many design techniques in modern control theory is based on the state feedback configuration. The block diagram of a system with state feedback control is shown in Figure 1.

The open-loop system, often called the plant, is described in state variable form as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{- - - (1)}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad \text{- - - (2)}$$

The equations which describe the state feedback problem are (1), (2) and the relation

$$u(t) = r(t) - \mathbf{K}\mathbf{x}(t) \quad \text{- - - (3)}$$

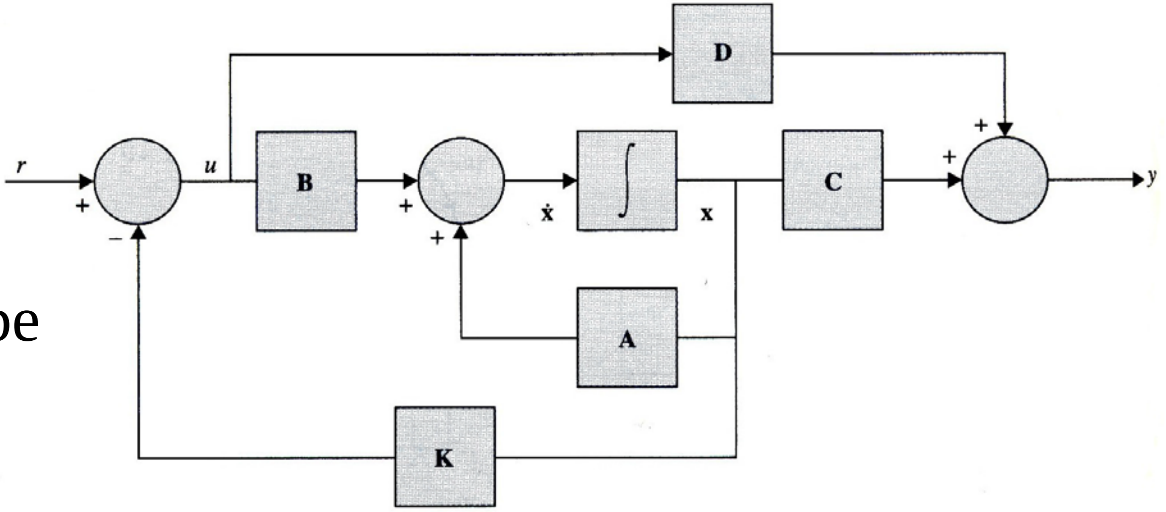


Figure 1: State variable feedback system.

Combining gives

$$\dot{x} = [A - BK]x + Br \quad \text{--- (4)}$$

$$y = [C - DK]x + Dr \quad \text{--- (5)}$$

With this setup in mind the question is: What changes in overall system characteristics can be achieved by the choice of K? Stability of the state feedback system depends on the eigenvalues of $(A - BK)$.

Controllability depends on the pair $([A - BK], B)$.

Observability depends on the pair $([A - BK], [C - DK])$.

Pole-Placement Design

Pole-Placement design is based on the state model of the system. The state model of the plant considered is as given in (1) and (2) with D = 0.

Initially, we will assume that $r(t) = 0$. A system of this type (input equal to zero) is called regulator control system. The purpose of such a system is to maintain the system output $y(t)$ at zero.

In general, in modern control design, the plant input $u(t)$ is made a function of the states, of the form

$$u(t) = f[\mathbf{x}(t)] \quad \text{- - - (6)}$$

This equation is called the control law. In pole-placement design, the control law is specified as a linear function of the states, in the form

$$u(t) = -\mathbf{K}\mathbf{x}(t)$$

$$= -\begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{---} (7)$$

We will show that this control law allows all poles of the closed-loop system to be placed in any desirable locations.

$$u(t) = -K_1 x_1(t) - K_2 x_2(t) - \cdots - K_n x_n(t) \quad \text{---} (8)$$

The design objective is: specify a desired root locations of the system characteristic equation, and then calculate the gains Ki to yield these desired root locations.

The closed-loop system can be represented as shown in Figure 2.

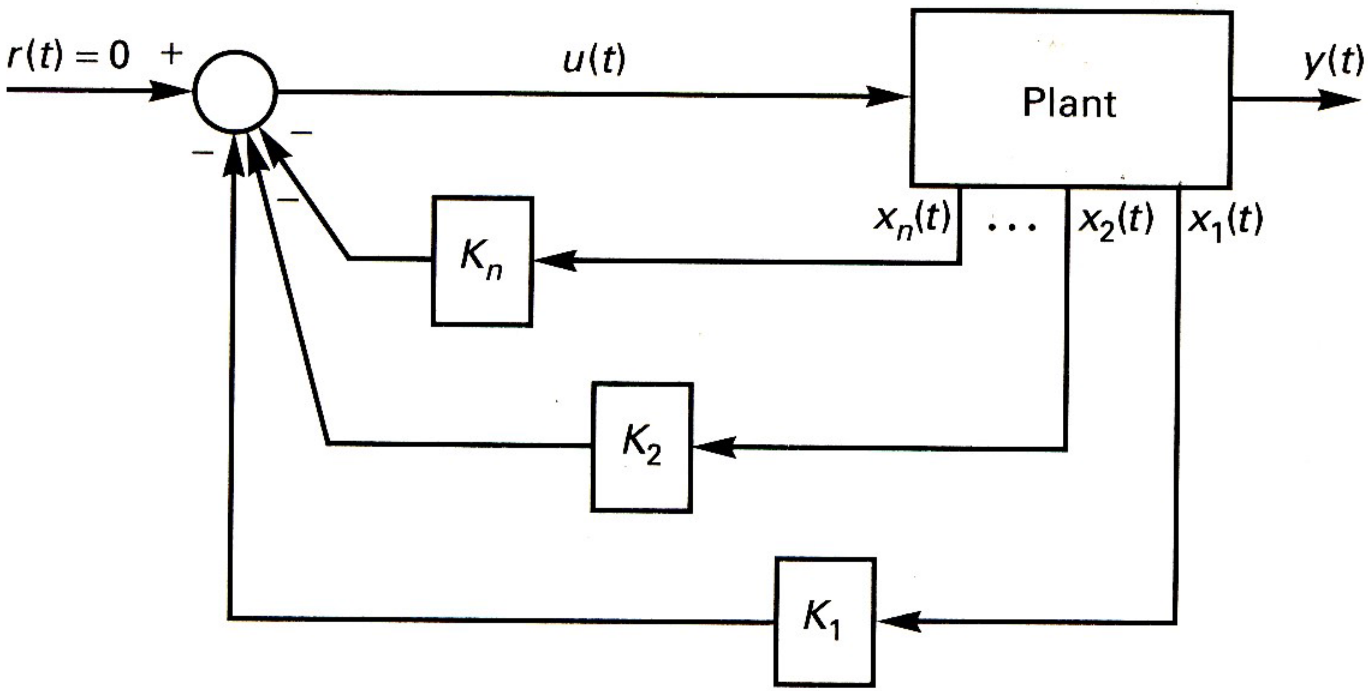


Figure 2: Pole-placement design.

- There are three approaches that can be used to determine the gain matrix **K** to place the poles at desired location.

  - Direct Substitution Method.
  - Ackermann's formula.
  - Using Transformation Matrix P.

- All those method yields the same result.

Pole Placement Procedure (Direct substitution Method)

The state equation of the plant is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{- - - - (10)}$$

The control law is chosen to be

$$u(t) = -\mathbf{K}\mathbf{x}(t) \quad \text{- - - - (11)}$$

with

$$\mathbf{K} = [K_1 \quad K_2 \quad \cdots \quad K_n] \quad \text{- - - - (12)}$$

and n is the order of the plant. Substitution of (11) into (10) yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) = \mathbf{A}_f\mathbf{x}(t) \quad \text{- - - - (13)}$$

where $\mathbf{A}_f = (\mathbf{A} - \mathbf{B}\mathbf{K})$ is the system matrix for the closed-loop system

The characteristic equation for the closed-loop system is then

$$|s\mathbf{I} - \mathbf{A}_f| = |s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = 0 \qquad ----(14)$$

Suppose that the design specifications require that the roots of the characteristic equation be at $-\lambda_1, -\lambda_2, \cdots, -\lambda_n$.

The desired characteristic equation for the system, which is denoted by $\alpha_c(s)$ is

$$\begin{aligned}\alpha_c(s) &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \\ &= (s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n) = 0 \qquad ----(15)\end{aligned}$$

The pole-placement design procedure results in a gain vector K such that (14) is equal to (15), that is,

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \alpha_c(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \quad ----(16)$$

- **Steps:**

1. Check the state controllability of the system.

$$C_T = [B \quad AB \quad A^2B \; \cdots \quad A^{n-1}B]$$

2. Define the state feedback gain matrix as

$$\boldsymbol{K} = [k_1 \quad k_2 \quad k_3 \cdots \quad k_n]$$

– And equating $|sI - A + BK|$ with desired characteristic equation.

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \alpha_c(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

Example : Consider the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$\mathbf{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

Find the control law that places the closed-loop poles of the system so that they are both at $s = -2$.

Solution From equation (15) we find that

$$\alpha_c(s) = (s+2)^2$$
$$= s^2 + 4s + 4 \qquad \text{- - - - (17)}$$

Equation (14) tells us that

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right|$$

or $s^2 + K_2 s + 1 + K_1 = 0$ - - - - (18)

Equating the coefficients with like powers of s in (18) and (17) yields the system of equations

$$s^2 + K_2 s + 1 + K_1 = s^2 + 4s + 4$$

$$K_2 = 4$$
$$1 + K_1 = 4$$

therefore,

$$K_1 = 3$$
$$K_2 = 4$$

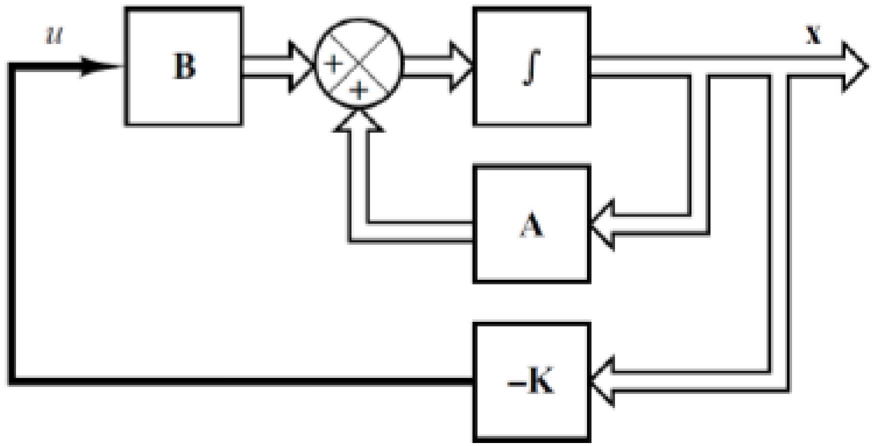
The control law is

$$\mathbf{K} = [K_1 \quad K_2] = [3 \quad 4]$$

Example 2

Consider the regulator system shown in following figure. The plant is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$



The system uses the state feedback control u=-**K**x. The desired eigenvalues are $\mu_1 = -2 + j4$, $\mu_2 = -2 - j4$,$\mu_3 = -1$. Determine the state feedback gain matrix **K**.

- Step-1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

- First, we need to check the controllability matrix of the system. Since the controllability matrix C_T is given by

$$C_T = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

- We find that rank(C_T)=3. Thus, the system is completely state controllable and arbitrary pole placement is possible.

- **Step-2:**
- Let **K** be

$$\boldsymbol{K} = [k_1 \quad k_2 \quad k_3]$$

$$|sI - A + BK| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right|$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

- Desired characteristic polynomial is obtained as

$$(s + 2 - 4j)(\, s + 2 + 4j)\, (s + 10) = s^3 + 14s^2 + 60s + 200$$

- Comparing the coefficients of powers of s

$$14 = (6 + k_3) \qquad k_3 = 8$$

$$60 = (5 + k_2) \qquad k_2 = 55$$

$$200 = 1 + k_1 \qquad k_1 = 199$$

```
%%%%%%%%%%%%%%  Pole placement[Direct method] 3rd order
clc
clear
t=0:0.001:20;
%%%%%%%%%%%%%%   poles location  %% p1=s+10 ,p2=s+2-4*i,
p3=s+2+4*i   syms K k1 k2 k3 s
p1=input('pole 1 p=[s+?]=')
p2=input('pole 2 p=[s+?]=')
p3=input('pole 2 p=[s+?]=')
%%% plant matrix
A=[0 1 0;0 0 1; -1 -5 -6];
b=[0; 0 ;1];
c=[1 0 0];
d=0;
I=[1 0 0;0 1 0;0 0 1];
%%%%%%%%%%%%%%%%%%%%%%%%%  Test Controbability
Co=[b  A*b  A^2*b];
n=length(A);
N=rank(Co);
if N==n
    disp('The system is controllable')
end
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ChEqCL=p1*p2*p3;   %   closed loop ch.eq
ChEqOL=det(s*I-A) ;   %%%%% open loop ch.eq
diF=ChEqCL-ChEqOL ;    %%% difference of CH.EQ
F=adjoint(s*I-A)*b;
KP=det(diF);
K1=[k1 k2 k3];
K1.*F==KP;
disp('Direct Method  K=[k1   k2  k3] =')
disp(coeffs(KP)); %%%% Note:KP=[k2 k1] %%%%%%% Direct
Method
K=coeffs(KP);
```

```
%%%%%%%%%%%%%%    AckerMan Method
pause
pcofficient=coeffs(ChEqCL);
pNote=[pcofficient(:,4)  pcofficient(:,3)  pcofficient(:,2)
pcofficient(:,1)]
p=input('coffecient of pNote [    ] =')
Yacker = polyvalm(p,A);
Coinv=inv(Co);
Kaker=[0 0 1]*Coinv;
KackerM=Kaker*Yacker;
%%%%%   AckerMan Method   [K1 K2]
disp('ackerman gain K=[K1 K2 K3]')
disp(KackerM)
%%%closed Loop Acl
Acl=A-b*KackerM;
bb=b*KackerM;
figure(1)
y1=step(A,b,c,d,1,t);
y2=step(Acl,bb,c,d,1,t);
plot (t,y1,t,y2)
title('Pole Placement Controller')
legend('Without pole placement','With pole placement')
xlabel('Time (sec)')
ylabel('Response')
grid
```

pole 1 p=[s+?]=s+2-4*i

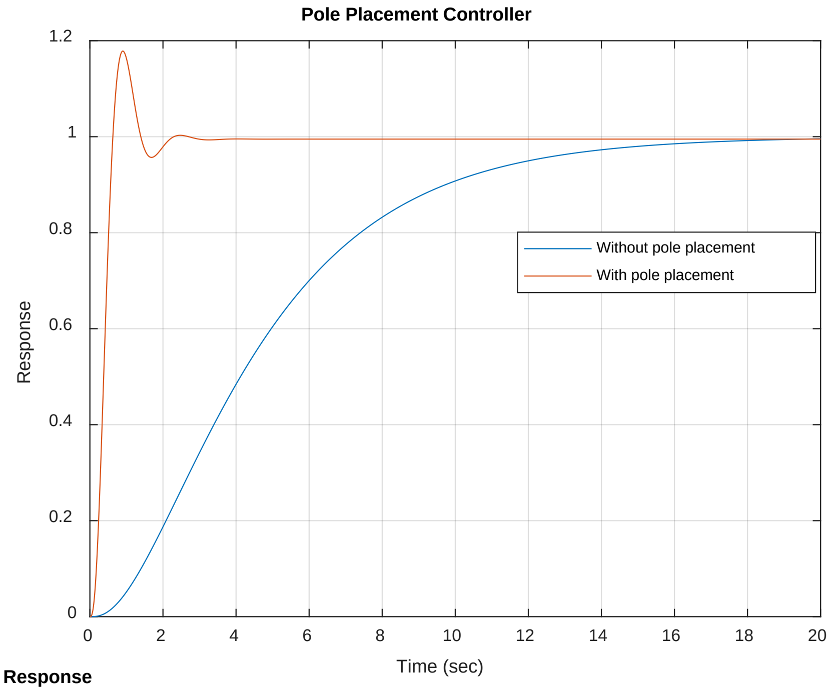
p1 = s + 2 - 4i

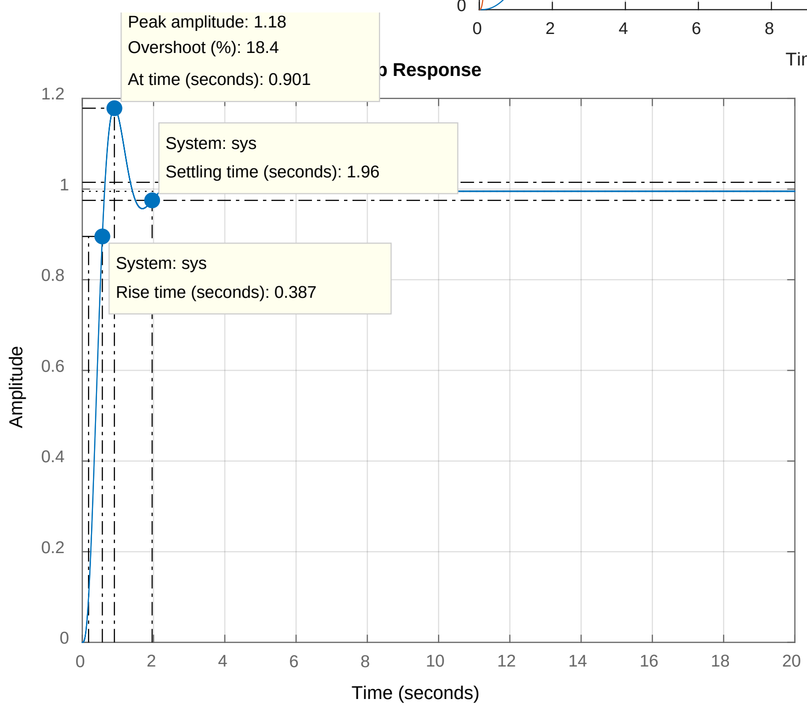
pole 2 p=[s+?]=s+2+4*i

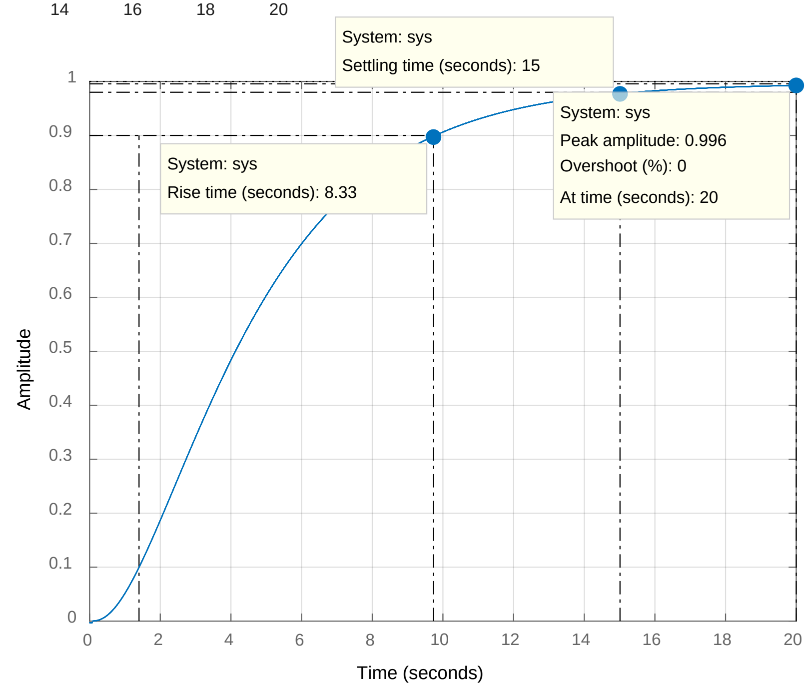
p2 =s + 2 + 4i

pole 2 p=[s+?]=s+10

p3 =s + 10
The system is controllable
Direct Method K=[k1 k2 k3] =
[199, 55, 8]
pNote =[1, 14, 60, 200]
coffecient of pNote [ ] =[1, 14, 60, 200]
p =[1 14 60 200]
ackerman gain K=[K1 K2 K3] =[199 55 8]


Pole Placement Controller
Response
Time (sec)
Without pole placement
With pole placement
0
0.2
0.4
0.6
0.8
1
1.2
0 2 4 6 8 10 12 14 16 18 20


Peak amplitude: 1.18
Overshoot (%): 18.4
At time (seconds): 0.901
System: sys
Settling time (seconds): 1.96
System: sys
Rise time (seconds): 0.387
Amplitude
Time (seconds)
0 0.2 0.4 0.6 0.8 1 1.2
0 2 4 6 8 10 12 14 16 18 20


System: sys
Settling time (seconds): 15
System: sys
Peak amplitude: 0.996
Overshoot (%): 0
At time (seconds): 20
System: sys
Rise time (seconds): 8.33
Amplitude
Time (seconds)
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 2 4 6 8 10 12 14 16 18 20

Advanced Control Systems

LECTURE 11

Pole placement

(Ackermann ’s Formula)

Prepared by: Mr. Abdullah I. Abdullah

Pole Placement (Ackermann's Formula)

- Following are the steps to be followed in this particular method.

1. Check the state controllability of the system

$$CM = [B \quad AB \quad A^2B \; \dots \quad A^{n-1}B]$$

Pole Placement (Ackermann's Formula)

- Following are the steps to be followed in this particular method.

2. Use Ackermann's formula to calculate **K**

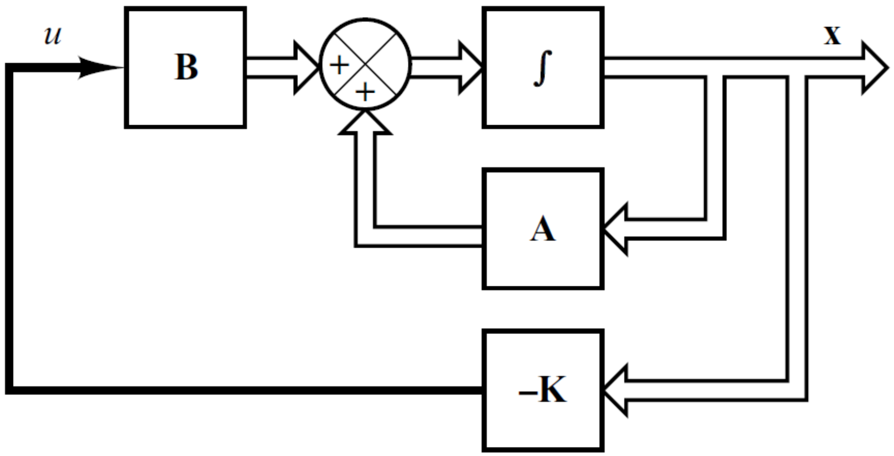
$$K = [0 \quad 0 \quad \cdots 0 \quad 1][B \quad AB \quad A^2B \; \cdots \quad A^{n-1}B]^{-1}\emptyset(A)$$

$$\emptyset(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_n I$$

Pole Placement (Ackermann's Formula)

- Example-1: Consider the regulator system shown in following figure. The plant is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$



- The system uses the state feedback control **u=-Kx**. The desired eigenvalues are $\mu_1 = -2 + j4$, $\mu_2 = -2 - j4$,$\mu_3 = -1$. Determine the state feedback gain matrix **K**.

Pole Placement (Using Transformation Matrix P)

- Example-1: Step-1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

- First, we need to check the controllability matrix of the system. Since the controllability matrix **CM** is given by

$$CM = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

- We find that rank(**CM**)=3. Thus, the system is completely state controllable and arbitrary pole placement is possible.

Pole Placement (Ackermann's Formula)

- Following are the steps to be followed in this particular method.

2. Use Ackermann's formula to calculate **K**

$$K = [0 \quad 0 \quad 1][B \quad AB \quad A^2B]^{-1} \emptyset(A)$$

$$\emptyset(A) = A^3 + \alpha_1 A^2 + \alpha_2 A + \alpha_3 I$$

- α_i are the coefficients of the desired characteristic polynomial.

$$(s + 2 - 4j)(s + 2 + 4j) (s + 10) = s^3 + 14s^2 + 60s + 200$$

$$\alpha_1 = 14, \qquad \alpha_2 = 60, \qquad \alpha_3 = 200$$

Pole Placement (Ackermann's Formula)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$\emptyset(A) = A^3 + 14A^2 + 60A + 200I$$

$$\emptyset(A) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\emptyset(A) = \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -34 & 117 \end{bmatrix}$$

Pole Placement (Ackermann's Formula)

$$[B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix} \qquad \emptyset(A) = \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -34 & 117 \end{bmatrix}$$

$$K = [0 \quad 0 \quad 1][B \quad AB \quad A^2]^{-1}\emptyset(A)$$

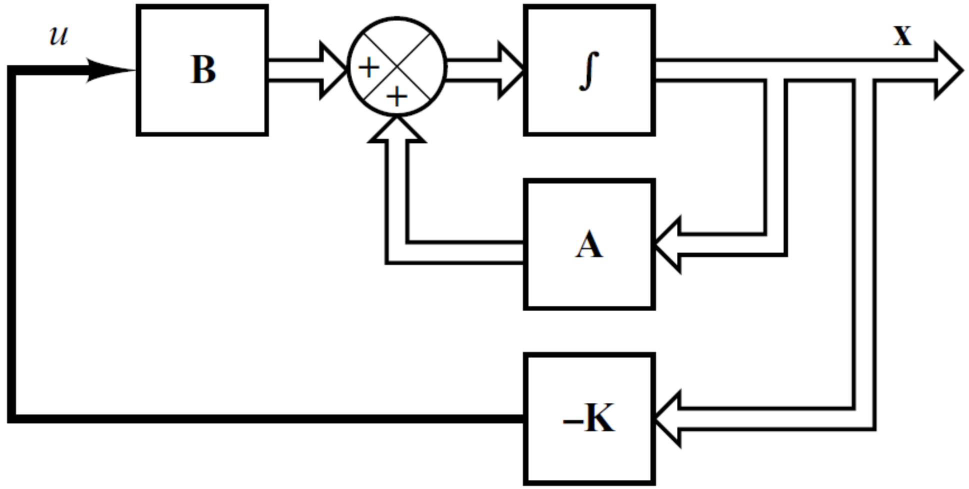
$$K = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}^{-1}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -34 & 117 \end{bmatrix}$$

$$K = [199 \quad 55 \quad 8]$$

Pole Placement

- Example-2: Consider the regulator system shown in following figure. The plant is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$



- Determine the state feedback gain for each state variable to place the poles at -1+j, -1-j,-3. (Apply all methods)