# Introduction to Robotics

Lecture One

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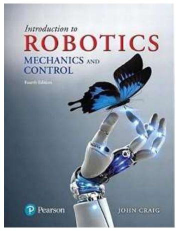
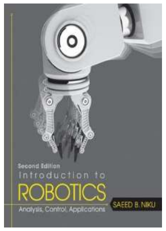
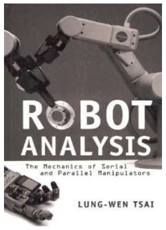
Class Code: mcpkee7

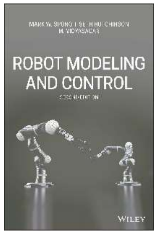
# References:

## Theory

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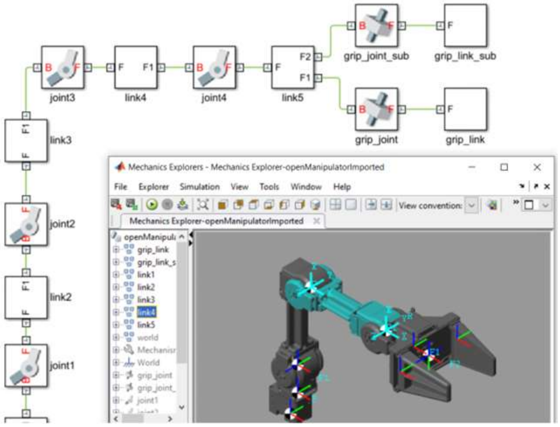
|---|---|---|---|
| [1] Introduction to Robotics Mechanics and Control by [John_J.Craig] | [2] Introduction to Robotics Analysis, systems, Applications by Saeed B. Niku | [3] L. W. Tsai, “ Robot analysis: the mechanics of serial and parallel manipulators,” John Wiley & Sons, Inc., 1999. | [4] Book - 2020 - Robot Modeling and Control - Mark W. Spong, Seth Hutchinson, M. Vidyasagar |

## Practical

| | | |
|---|---|---|
| RoboDK Simulation Platform | Overview to Simscape Multibody with Simulink in MATLAB with Integration of Robotics Toolbox | RVC Matlab Tool |



# Who Coined the Terms Robot & Robotics

- The term **robot** was first introduced by the Czech playwright Karel Capek in his 1920 play Rossum's Universal Robots, the word **robota** (meaning "**worker**.") being the Czech word for worker. Since then the term has been applied to a great variety of mechanical devices, such as teleoperators, underwater vehicles, autonomous cars, drones, etc. Virtually anything that operates with some degree of autonomy under computer control has at some point been called a **robot**. [4]

- ***robot*** and ***robotics*** were coined by science fiction writers. Karel Capek gave us ***robot*** in his 1922 play ***Rossum's Universal Robots*** **(RUR)**, and Isaac Asimov coined the word ***robotics*** in the early 1940s to describe the art and science in which we roboticists are engaged today. [5]

# Who Coined the Terms Robot & Robotics

- There is an important distinction between these two science fiction writers. Capek decided that robots would ultimately become malevolent and take over the world, while Asimov from the outset built circuits into his robots to assure mankind that robots would always be benevolent. [5]

- In summary, a **robot** is a programmable machine that can complete a task, while the term **robotics** describes the field of study focused on developing robots and automation.

# Definitions of Industrial Robot

- Any automatic machine cannot be considered as a robot. Robot is to have a specific set of characteristics. Interestingly, a 3-axis computer numerical control (CNC) milling machine may have a very similar configuration and control system of a robot arm. [7]

- However, the CNC machine is just a machine. It cannot do jobs other than milling. But the robot must do something more. That is why the definitions are proposed for a machine to be a robot. Different countries have different definitions for a robot. [7]

# Definitions of Industrial Robot

- The Robot Institute of America (RIA, 1985) defines the robot as

***A robot is a reprogrammable multi-function manipulator designed to move materials, parts, or specialized devices through variable programmable motions for the performance of a variety of tasks.***

This definition restricts robots in industrial applications. The two important key words are ‘ reprogrammable’ and ‘ multi-functional’. If the machine is single functional, it cannot be reprogrammable. Reprogrammable means that

**(i) the robot motion is controlled by a written program and**

**(ii) the program can be modified to change significantly the robot motion.**

Multi-functional implies that a robot is able to perform many different tasks depending on the program in the memory and tooling at the end of arm. This means that the robot can be programmed for welding with a welding tool at the end of arm and can be reprogrammed if the end of arm has a totally new facility such as for gripping. [7]

# Definitions of Industrial Robot

- Another, a little broader definition is proposed by McKerrow (1986) as

***A robot is a machine which can be programmed to do a variety of tasks in the same way that a computer is an electronic circuit which can be programmed to do a variety of tasks.***

- This definition excludes numerical control machines because they can be programmed for variations within only one task. Teleoperators are also not considered as robots because there is a human in the control system. They provide extended capabilities, not a replacement of a human. [7]

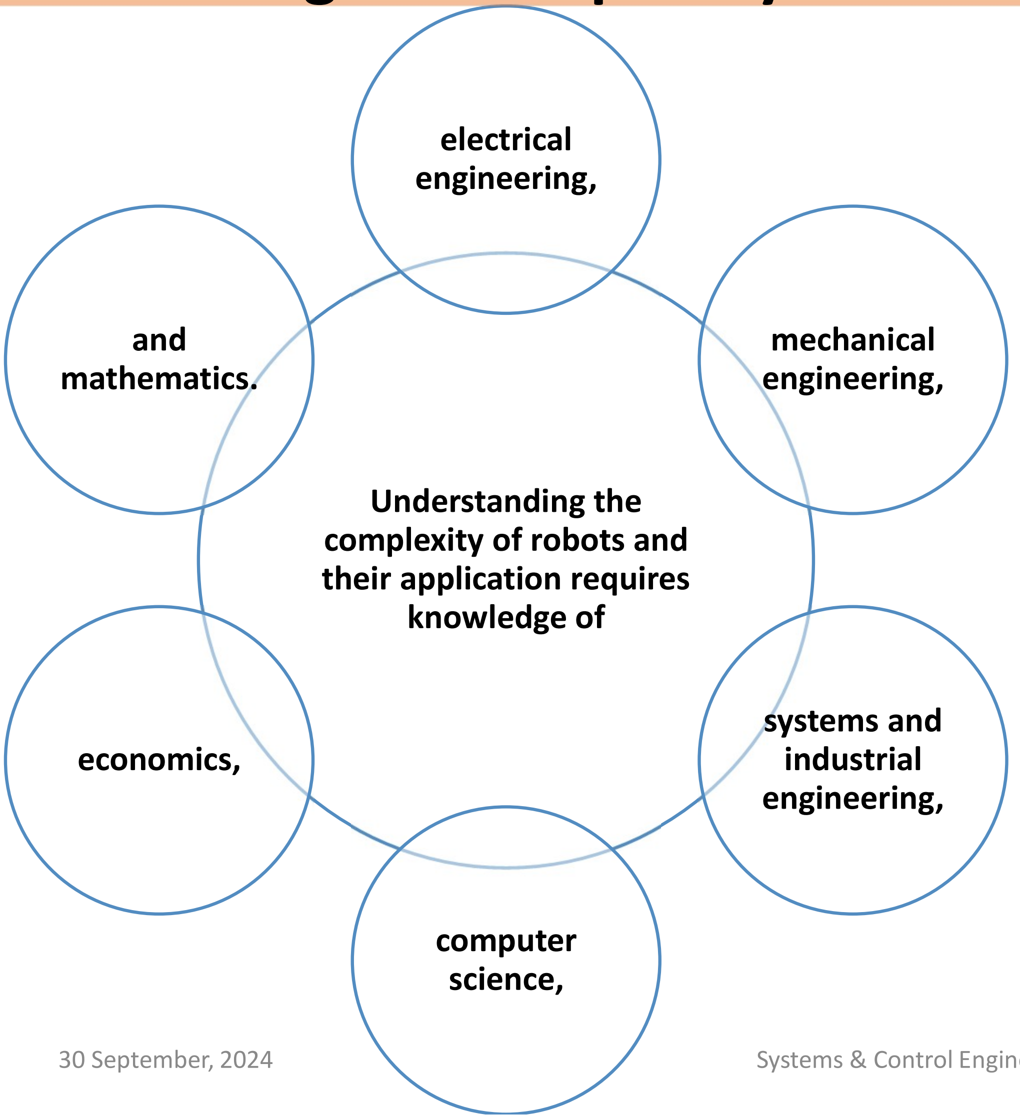
# Definitions of Industrial Robot [7]

- The International Standards Organization (ISO 8373) defines a robot in a similar way as follows:

**A robot is an automatically controlled, reprogrammable, multi-purpose, manipulative machine with several reprogrammable axes, which may be either fixed in place or mobile for use in industrial automation applications.**

- This definition specifically mentions 'reprogrammable axes' for industrial tasks. Such a definition particularly points out that industrial robots are very suitable to modern industries. [7]
- There are several such definitions on robots – industrial robots in particular. One way or other, each definition has to be expanded to suit the functioning of the modern industrial robots. In most cases, the definition given by RIA is accepted to be closer to industrial robots of modern times and such a definition is considered worth designing industrial robots. [7]

# Understanding the complexity of robots




New disciplines of engineering, such as manufacturing engineering, applications engineering, and knowledge engineering have emerged to deal with the complexity of the field of robotics and factory automation. More recently, mobile robots are increasingly important for applications like autonomous vehicles and planetary exploration. [4]

# Why do we use robots?

- An official definition of such a robot comes from the **Robot Institute of America (RIA)**:

*A **robot** is a reprogrammable, multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks. [4]*

The key element in the above definition is the reprogrammability, which gives a robot its utility and adaptability. The so-called **robotics revolution** is, in fact, part of the larger computer revolution. [4]

- Even this restricted definition of a robot has several features that make it attractive in an industrial environment. Among the advantages often cited in favor of the introduction of robots are **decreased labor costs**, **increased precision** and **productivity, increased flexibility compared with specialized machines**, and more humane working conditions as **dull**, **repetitive**, or **hazardous jobs** are performed by robots. [4]

# Automation & Industrial Automation

- Automation is based on foundations of automatic control and obviously on **feedback** theory. [7]
- Automation helps the industrial manufacturers in achieving **high productivity**, **high level of accuracy**, **consistent quality** and **increased labour saving**. [7]
- The term '**Industrial automation**' is defined as the technology concerned with control of systems in the process of achieving an end product. The process of achieving the end product has to be with minimum or no human intervention. [7]
- Intuitive inventions were contributing to the development of automatic control and hence automation till 1868 when Maxwell formulated **the mathematical model** (in terms of differential equations) to describe a system. He demonstrated the effect of parameters on system performance. The concepts of accuracy and stability were understood. [7]

# Automation & Industrial Automation

- **The mathematical models** in different forms such as transfer function, pulse transfer function, describing function and state variable modelling were considered as inevitable tools for analysing and designing of control systems.[7]
- The idea of using computers in automation emerged during 1950s.
- Automation without a computer is now hard to imagine.
- Analogue computers were used as on-line controllers in continuous processes such as steel and paper industries.
- The cost of analogue controllers increased linearly with increased control loops. On the other hand, even though initial cost of digital computer was large, the cost of adding additional loops was small.

# Automation & industrial robots

- ***Automation*** and ***industrial robots*** are two closely related technologies. According to definition of automation, an industrial robot can be considered itself as a form of automation.
- A **robot (industrial robots)** is a general purpose programmable machine which possesses the characteristics of a human arm. The robot can be programmed by its computer to move its arm through sequences of motion in order to perform some useful tasks. It repeats the motions over and over until it is reprogrammed to perform some other task. Many industrial operations involve robots working together with other equipment.

# The Laws of Robotics

- In 1942, Sir Isaac Asimov developed the famous three laws (Law one, Law two and Law three) of robotics which still remain as worthy industrial design standard. However, the laws have been extended and revised by him and others since 1985 to accommodate his creations, his attitude to robotics and the modern requirements of humanity. The extended set of laws is as follows:
- **The Meta-Law**

A robot may not act unless its actions are subject to the laws of robotics.

- **Law Zero**

A robot may not injure humanity, or through inaction, allow a humanity to come to harm (humanity is the family of all human beings and other biologically living things).

- **Law One**

A robot may not injure a human being, or through inaction, allow a human being to come to harm, unless this would violate a higher order (an earlier stated) law.

# The Laws of Robotics

- **Law Two**
- A robot must obey orders given by human being, except where such orders would conflict with a higher order law.
- A robot must obey orders given by subordinate robots, except where such orders would conflict with a higher order law.

  **Law Three**
- A robot must protect the existence of a subordinate robot as long as such protection does not conflict with a higher order law.
- A robot must protect its own existence as long as such protection does not conflict with a higher order law.
- **Law Four**
- A robot must perform the duties for which it has been programmed, except where that would conflict with a higher order law.

# The Laws of Robotics

- **The Procreation Law**
- The robot may not take any part in the design or manufacture of a robot unless the new robot's actions are subject to the laws of robotics.
- The robots which are strictly manufactured in accordance with the above rules do behave better than human beings.
- When the concept of robot was introduced and strengthened, the necessity of industrial automation was also deeply felt. Moreover, the technological progress in thermionic valve (1904), hydraulic and pneumatic systems (1906), logic circuits (1943), digital computer (1946), transistor (1947), microelectronics (1970) and microcomputer (1977) have all made automation and robotics a reality. The first commercial robot, controlled by limit switches and cams, was introduced in 1959. Since then, the development in robot technology has been in constant growth. Nowadays, the service robot within industry and in other areas of applications has made a breakthrough in robot applications.

**Table 1.1** Chronological Developments of Robot Technology

| Year | Development |
|---|---|
| 1921 | The word 'Robot' was coined |
| 1939 | Early humanoid robot exhibited in 1939, 1940 World Fairs by Westinghouse Electric Corporation |
| 1942 | The word 'robotics' appears in Sir Isaac Asimov story 'Runaround' |
| 1952 | Numerical control machine demonstrated at Massachusetts Institute of Technology, USA |
| 1954 | George Devol designed the programmable, teachable and digitally controlled article transfer robot |
| 1956 | First robot company UNIMATION formed |
| 1959 | First commercial robot controlled by limit switches and cams |
| 1961 | First hydraulic drive robot UNIMATE in die casting machine in Ford Motors |
| 1968 | Mobile robot designed by Stanford Research Institute |
| 1971 | Electrically powered 'Stanford Arm' developed by Stanford University |
| 1974 | ASEA introduced all electric drive IRb6 robot |
| 1974 | KAWASAKI installed arc welding robot |
| 1974 | Cincinnati Milacron introduced $T^3$ robot with computer control |
| 1978 | PUMA robot introduced by UNIMATION for assembly applications |
| 1978 | Cincinnati Milacron's $T^3$ robot applied in drilling and routing operations |
| 1979 | SCARA robot introduced for assembly applications at Yamanashi University |
| 1980 | Bin-picking robotic applications demonstrated at the University of Rhode Island |
| 1983 | Flexible automated assembly line using robots by Westinghouse Corporation |
| 1986 | 'ORACLE' robot used in commercial wool harvesting from sheep, Australia |
| 1992 | Flexible hydraulic microactuator for robotic mechanism developed in Japan |
| 1992 | First Humanoid by Honda Corporation, Japan, recognizes human faces |
| 2000 | Humanoid robot, ASIMO, put in service to society |
| 2004 | NASA in USA developed RED BALL robot with an intention of protecting the astronaut coming out of space vehicle for repairs |
| 2006 | Jumping robot has been developed to investigate the surface of any unknown areas |
| 2010 | RED BALL has been manufactured for protecting the astronaut |

# Spatial Descriptions and transformations

(DESCRIPTIONS, AND MAPPINGS)

## Lecture Two

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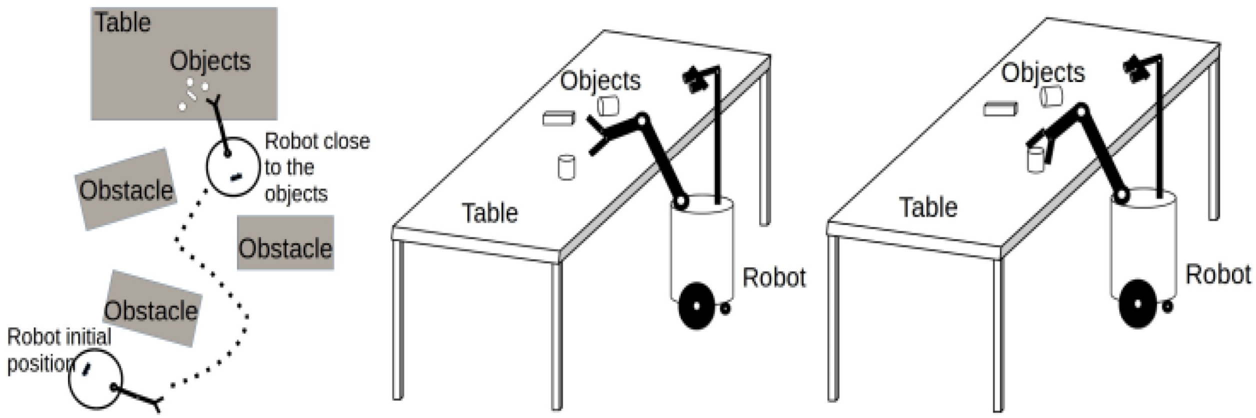
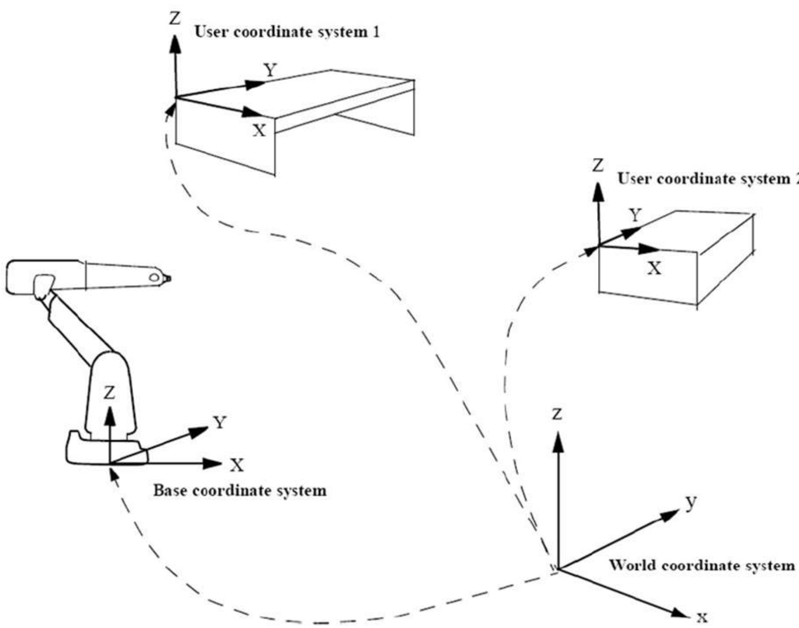



(a) (b) (c)

Contents

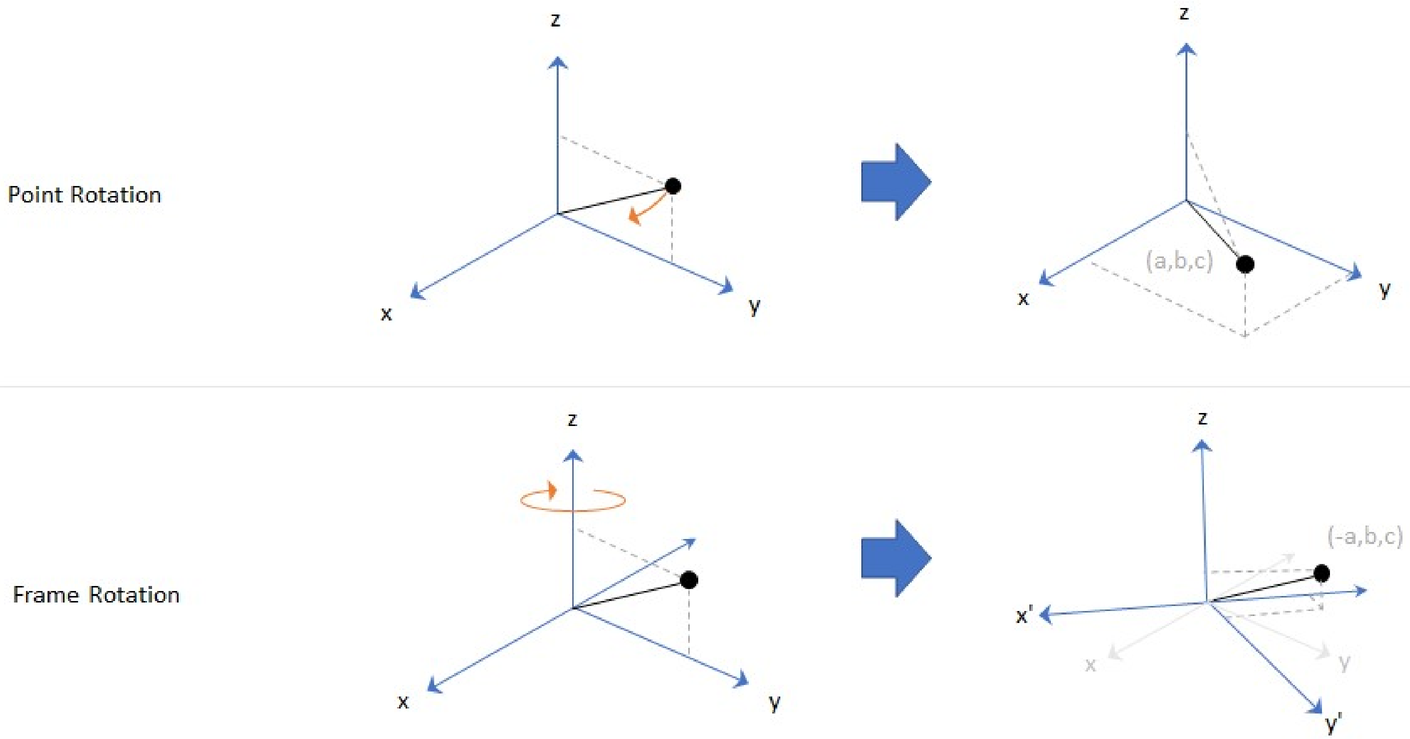
# 2.1 INTRODUCTION

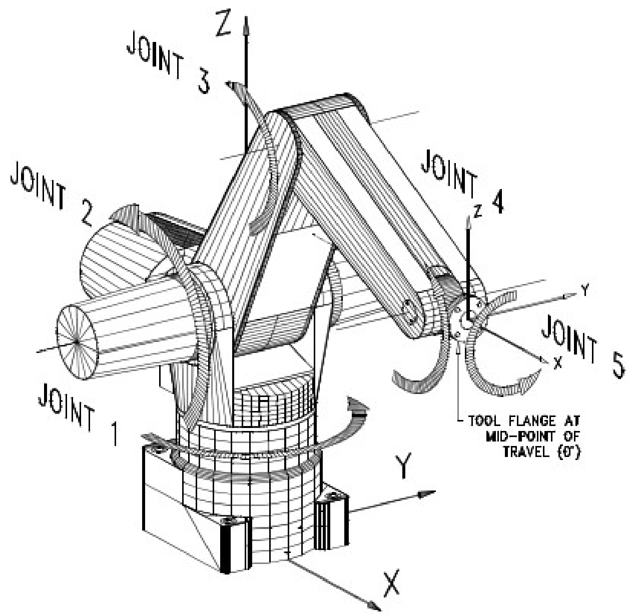
- **Robotic manipulation**, by definition, implies that parts and tools will be moved around in space by some sort of mechanism. [1]
- This naturally leads to a need for **representing positions and orientations** of parts, of tools, and of the mechanism itself. [1]
- To define and manipulate mathematical quantities that represent position and orientation, we must define coordinate systems and develop conventions for representation. [1]
- We will describe all positions and orientations with respect to the **universe coordinate system** or **with respect to other Cartesian coordinate systems that are** (or could be) **defined relative to the universe system**. [1]



## 2.2 DESCRIPTIONS: POSITIONS, ORIENTATIONS, AND FRAMES

- A **description** is used to specify attributes of **various objects** with which a manipulation system deals. **These objects are parts, tools, and the manipulator itself**.



# Description of a position

- Once a coordinate system is established, **we can locate any point in the universe with a 3 × 1 position vector**.
- Because we will often define many coordinate systems in addition to the universe coordinate system, vectors must be tagged with information identifying which coordinate system they are defined within.
- vectors are written with a leading superscript indicating the coordinate system to which they are referenced — for example, ${}^{A}\boldsymbol{P}$.
- This means that the components of ${}^{A}\boldsymbol{P}$ have numerical values that indicate distances along the axes of **{A}**.

$$ {}^{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_x \\ \boldsymbol{P}_y \\ \boldsymbol{P}_z \end{bmatrix} ---(2.1) $$

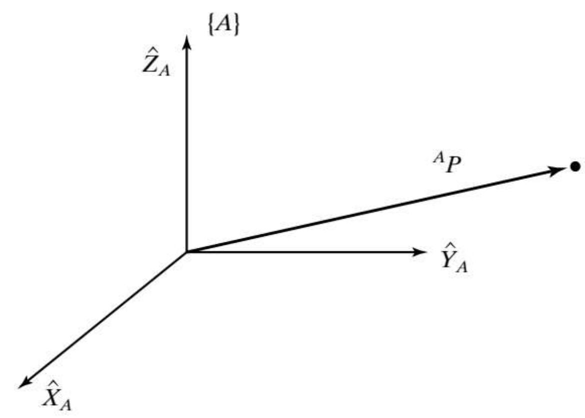



FIGURE 2.1: Vector relative to frame (example).

# Description of a position

- Figure 2.1 pictorially represents a coordinate system, **{A}**, with **three mutually orthogonal** unit vectors with solid heads.

- A point $^{A}P$ is represented as a vector and can equivalently be thought of as a position in space, or simply as an ordered set of three numbers.

$$^{A}P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} ---(2.1)$$

- **For example**, in a 3D space, let's consider a **point P(3, 2, 1)**. The position vector of this point from the **origin O(0, 0, 0)** would be: $^{O}P = 3\hat{\imath} + 2\,\hat{\jmath} + \hat{k}$

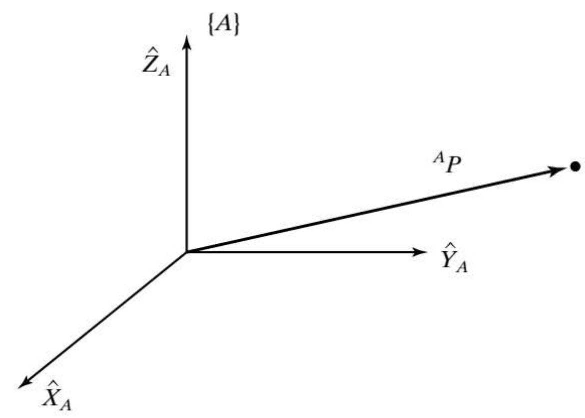



FIGURE 2.1: Vector relative to frame (example).

# Vectors and Geometry of Space

For the following equations, assume that vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and are defined as:

$$\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3), \vec{c} = (c_1, c_2, c_3),$$

Length of vector $\vec{a}$:

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

The unit vector corresponding to vector $\vec{a}$ is:

$$\frac{\vec{a}}{|\vec{a}|} = \frac{a_1\vec{i} + a_2\vec{j} + a_3\vec{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

The dot product between vectors $\vec{a}$ and $\vec{b}$:

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

The angle between the two vectors θ is:

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

# Description of an orientation

- Often, we will find it necessary not only to represent a point in space but also to describe the **orientation of a body in space**.

- For example, if vector $^{A}P$ in **Fig. 2.2** locates the point directly between the fingertips of a manipulator's hand, the complete location of the hand is still not specified until its orientation is also given. **Assuming that the manipulator has a sufficient number of joints**, the hand could be **oriented** arbitrarily while keeping the point between the fingertips at the same position in space.

- **In order to describe the orientation of a body**, we will *attach a coordinate system to the body and then give a description of this coordinate system relative to the reference system*. In **Fig. 2.2**, **coordinate system {B}** has been attached to the body in a known way. A description of **{B}** relative to **{A}** now suffices to give the orientation of the body.

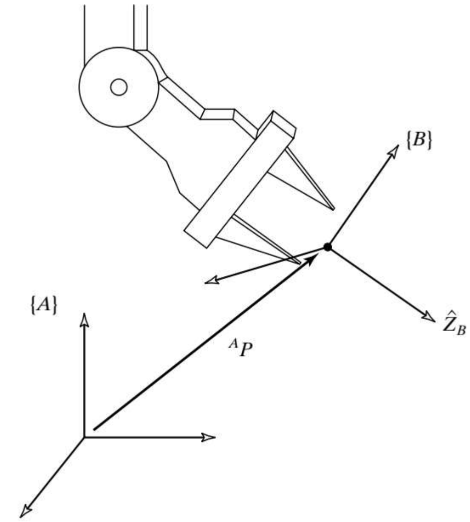


FIGURE 2.2: Locating an object in position and orientation.

# Description of an orientation

- Thus, positions of points are described with vectors and orientations of bodies are described with an attached coordinate system. One way to describe the **body attached coordinate system, {B}**, is to write the unit vectors of its three principal axes in terms of the **coordinate system {A}**.

- We denote the unit vectors giving the principal directions of **coordinate system {B}** as $\hat{X}_B$, $\hat{Y}_B$, **and** $\hat{Z}_B$. When written in terms of **coordinate system {A}**, they are called $\hat{X}_A$, $\hat{Y}_A$, **and** $\hat{Z}_A$.

- It will be convenient if we stack these three unit vectors together as the columns of a **3 × 3 matrix**, in the order ${}^A\hat{X}_B$, ${}^A\hat{Y}_B$, ${}^A Z_B$. We will **call this matrix a rotation matrix**, and, because this particular rotation matrix describes **{B}** relative to **{A}**, we name it with the notation ${}^A R_B$ (the choice of leading sub and superscripts in the definition of rotation matrices will become clear in following sections):

$$ {}^A_B R = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A Z_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad ---(2.2) $$

# Description of an orientation

- Hence, whereas the position of a point is represented with a vector, the orientation of a body is represented with a matrix.

$$ {}^{A}_{B}R = \left[ {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}Z_B \right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad ---(2.2) $$

$$ {}^{A}_{B}R = \left[ {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}Z_B \right] = \begin{bmatrix} \hat{X}_B.\hat{X}_A & \hat{Y}_B.\hat{X}_A & \hat{Z}_B.\hat{X}_A \\ \hat{X}_B.\hat{Y}_A & \hat{Y}_B.\hat{Y}_A & \hat{Z}_B.\hat{Y}_A \\ \hat{X}_B.\hat{Z}_A & \hat{Y}_B.\hat{Z}_A & \hat{Z}_B.\hat{Z}_A \end{bmatrix} \quad ---(2.3) $$

- For brevity, we have omitted the leading **superscripts** in the rightmost matrix of (2.3).

- The **dot product of two unit vectors yields the cosine of the angle between them**, so it is clear why the components of rotation matrices are often referred to as direction cosines.

$$ {}^{A}_{B}R = \left[ {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}Z_B \right] = \begin{bmatrix} {}^{B}\hat{X}_A^T \\ {}^{B}\hat{Y}_A^T \\ {}^{B}\hat{Z}_A^T \end{bmatrix} \quad ---(2.4) $$

$$ {}^{A}_{B}R = {}^{B}_{A}R^T \quad ---(2.5) $$

# Description of an orientation

$${}^{A}_{B}\boldsymbol{R} = \left[\,{}^{A}\hat{\boldsymbol{X}}_{B} \quad {}^{A}\hat{\boldsymbol{Y}}_{B} \quad {}^{A}\boldsymbol{Z}_{B}\,\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} ---(2.2)$$

$${}^{A}_{B}\boldsymbol{R} = \left[\,{}^{A}\hat{\boldsymbol{X}}_{B} \quad {}^{A}\hat{\boldsymbol{Y}}_{B} \quad {}^{A}\boldsymbol{Z}_{B}\,\right] = \begin{bmatrix} \hat{X}_{B}.\hat{X}_{A} & \hat{Y}_{B}.\hat{X}_{A} & \hat{Z}_{B}.\hat{X}_{A} \\ \hat{X}_{B}.\hat{Y}_{A} & \hat{Y}_{B}.\hat{Y}_{A} & \hat{Z}_{B}.\hat{Y}_{A} \\ \hat{X}_{B}.\hat{Z}_{A} & \hat{Y}_{B}.\hat{Z}_{A} & \hat{Z}_{B}.\hat{Z}_{A} \end{bmatrix} ---(2.3)$$

$${}^{A}_{B}\boldsymbol{R} = \left[\,{}^{A}\hat{\boldsymbol{X}}_{B} \quad {}^{A}\hat{\boldsymbol{Y}}_{B} \quad {}^{A}\boldsymbol{Z}_{B}\,\right] = \begin{bmatrix} {}^{B}\hat{\boldsymbol{X}}_{A}^{T} \\ {}^{B}\hat{\boldsymbol{Y}}_{A}^{T} \\ {}^{B}\hat{\boldsymbol{Z}}_{A}^{T} \end{bmatrix} ---(2.4)$$

$${}^{B}_{A}\boldsymbol{R} = {}^{A}_{B}\boldsymbol{R}^{T} ---(2.5)$$

$${}^{A}_{B}\boldsymbol{R} \;\; {}^{A}_{B}\boldsymbol{R}^{T} = \begin{bmatrix} {}^{A}\hat{\boldsymbol{X}}_{B}^{T} \\ {}^{A}\hat{\boldsymbol{Y}}_{B}^{T} \\ {}^{A}\hat{\boldsymbol{Z}}_{B}^{T} \end{bmatrix} \left[\,{}^{A}\hat{\boldsymbol{X}}_{B} \quad {}^{A}\hat{\boldsymbol{Y}}_{B} \quad {}^{A}\boldsymbol{Z}_{B}\,\right] = \boldsymbol{I}_{3} ---(2.6)$$

$${}^{A}_{B}\boldsymbol{R} = {}^{B}_{A}\boldsymbol{R}^{-1} = {}^{B}_{A}\boldsymbol{R}^{T} ---(2.7)$$

# Description of a frame

- For convenience, the point whose position we will describe is chosen as the origin of the body-attached frame.
- The situation of a **position** and an **orientation** pair arises so often in robotics that we define an entity **called a frame**, which is a set of four vectors giving **position** and **orientation** information.
- Note that a frame is a coordinate system where, in addition to the orientation, we give a position vector which locates its origin relative to some other embedding frame. For example, frame $\{B\}$ is described by ${}^{A}_{B}\boldsymbol{R}$ and ${}^{A}\boldsymbol{P}_{BORG}$, where ${}^{A}\boldsymbol{P}_{BORG}$ is the vector that locates the origin of the frame

$$\{B\} = \{{}^{A}_{B}\boldsymbol{R} \;,\; {}^{A}\boldsymbol{P}_{BORG}\} --- (2.8)$$

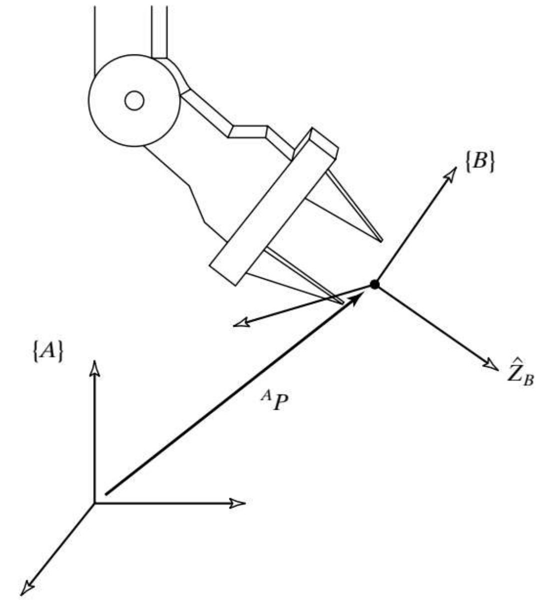



FIGURE 2.2: Locating an object in position and orientation.

# Description of a frame

- In **Fig. 2.3**, there are three frames that are shown along **with the universe coordinate system**. Frames **{A}** and **{B}** are known relative to the universe coordinate system, and frame **{C}** is known relative to frame **{A}**.

- In **Fig. 2.3**, we introduce a ***graphical representation of frames***, which is convenient in visualizing frames. A frame is depicted by three arrows representing unit vectors defining the principal axes of the frame. An arrow representing a vector is drawn from one origin to another. This vector represents the position of the origin at the head of the arrow in terms of the frame at the tail of the arrow. The direction of this locating arrow tells us, for example, in **Fig. 2.3**, that **{C}** is known relative to **{A}** and not vice versa.

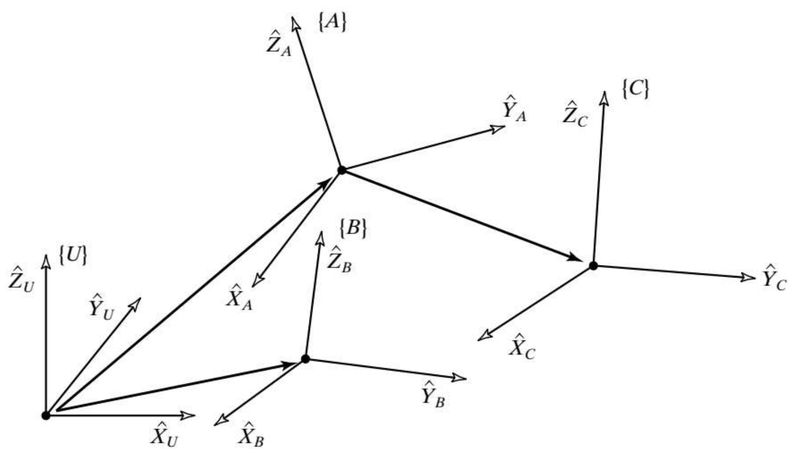
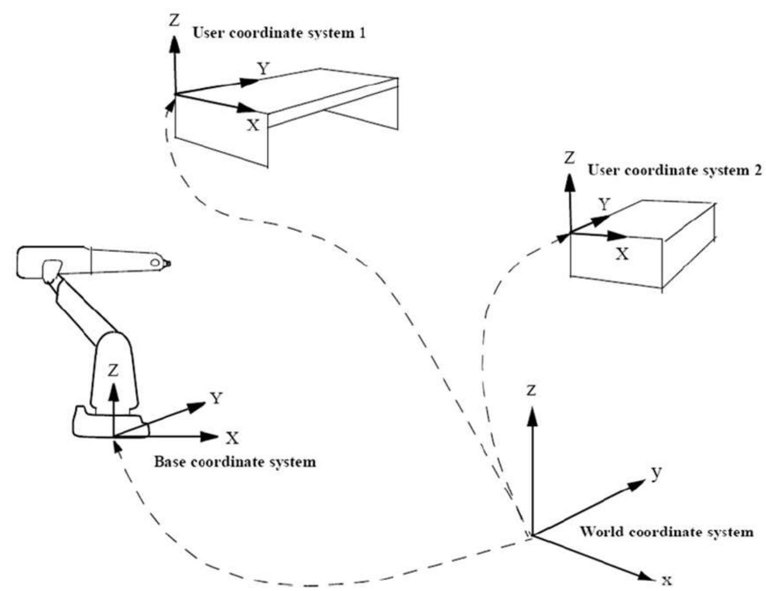



FIGURE 2.3: Example of several frames.



# Description of a frame

- In summary, a frame can be used as a description of one coordinate system relative to another.
- A frame encompasses two ideas by representing both position and orientation and so may be thought of as a generalization of those two ideas.
- Positions could be represented by a frame whose rotation-matrix part is the identity matrix and whose position-vector part locates the point being described.
- Likewise, an orientation could be represented by a frame whose position-vector part was the zero vector.

# MAPPINGS: CHANGING DESCRIPTIONS FROM FRAME TO FRAME

- In a great many of the problems in robotics, we are concerned with expressing the same quantity in terms of various reference coordinate systems. The previous section introduced descriptions of positions, orientations, and frames; we now consider the mathematics of **mapping** in order to change descriptions from frame to frame.

# Mappings involving translated frames

- In **Fig. 2.4**, we have a **position** defined by the vector $^{B}P$. We wish to express this point in space in terms of **frame {A}**, when **{A}** has the same **orientation** as **{B}**. In this case, **{B}** differs from **{A}** only by a ***translation***, which is given by $^{A}P_{BORG}$, a vector that locates the origin of **{B}** relative to **{A}**

$$^{A}P = {}^{B}P + {}^{A}P_{BORG} \quad ---(2.9)$$

- Note that only in the special case of equivalent orientations may we add vectors that are defined in terms of different

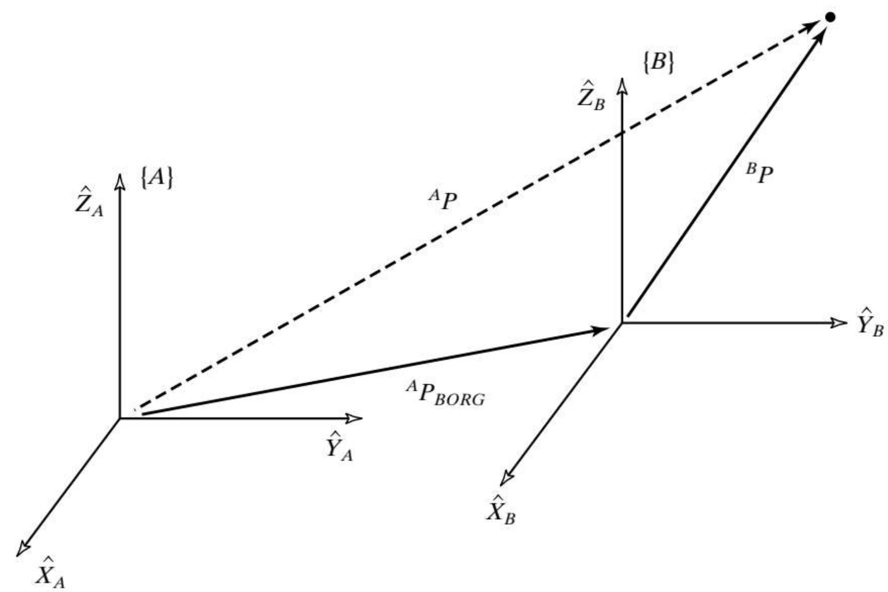





FIGURE 2.4: Translational mapping.

# Mappings involving rotated frames

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} \quad ---(2.10) $$

$$ {}^{A}_{B}R = \left[ {}^{A}\hat{X}_{B} \quad {}^{A}\hat{Y}_{B} \quad {}^{A}Z_{B} \right] = \begin{bmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{bmatrix} ---(2.11) $$

$$ {}^{A}p_{x} = {}^{B}\hat{X}_{A} \,.\, {}^{B}P \quad , $$

$$ {}^{A}p_{y} = {}^{B}\hat{Y}_{A} \,.\, {}^{B}P \quad ,\text{- - - (2.12)} $$

$$ {}^{A}p_{z} = {}^{B}\hat{Z}_{A} \,.\, {}^{B}P \quad , $$

$$ {}^{A}P = {}^{A}_{B}R \quad {}^{B}P \quad ---(2.13) $$

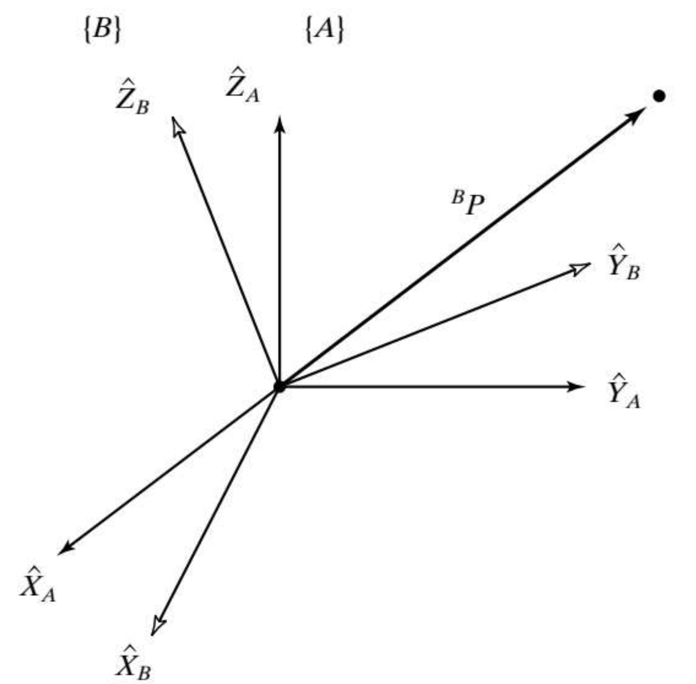



FIGURE 2.5: Rotating the description of a vector.

# Mappings involving general frames

$$^{A}P = {}^{A}_{B}R \; {}^{B}P + {}^{A}P_{BORG} \quad ---(2.17)$$

$$^{A}P = {}^{A}_{B}T \; {}^{B}P \quad ---(2.18)$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix} \quad ---(2.19)$$ **called a homogeneous transform**

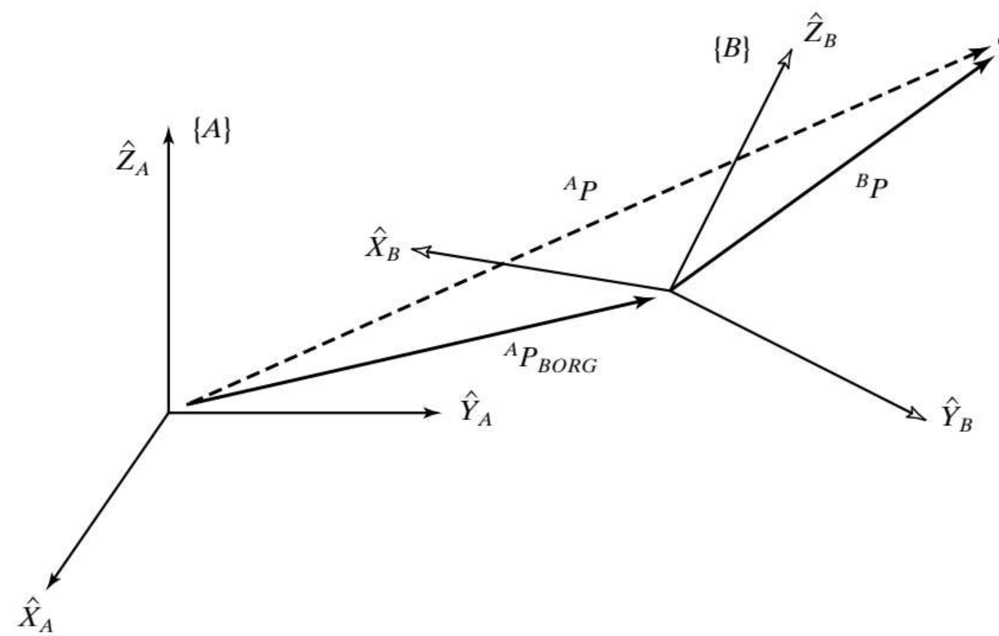





FIGURE 2.7: General transform of a vector.

# Mappings involving general frames

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} ---(2.19)$$

1. *a "1"* is added as the last element of the 4 × 1 vectors;
2. a row ''[0 0 0 1]'' is added as the last row of the 4 × 4 matrix.

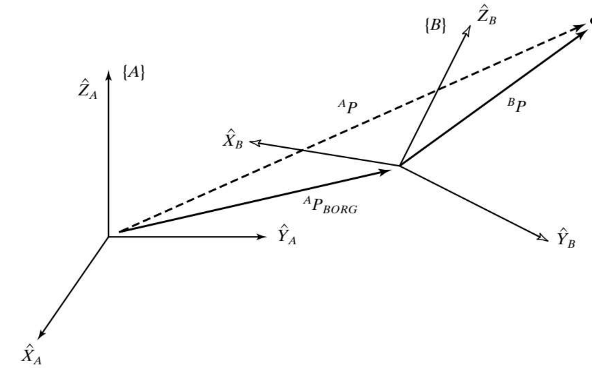


FIGURE 2.7: General transform of a vector.

The 4×4 matrix in (2.19) is called a **homogeneous transform**. For our purposes, it can be regarded purely as a construction used to cast the rotation and translation of the general transform into a single matrix form. In other fields of study, it can be used to compute perspective and scaling operations (when the last row is other than ''[0 0 0 1]'' or the rotation matrix is not orthonormal).

$${}^{A}P = {}^{A}_{B}R \; {}^{B}P + {}^{A}P_{BORG} ---(2.20)$$

1=1

# Rotation Matrix

- $${}^{A}_{B}R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} ---(about\ X)$$

- $${}^{A}_{B}R_y = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} ---(about\ Y)$$

- $${}^{A}_{B}R_z = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} ---(about\ Z)$$

# Example 2.1 / Page 26

**Figure 2.6 shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees. Find the rotation matrix?**

**Then find the translation matrix of** $^{B}P = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ **relative to {A}?**

**Solution:**

$$R = ROT(\hat{z}, \theta)$$

$$^{A}_{B}R_z = \begin{bmatrix} ^{A}\hat{X}_B & ^{A}\hat{Y}_B & ^{A}Z_B \end{bmatrix} = \begin{bmatrix} \hat{X}_B.\hat{X}_A & \hat{Y}_B.\hat{X}_A & \hat{Z}_B.\hat{X}_A \\ \hat{X}_B.\hat{Y}_A & \hat{Y}_B.\hat{Y}_A & \hat{Z}_B.\hat{Y}_A \\ \hat{X}_B.\hat{Z}_A & \hat{Y}_B.\hat{Z}_A & \hat{Z}_B.\hat{Z}_A \end{bmatrix}$$

$$^{A}_{B}R_z = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C30 & -S30 & 0 \\ S30 & C30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{A}P = {}^{A}_{B}R_z \; {}^{B}P = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix}$$

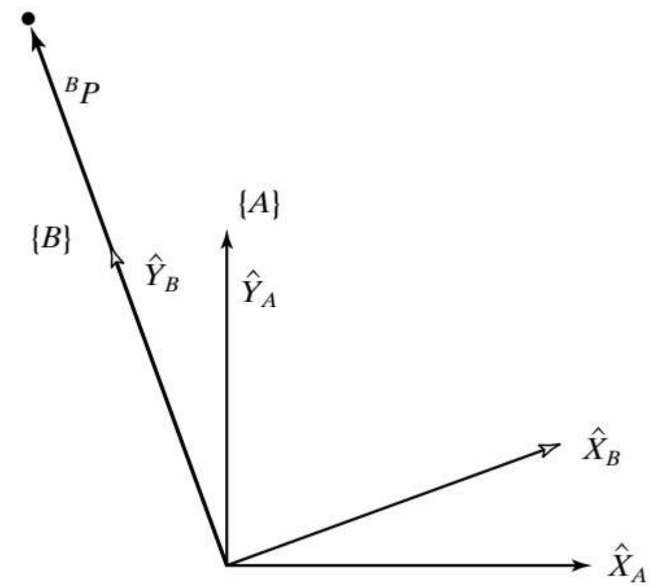



FIGURE 2.6: $\{B\}$ rotated 30 degrees about $\hat{Z}$.

$^{A}_{B}R_z$ acts as a mapping that is used to describe $^{B}P$ relative to frame {A}, $^{A}P$ .

As was introduced in the case of translations, it is important to remember that, viewed as a mapping, **the original vector P is not changed in space**.

# Example 2.2 / Page 29

**Figure 2.8 shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$ .Find the $^{A}P$ , where $^{B}P = [3 \quad 7 \quad 0]^T$.**

**Solution:**

$$R = ROT(\hat{z}, \theta)$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \qquad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3\times 2}$$

The definition of frame {B} is

$$^{A}_{B}T = \begin{bmatrix} C\theta & -S\theta & 0 & X \\ S\theta & C\theta & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Vector (origin to origin)

Scale

$$^{A}P = {}^{A}_{B}T \; {}^{B}P \quad ---(2.18)$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} ^{A}_{B}R & ^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix} \quad ---(2.19)$$

$$^{A}P = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}$$

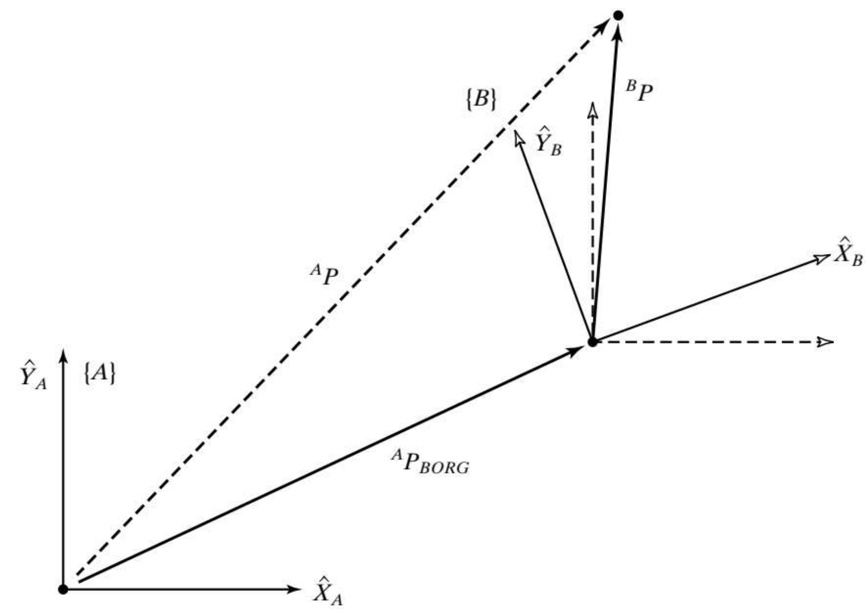
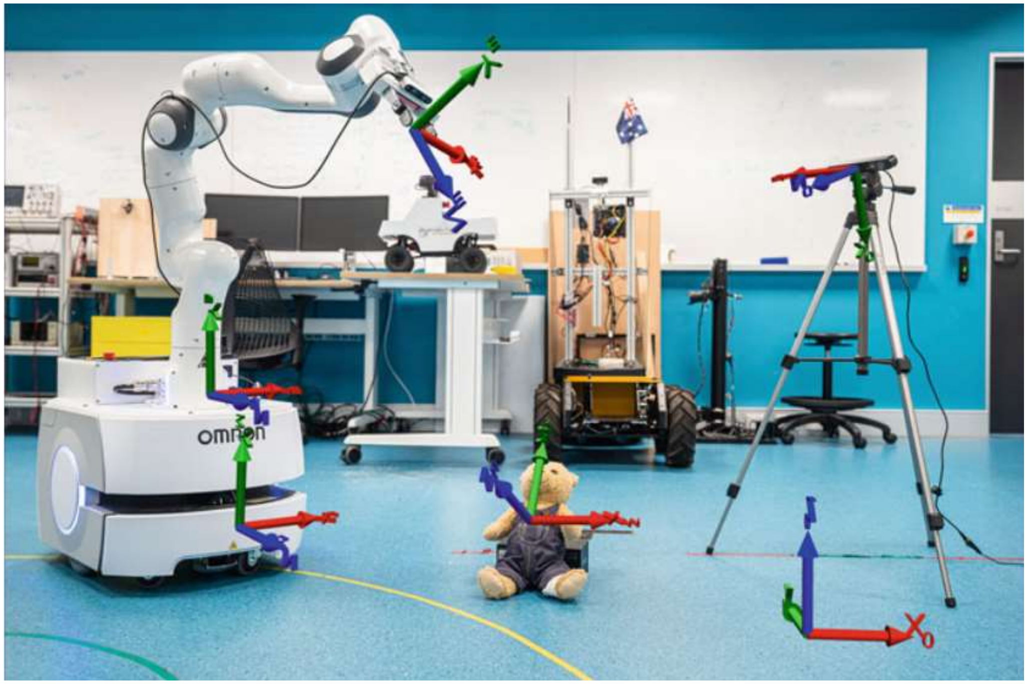



FIGURE 2.8: Frame {B} rotated and translated.

# Spatial Descriptions and transformations

(OPERATORS: TRANSLATIONS, ROTATIONS, AND TRANSFORMATIONS)




Lecture Three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Prepared by : Yazen Hudhaifa

Contents

## 2.4 OPERATORS: TRANSLATIONS, ROTATIONS, AND TRANSFORMATIONS

- **Translational operators**
- **Rotational operators**
- **Transformation operators**

## 2.5 SUMMARY OF INTERPRETATIONS

## 2.6 TRANSFORMATION ARITHMETIC

- **Compound transformations**
- **Inverting a transform**

## 2.7 TRANSFORM EQUATIONS

## 2.4 OPERATORS: TRANSLATIONS, ROTATIONS, AND TRANSFORMATIONS

- The same mathematical forms used to map points between frames can also be interpreted as operators that translate points, rotate vectors, or do both.
- Translational operators
- Rotational operators

# Example 2.3 / Page 32

**Figure 2.10 shows a vector $^{A}P_{1}$ . Compute the vector obtained by rotating this vector about $\hat{Z}$ by 30 degrees. Call the new vector $^{A}P_{2}$ .**

Solution:

$$R_z\ (30,0) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C30 & -S30 & 0 \\ S30 & C30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{A}P_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$^{A}P_2 = R_z\ (30,0)\ ^{A}P_1 = \begin{bmatrix} -1 \\ 1.732 \\ 0 \end{bmatrix}$$

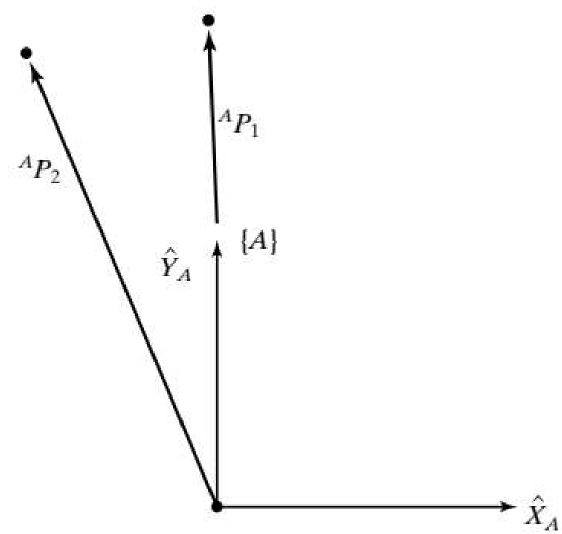


FIGURE 2.10: The vector $^{A}P_1$ rotated 30 degrees about $\hat{Z}$.

# Example 2.4 / Page 33

**Figure 2.11 shows a vector $^{A}P_{1}$. Compute the vector obtained by rotating this vector about $\hat{Z}$ by 30 degrees and translate it 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$. Find $^{A}P_{2}$, where $^{A}P_{1} = [3 \quad 7 \quad 0]^{T}$.**

Solution:

$$^{A}P_{1} = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$$

$$T = \begin{bmatrix} C\theta & -S\theta & 0 & X \\ S\theta & C\theta & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{A}P_{2} = T\ ^{A}P_{1} = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix}$$

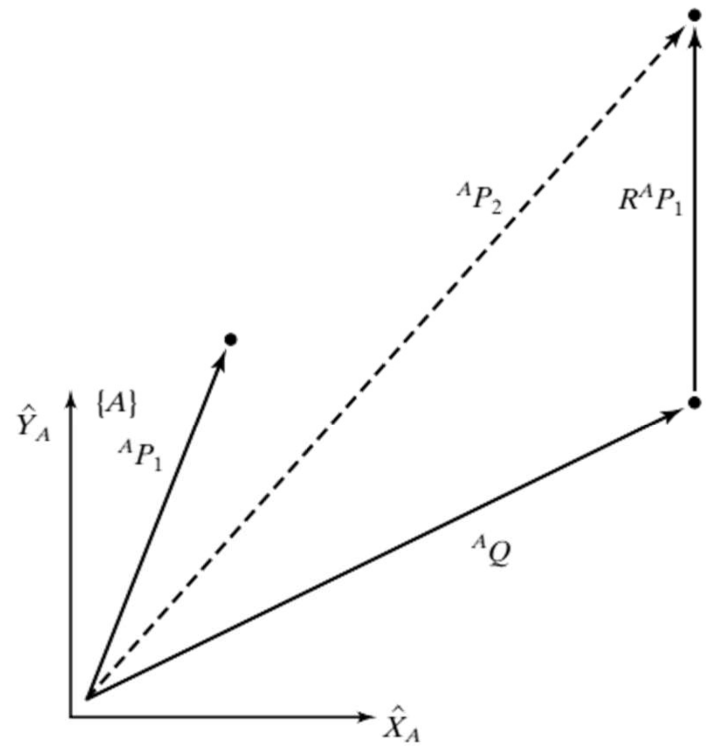


FIGURE 2.11: The vector $^{A}P_{1}$ rotated and translated to form $^{A}P_{2}$.

## 2.5 SUMMARY OF INTERPRETATIONS

- We have introduced concepts first for the case of translation only, then for the case of rotation only, and finally for the general case of rotation about a point and translation of that point.

- As a general tool to represent frames, we have introduced the ***homogeneous transform*****, a 4 × 4 matrix** containing orientation and position information. We have introduced three interpretations of this homogeneous transform: **1.** It is a *description of a frame*. ${}^{A}_{B}T$ describes the frame **{B}** relative to the frame **{A}**. Specifically, the columns of ${}^{A}_{B}R$ are unit vectors defining the directions of the principal axes of **{B}**, and ${}^{A}P_{BORG}$ locates the position of the origin of **{B}**.

- **2.** It is a ***transform mapping***. ${}^{A}_{B}T$ maps ${}^{B}P \rightarrow {}^{A}P$ .

- **3.** It is a ***transform operator***. **T** operates on ${}^{A}P_1$ to create ${}^{A}P_2$ .

- From this point on, the **terms *frame*** and ***transform*** will both be used to refer to a position vector plus an orientation. ***Frame*** is the term favored in speaking of a description, and ***transform*** is used most frequently when function as a **mapping** or **operator** is implied. Note that transformations are generalizations of (and subsume) translations and rotations; we will often use the **term *transform*** when speaking of a pure rotation (or translation).

## 2.6 TRANSFORMATION ARITHMETIC - Compound transformations

In **Fig. 2.12**, we have ${}^{C}\boldsymbol{P}$ and wish to find ${}^{A}\boldsymbol{P}$ .

Frame **{C}** is known relative to frame **{B}**, and frame **{B}** is known relative to frame **{A}**. We can transform ${}^{C}\boldsymbol{P}$ into ${}^{B}\boldsymbol{P}$ as:

$$ {}^{B}\boldsymbol{P} = {}^{B}_{C}\boldsymbol{T}\ {}^{C}\boldsymbol{P} \quad ---(\mathbf{2.37}) $$

then we can transform ${}^{B}\boldsymbol{P}$ into ${}^{A}\boldsymbol{P}$ as

$$ {}^{A}\boldsymbol{P} = {}^{A}_{B}\boldsymbol{T}\ {}^{B}\boldsymbol{P} \quad --(\mathbf{2.38}) $$

Combining (2.37) and (2.38), we get the (not unexpected) result

$$ {}^{A}\boldsymbol{P} = {}^{A}_{B}\boldsymbol{T}\ {}^{B}_{C}\boldsymbol{T}\ {}^{C}\boldsymbol{P} \quad ---(\mathbf{2.39}) $$

$$ {}^{A}_{C}\boldsymbol{T} = {}^{A}_{B}\boldsymbol{T}\ {}^{B}_{C}\boldsymbol{T} \quad ---(\mathbf{2.40}) $$

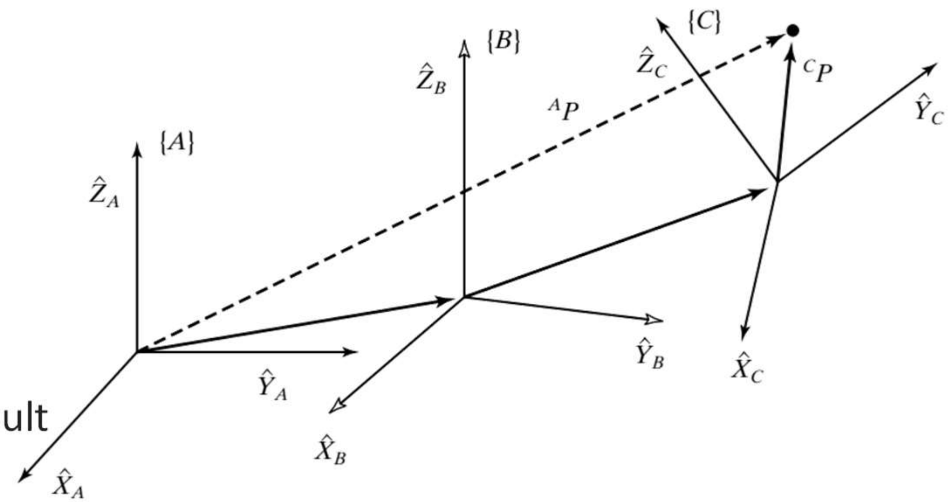


FIGURE 2.12: Compound frames: Each is known relative to the previous one.

## 2.6 TRANSFORMATION ARITHMETIC - Compound transformations

- Again, note that familiarity with the sub- and superscript notation makes these manipulations simple. In terms of the known descriptions of **{B}** and **{C}**, we can give the expression for ${}^{A}_{C}T$ as

$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R \quad {}^{B}_{C}R & {}^{A}_{B}R \ {}^{B}P_{CORG} + {}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} ---(2.41) $$

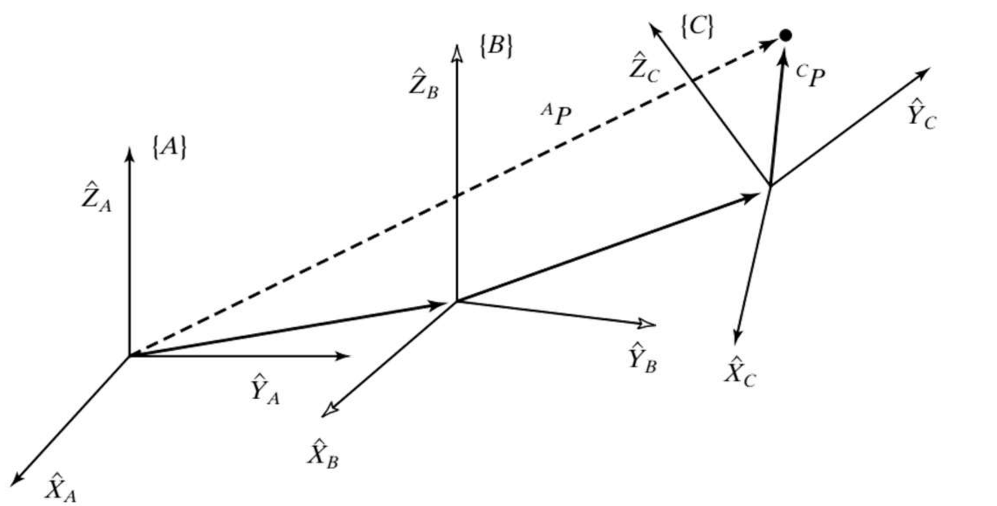




FIGURE 2.12: Compound frames: Each is known relative to the previous one.

## 2.6 TRANSFORMATION ARITHMETIC - Inverting a transform

Consider a frame **{B}** that is known with respect to a frame **{A}**—that is, we know the value of ${}^{A}_{B}T$ . Sometimes we will wish to invert this transform, in order to get a description of **{A}** relative to **{B}**—that is, ${}^{B}_{A}T$ . A straightforward way of calculating the inverse is to compute the inverse of the 4×4 homogeneous transform.

To find ${}^{B}_{A}T$ , we must compute ${}^{B}_{A}R$ and ${}^{B}P_{AORG}$ from ${}^{A}_{B}R$ and ${}^{A}P_{BORG}$. First, recall from our discussion of rotation matrices that

${}^{B}_{A}R = {}^{A}_{B}R^{T} \quad ---(2.42)$ || Next, we change the description of ${}^{A}P_{BORG}$ into {B} by using (2.13):

${}^{B}({}^{A}P_{BORG}) = {}^{B}_{A}R \; {}^{A}P_{BORG} + {}^{B}P_{AORG} \quad ---(2.43)$ || The left-hand side of (2.43) must be zero, so we have

${}^{B}P_{AORG} = -{}^{B}_{A}R \; {}^{A}P_{BORG} = -{}^{A}_{B}R^{T} \; {}^{A}P_{BORG} \quad ---(2.44)$ || Using (2.42) and (2.44), we can write the form of B AT as

${}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T} \; {}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \quad ---(2.45)$ || Note that, with our notation, ${}^{B}_{A}T = {}^{A}_{B}T^{-1}$

## Example 2.5 / Page 36

**Figure 2.13 shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees and translated four units in $\hat{X}_A$ and three units in $\hat{Y}_A$. Thus, we have a description of ${}^{A}_{B}T$. Find ${}^{B}_{A}T$.**

**The frame defining {B} is**

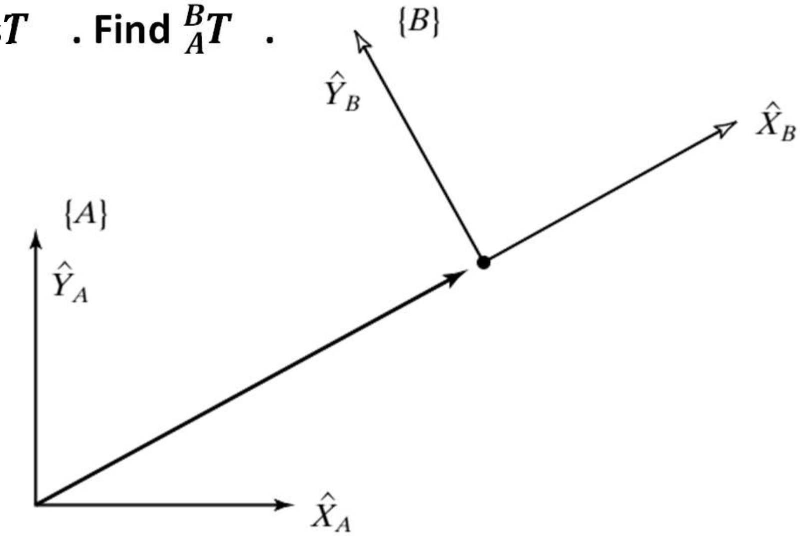


FIGURE 2.13: $\{B\}$ relative to $\{A\}$.

Solution:

$$ {}^{A}_{B}T = \begin{bmatrix} C\theta & -S\theta & 0 & X \\ S\theta & C\theta & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^T & -{}^{A}_{B}R^T \; {}^{A}P_{BORG} \\ \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} & \mathbf{1} \end{bmatrix} ---(2.45) \;||\; {}^{B}_{A}T = {}^{A}_{B}T^{-1} $$

$$ {}^{B}P_{AORG} = -{}^{A}_{B}R^T \; {}^{A}P_{BORG} = -\begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.964 \\ -0.598 \\ 0 \end{bmatrix} $$

$$ {}^{B}_{A}T = \begin{bmatrix} 0.866 & 0.5 & 0 & -4.964 \\ -0.5 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

|| Note: ${}^{B}P_{AORG} = -{}^{B}_{A}R \; {}^{A}P_{BORG} = -{}^{A}_{B}R^T \; {}^{A}P_{BORG}$

# Example 2.5 / Page 36

**Figure 2.13 shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees and translated four units in $\hat{X}_A$ and three units in $\hat{Y}_A$. Thus, we have a description of ${}^{A}_{B}T$. Find ${}^{B}_{A}T$.**

**The frame defining {B} is**

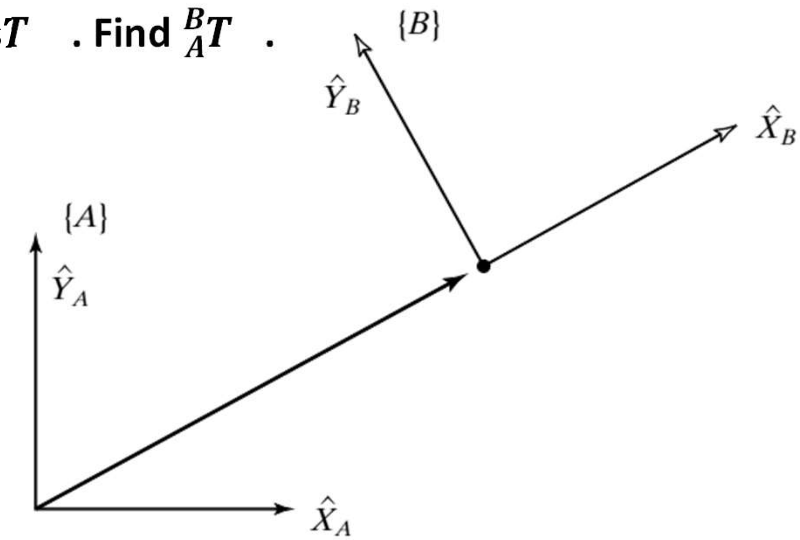


FIGURE 2.13: $\{B\}$ relative to $\{A\}$.

Solution:

$$ {}^{A}_{B}T = \begin{bmatrix} C\theta & -S\theta & 0 & X \\ S\theta & C\theta & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\ {}^{A}P_{BORG} \\ \mathbf{0 \quad 0 \quad 0} & \mathbf{1} \end{bmatrix} ---(2.45) \quad || \quad {}^{B}_{A}T = {}^{A}_{B}T^{-1} $$

$$ {}^{B}P_{AORG} = -{}^{A}_{B}R^{T}\ {}^{A}P_{BORG} = -\begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.964 \\ -0.598 \\ 0 \end{bmatrix} $$

$$ {}^{B}_{A}T = \begin{bmatrix} 0.866 & 0.5 & 0 & -4.964 \\ -0.5 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

|| Note: ${}^{B}P_{AORG} = -{}^{B}_{A}R\ \ {}^{A}P_{BORG} = -{}^{A}_{B}R^{T}\ {}^{A}P_{BORG}$

## Orthogonal Matrix

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A Square matrix 'A' is orthogonal if

$$A^T = A^{-1}$$

(OR)

$$AA^T = A^TA = I$$ , where

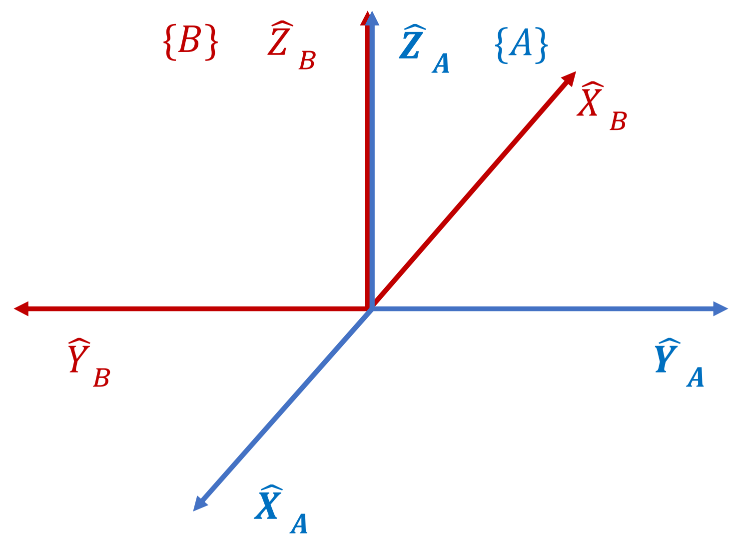
- $A^T$ = Transpose of A
- $A^{-1}$ = Inverse of A
- $I$ = Identity matrix of same order as 'A'

## Review on rotation Matrix

Rotation matrix represents the component of each axis of a coordinate system with respect to a reference frame.

$$ {}^{A}_{B}R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} {}^{B}\hat{X}_A^T \\ {}^{B}\hat{Y}_A^T \\ {}^{B}\hat{Z}_A^T \end{matrix} $$

$$ \begin{matrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{matrix} $$

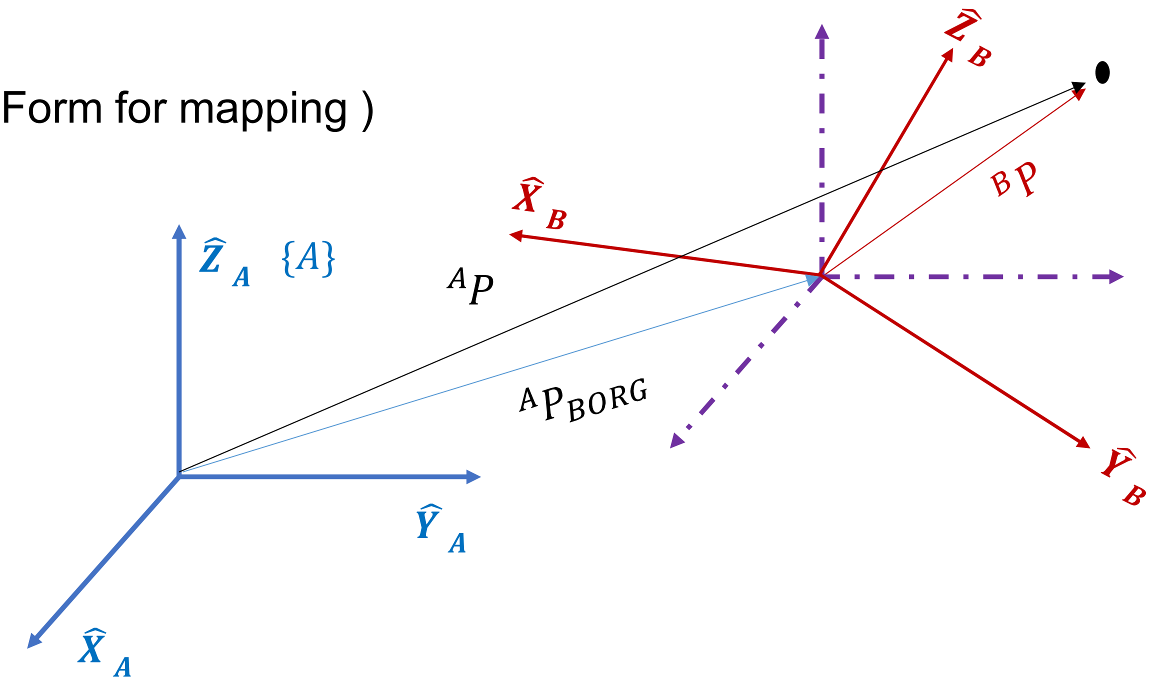


# General Transform

General Transform between Frames means that we include (Rotation+ Translation) to map or describe a point from one frame to another

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

( General Form for mapping )




Review Lecture One and Two

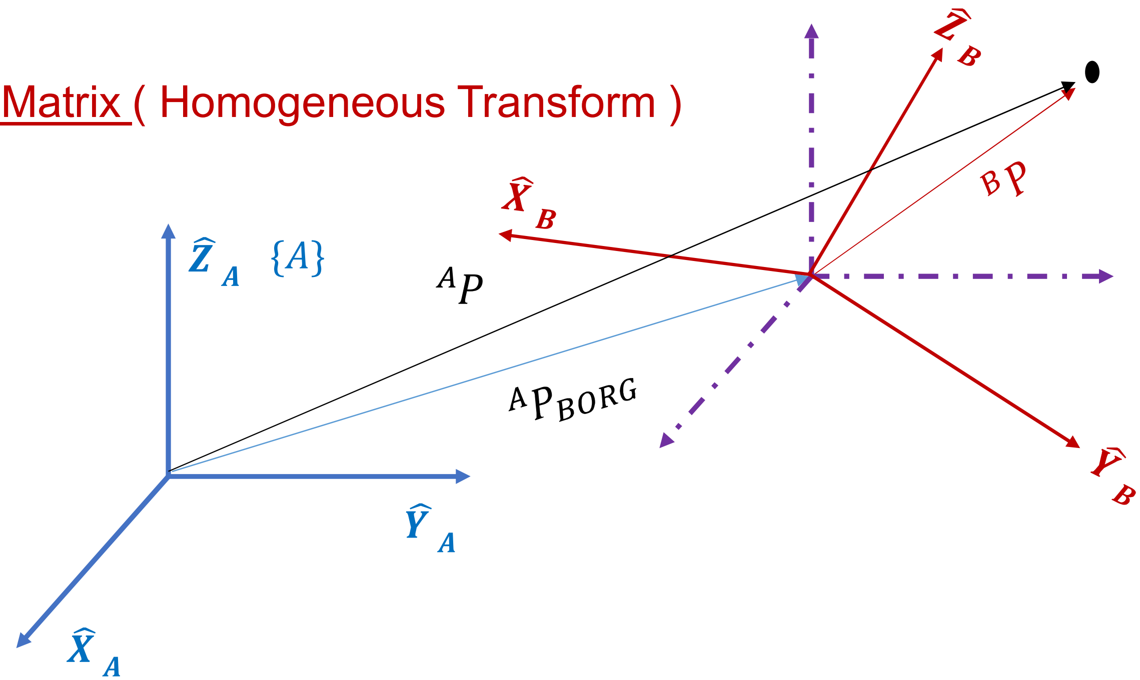
# Homogeneous Transform

We would like to think of a mapping from one frame to another as an operator in matrix form.

This aids in writing compact equations and is conceptually clearer than general form.

$$\begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix}$$

4X4 Matrix ( Homogeneous Transform )



# Homogeneous Transform

1. a "1" is added as the last element of the 4 x 1 vectors;

2. a row "[0001]" is added as the last row of the 4 x 4 matrix.

3. Normally we will write the Homogenous Transform as follow:

$$^{A}P = {}^{A}_{B}T\,{}^{B}P$$

$$\text{Where } {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BORG} \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

rotation matrices used to specify an orientation, we will use transforms (usually in homogeneous representation) to specify a frame. Observe that, although we have introduced homogeneous transforms in the context of mappings, they also serve as descriptions of frames. The description of frame {B} relative to (A} is ${}^{A}_{B}T$

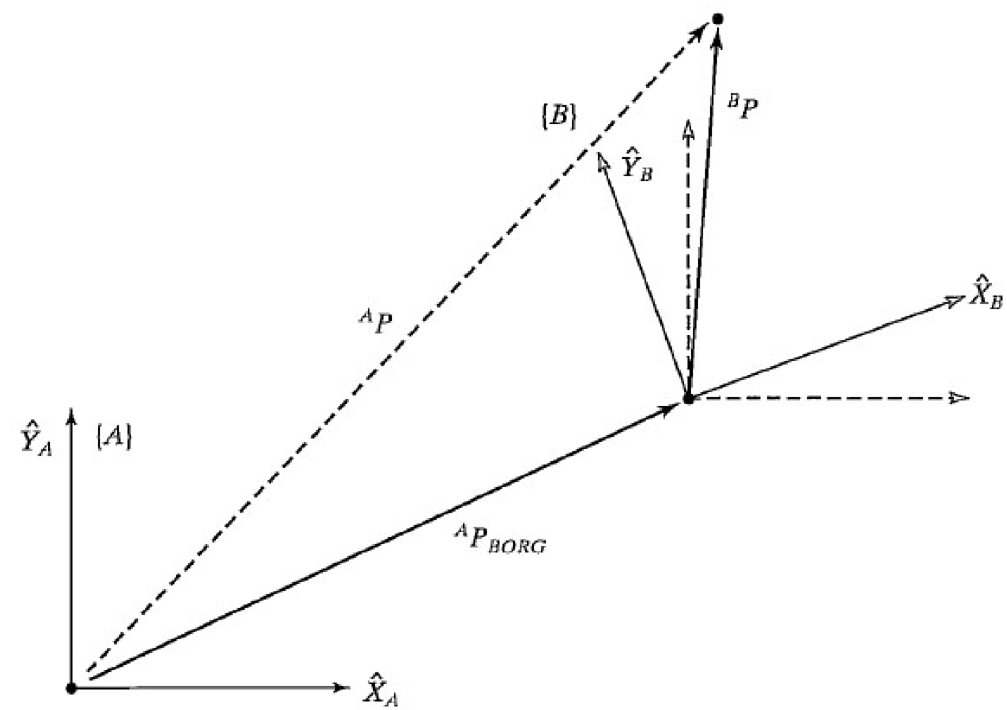
# Example

The Figure shows a frame {B}, which is rotated relative to frame (A} about $\hat{Z}$ by 30° degrees, translated 10 units in $\widehat{\boldsymbol{X}}_{\boldsymbol{A}}$, and translated 5 units in $\widehat{\boldsymbol{Y}}_{\boldsymbol{A}}$ Find ${}^{A}P$ where ${}^{B}P = [3.0\ \ 7.0\ \ 0.0]^T$.

$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} $$

Given ${}^{B}P = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$

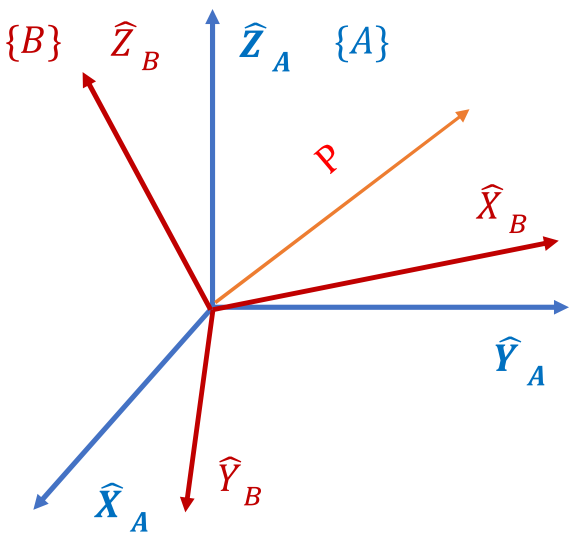
$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}P = \begin{bmatrix} 9.09 \\ 12.56 \\ 0.0 \end{bmatrix} $$



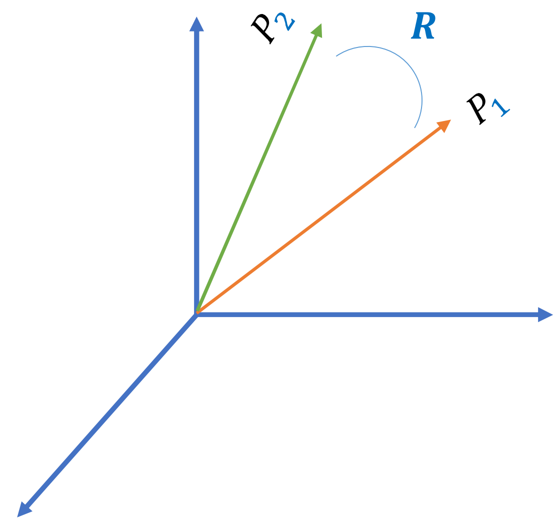

Review Lecture One and Two

# Operators

- Mapping: changing descriptions from frame to frame
- Operators: moving points (within the same frame)




$$Mapping\ {}^{A}P = {}^{A}_{B}R\,{}^{B}P$$




*Rotational Operator*

$$P_2 = R P_1$$

Review Lecture One and Two

# Rotational Operators

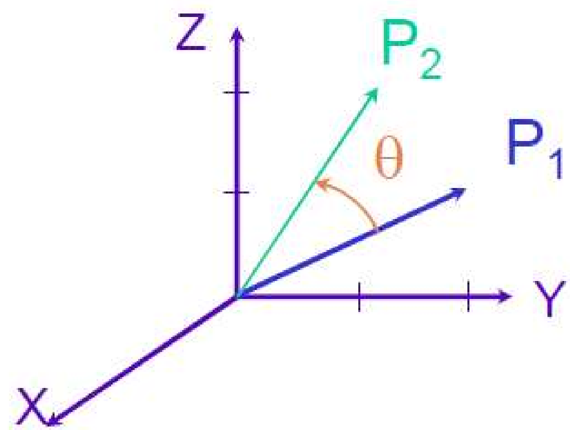
$R_k$ (θ): $P_1 \longrightarrow P_2$

$P_2 = R_k$ (θ) $P_1$ where k: ( X, Y, Z) .

$$R_X\,(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$P_2 = R_X\,(30)\,P_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$




Review Lecture One and Two

## Translations

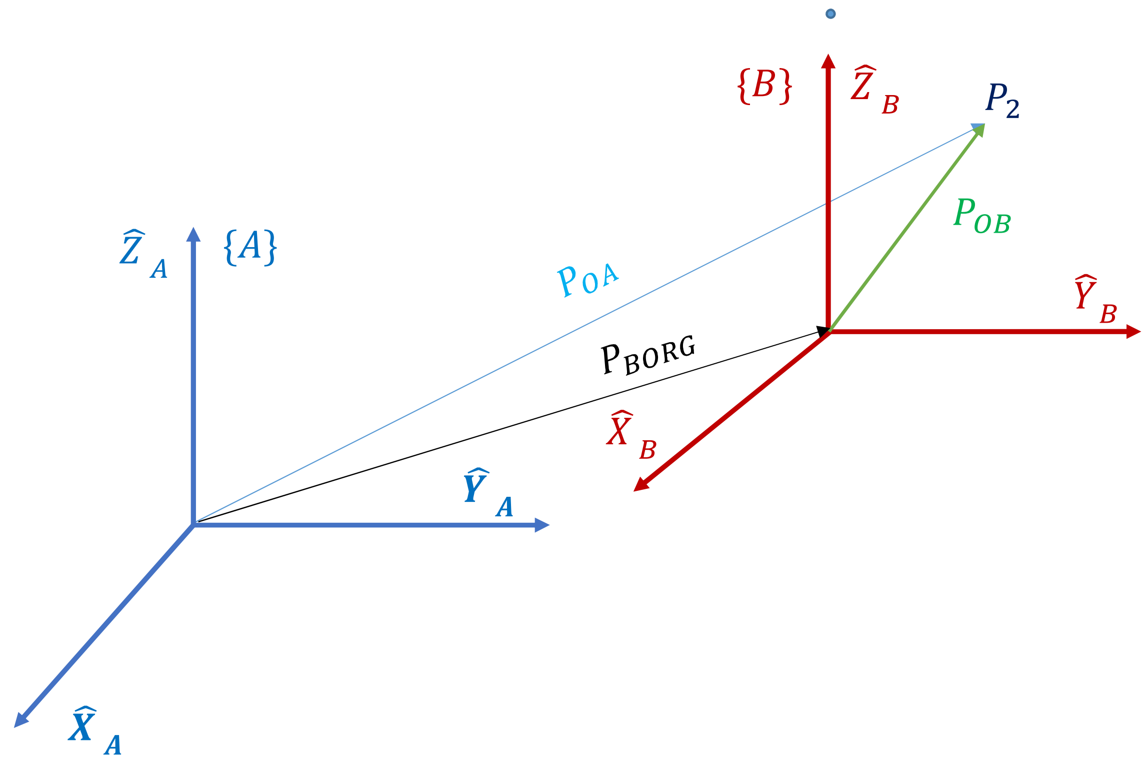
Mapping approach :

$P_{OB}$ ———————— $P_{OA}$

$P_{OA} = P_{OB} + P_{BORG}$

In this method two vectors

For the same point $P_2$




Review Lecture One and Two

# Translations Operator

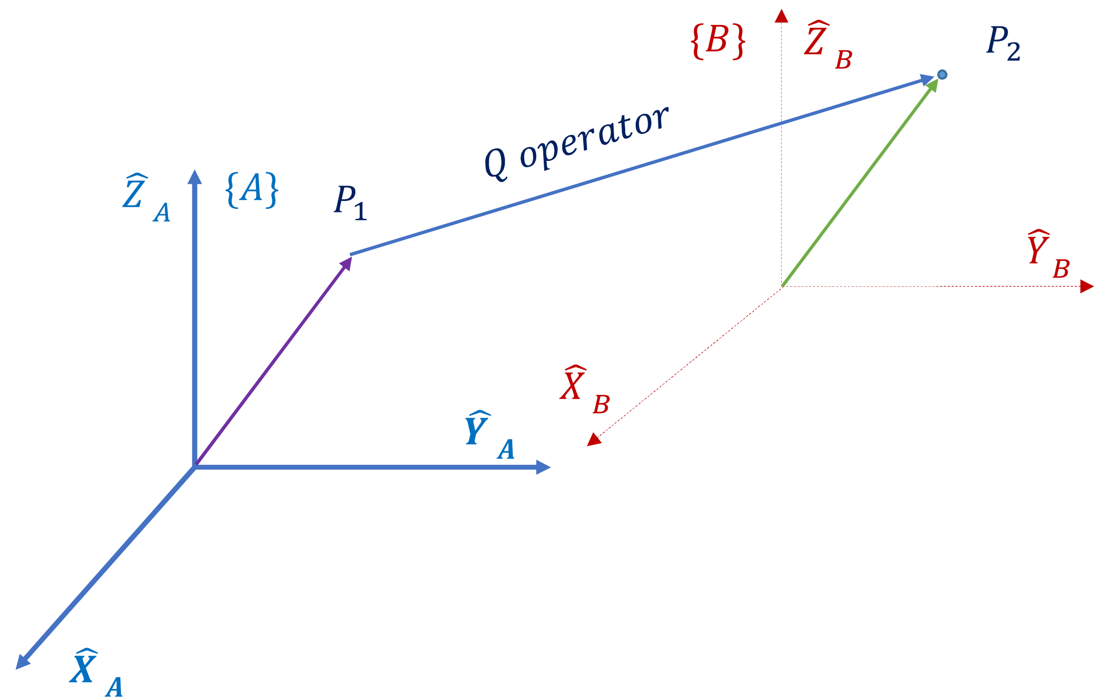
Mapping approach :

$P_1$———————$P_2$

$P_2 = P_1 + Q$

In this method

Two different vectors (2 point)




Review Lecture One and Two

# Translations Operator

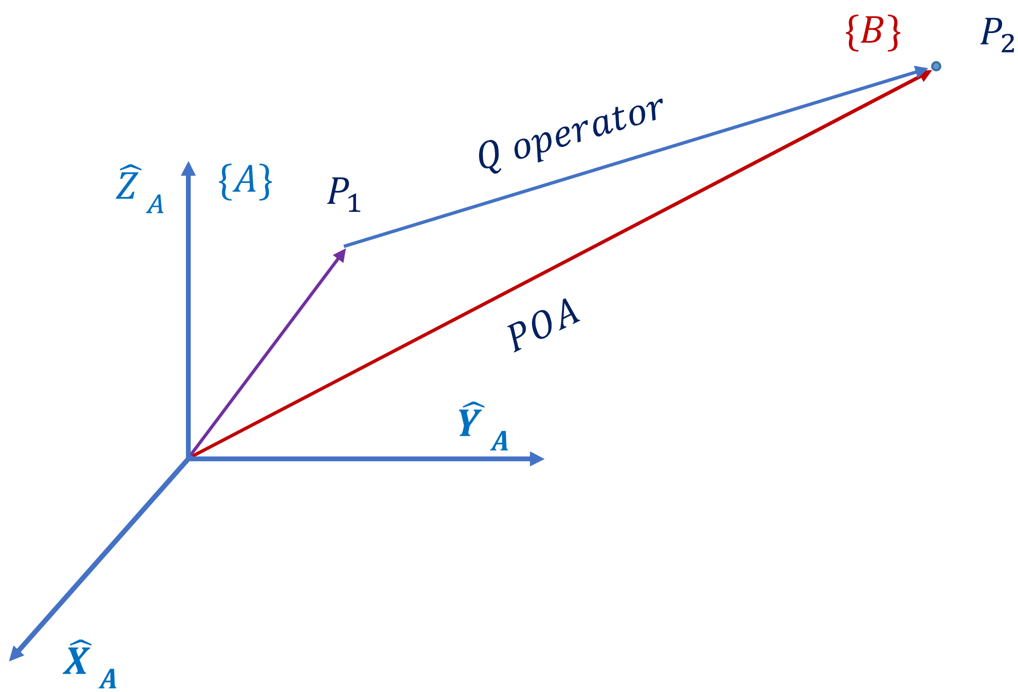
Operator approach :

$P_1$ ———————— $P_2$

$P_2 = P_1 + Q$

In this method

Two different vectors (2 point)




Review Lecture One and Two

# Translations Operator



operator approach :

$P_1$ ———————— $P_2$

operator ${}^{A}P_2 = {}^{A}P_1 + {}^{A}Q$

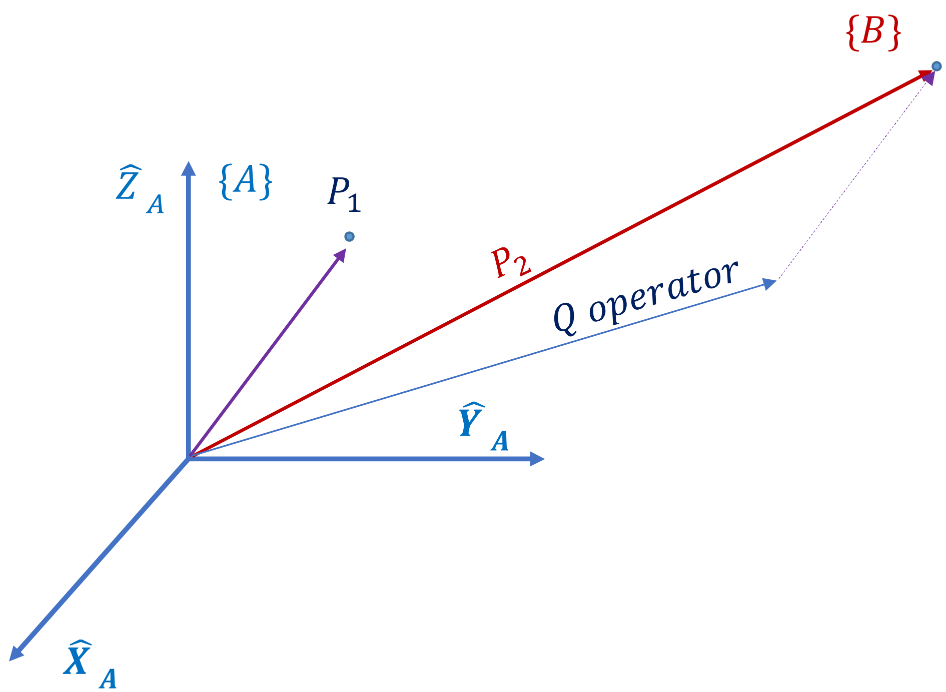
In this method

Two different vectors (2 point)

Homogeneous Transform:

$$D_Q = \begin{bmatrix} 1 & 0 & 0 & q_X \\ 0 & 1 & 0 & q_Y \\ 0 & 0 & 1 & q_Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}P_2 = {}^{A}D_Q \; {}^{A}P_1$$



## General Operators

To represent the operator transformation in a general homogeneous form

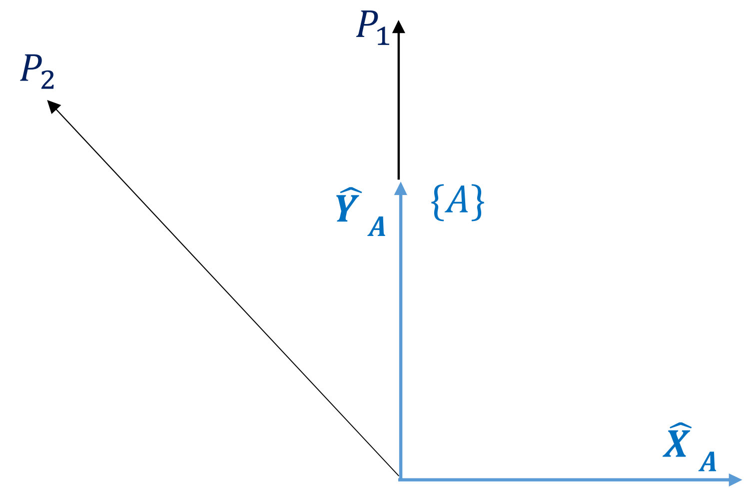
$$P_2 = \begin{bmatrix} R_k(\theta) & Q \\ 0\ 0\ 0 & 1 \end{bmatrix} \times P_1$$

$$P_2 = T \times P_1$$

only one coordinate system is involved, and so the **symbol T** is used without sub- or superscripts.

## Example on Rotational Operator

The Figure shows a vector ${}^{A}P_1$. We wish to compute the vector obtained by rotating this vector about Z by 45°. Call the new vector ${}^{A}P_2$.The rotation matrix that rotates vectors by 45° about Z is the same as the rotation matrix that describes a frame rotated 45° about Z relative to the reference frame. Thus, the correct rotational operator is




$$R_z\,(45.0)=\begin{bmatrix}\cos 45 & -\sin 45 & 0\\ \sin 45 & \cos 45 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0.707 & -0.707 & 0\\ 0.707 & 0.707 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Given ${}^{A}P_1=\begin{bmatrix}0.0\\ 2.0\\ 0.0\end{bmatrix}$

$${}^{A}P_2=R_z\,(45.0)\;{}^{A}P_1$$

$${}^{A}P_2=\begin{bmatrix}-1.414\\ 1.414\\ 0.0\end{bmatrix}$$

Review Lecture One and Two

## Example on Transformation Operator

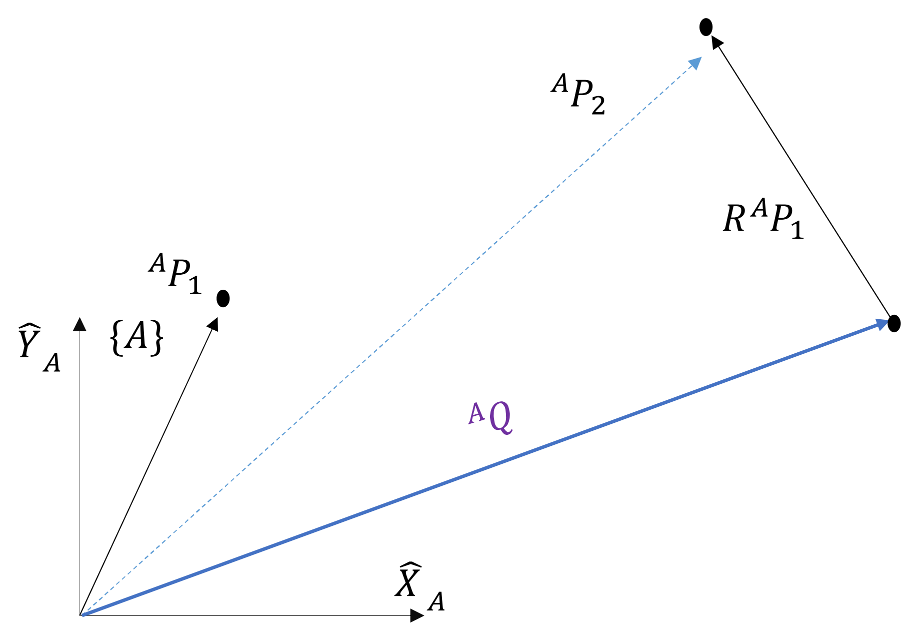
The figure shows a vector ${}^{A}P_1$. We wish to rotate it about Z by 60 and translate it 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$

Find ${}^{A}P_2$ where ${}^{A}P_1$ =$[3\ \ 7\ 0]^T$ The operator T, which performs the translation and rotation, is :

$$T=\begin{bmatrix} 0.5 & -0.866 & 0 & 10 \\ 0.866 & 0.5 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}P_2 = R\,{}^{A}P_1 + {}^{A}Q = T\,{}^{A}P_1$$

$${}^{A}P_2 = \begin{bmatrix} 0.5 & -0.866 & 0 & 10 \\ 0.866 & 0.5 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5.438 \\ 11.09 \\ 0 \end{bmatrix}$$




Review Lecture One and Two

# Inverse Transform

$$^{A}_{B}T = \begin{bmatrix} ^{A}_{B}R & ^{A}P_{BORG} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}$$

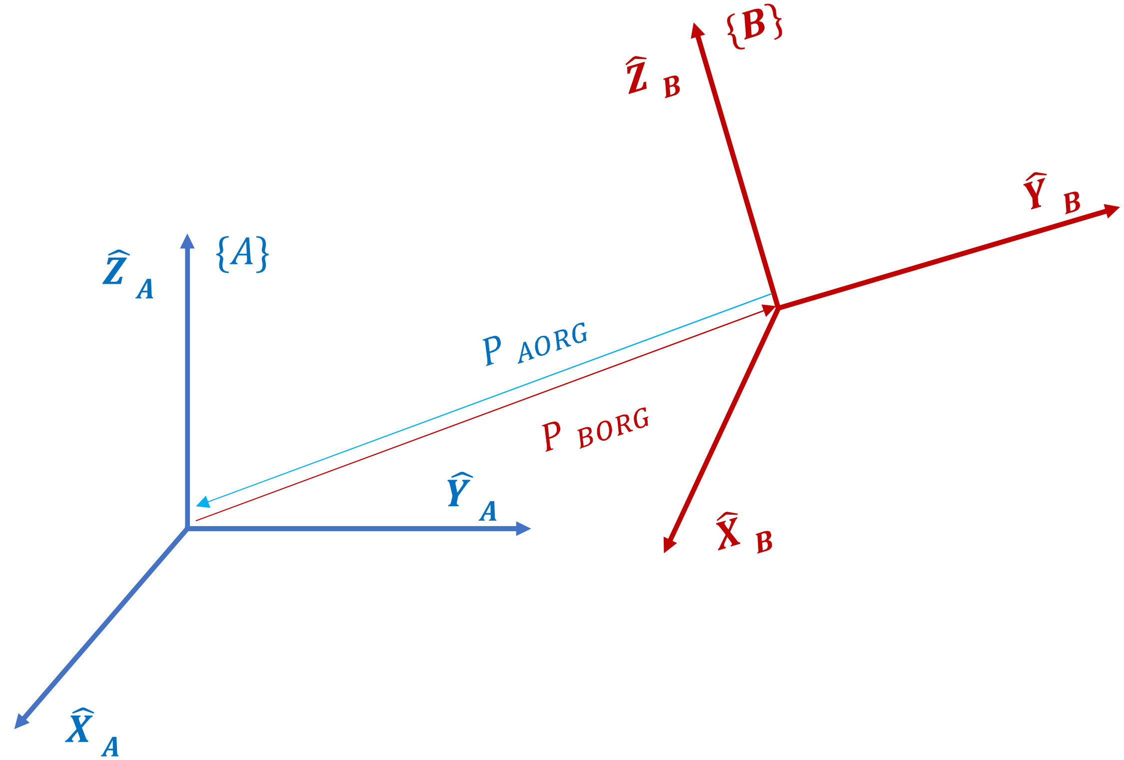
Rotation Matrix is orthonormal

$$^{A}_{B}R^{-1} = {}^{B}_{A}R = {}^{A}_{B}R^{T}$$

However, $^{A}_{B}T^{-1} \neq {}^{A}_{B}T^{T}$, Thus

$$^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} ^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}$$

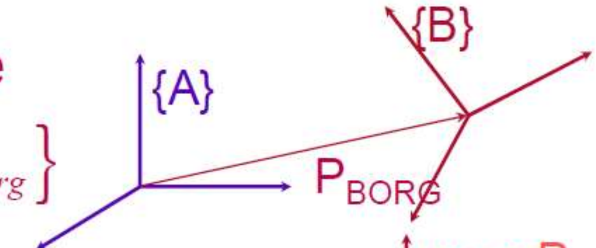
$-{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} = {}^{B}P_{AORG}$



# Homogenous Transformation summary

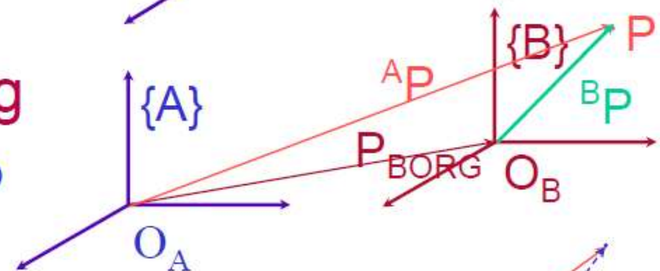
**Description of a frame**

$$ {}^{A}_{B}T: \ \{B\} = \left\{ {}^{A}_{B}R \quad {}^{A}P_{Borg} \right\} $$



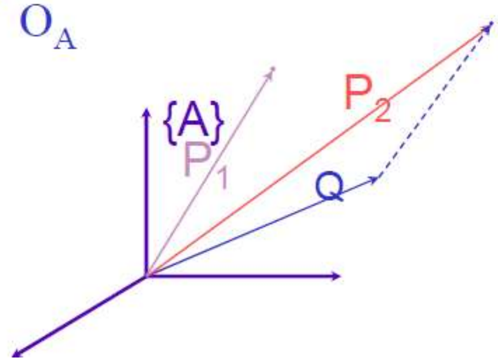

**Transform mapping**

$$ {}^{A}_{B}T: \ {}^{B}P \rightarrow {}^{A}P $$




**Transform operator**

$$ T: \ P_1 \rightarrow P_2 $$




Review Lecture One and Two

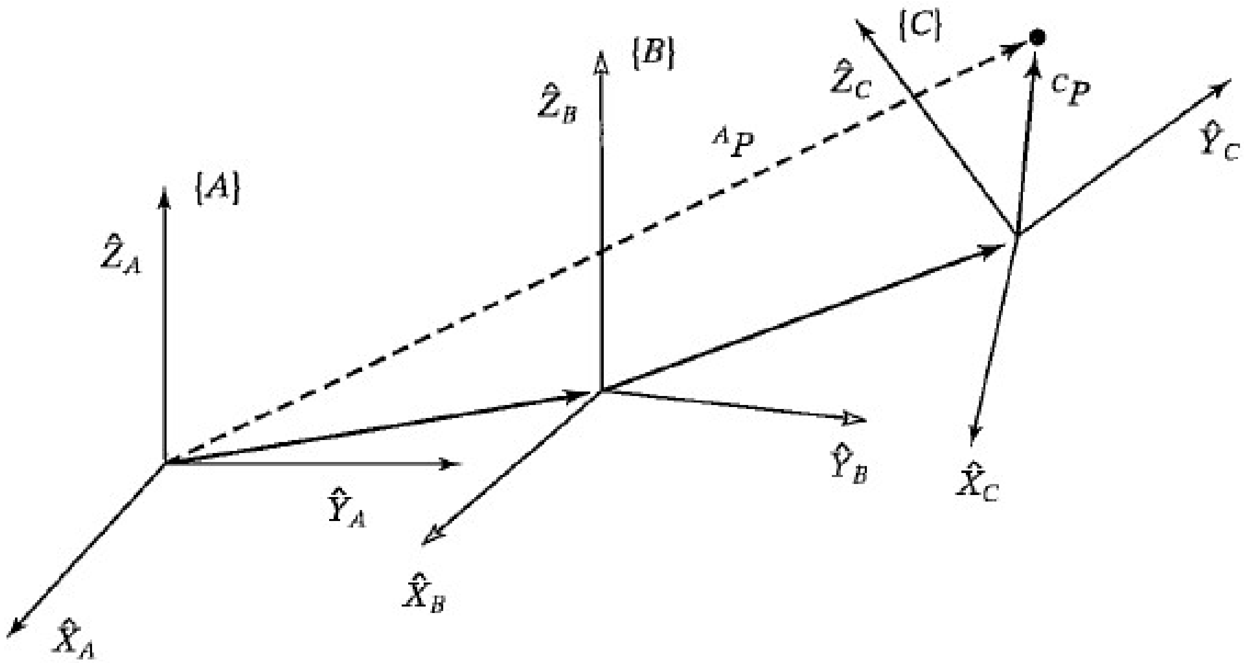
# Compound Transformations

$$^{B}P = {}^{B}_{C}T\,{}^{C}P$$

$$^{A}P = {}^{A}_{B}T\,{}^{B}P$$

$$^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P$$

$$^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T$$




Review Lecture One and Two

# Homogenous form of Compound transformation

$$ {}^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T $$

$$ {}^{A}_{C}T = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

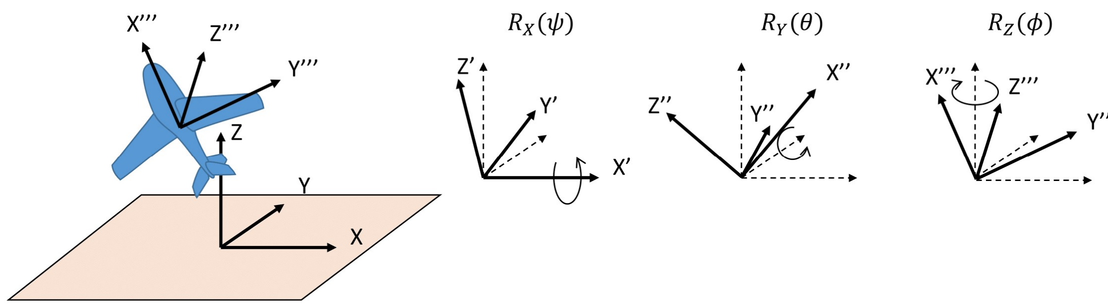
Review Lecture One and Two

# Spatial Descriptions and transformations

## Lecture Four


**Lecturer : Abdurahman B. Ayoub**

**Prepared by : Yazen Hudhaifa**

Contents

## 2.7 TRANSFORM EQUATIONS

## 2.8 MORE ON REPRESENTATION OF ORIENTATION

- **X–Y–Z fixed angles**
- **Z–Y–X Euler angles**
- **Z–Y–Z Euler angles**

## 2.7 TRANSFORM EQUATIONS

**Figure 2.14** indicates a situation in which a **frame {D}** can be expressed as products of transformations in **two different ways**. First,

$$ {}^{U}_{D}T = {}^{U}_{A}T \; {}^{A}_{D}T \quad ---(2.48) $$

Second;

$$ {}^{U}_{D}T = {}^{U}_{B}T \; {}^{B}_{C}T \; {}^{C}_{D}T \quad ---(2.49) $$

We can set these two descriptions of ${}^{U}_{D}T$ equal to construct a **transform equation**:

$$ {}^{U}_{A}T \; {}^{A}_{D}T = {}^{U}_{B}T \; {}^{B}_{C}T \; {}^{C}_{D}T \quad ---(2.50) $$ --- **(Transform Equations)**

**Transform equations** can be used to solve for transforms in the case of **n** unknown transforms and **n** transform equations.

$$ {}^{B}_{C}T = {}^{U}_{B}T^{-1} \, {}^{U}_{A}T \; {}^{A}_{D}T \; {}^{C}_{D}T^{-1} \quad ---(2.51) $$

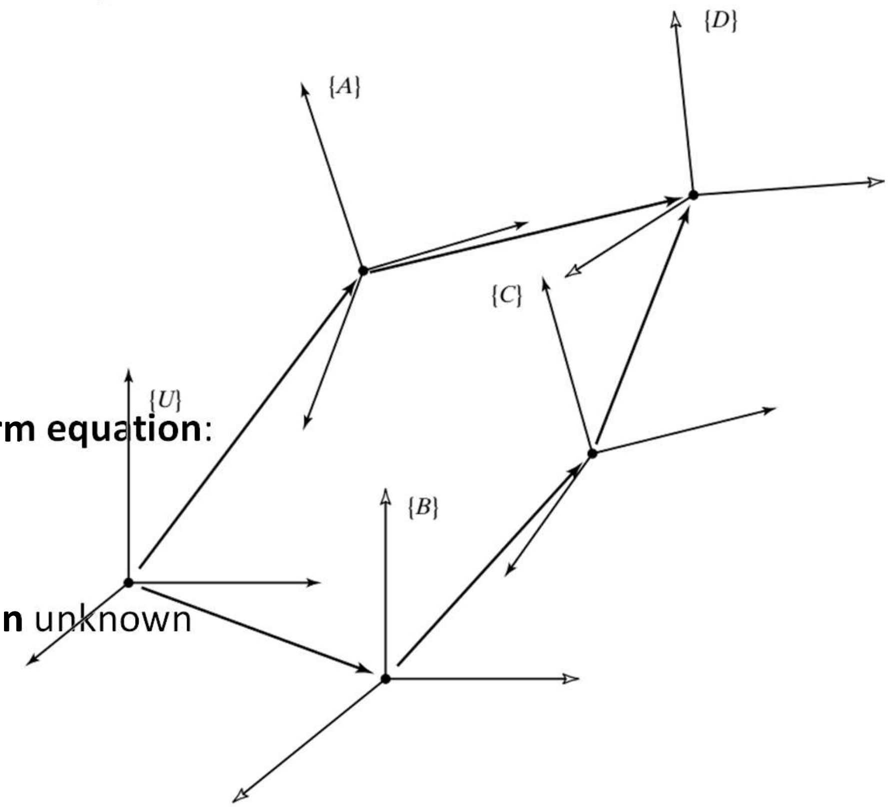



FIGURE 2.14: Set of transforms forming a loop.

The arrow's direction indicates which way the frames are defined: In Fig. 2.14, frame {D} is defined relative to {A}

## 2.7 TRANSFORM EQUATIONS

In **Fig. 2.15**, two possible descriptions of **{C}** are

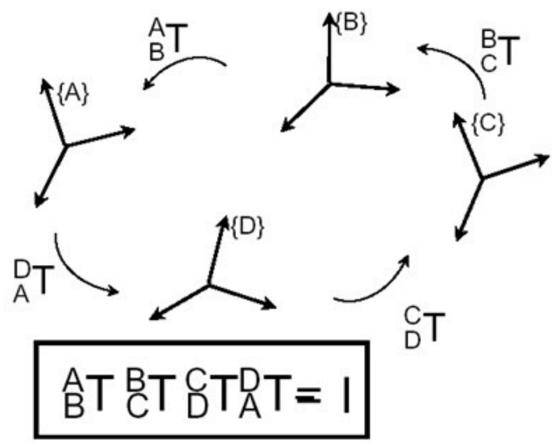
$$ {}^{U}_{C}T = {}^{U}_{A}T \; {}^{D}_{A}T^{-1} \; {}^{D}_{C}T \quad ---(2.52) $$

and

$$ {}^{U}_{C}T = {}^{U}_{B}T \; {}^{B}_{C}T \quad ---(2.53) $$

To find ${}^{U}_{A}T$

$$ {}^{U}_{A}T = {}^{U}_{B}T \; {}^{B}_{C}T \; {}^{D}_{C}T \; {}^{D}_{A}T $$



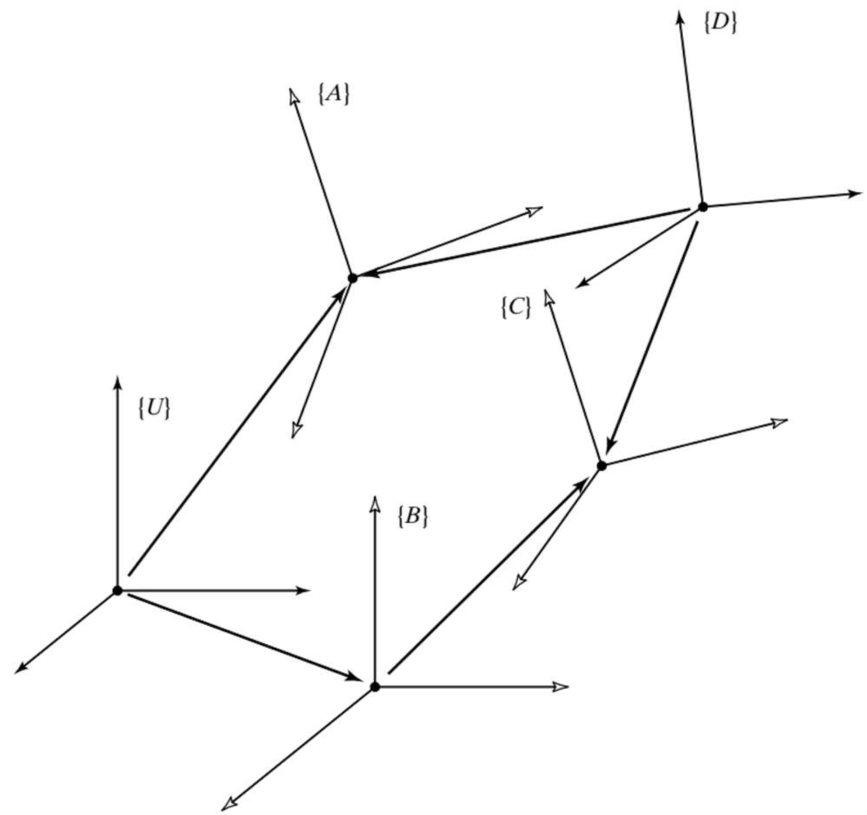




FIGURE 2.15: Example of a transform equation.

# EXAMPLE 2.6 / Page 39

Assume that we know the transform ${}^{B}_{T}T$ in **Fig. 2.16**, which describes the frame at the manipulator's fingertips **{ T }** relative to the base of the manipulator, **{B}**, that we know where the tabletop is located in space relative to the manipulator's base (because we have a description of the frame **{S}** that is attached to the table as shown, ${}^{B}_{S}T$ ), and that we know the location of the frame attached to the bolt lying on the table relative to the table frame—that is, ${}^{S}_{G}T$ . ***Calculate the position and orientation of the bolt relative to the manipulator's hand,*** ${}^{T}_{G}T$ .

**Solution:**

$$ {}^{T}_{G}T = {}^{B}_{T}T^{-1}\,{}^{B}_{S}T\,{}^{S}_{G}T $$

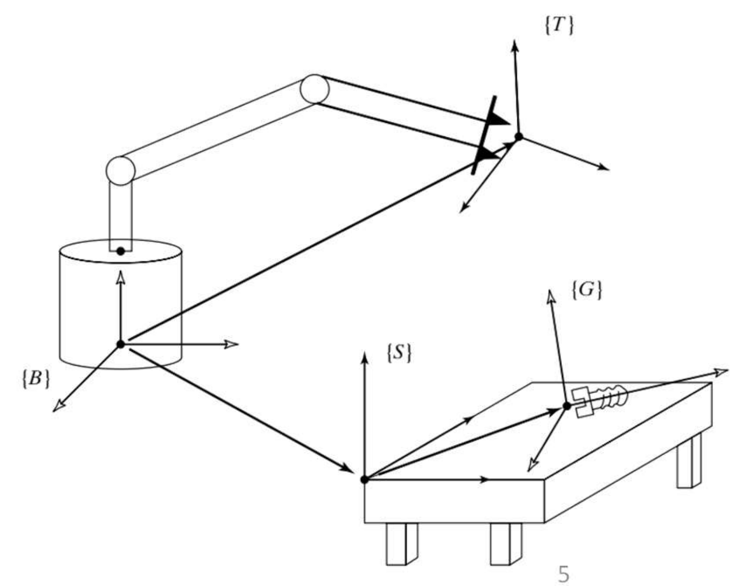



FIGURE 2.16: Manipulator reaching for a bolt.

# EXAMPLE 2.7 / Page 41

Consider two rotations, one about Zˆ by 30 degrees and one about Xˆ by 30 degrees:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Solution:**

$$R_z\,(30) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C30 & -S30 & 0 \\ S30 & C30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

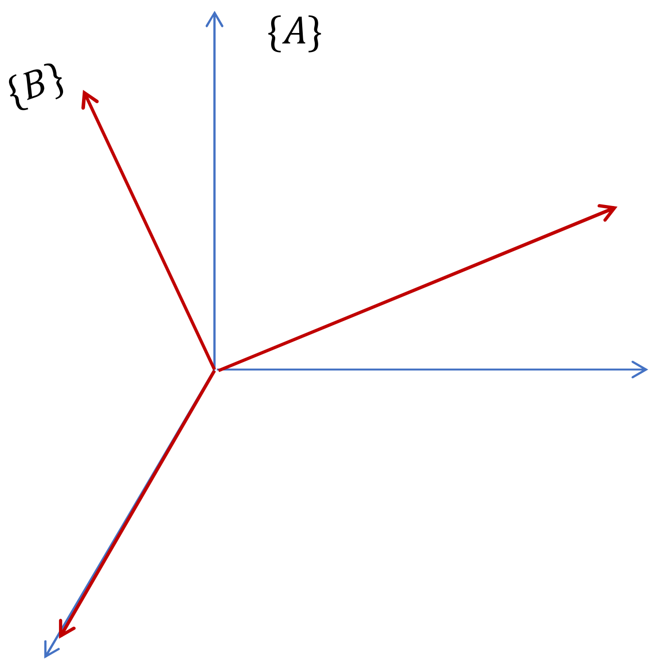
$$R_x\,(30) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

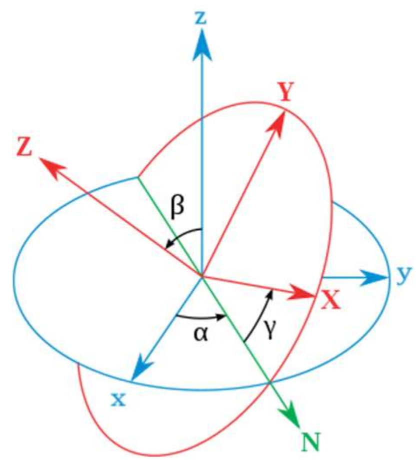
$$R_z\,(30) \quad R_x\,(30) = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix} = \begin{bmatrix} 0866 & -0.433 & 0.25 \\ 0.5 & 0.75 & -0.433 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

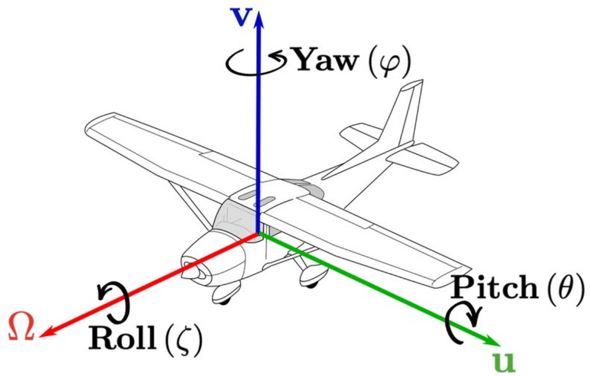
$$\neq \quad R_x\,(30) \quad R_z\,(30) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.433 & 0.75 & -0.5 \\ 0.25 & 0.433 & 0.866 \end{bmatrix}$$

## Three Angle Representations

Instead to use the three vectors to describe the orientation , we will use angles of the rotated frame



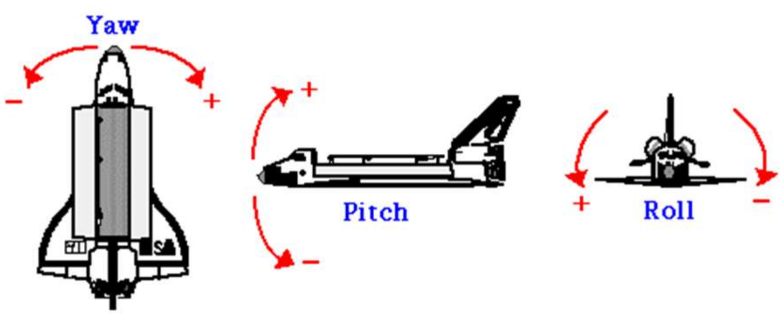



# X–Y–Z fixed angles

- One method of describing the orientation of a **frame {B}** is as follows: Start with the frame coincident with a known **reference frame {A}**. Rotate {B} first about $\hat{X}_A$ by an angle $\gamma$, then about $\hat{Y}_A$ by an angle $\beta$, and, finally, about $\hat{Z}_A$ by an angle $\alpha$.

- Each of the three rotations takes place about an axis in the **fixed reference frame {A}**. We will call this convention for specifying an orientation **X–Y–Z fixed angles**. The word **"fixed"** refers to the fact that the rotations are specified about the fixed (i.e., nonmoving) reference frame (**Fig. 2.17**). Sometimes this convention is referred to as **roll, pitch, yaw angles**, but care must be used, as this name is often given to other related but different conventions.



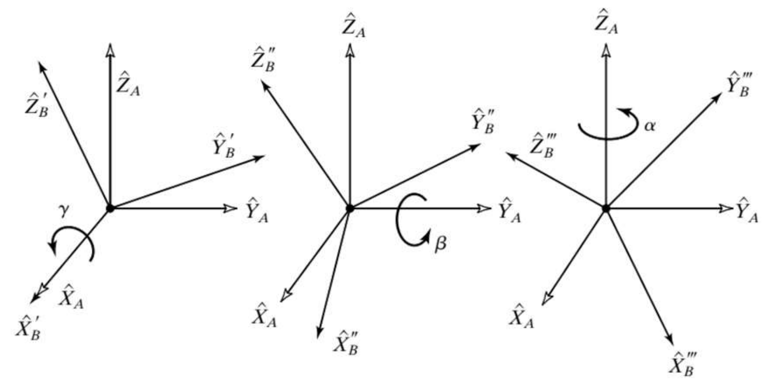



FIGURE 2.17: X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

# X–Y–Z fixed angles

- The derivation of the equivalent rotation matrix, ${}^{A}_{B}\mathbf{R}_{XYZ}\,(\boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\alpha})$, is straightforward, because all rotations occur about axes of the reference frame; that is,

- ${}^{A}_{B}R_{XYZ}\,(\gamma, \beta, \alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma)$

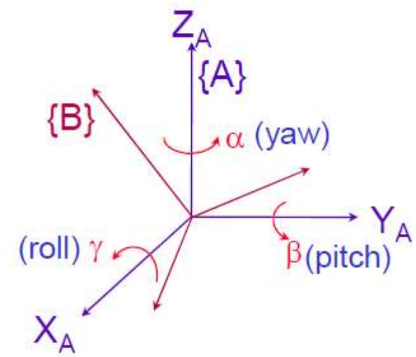
- $= \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix}$

Rotate the frame by :

$$\gamma \ around \ X_A$$

$$\beta \ around Y_A$$

$$\alpha \ around \ Z_A$$




$R_X(\gamma)$: $v \rightarrow R_X(\gamma).v$

$R_Y(\beta)$: $(R_X(\gamma).v) \rightarrow R_Y(\beta).(R_X(\gamma).v)$

$R_Z(\alpha)$: $(R_Y(\beta).R_X(\gamma).v) \rightarrow R_Z(\alpha).(R_Y(\beta).R_X(\gamma).v)$

$${}^{A}_{B}R = {}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

# Z-Y-X Euler Angles

- Another possible description of a **frame {B}** is as follows: Start with the frame coincident with a known **frame {A}**. Rotate **{B}** first about $\hat{Z}_B$ by an angle **α**, then about $\hat{Y}_B$ by an angle **β**, and, finally, about $\hat{X}_B$ by an angle **γ**.
- In this representation, each rotation is performed about an axis of the moving system **{B}** rather than one of the fixed reference **{A}**. Such sets of three rotations are called **Euler angles**.
- Note that each rotation takes place about an axis whose location depends upon the preceding rotations. Because the three rotations occur about the axes $\hat{Z}$, $\hat{Y}$, and $\hat{X}$ , we will call this representation **Z–Y–X Euler angles**.

$$ {}^{A}_{B}R_{Z'Y'X'}\,(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

- Note that we have added ‘‘**primes**’’ to the subscripts to indicate that this rotation is described by **Euler angles**.
- $ {}^{A}_{B}R_{Z'Y'X'}\,(\alpha,\beta,\gamma) = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\gamma & -S\gamma \\ 0 & S\gamma & C\gamma \end{bmatrix} $
- $ {}^{A}_{B}R_{Z'Y'X'}\,(\alpha,\beta,\gamma) = \begin{bmatrix} c_{\propto}c_{\beta} & c_{\propto}s_{\beta}c_{\beta} - s_{\propto}c_{\gamma} & c_{\propto}s_{\beta}c_{\gamma} + s_{\propto}s_{\gamma} \\ s_{\propto}c_{\beta} & s_{\propto}s_{\beta}s_{\gamma} + c_{\propto}c_{\gamma} & s_{\propto}s_{\beta}c_{\gamma} + c_{\propto}s_{\gamma} \\ -s_{\beta} & c_{\beta}s_{\gamma} & c_{\beta}c_{\gamma} \end{bmatrix} $

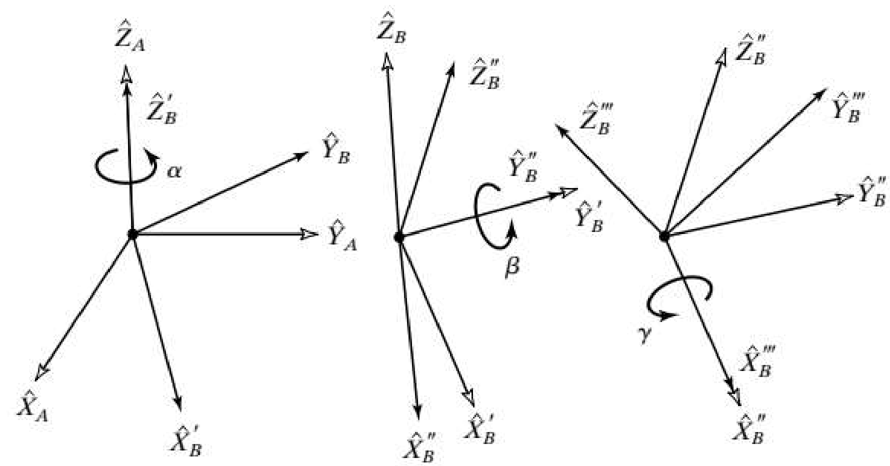



FIGURE 2.18: Z−Y−X Euler angles.

## Z-Y-Z Euler Angles

- Another possible description of a **frame {B}** is Start with the frame coincident with a known **frame {A}**. Rotate **{B}** first about $\hat{\boldsymbol{Z}}_{\boldsymbol{B}}$ by an angle **α**, then about $\hat{\boldsymbol{Y}}_{\boldsymbol{B}}$ by an angle **β**, and, finally, about $\hat{\boldsymbol{Z}}_{\boldsymbol{B}}$ by an angle γ.

- $${}^{A}_{B}R_{Z'Y'Z'}\,(\alpha,\beta,\gamma) = \begin{bmatrix} c_{\propto}c_{\beta}c_{\gamma} - s_{\propto}s_{\gamma} & -c_{\propto}c_{\beta}ss_{\gamma} - s_{\propto}s_{\gamma} & c_{\propto}s_{\beta} \\ s_{\propto}c_{\beta}c_{\gamma} + c_{\propto}c_{\gamma} & -s_{\propto}c_{\beta}s_{\gamma} + c_{\propto}c_{\gamma} & s_{\propto}s_{\beta} \\ -s_{\beta}c\gamma & s_{\beta}s_{\gamma} & c_{\beta} \end{bmatrix}$$

$${}^{A}_{B}R = R_{Z'}(\alpha).R_{Y'}(\beta).R_{Z'}(\gamma)$$

# Fixed & Euler Angles

## X-Y-Z Fixed Angles

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

## Z-Y-X Euler Angles

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

# Exercises

## Lecture Five

**Lecturer : Abdurahman B. Ayoub**

**Prepared by : Yazen Hudhaifa**

# Fixed & Euler Angles

## X-Y-Z Fixed Angles

$$R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

## Z-Y-X Euler Angles

$$R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_Z(\alpha).R_Y(\beta).R_X(\gamma)$$

$$\boxed{R_{Z'Y'X'}(\alpha,\beta,\gamma) = R_{XYZ}(\gamma,\beta,\alpha)}$$

**2.1) A vector $^{A}P$ is rotated about $\hat{Z}_A$ by θ degrees and is subsequently rotated about $\hat{X}_A$ by $\varphi$ degrees. Give the rotation matrix that accomplishes these rotations in the given order.**

**Sol:**

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

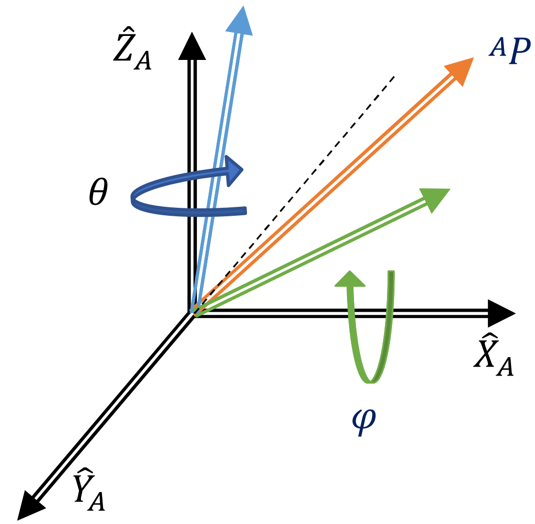
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R= rot($\hat{X}_A$, $\varphi$) rot($\hat{Z}_A$, $\theta$)

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \end{bmatrix}$$




Exercises

**2.2) A vector $^{A}P$ is rotated about $\hat{Y}_A$ by 30 degrees and is subsequently rotated about $\hat{X}_A$ by 45 degrees. Give the rotation matrix that accomplishes these rotations in the given order.**

**Sol:**

R= rot($\hat{X}_A$, 45) rot($\hat{Y}_A$, 30)

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{bmatrix} \begin{bmatrix} \cos 30 & 0 & \sin 30 \\ 0 & 1 & 0 \\ -\sin 30 & 0 & \cos 30 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0.353 & 0.707 & -0.612 \\ -0.353 & 0.707 & 0.612 \end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercises

**2.3) A frame {B} is located initially coincident with a frame {A}. We rotate {B} about $\hat{Z}_B$ by $\theta$ degrees, and then we rotate the resulting frame about $\hat{X}_B$ by $\emptyset$ degrees. Give the rotation matrix that will change the description of vectors from ${}^{B}P$ to ${}^{A}P$.**

**Sol:**

since the rotations are about the frame being rotated , then Euler angles will be applied.

$$R = rot(\hat{Z}_B, \theta^{\circ})\, rot(\hat{X}_B, \emptyset^{\circ})$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\emptyset & -\sin\emptyset \\ 0 & \sin\emptyset & \cos\emptyset \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & -S\theta\, C\emptyset & S\theta\, S\emptyset \\ S\theta & C\theta\, C\emptyset & -C\theta\, S\emptyset \\ 0 & S\emptyset & C\emptyset \end{bmatrix}$$

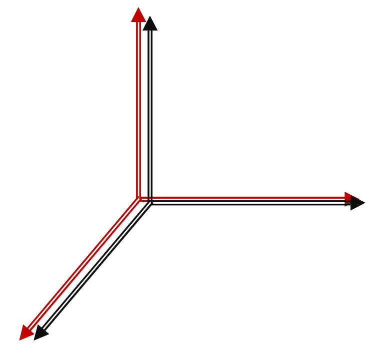
Exercises

**2.4) A frame {B} is located initially coincident with a frame {A}. We rotate {B} about $\hat{Z}_B$ by 30 degrees, and then we rotate the resulting frame about $\hat{X}_B$ by 45 degrees. Give the rotation matrix that will change the description of vectors from ${}^{B}P$ to ${}^{A}P$.**

**Sol:**

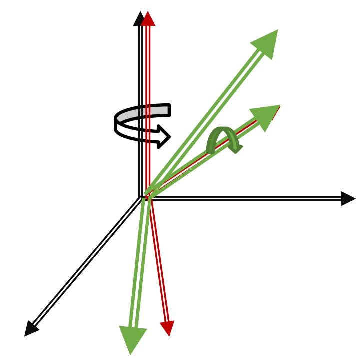
since the rotations are about the frame being rotated , then Euler angles will be applied.

R= rot($\hat{Z}_A$, 30°) rot($\hat{X}_B$,45°)



$$= \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.707 & -0.707 \\ 0 & 0.707 & 0.707 \end{bmatrix}$$



$$= \begin{bmatrix} 0.866 & -0.353 & 0.353 \\ 0.5 & 0.612 & -0.612 \\ 0 & 0.707 & 0.707 \end{bmatrix}$$

**2.13) The following frame definitions are given as known:**

$$ {}^{U}_{A}T = \begin{bmatrix} 0.866 & -0.5 & 0.0 & 11.0 \\ 0.5 & 0.866 & 0.0 & -1 \\ 0.0 & 0.0 & 1.0 & 8.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} $$

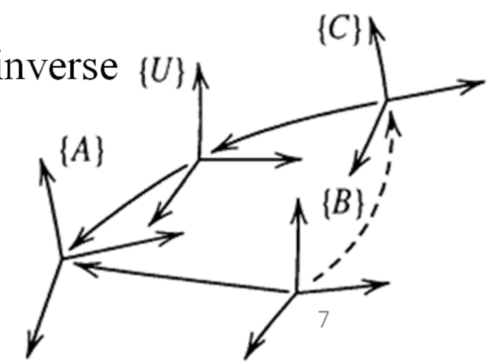
$$ {}^{B}_{A}T = \begin{bmatrix} 1.00 & -0.5 & 0.0 & 0 \\ 0.0 & 0.866 & -0.5 & 10 \\ 0.0 & 0.5 & 0.866 & -20 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} $$

$$ {}^{C}_{U}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & -3 \\ 0.433 & 0.750 & -0.500 & -3 \\ 0.250 & 0.433 & 0.866 & 3 \\ 0.00 & 0.00 & 0.000 & 1.0 \end{bmatrix}, $$

**Draw a frame diagram to show their arrangement qualitatively, and solve for ${}^{B}_{C}T$**

**Sol:** by just following the arrows and inverting when needed , for rapid calculation for inverse matrices:

$$ {}^{B}_{C}T = {}^{B}_{A}T \; {}^{U}_{A}T^{-1} \; {}^{C}_{U}T^{-1} $$




Exercises

**2.27 – 2.31) Referring to Fig. Below , give the value of ${}^{A}_{B}T$ , ${}^{A}_{C}T$, ${}^{B}_{C}T$, AND ${}^{C}_{A}T$.**

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Sol:**

$${}^{A}_{B}T = \begin{bmatrix} \cos(180) & -\sin(180) & 0 & 3 \\ \sin(180) & \cos(180) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}_{C}R = \begin{bmatrix} \cos 90 & 0 & \sin 90 \\ 0 & 1 & 0 \\ -\sin 90 & 0 & \cos 90 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos -30 & -\sin -30 \\ 0 & \sin -30 & \cos -30 \end{bmatrix}$$

$${}^{A}_{C}T = \begin{bmatrix} 0 & -0.5 & 0.866 & 3 \\ 0 & 0.866 & 0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

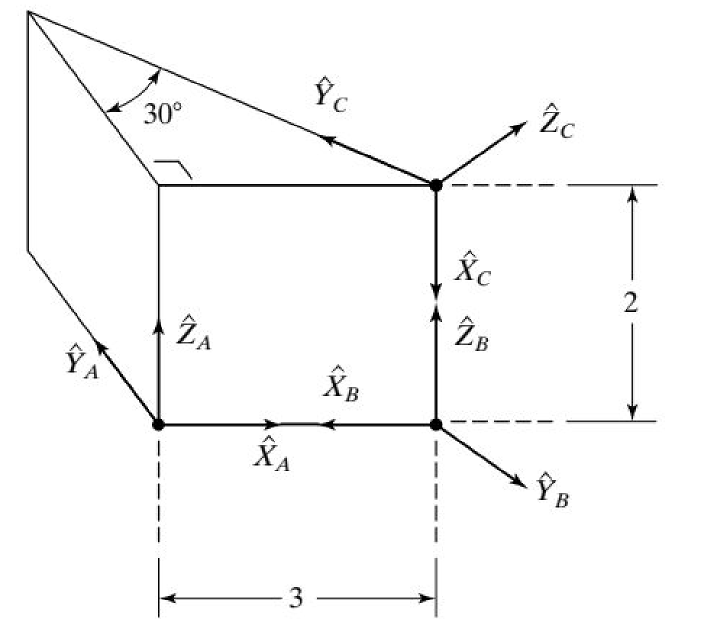



FIGURE 2.25: Frames at the corners of a wedge.

Exercises

**2.27 – 2.31) Referring to Fig. Below , give the value of ${}^{A}_{B}T$ , ${}^{A}_{C}T$, ${}^{B}_{C}T$, AND ${}^{C}_{A}T$.**

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{B}_{C}R = \begin{bmatrix} cos\,90 & 0 & sin\,90 \\ 0 & 1 & 0 \\ -sin\,90 & 0 & cos\,90 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\,150 & -sin\,150 \\ 0 & sin\,150 & cos\,150 \end{bmatrix}$$

$${}^{B}_{C}T = \begin{bmatrix} 0 & 0.5 & -0.866 & 0 \\ 0 & -0.866 & -0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

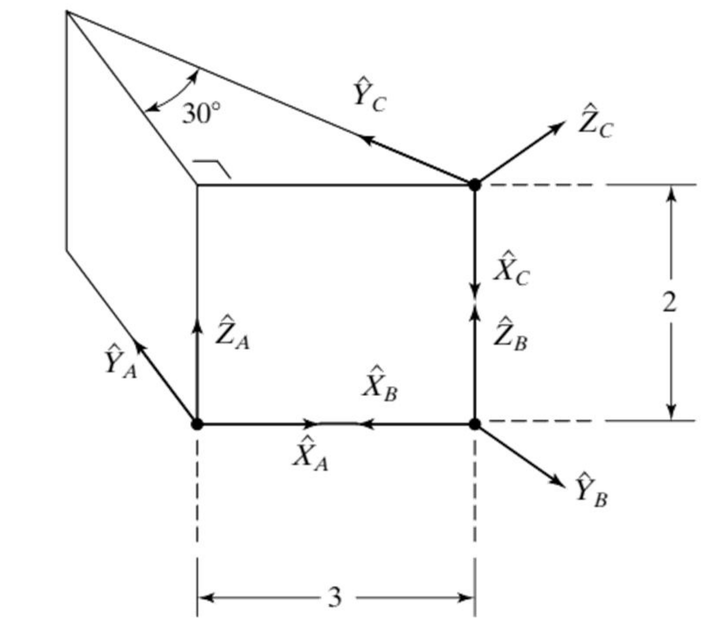



FIGURE 2.25: Frames at the corners of a wedge.

Exercises

## 2.27 – 2.31) Referring to Fig. Below , give the value of ${}^{A}_{B}T$ , ${}^{A}_{C}T$, ${}^{B}_{C}T$, AND ${}^{C}_{A}T$.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}_{C}T = \begin{bmatrix} 0 & -0.5 & 0.866 & 3 \\ 0 & 0.866 & 0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{C}_{A}T = \begin{bmatrix} {}^{A}_{C}R^T & -{}^{A}_{C}R^T \; {}^{A}P_{CORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \text{ —— —Inverting a transform}$$

$${}^{C}_{A}T = \left[\begin{array}{ccc|c} 0 & 0 & -1 & 2 \\ -0.5 & 0.866 & 0 & 3 * 0.5 \\ 0.866 & -0.5 & 0 & -3 * 0.866 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

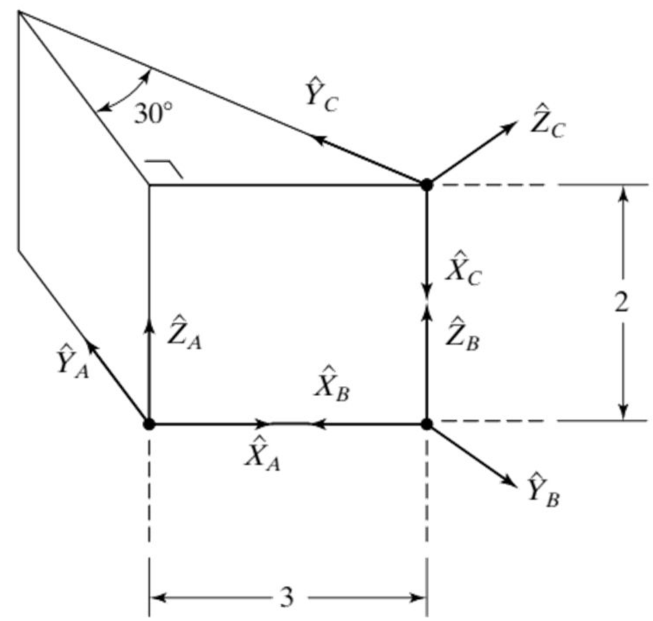



FIGURE 2.25: Frames at the corners of a wedge.

Exercises

## 2.32 – 2.34) Referring to Fig. Below , give the value of ${}^{A}_{B}T$ , ${}^{A}_{C}T$, ${}^{B}_{C}T$, AND ${}^{C}_{A}T$.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:

$${}^{A}_{B}R = R(Y,-180)R(X,90) \rightarrow {}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{B}_{C}R = R(X,-90)R(Z,150) \rightarrow {}^{B}_{C}T = \begin{bmatrix} -0.866 & -0.5 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ -0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

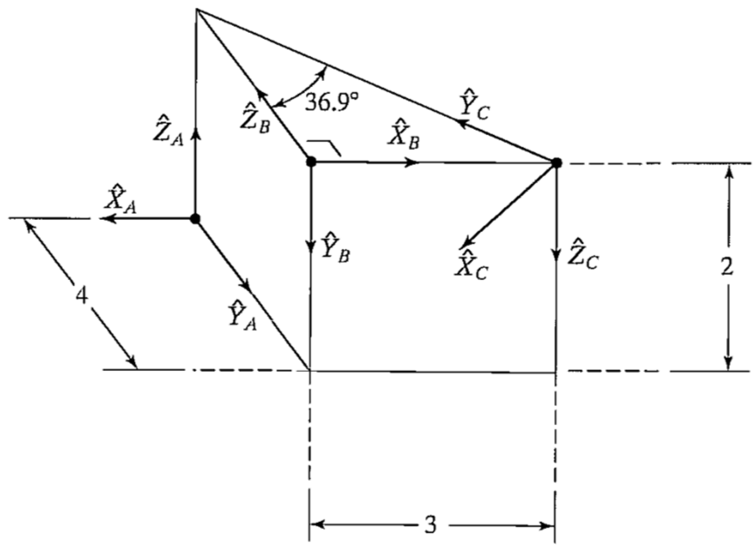



FIGURE 2.26: Frames at the corners of a wedge.

Exercises

**2.37) Given**

$$ {}^{A}_{B}T = \begin{bmatrix} 0.25 & 0.43 & 0.86 & 5 \\ 0.87 & -0.50 & 0 & -4 \\ 0.43 & 0.75 & -0.50 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

**, what is the ${}^{B}_{A}T$ ?**

**Sol:**

$$ {}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\ {}^{A}P_{CORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \text{ --- Inverting a transform} $$

$$ {}^{B}_{A}T = \begin{bmatrix} 0.25 & 0.87 & 0.43 & 2.11 \\ 0.43 & -0.50 & 0.75 & -6.35 \\ 0.86 & 0 & -0.50 & -2.35 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Exercises

# $1^{st}$ Quiz

Q) Referring to Fig. Below, give the value of ${}^{A}_{C}T$ and ${}^{C}_{A}T$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}_{C}T = \begin{bmatrix} 0.866 & 0.5 & 0 & -3 \\ 0.5 & -0.866 & 0 & 4 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

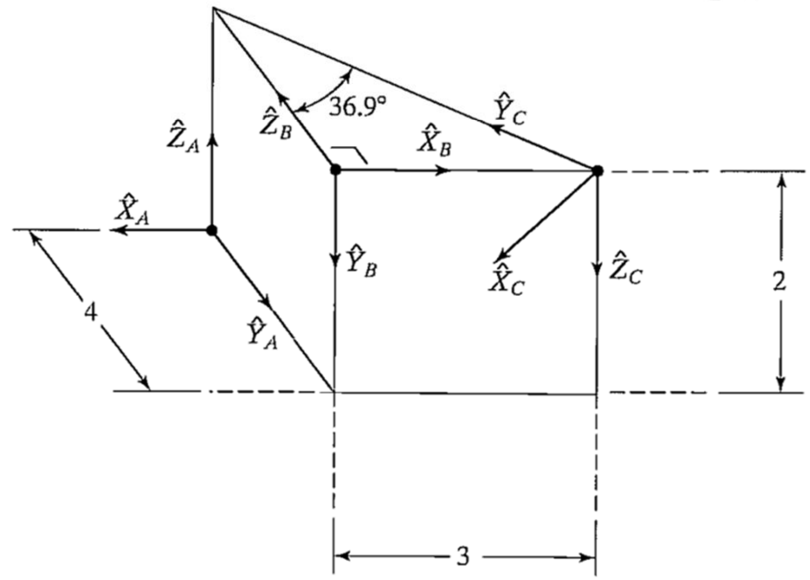
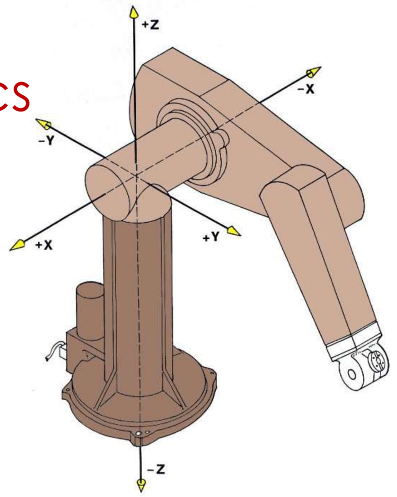



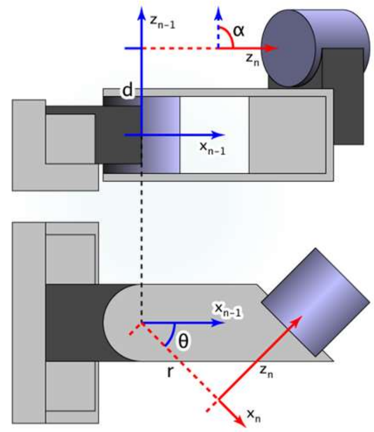

FIGURE 2.26: Frames at the corners of a wedge.

1

# Lecture 6 Robotics Forward Kinematics

## Classical and Modified DH-Convention




LECTURE 6

NINEVEH UNIVERSITY – SYSTEMS & CONTROL DEP.

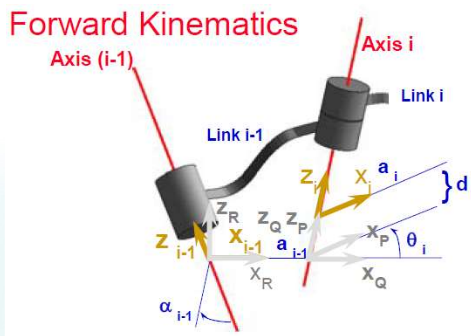
PREPARED BY : YAZEN H SHAKIR

DATE : 27/11/2022

HTTP://RUTHERFORD-ROBOTICS.COM/PUMA/

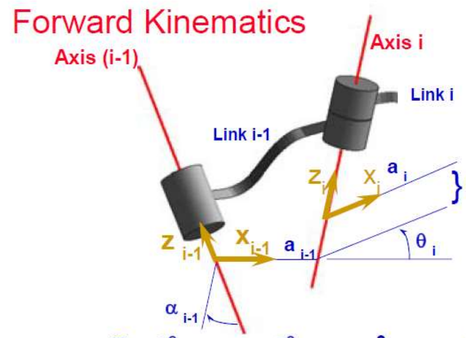
2

# Recap from previous Lecture




$${}^{i-1}_{i}T = {}^{i-1}_{R}T \; {}^{R}_{Q}T \; {}^{Q}_{P}T \; {}^{P}_{i}T$$

$${}^{i-1}_{i}T_{(\alpha_{i-1}, a_{i-1}, \theta_i, d_i)} = R_x(\alpha_{i-1})\, D_x(a_{i-1})\, R_z(\theta_i)\, D_z(d_i)$$

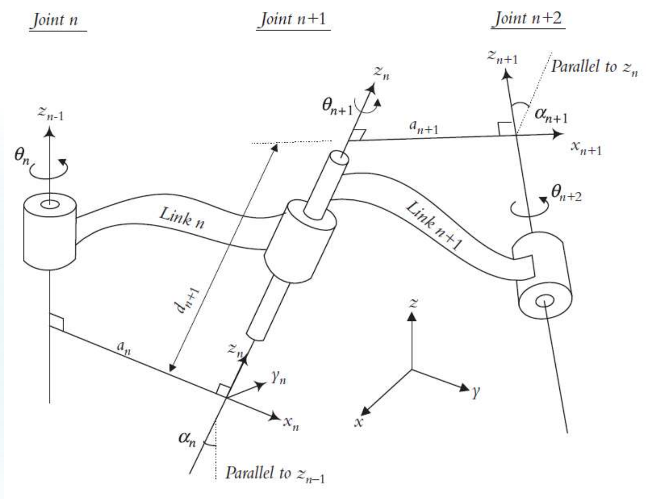



$${}^{i-1}_{1}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

******* Note: This derivation in Introduction to Robotics: Mechanics and Control Book by John J Craig which is called ( Modified DH parameters )***

Another representation of Forward kinematics by Denavit-Hartenberg representation of a general purpose joint-link combination. (Drawn by S. Niku.) ( Classical DH- Matrix)

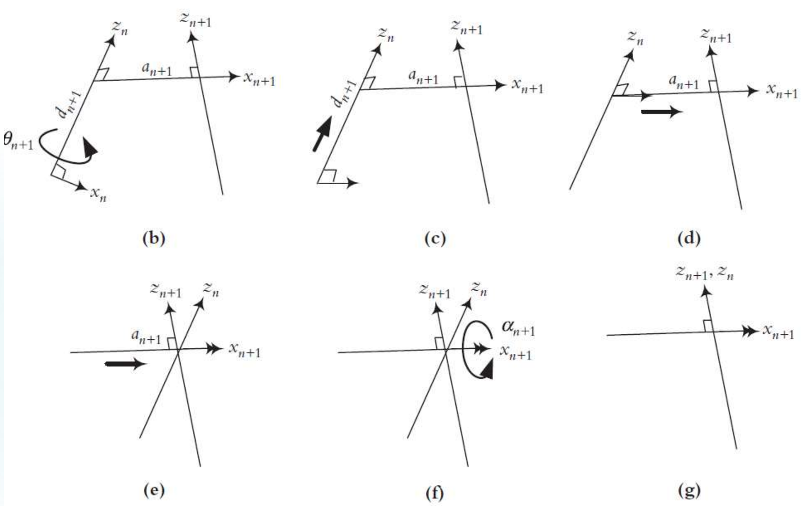
3




(a)

Classical DH-representation according S. Niku. Textbook

4



5

The transformation ${}_{n+1}^{n}T$ (called $A_{n+1}$) between two successive frames representing the preceding four movements is the product of the four matrices representing them. Since all transformations are relative to the current frame (they are measured and performed relative to the axes of the current local frame), all matrices are post-multiplied. The result is:

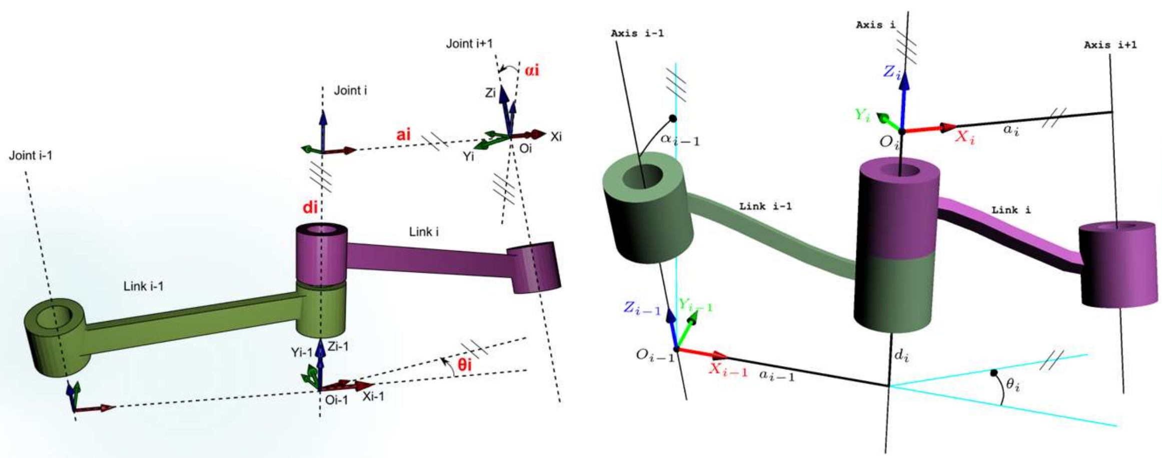
$$ {}^{n}T_{n+1} = A_{n+1} = Rot(z, \theta_{n+1}) \times Trans(0, 0, d_{n+1}) \times Trans(a_{n+1}, 0, 0) \times Rot(x, \alpha_{n+1}) $$

$$ = \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1} & 0 & 0 \\ S\theta_{n+1} & C\theta_{n+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & a_{n+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{n+1} & -S\alpha_{n+1} & 0 \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ A_{n+1} = \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1} \\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1} \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

# Difference between Classical and Modified DH-Convention

6




Some books such as *Introduction to Robotics: Mechanics and Control (3rd Edition)* use modified DH parameters. The difference between the classic DH parameters and the modified DH parameters are the locations of the coordinates system attachment to the links and the order of the performed transformations.

## Recap from previous Lect.

7

- $a_i$ = the Distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$;
- $\alpha_i$ = the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$;
- $d_i$ = the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$;
- $\theta_i$ = the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$;

8

Ex7.1) Figure below shows a robot having three degrees of freedom and one prismatic joint. This manipulator can be called an "RPR mechanism," in a notation that specifies the type and order of the joints. It is a "cylindrical" robot whose first two joints are analogous to polar coordinates when viewed from above. The last joint (joint 3) provides "roll" for the hand.

Solution :

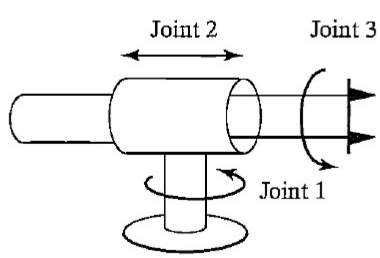
1- draw first the Joint axes

2- If we have two intersect axes , the intersection point

Represents the origin of the frame.

3- Revolute Joint has variable $\theta$ , d=0.

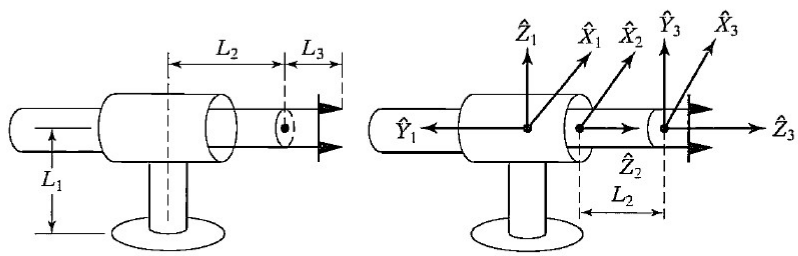
4- Prismatic Joint has variable offset (d) and $\theta$=0.




Manipulator with 3 DOF , one Joint is Prismatic

## Link Assignment

9




1- frame {0} and frame {1} are shown as exactly coincident in this figure, because the robot is drawn for the position $\theta$= 0.

2- it is sufficient that frame {0} be attached anywhere to the non-moving link 0

## DH- Parameter

10

1- Note that rotational joints rotate about the Z axis of the associated frame, but prismatic joints slide along Z.

2- Frame 2 should be attached to the point where the minimum $d_2$ is zero .

3- $\theta_2$ is zero for this robot and that $d_2$ is a variable.

4- Axes 1 and 2 intersect, so $a_1$ is zero.

5- Angle $\alpha_1$ must be 90 degrees in order to rotate $Z_1$ so as to align with $Z_2$, (about X1).

| i | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | 90 ° | 0 | $d_2$ | $\theta_2$ |
| 3 | 0 | 0 | $L_2$ | $\theta_3$ |

## Forward Kinematics for Ex7.1

11

$$ {}^{i-1}_{i}T = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & \alpha_{i-1} \\ S\theta_i C\alpha_{i-1} & C\theta_i C\alpha_{i-1} & -S\alpha_{i-1} & -S\alpha_{i-1} d_i \\ S\theta_i S\alpha_{i-1} & C\theta_i S\alpha_{i-1} & C\alpha_{i-1} & C\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{1}T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}T = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

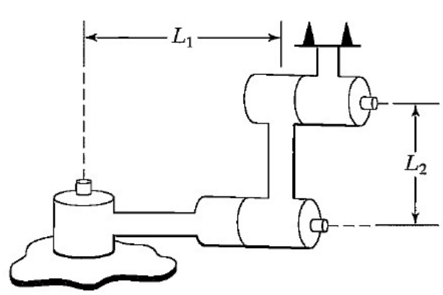
12

Ex7.2: Figure below shows a three-link, 3R manipulator for which joint axes 1 and 2 intersect and axes 2 and 3 are parallel. Demonstrate the non-uniqueness of frame assignments and of the Denavit— Hartenberg parameters by showing several possible correct assignments of frames {1} and {2}.

1- Two possible frame assignments and corresponding parameters for the two possible choices of direction of $Z_2$.

2- In general, when $Z_i$, and $Z_{i+1}$ intersect, there are two choices for $X_i$. In this example, joint axes 1 and 2 intersect, so there are two choices for the direction of X1.

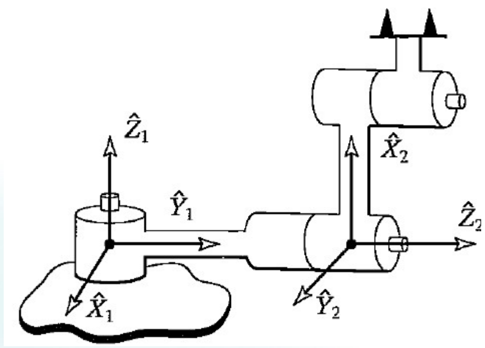
3- In fact, If we take the rotation Axis $Z_1$ downwards , so two more possible assignments



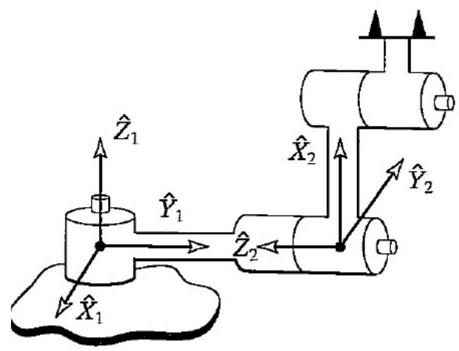

Three-link, non-planar manipulator

## Frame assignments for the Example 7.2 ($Z_2$)

13



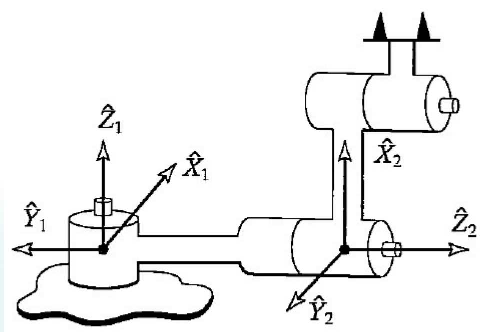

$a_1 = 0$ --------- $a_2 = \boldsymbol{L_2}$
$\alpha_1 = -90$ -------- $\alpha_2 = 0$
$d_1 = 0$ ---------- $d_2 = \boldsymbol{L_1}$
$\theta_2 = -90$



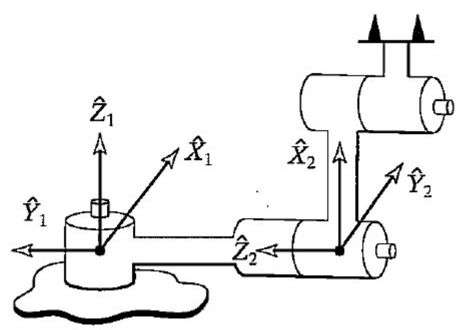

$a_1 = 0$ --------- $a_2 = \boldsymbol{L_2}$
$\alpha_1 = 90$ ---------- $\alpha_2 = 0$
$d_1 = 0$ ---------- $d_2 = -\boldsymbol{L_1}$
$\theta_2 = 90$

## Frame assignments for the Example 7.2 ($X_2$)

14



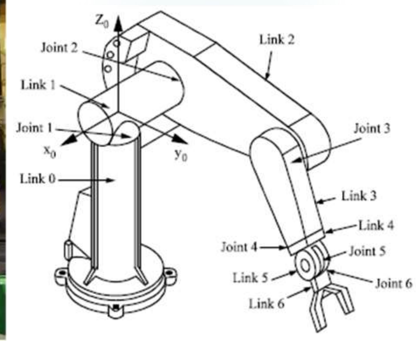

$a_1 = 0$ --------- $a_2 = \boldsymbol{L_2}$
$\alpha_1 = 90$ -------- $\alpha_2 = 0$
$d_1 = 0$ ---------- $d_2 = \boldsymbol{L_1}$
$\theta_2 = 90$




$a_1 = 0$ --------- $a_2 = \boldsymbol{L_2}$
$\alpha_1 = -90$ ---------- $\alpha_2 = 0$
$d_1 = 0$ ---------- $d_2 = -\boldsymbol{L_1}$
$\theta_2 = 90$

## Case Studies on Industrial Robots (PUMA 560)

15

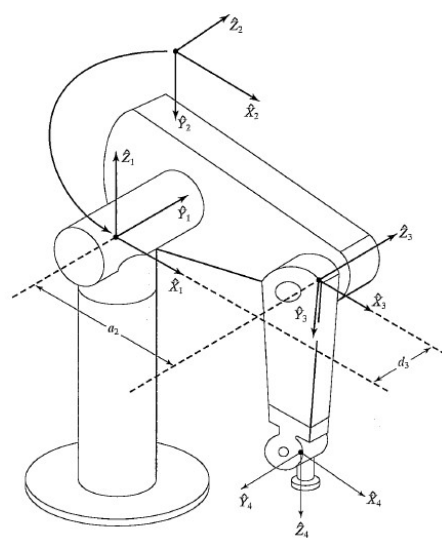




## Frame assignment for (PUMA 560)

16

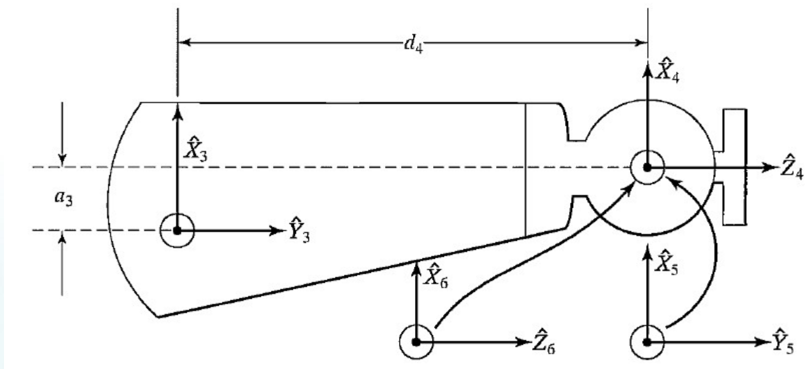
Assumptions:

- 1-All joint angles equal to zero.
- 2- The frame {0} (not shown) is coincident with frame [1} when $\theta_1$ is zero.
- The joint axes of joints 4, 5, and 6 all intersect at a common point, and this point of intersection coincides with the origin of frames {4}, {5}, and {6}.
- We will consider only the kinematics from joint space to Cartesian space. However, that gearing arrangement in the wrist of the manipulator couples together the motions of joints 4, 5, and 6.



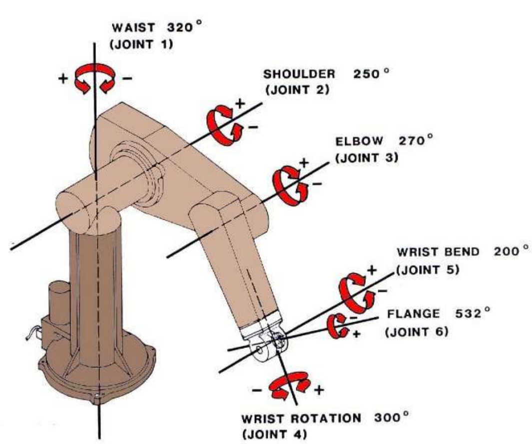
# Forearm frame assignment for (PUMA 560)

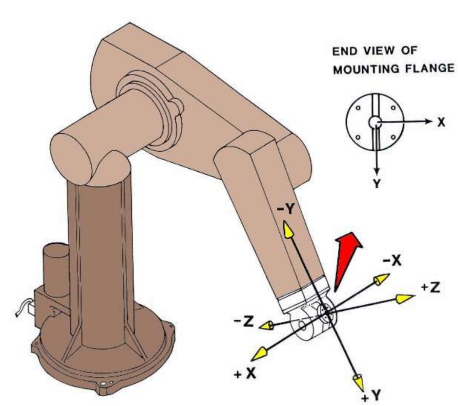
17




# Multiple views for PUMA

18



## Link Parameters or DH- Parameters for PUMA 560

19

| i | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | **0** | **0** | **0** | $\theta_1$ |
| 2 | **-90** ° | **0** | **0** | $\theta_2$ |
| 3 | **0** | $a_2$ | $d_3$ | $\theta_3$ |
| 4 | **-90** ° | $a_3$ | $d_4$ | $\theta_4$ |
| 5 | **90** ° | **0** | **0** | $\theta_5$ |
| 6 | **-90** ° | **0** | **0** | $\theta_6$ |

## Forward Kinematics Transformation

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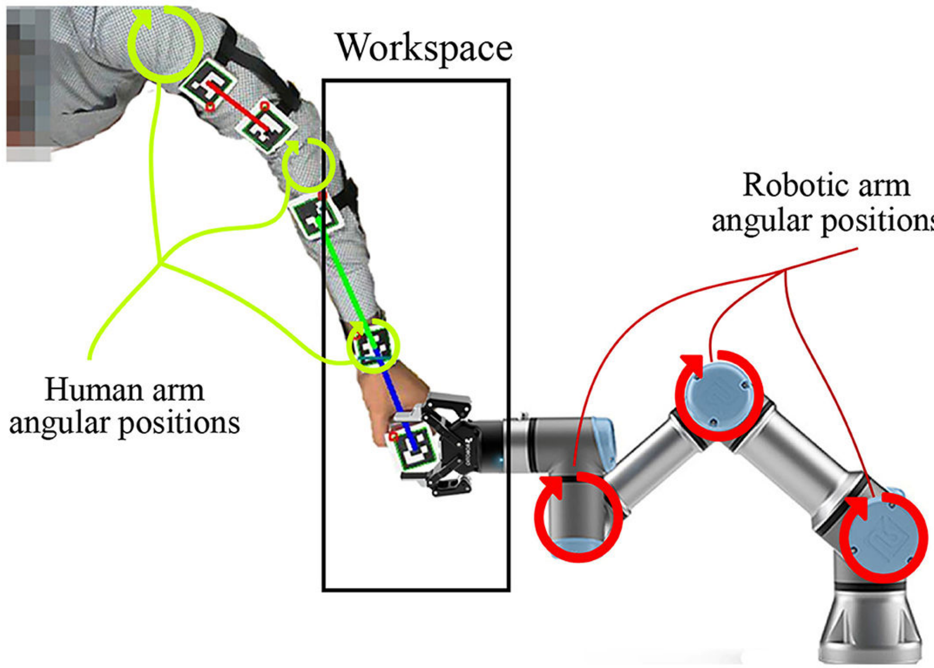
$$ {}^0_1T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -S\theta_1 & -C\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_3T = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^3_4T = \begin{bmatrix} C\theta_4 & -S\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -S\theta_4 & -C\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^4_5T = \begin{bmatrix} C\theta_5 & -S\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S\theta_5 & C\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^5_6T = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S\theta_5 & -C\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

We now form by matrix multiplication of the individual link matrices. While forming this product, we will derive some sub results that will be useful when solving the inverse kinematic problem in Chapter 4.

# Manipulator kinematics




**Lecture Five**


**Lecturer : Abdurahman B. Ayoub**

**Prepared by : Yazen Hudhaifa**

# Content (Denavit–Hartenberg notation)

- **3.1 INTRODUCTION**
- **3.2 LINK DESCRIPTION**
- **3.3 LINK-CONNECTION DESCRIPTION**
- **3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS**

## 3.1 INTRODUCTION

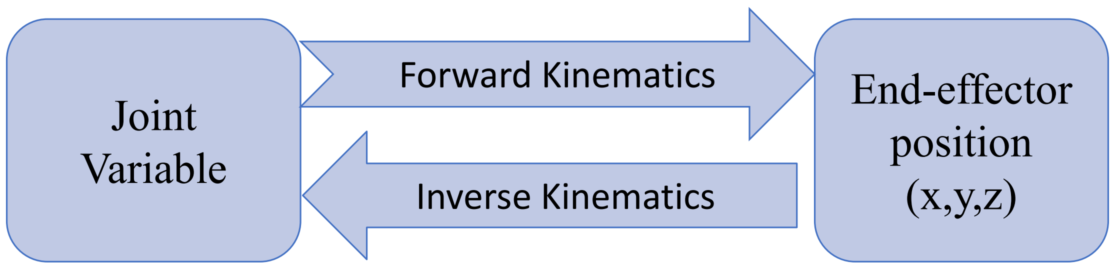
- A **manipulator** is a mechanical structure consisting of **rigid bodies**, or **links**, connected together through **joints**. The manipulator part that most interacts with the surrounding environment, the last body in the manipulator's structure, is **called the end-effector**. The first part of the manipulator, **the base**, is <u>typically fixed in the environment</u>. [8]
- **Kinematics** is the science of motion that treats the subject without regard to the forces that cause it.
- **Dynamics** is the study of how forces affect the motion of objects.
- The model describing the relationships between the manipulator configuration and the end-effector configuration is **called the forward kinematics of the manipulator**.

### Kinematics vs. Dynamics

| kinematics | dynamics |
| --- | --- |
| The effect of a robot's geometry on its motion. | The effect of all forces (internal and external) on a robot's motion. |
| If the motors move ***this*** much, where will the robot be? | If the motors apply ***this*** much force, where will the robot be? |
| Assumes that we control ***encoder readings***… | Assumes that we control ***motor current***… |

## 3.1 INTRODUCTION

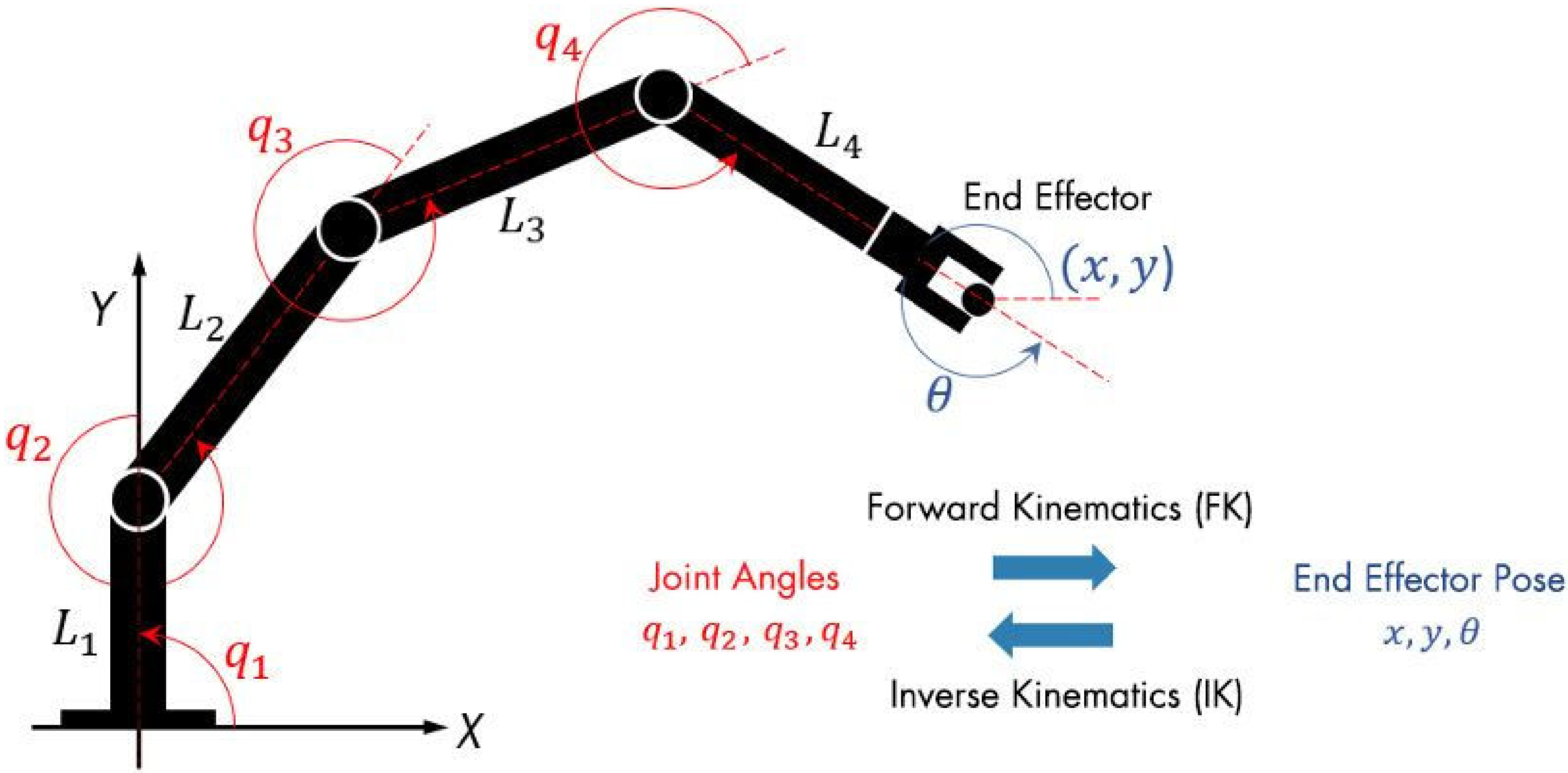
- Forward Kinematics: determine where the end-effector or the robot hand is located ( All the Joint Variables are known)
  - ➢You know already : length of each link , angle of each Joint.
  - ➢You are going to compute : position of any point in 3D- space .




- Inverse Kinematics: the inverse process of forward kinematics in which calculate what each joint variable is (If we desire that the hand be located at a particular point)
  - ➢you are given: length of each link, position of some point on the robot
  - ➢you find: The angles of each joint needed to obtain that position

## 3.1 INTRODUCTION

**Forward Vs Inverse Kinematics for the manipulator**




https://www.mathworks.com/discovery/inverse-kinematics.html

## 3.2 LINK DESCRIPTION

- A manipulator may be thought of as a set of bodies connected in a chain by **joints**. These bodies are **called links**.
- The term **lower pair** is used to describe the connection between a pair of bodies when the relative motion is characterized by two surfaces sliding over one another.
- Most manipulators have **revolute joints** or have sliding joints **called prismatic joints**.
- In the rare case that a mechanism is built with a **joint having n degrees of freedom**, it can be modeled as **n joints** of one degree of freedom connected with **n - 1 links** of zero length.

***In general, a manipulator has n joints and n+1 links (including the base and the end-effector).***

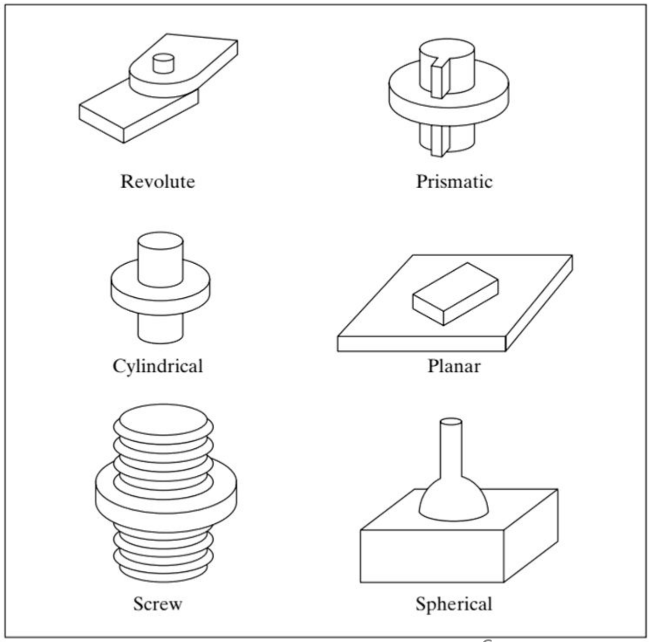
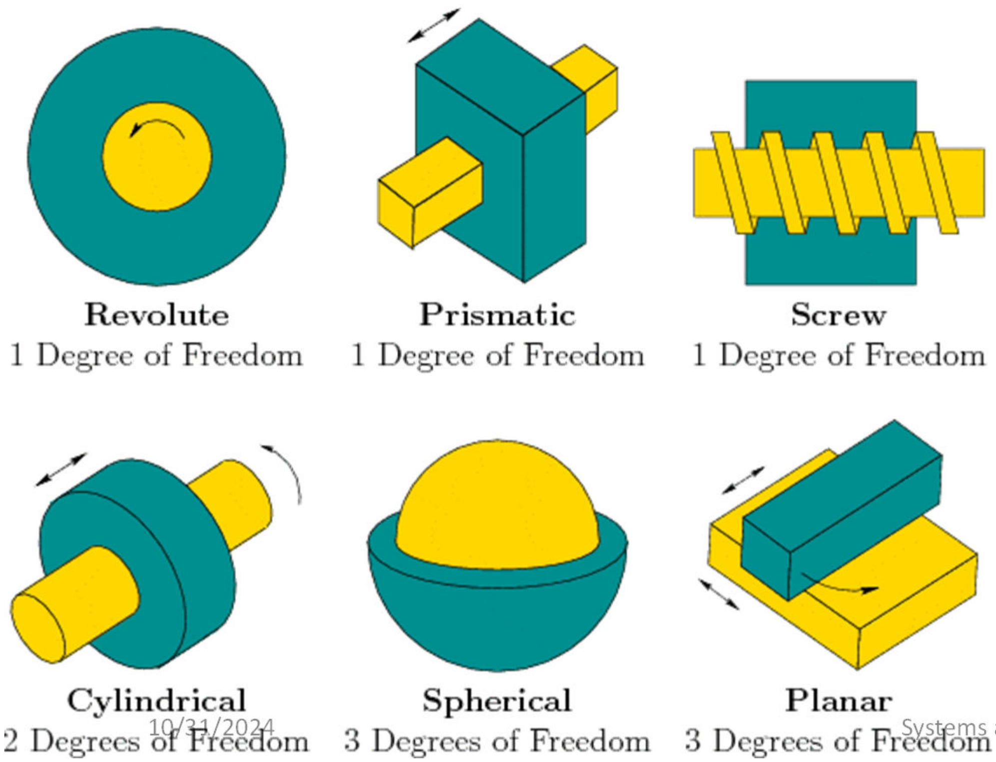
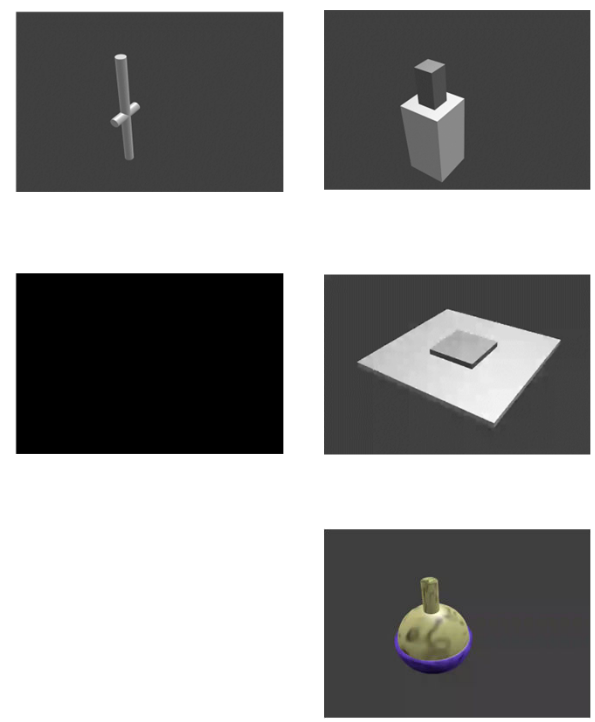



FIGURE 3.1: The six possible lower-pair joints.

## 3.2 LINK DESCRIPTION

A lower pair is an ideal joint that constrains contact between a surface in the moving body to a corresponding in the fixed body. A lower pair is one in which there occurs a surface or area contact between two members, e.g. nut and screw, universal joint used to connect two propeller shafts.



## 3.2 LINK DESCRIPTION

- The links are numbered starting from the immobile base of the arm, which might be **called link 0**. The first moving body is **link 1**, and so on, out to the free end of the arm, which is **link n**.
- In order to position an end-effector generally in 3-space, a minimum of six joints is required.
- ***a link is considered only as a rigid body that defines the relationship between two neighboring joint axes of a manipulator.***
- **Joint axes** are defined by lines in space. **Joint axis i** is defined by a line in space, or a vector direction, about which **link i** rotates relative to **link i − 1**.

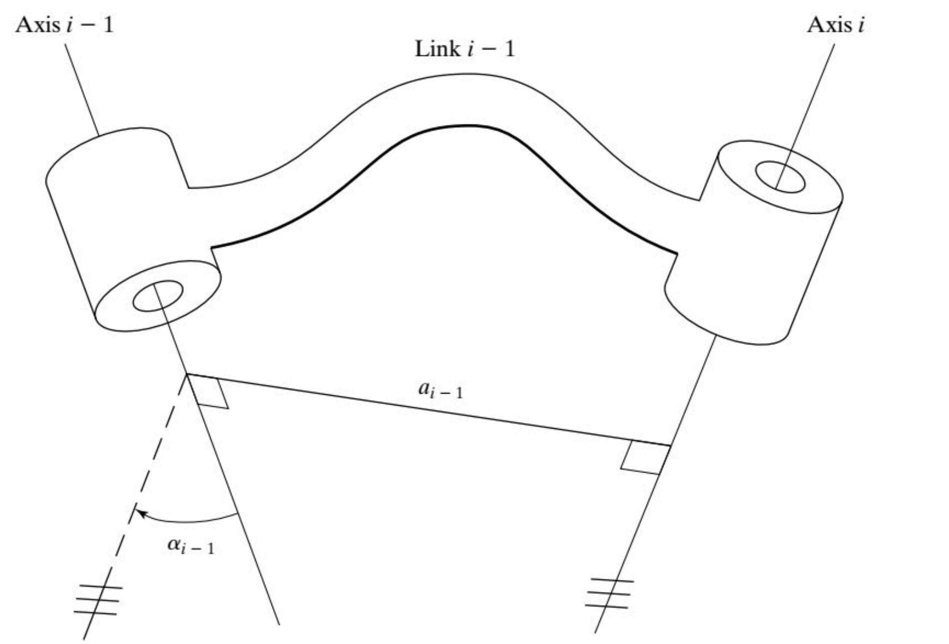



FIGURE 3.2: The kinematic function of a link is to maintain a fixed relationship between the two joint axes it supports. This relationship can be described with two parameters: the link length, $a$, and the link twist, $\alpha$.

## 3.2 LINK DESCRIPTION

- For any two axes in 3-space, there exists a well-defined measure of distance between them. This distance is measured along a line that is mutually perpendicular to both axes.
- This mutual perpendicular always exists; it is unique except when both axes are parallel, in which case there are many mutual perpendiculars of equal length.
- **Figure 3.2** shows **link** $\boldsymbol{i-1}$ and the mutually perpendicular line along which the **link length**, $\boldsymbol{a_{i-1}}$, is measured.

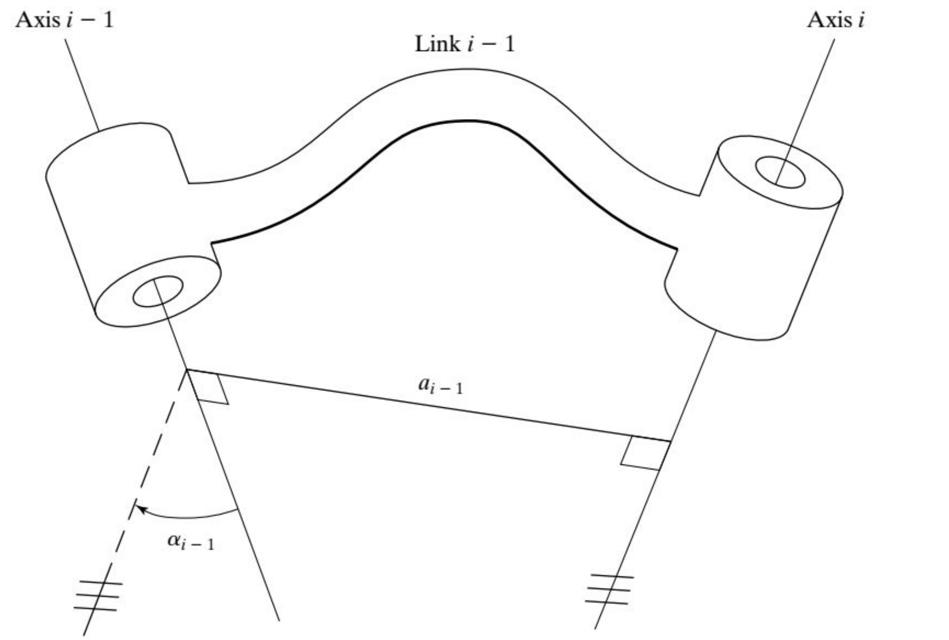
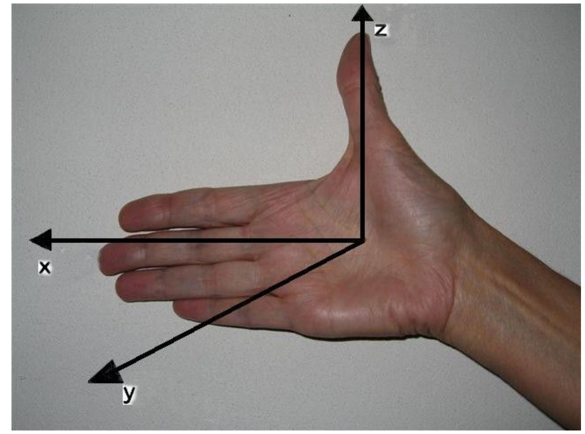



FIGURE 3.2: The kinematic function of a link is to maintain a fixed relationship between the two joint axes it supports. This relationship can be described with two parameters: the link length, $a$, and the link twist, $\alpha$.

## 3.2 LINK DESCRIPTION

- The second parameter needed to define the relative location of the two axes is **called the link twist**.
- If we imagine a plane whose normal is the mutually perpendicular line just constructed, we can project the **axes** $\boldsymbol{i-1}$ and $\boldsymbol{i}$ onto this plane and measure the angle between them. This angle is measured from axis $\boldsymbol{i-1}$ to axis $\boldsymbol{i}$ in the right-hand sense about $\boldsymbol{a_{i-1}}$.
- **Fig. 3.2**, $\boldsymbol{\alpha_{i-1}}$ is indicated as the angle between **axis** $\boldsymbol{i-1}$ and **axis** $\boldsymbol{i}$.
- In the case of intersecting axes, twist is measured in the plane containing both axes, but the sense of $\boldsymbol{\alpha_{i-1}}$ is lost. In this special case, one is free to assign the sign of $\boldsymbol{\alpha_{i-1}}$ arbitrarily.



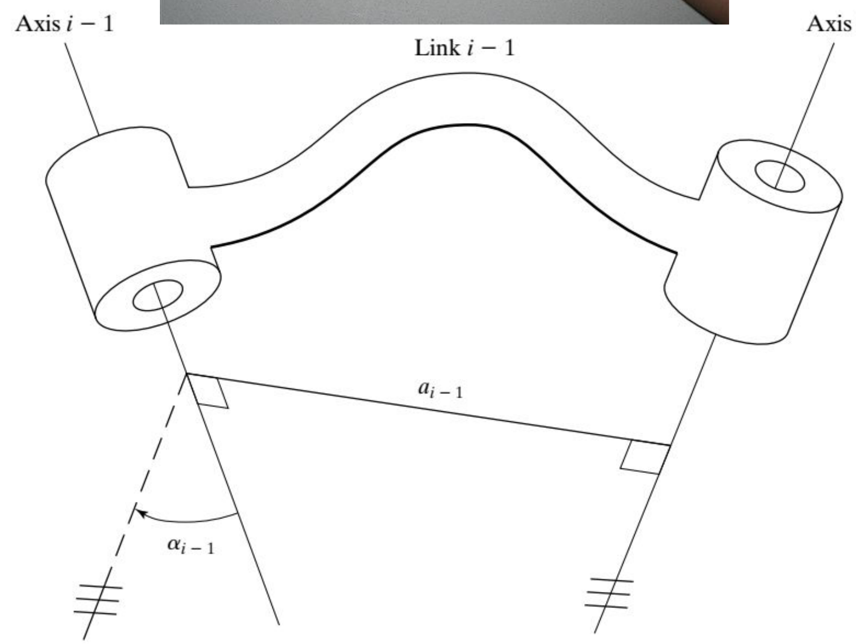




FIGURE 3.2: The kinematic function of a link is to maintain a fixed relationship between the two joint axes it supports. This relationship can be described with two parameters: the link length, $a$, and the link twist, $\alpha$.

## 3.2 LINK DESCRIPTION – Summary

- Most manipulators have revolute joints or have sliding joints called prismatic or linear joints.
- We will consider the Joint is one degree of freedom.
- Base frame has link numbered ''0''.
- The first moving body is link 1.
- Link n denoted to the free end arm.
- Joint axes are defined by lines in space.

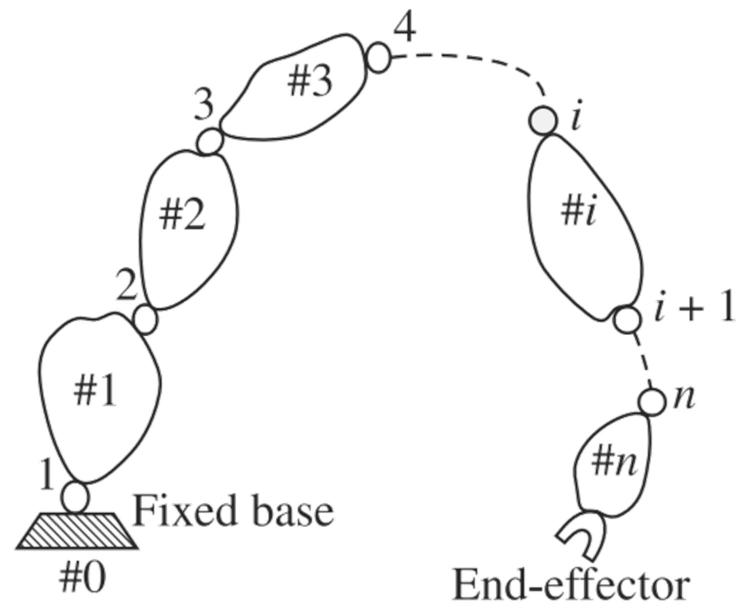
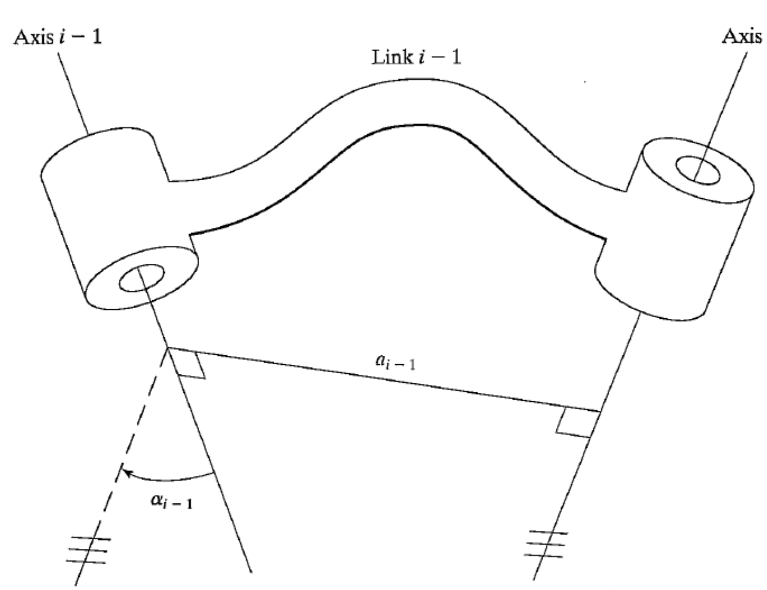


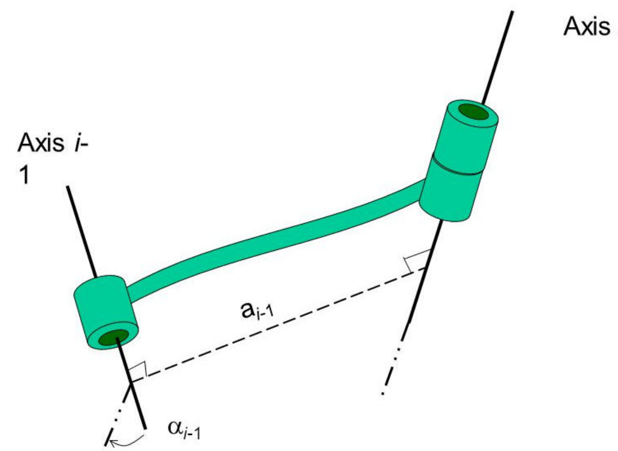

**Fig. 5.27** Serial manipulator

## 3.2 LINK DESCRIPTION – Summary

- **link i** rotates relative to **link i -1**.
- Two parameter or two numbers can give the **position** and **orientation** of the **two joint axes relative each other**:

1. $\boldsymbol{a_{i-1}}$ refers to the ***link length*** or mutual perpendicular from **axis** $\boldsymbol{i-1}$ to **axis** $\boldsymbol{i}$. It is unique except for parallel axis.
2. $\boldsymbol{\alpha_{i-1}}$ states the twist between two neighbouring Joints which is called ***link twist .*** measured in the right-hand sense about $\boldsymbol{a_{i-1}}$



## 3.3 LINK-CONNECTION DESCRIPTION

- Neighboring links have a common joint axis between them. One parameter of interconnection has to do with the distance along this common axis from one link to the next. This parameter is **called the link offset**.
- The offset at **joint axis $i$** is **called $d_i$**. The second parameter describes the amount of rotation about this common axis between one link and its neighbor. This is **called the joint angle, $\theta_i$**.
- The **link offset $d_i$** is variable if **joint $i$** is **prismatic**.
- The **joint angle $\theta_i$** is variable if **joint $i$** is a **revolute joint**.

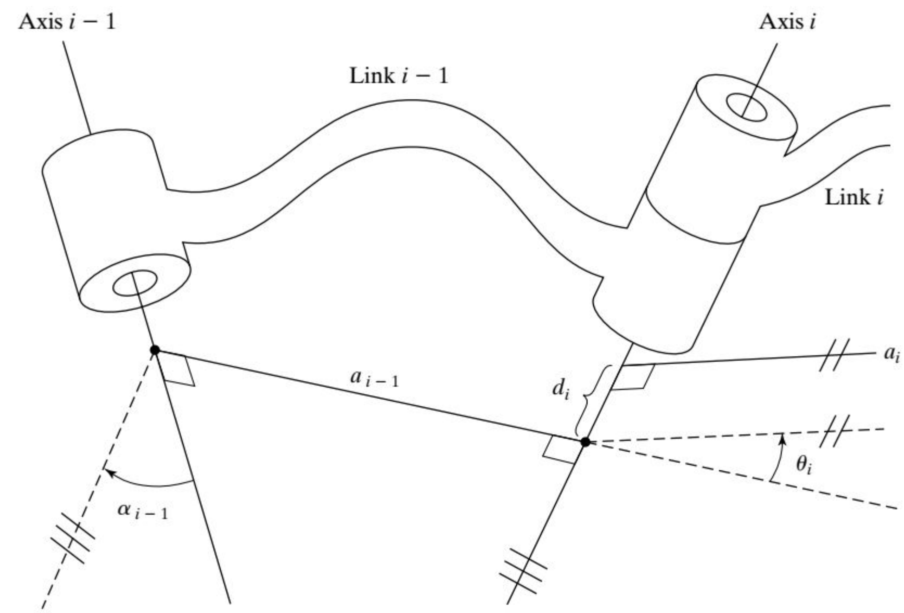



FIGURE 3.4: The link offset, $d$, and the joint angle, $\theta$, are two parameters that may be used to describe the nature of the connection between neighboring links.

## 3.3 LINK-CONNECTION DESCRIPTION

**First and last links in the chain**

- Link length, $a_i$, and link twist, $\alpha_i$, depend on joint axes $i$ and $i+1$. Hence, $a_1$ through $a_{n-1}$ and $\alpha_1$ through $\alpha_{n-1}$ are defined as was discussed in this section.
- At the ends of the chain, it will be our convention to assign zero to these quantities. That is, $a_0 = a_n = 0$ and $\alpha_0 = \alpha_n = 0$.
- Link offset, $d_i$, and joint angle, $\theta_i$, are well defined for **joints 2** through $n-1$ according to the conventions discussed in this section.

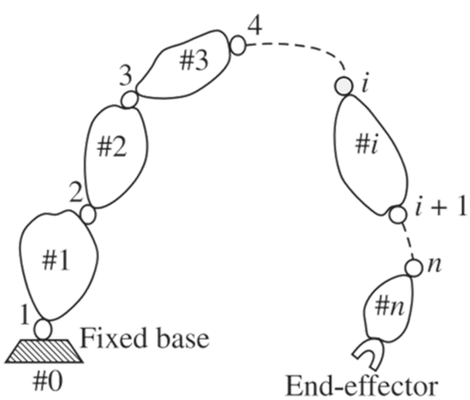



**Fig. 5.27** Serial manipulator

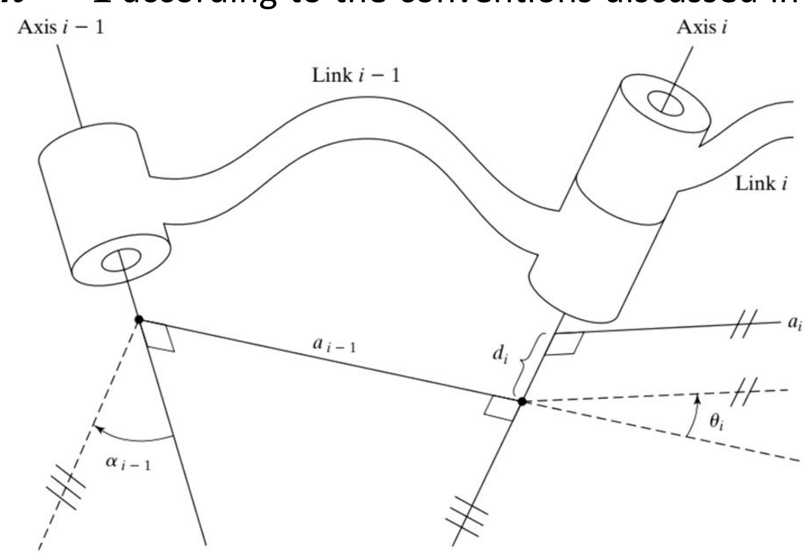



FIGURE 3.4: The link offset, $d$, and the joint angle, $\theta$, are two parameters that may be used to describe the nature of the connection between neighboring links.

## 3.3 LINK-CONNECTION DESCRIPTION

**First and last links in the chain**

- **If joint 1 is revolute**, the **zero position** for **θ1** may be chosen arbitrarily; **d1 = 0.0** will be our convention. Similarly, if **joint 1** is prismatic, the **zero position of d1** may be chosen arbitrarily; **θ1 = 0.0** will be our convention. Exactly the same statements apply to **joint n**.
- These conventions have been chosen so that, in a case where a quantity could be assigned arbitrarily, a zero value is assigned so that later calculations will be as simple as possible.

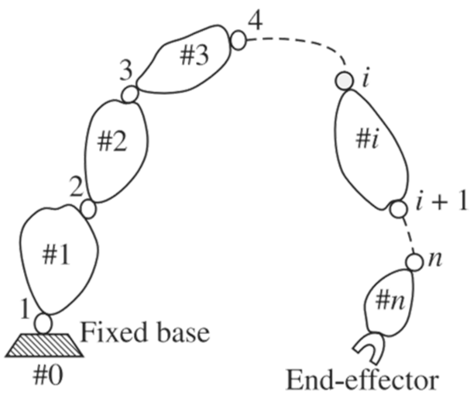


**Fig. 5.27** Serial manipulator

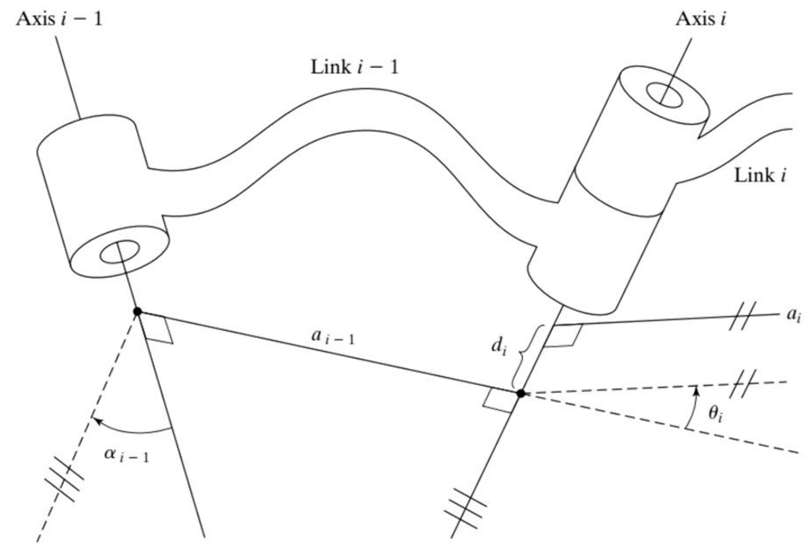


FIGURE 3.4: The link offset, $d$, and the joint angle, $\theta$, are two parameters that may be used to describe the nature of the connection between neighboring links.

$a_i$ and $\alpha_i$ depend on joint axes i and i+1.

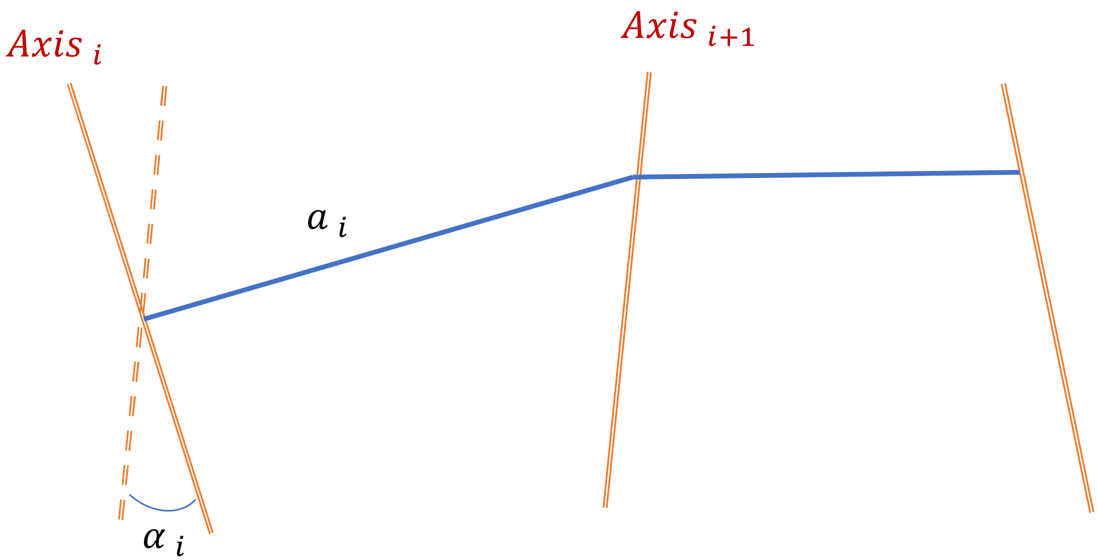
$a_i$ and $\alpha_i$ depend on joint axes i and i+1

$a_1, a_2 \dots a_{n-1}$ and $\alpha_1, \alpha_2 \dots \alpha_{n-1}$

For simplicity :

$a_0 = a_n = 0$

$\alpha_0 = \alpha_n = 0$

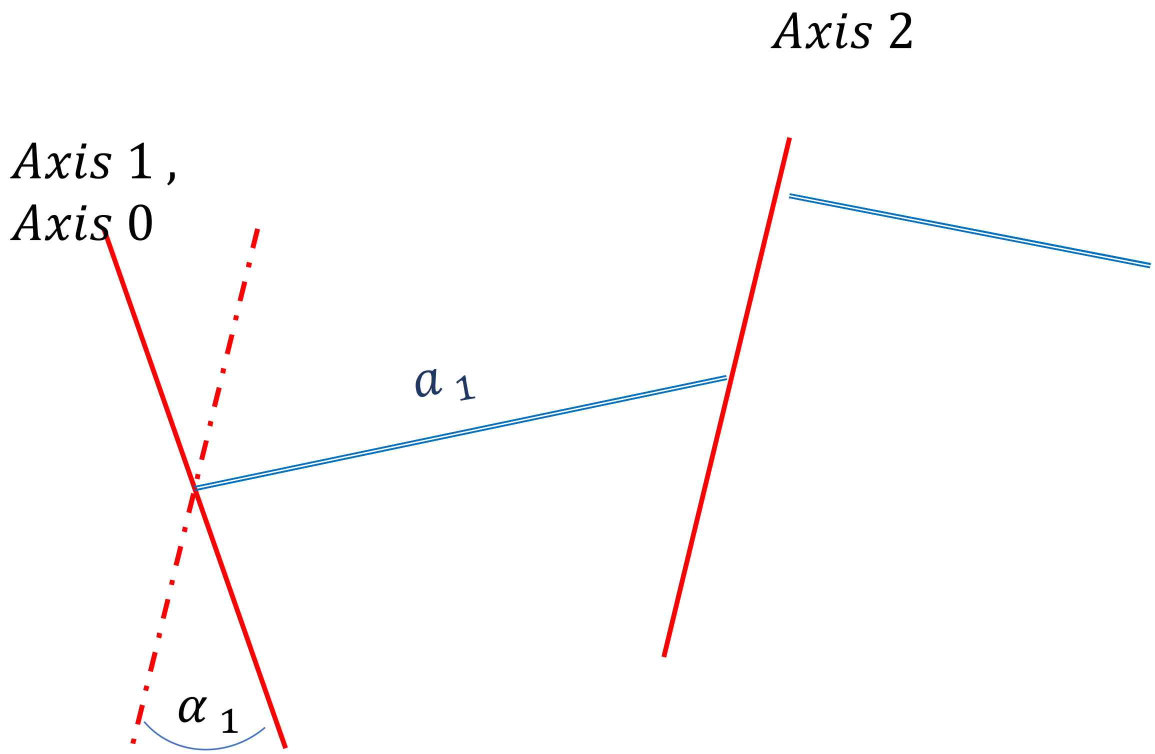


## 3.3 LINK-CONNECTION DESCRIPTION - First and last links in the chain – Summary

**Axis 0 is connected to the base, it can be put it anywhere ,however, for simplicity we will put is coincident with Axis 1. thus :**

$\boldsymbol{a_0 = 0}$

$\boldsymbol{\alpha_0 = 0}$



## 3.3 LINK-CONNECTION DESCRIPTION - Link parameters - Denavit–Hartenberg notation

- Any robot can be described kinematically by giving the values of **four quantities for each link**.
- **Two** describe the link itself, and **two** describe the link's connection to a neighboring link.
- In the usual case of a revolute joint, $\boldsymbol{\theta_i}$ is called the **joint variable**, and the other three quantities would be fixed link parameters.
- For **prismatic joints**, $\boldsymbol{d_i}$ is the joint variable, and the other three quantities are fixed link parameters.
- The definition of mechanisms by means of these quantities is a convention usually called the **Denavit–Hartenberg notation.**

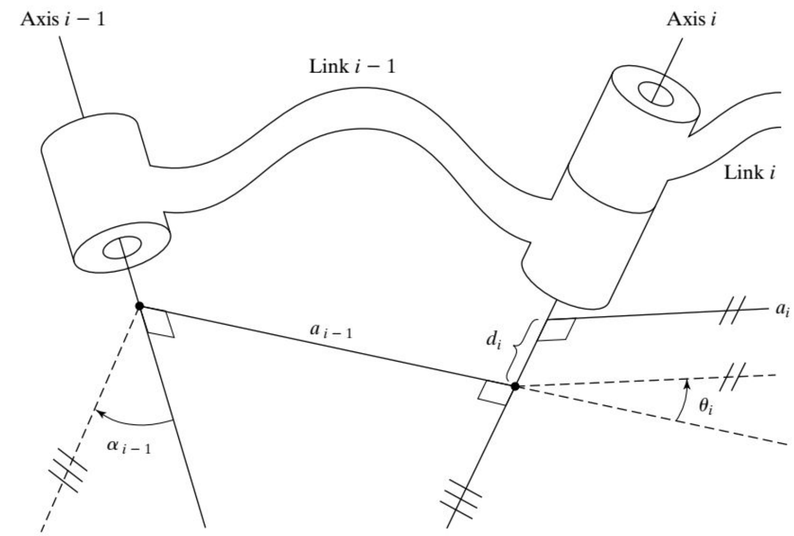



FIGURE 3.4: The link offset, $d$, and the joint angle, $\theta$, are two parameters that may be used to describe the nature of the connection between neighboring links.

## 3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS

In order to describe the location of each link relative to its neighbors, we define a frame attached to each link. The link frames are named by number according to the link to which they are attached. That is, **frame {i}** is attached rigidly to **link i**.

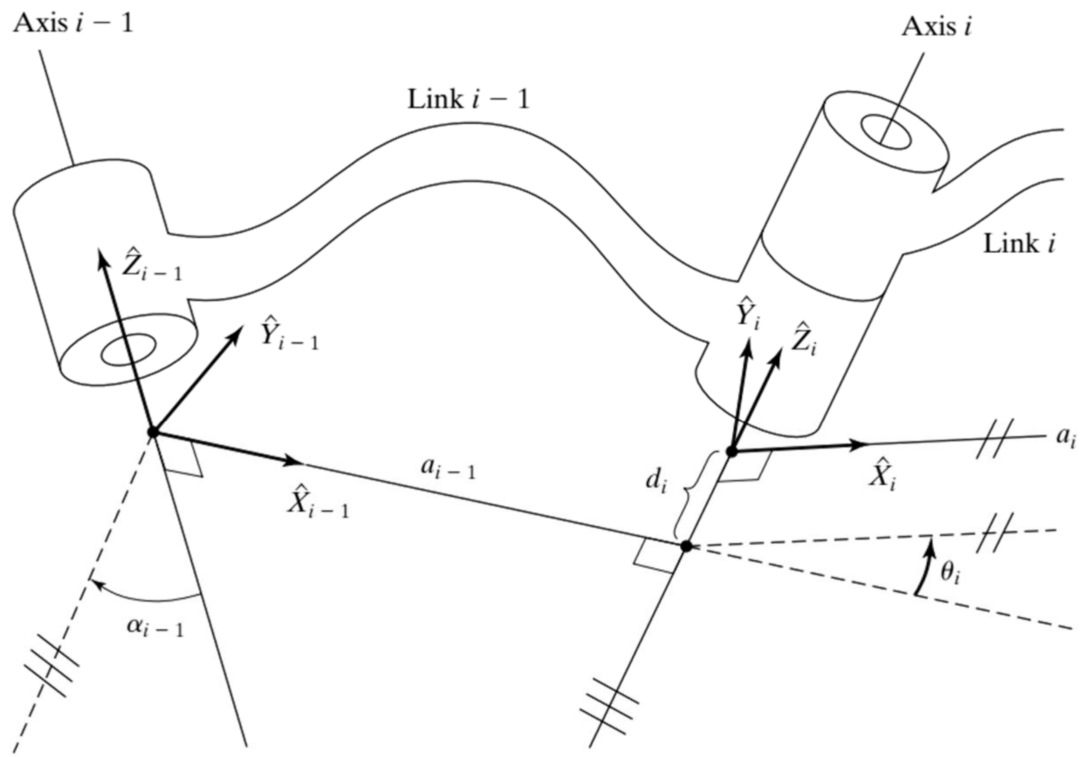





FIGURE 3.5: Link frames are attached so that frame $\{i\}$ is attached rigidly to link $i$.

## 3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS - Intermediate links in the chain

- The convention we will use to locate frames on the links is as follows: The $\hat{\mathbf{Z}}-$**axis** of **frame** $\{\mathbf{i}\}$, called $\hat{\mathbf{Z}}_\mathbf{i}$, is coincident with the **joint axis i**.

- The origin of **frame {i}** is located where the $\mathbf{a_i}$ perpendicular intersects the **joint i axis**. $\hat{\mathbf{X}}_\mathbf{i}$ points along $\mathbf{a_i}$ in the direction from **joint i** to **joint i + 1**.

- In the case of $\mathbf{a_i} = \mathbf{0}$, $\hat{\mathbf{X}}_\mathbf{i}$ is normal to the plane of $\hat{\mathbf{Z}}_\mathbf{i}$ and $\hat{\mathbf{Z}}_{\mathbf{i+1}}$. We define $\boldsymbol{\alpha}_\mathbf{i}$ as being measured in the right-hand sense about $\hat{\mathbf{X}}_\mathbf{i}$, and so we see that the freedom of choosing the sign of $\boldsymbol{\alpha}_\mathbf{i}$ in this case corresponds to two choices for the direction of $\hat{\mathbf{X}}_\mathbf{i}$, $\hat{\mathbf{Y}}_\mathbf{i}$ is formed by the right-hand rule to complete the **ith** frame. **Figure 3.5** shows the location of **frames** $\{\mathbf{i}-\mathbf{1}\}$ and $\{\mathbf{i}\}$ for a general manipulator.

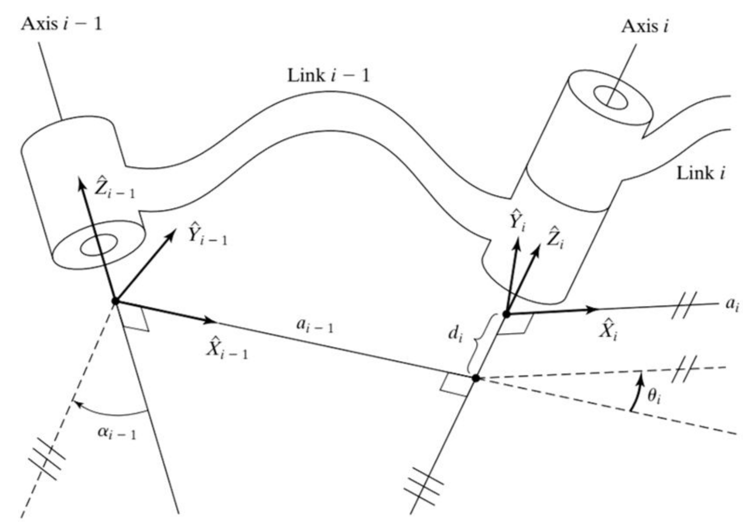
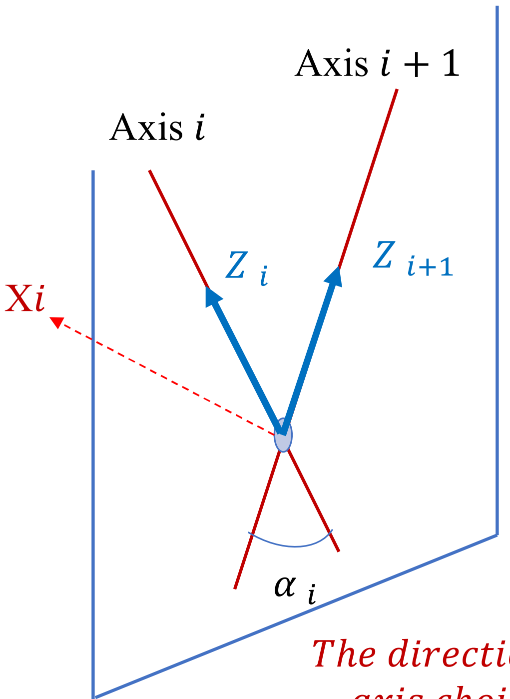
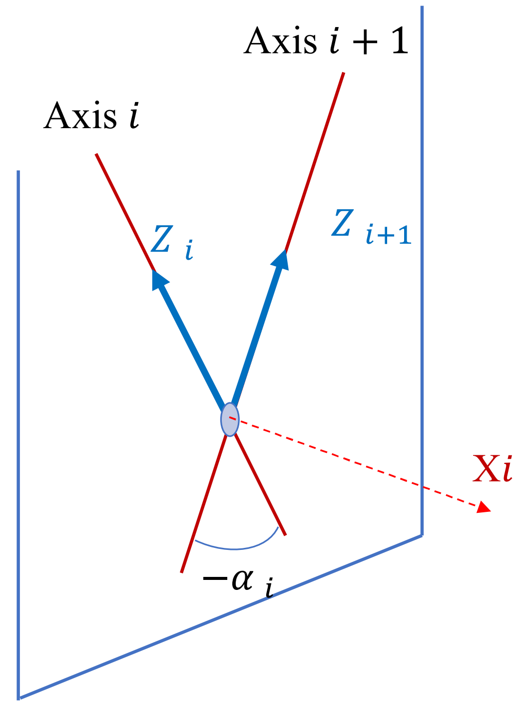


FIGURE 3.5: Link frames are attached so that frame $\{i\}$ is attached rigidly to link $i$.

# Intersecting Joint Axes




*The direction of angle $\alpha_i$ is slected based on X − axis choice*

## 3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS - First and last links in the chain

- We attach a frame to the base of the robot, or **link 0**, called **frame {0}**. This frame does not move; for the problem of arm kinematics, it can be considered the reference frame. We may describe the position of all other link frames in terms of this frame.
- **Frame {0}** is arbitrary, so it always simplifies matters to choose $\hat{\mathbf{Z}}_0$ along **axis 1** and to locate **frame {0}** so that it coincides with **frame {1}** when **joint variable 1** is **zero**.
- Using this convention, we will always have $\mathbf{a_0 = 0}, \boldsymbol{\alpha_0 = 0}$. Additionally, this ensures that $\boldsymbol{d_1 = 0}$ if **joint 1** is **revolute**, or $\boldsymbol{\theta_1 = 0}$ if **joint 1** is **prismatic**.

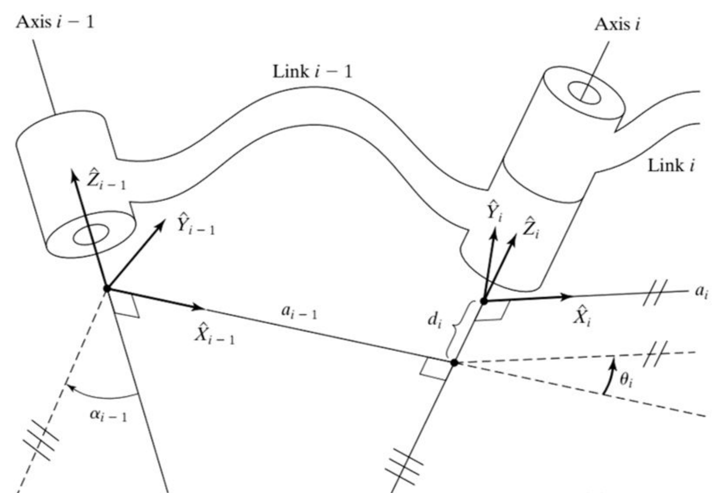



FIGURE 3.5: Link frames are attached so that frame $\{i\}$ is attached rigidly to link $i$.

## 3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS - First and last links in the chain

- **For joint n revolute**, the direction of $\hat{X}_N$ is chosen so that it aligns with $\hat{X}_{N-1}$ when $\theta_n = 0$, and the origin of **frame {N}** is chosen so that $d_n = 0$.
- **For joint n prismatic**, the direction of $\hat{X}_N$ is chosen so that $\theta_n = 0$, and the origin of **frame {N}** is chosen at the intersection of $\hat{X}_{N-1}$ and **joint** $\boldsymbol{axis\ n}$ when $d_n = 0$.

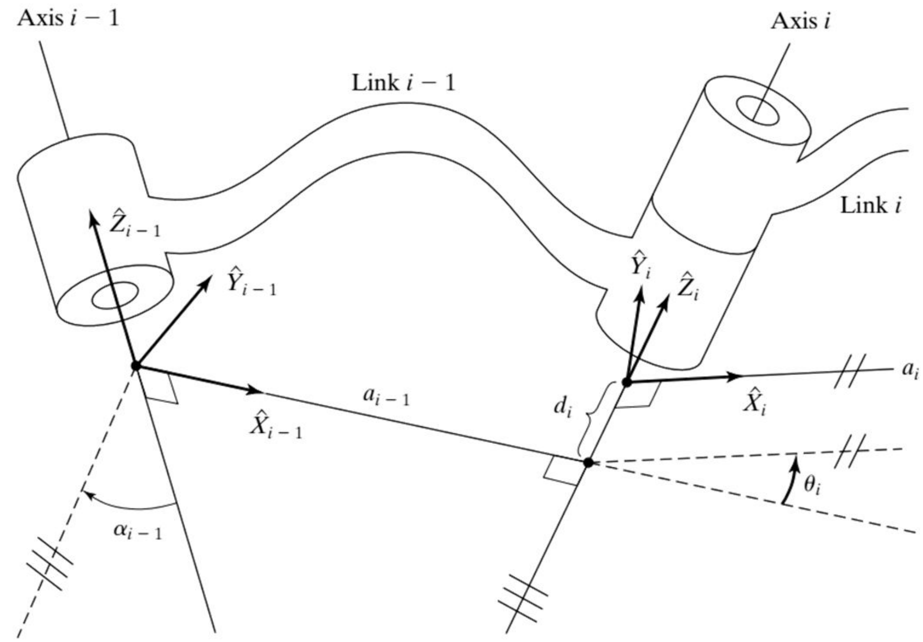





FIGURE 3.5: Link frames are attached so that frame $\{i\}$ is attached rigidly to link $i$.

## Summary of the link parameters in terms of the link frames

- If the link frames have been attached to the links according to our convention, the following definitions of the link parameters are valid:
- $\boldsymbol{a_i}$ = the distance from $\widehat{\boldsymbol{Z}}_{\boldsymbol{i}}$ to $\widehat{\boldsymbol{Z}}_{\boldsymbol{i+1}}$ measured along $\widehat{\boldsymbol{X}}_{\boldsymbol{i}}$;
- $\boldsymbol{\alpha_i}$ = the angle from $\widehat{\boldsymbol{Z}}_{\boldsymbol{i}}$ to $\widehat{\boldsymbol{Z}}_{\boldsymbol{i+1}}$ measured about $\widehat{\boldsymbol{X}}_{\boldsymbol{i}}$;
- $\boldsymbol{d_i}$ = the distance from $\widehat{\boldsymbol{X}}_{\boldsymbol{i-1}}$ to $\widehat{\boldsymbol{X}}_{\boldsymbol{i}}$ measured along $\widehat{\boldsymbol{Z}}_{\boldsymbol{i}}$; and
- $\boldsymbol{\theta_i}$ = the angle from $\widehat{\boldsymbol{X}}_{\boldsymbol{i-1}}$ to $\widehat{\boldsymbol{X}}_{\boldsymbol{i}}$ measured about $\widehat{\boldsymbol{Z}}_{\boldsymbol{i}}$.
- We usually choose $\boldsymbol{a_i} > \boldsymbol{0}$, because it corresponds to a distance; however, $\boldsymbol{\alpha_i}, \boldsymbol{d_i}$, and $\boldsymbol{\theta_i}$ are signed quantities.

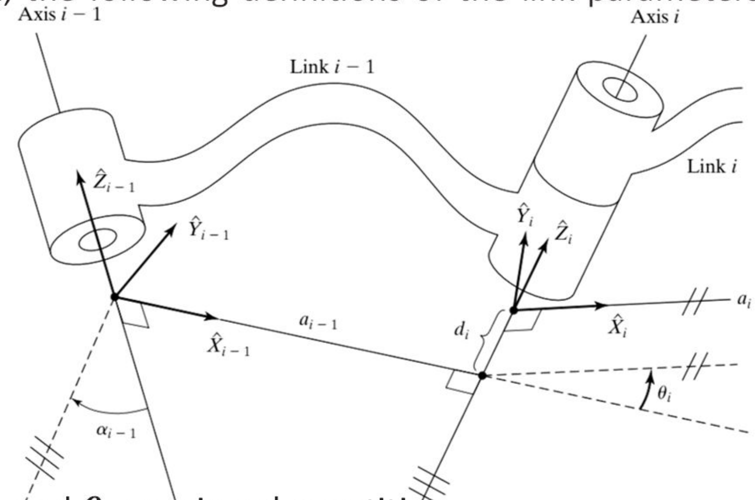



FIGURE 3.5: Link frames are attached so that frame $\{i\}$ is attached rigidly to link $i$.

- A final note on uniqueness is warranted. The convention outlined above does not result in a unique attachment of frames to links. First of all, when we first align the $\widehat{\boldsymbol{Z}}_{\boldsymbol{i}}$ **axis** with **joint axis i**, there are two choices of direction in which to point $\widehat{\boldsymbol{Z}}_{\boldsymbol{i}}$.
- Furthermore, in the case of **intersecting joint axes** (i.e., $\boldsymbol{a_i} = \boldsymbol{0}$), there are two choices for the direction of $\widehat{\boldsymbol{X}}_{\boldsymbol{i}}$, corresponding to the choice of signs for the normal to the plane containing $\widehat{\boldsymbol{Z}}_{\boldsymbol{i}}$ and $\widehat{\boldsymbol{Z}}_{\boldsymbol{i+1}}$.
- When **axes i** and **i + 1** are **parallel**, the choice of origin location for **{i}** is arbitrary (though generally chosen in order to cause $\boldsymbol{d_i}$ to be zero). Also, when prismatic joints are present, there is quite a bit of freedom in frame assignment. (See also Example 3.5.)

## Summary of link-frame attachment procedure

- The following is a summary of the procedure to follow when faced with a new mechanism, in order to properly attach the link frames:
- **1.** Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at $\boldsymbol{axes\ i}$ and $\boldsymbol{i + 1}$).
- **2.** Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the $\boldsymbol{i\ th}$ axis, assign the link-frame origin.
- **3.** Assign the $\boldsymbol{\hat{Z}_i}$ axis pointing along the $\boldsymbol{i\ th}$ joint axis.
- **4.** Assign the $\boldsymbol{\hat{X}_i}$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\boldsymbol{\hat{X}_i}$ to be normal to the plane containing the two axes.
- **5.** Assign the $\boldsymbol{\hat{Y}_i}$ axis to complete a right-hand coordinate system.
- **6.** Assign **{0}** to match **{1}** when the **first joint variable is zero**. For **{N}**, choose an origin location and $\boldsymbol{\hat{X}_N}$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

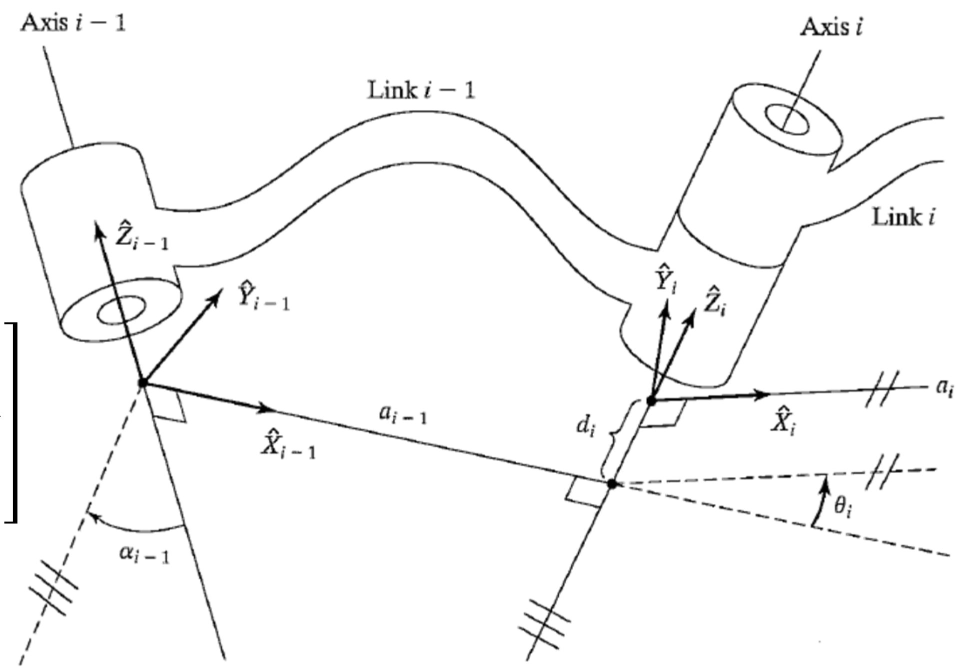
# Denavit-Hartenberg Notation (D-H convention)

Kinematically can describe the robot by four parameters:-$a_i, \alpha_i, d_i, \theta_i$

- Three of those parameters are constant , ( 3 Fixed parameters)
- One Joint Variable ($\theta_i$ for revolute joint while $d_i$ for prismatic Joint).
- $a_i, \alpha_i$ gives the description for the link i
- $d_i, \theta_i$ describe the connection between links.

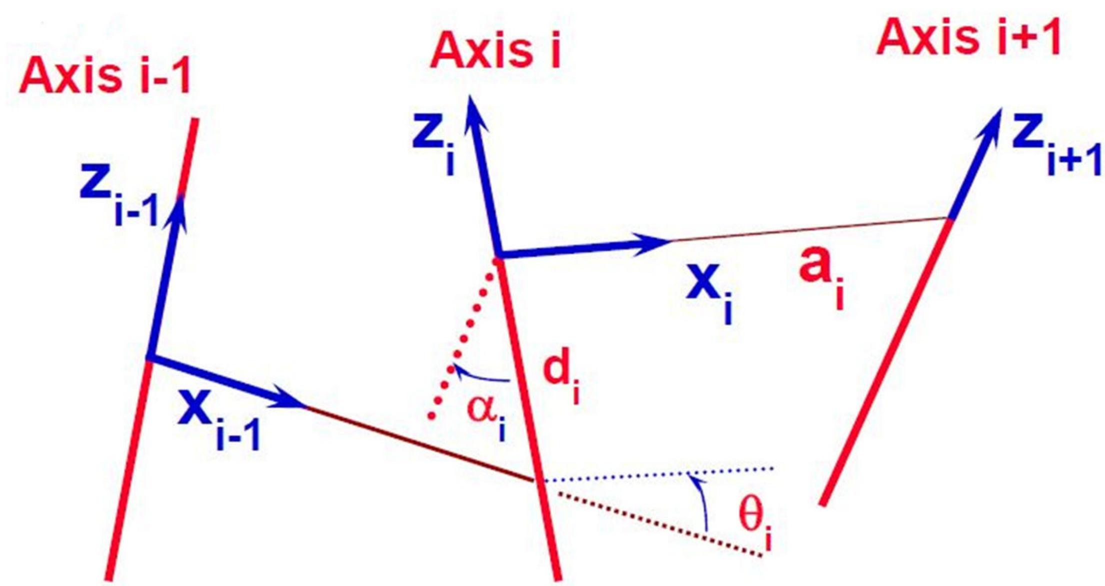
$$ {}^{i-1}_{\ i}T = R\left(X, \alpha_{(i-1)}\right) T\left(X, a_{(i-1)}\right) R\left(Z, \theta_i\right) T\left(Z, d_{(i)}\right) $$

$$ {}^{i-1}_{\ i}T = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{(i-1)} \\ S\theta_i\, C\alpha_{(i-1)} & C\theta_i\, C\alpha_{(i-1)} & -s\alpha_{(i-1)} & -s\alpha_{(i-1)}\, d_i \\ S\theta_i\, S\alpha_{(i-1)} & C\theta_i\, S\alpha_{(i-1)} & C\alpha_{(i-1)} & C\alpha_{(i-1)}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$




$$ {}^{0}_{N}T = {}^{0}_{1}T \quad {}^{1}_{2}T \quad {}^{2}_{3}T \quad \dots \quad {}^{N-1}_{\ N}T $$

# Summary of DH




**$a_i$ : distance ($z_i$ , $z_{i+1}$) along $x_i$**

**$\alpha_i$ : angle ($z_i$ , $z_{i+1}$) about $x_i$**

**$d_i$ : distance (xi-1 , xi ) along zi**

**$\theta_i$ : angle ($x_{i-1}$ , $x_i$ ) about $z_i$**

# EXAMPLE 3.3 / Page 69

**Figure 3.6(a)** shows a three-link planar arm. Because all three joints are revolute, this manipulator is sometimes called an ***RRR*** (or **3*R***) **mechanism**. **Fig. 3.6(b)** is a schematic representation of the same manipulator. Note the double hash marks indicated on each of the three axes, which indicate that these axes are parallel. Assign link frames to the mechanism and give the **Denavit-Hartenberg parameters**.

**Solution:**

$a_i$ = the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$;

$\alpha_i$ = the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$;

$d_i$ = the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$; and

$\theta_i$ = the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$.

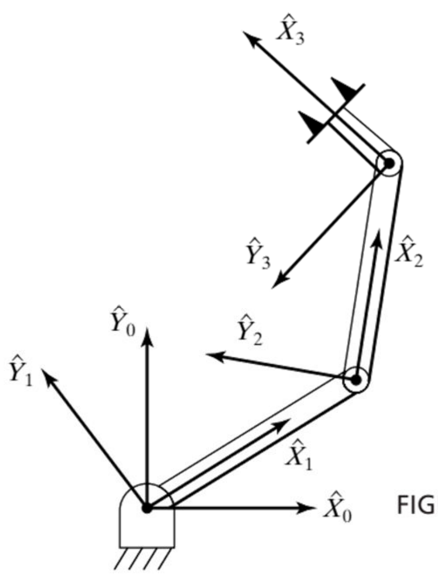



FIGURE 3.7: Link-frame assignments.

| $i$ | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | 0 | $L_1$ | 0 | $\theta_2$ |
| 3 | 0 | $L_2$ | 0 | $\theta_3$ |

FIGURE 3.8: Link parameters of the three-link planar manipulator.

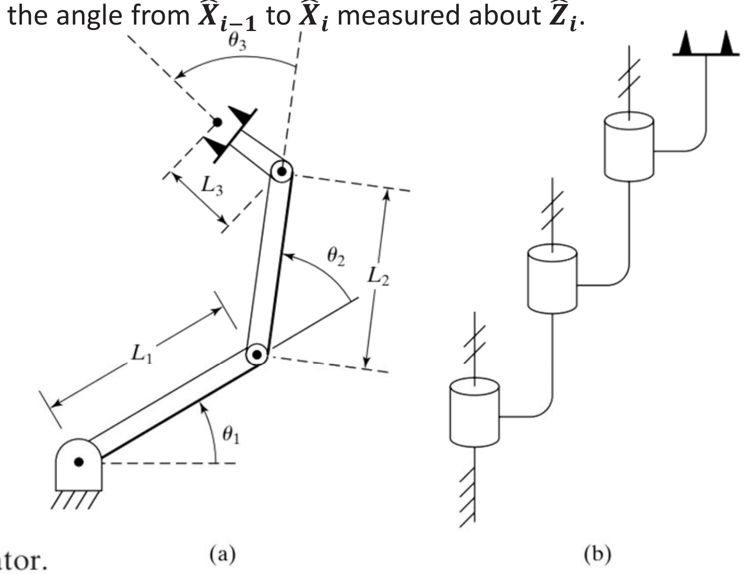



FIGURE 3.6: A three-link planar arm. On the right, we show the same manipulator by means of a simple schematic notation. Hash marks on the axes indicate that they are mutually parallel.

# EXAMPLE 3.4 / Page 71

**Figure 3.9(a) shows a robot having three degrees of freedom and one prismatic joint. This manipulator can be called an ‘‘RPR mechanism,’’ in a notation that specifies the type and order of the joints. It is a ‘‘cylindrical’’ robot whose first two joints are analogous to polar coordinates when viewed from above. The last joint (joint 3) provides ‘‘roll’’ for the hand. Figure 3.9(b) shows the same manipulator in schematic form. Note the symbol used to represent prismatic joints, and note that a ‘‘dot’’ is used to indicate the point at which two adjacent axes intersect. Also, the fact that axes 1 and 2 are orthogonal has been indicated.**

**Solution:**

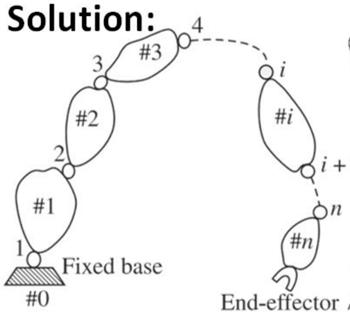
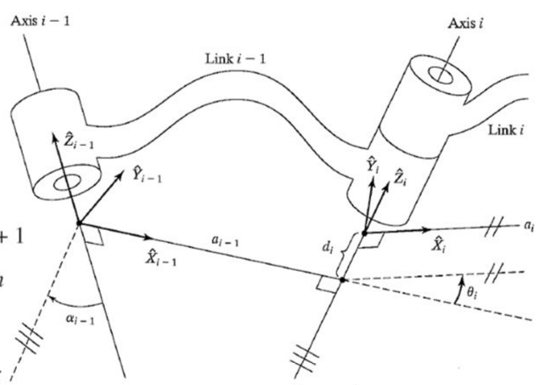



Fig. 5.27 Serial manipulator




$a_i$ = the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$;

$\alpha_i$ = the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$;

$d_i$ = the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$; and

$\theta_i$ = the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$.

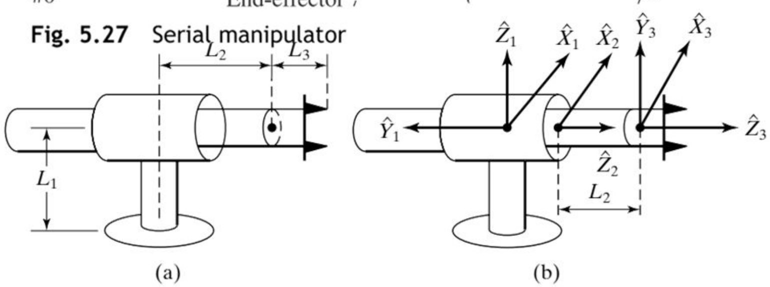



FIGURE 3.10: Link-frame assignments.

| $i$ | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | 90° | 0 | $d_2$ | 0 |
| 3 | 0 | 0 | $L_2$ | $\theta_3$ |

FIGURE 3.11: Link parameters for the *RPR* manipulator of Example 3.4.

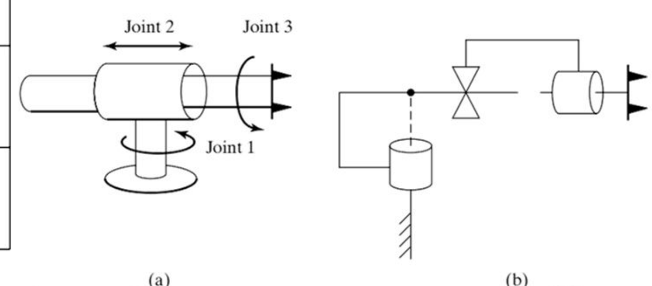



FIGURE 3.9: Manipulator having three degrees of freedom and one prismatic joint.

1) [2] Page (34) **The frame F shown in Figure 2.7 is located at 3, 5, 7 units, with its n-axis parallel to x, its o-axis at 45° relative to the y-axis, and its a-axis at 45° relative to the z-axis. The frame can be described by:**

Solution:

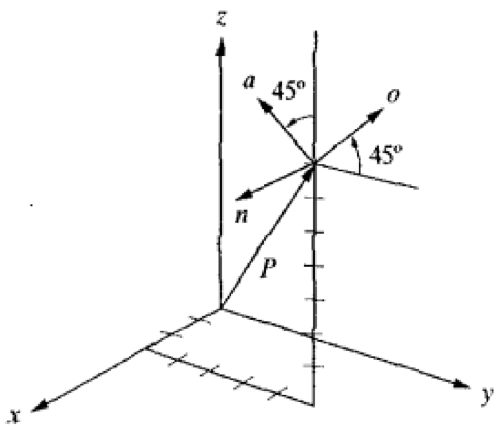



**Figure 2.7** An example of represe tion of a frame in space.

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0.707 & -0.707 & 5 \\ 0 & 0.707 & 0.707 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) [2] Page (40) **A frame F has been moved nine units along the x-axis and five units along the z-axis of the reference frame. Find the new location of the frame:**

$$\mathbf{F} = \begin{bmatrix} \mathbf{0.527} & \mathbf{0.574} & \mathbf{0.628} & \mathbf{5} \\ \mathbf{0.369} & \mathbf{0.819} & \mathbf{0.439} & \mathbf{3} \\ \mathbf{-0.766} & \mathbf{0} & \mathbf{0.643} & \mathbf{8} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution:

$$F_{new} = Trans(d_x, d_y, d_z)\ X\ F_{old} = Trans(9, 0, 5)\ X\ F_{old}$$

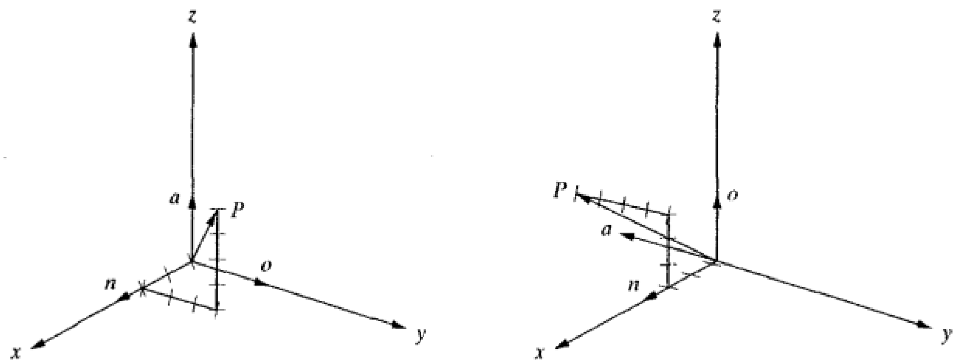
$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.527 & 0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{new} = \begin{bmatrix} 0.527 & 0.574 & 0.628 & 14 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**3) [2] Page (43) A point P $(2, 3, 4)^T$ is attached to a rotating frame. The frame rotates 90° about the x-axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation, and verify the result graphically.**

Solution:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} X \begin{bmatrix} P_n \\ P_o \\ P_a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} X \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$




$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Figure 2.12** Rotation of a frame relative to the reference frame.

4) [2] Page (44) **A point $P(7,3,2)^T$ is attached to a frame $(\bar{n}, \bar{o}, \bar{a})$ and is subjected to the transformations described next. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.**

a) **Rotation of 90° about the z-axis,**

b) **Followed by a rotation of 90° about the y-axis,**

c) **Followed by a translation of [4, —3, 7].**

Solution:

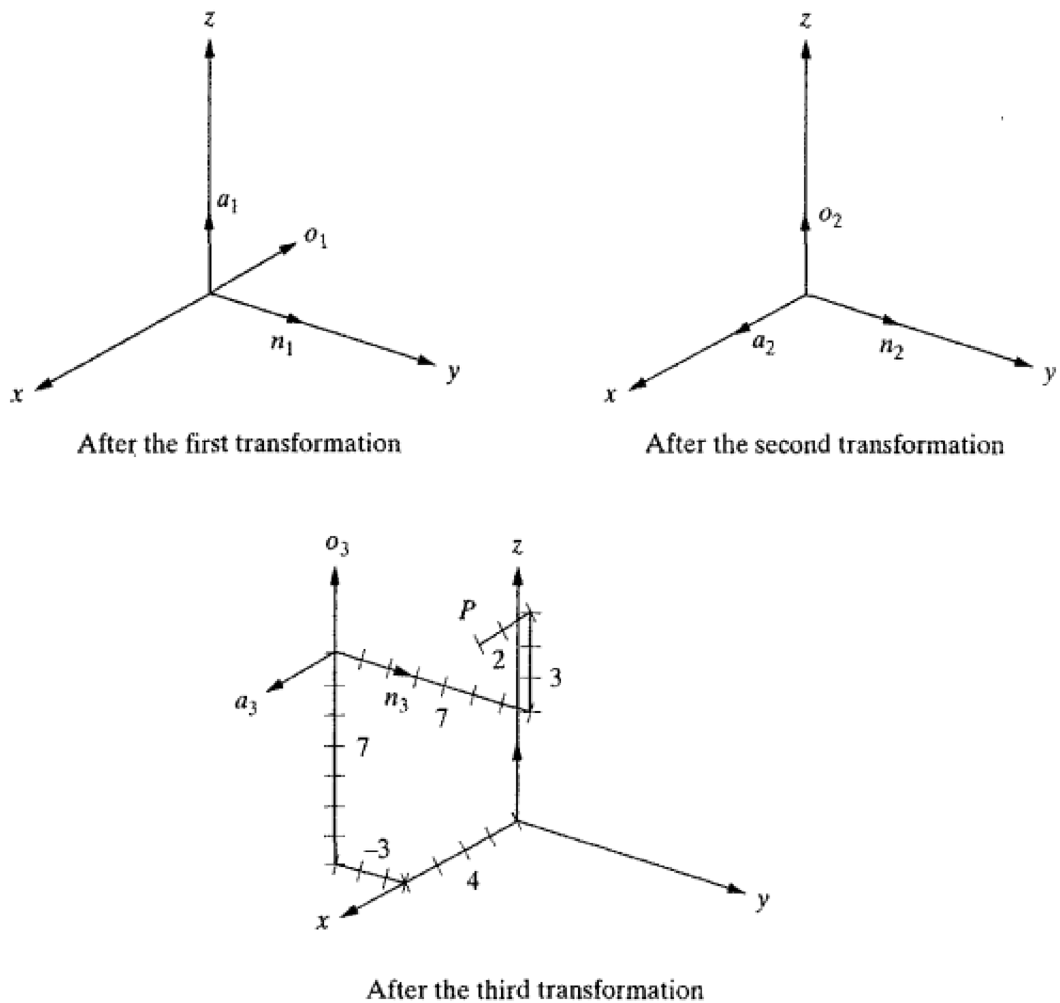



**Figure 2.13** Effects of three successive transformations.

$$P_{xyz} = \text{Trans}(4,-3,7)\ \text{Rot}(y,90)\text{Rot}(z,90)P_{noa} =$$

$$P_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

5) [2] Page (45) **In this case, assume that the same point $P(7, 3, 2)^T$, attached to a frame $(\bar{n}, \bar{o}, \bar{a})$, is subjected to the same transformations, but that the transformations are performed in a different order, as shown. Find the coordinates of the point relative to the reference frame at the conclusion of transformations:**

**a) A rotation of 90° about the z-axis,**

**b) Followed by a translation of [4, - 3 , 7],**

**c) Followed by a rotation of 90° about the y-axis.**

Solution:

$$P_{xyz} = Rot(y, 90)\ Trans(4, -3, 7)\ Rot(z, 90) P_{noa} =$$

$$P_{xyz} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

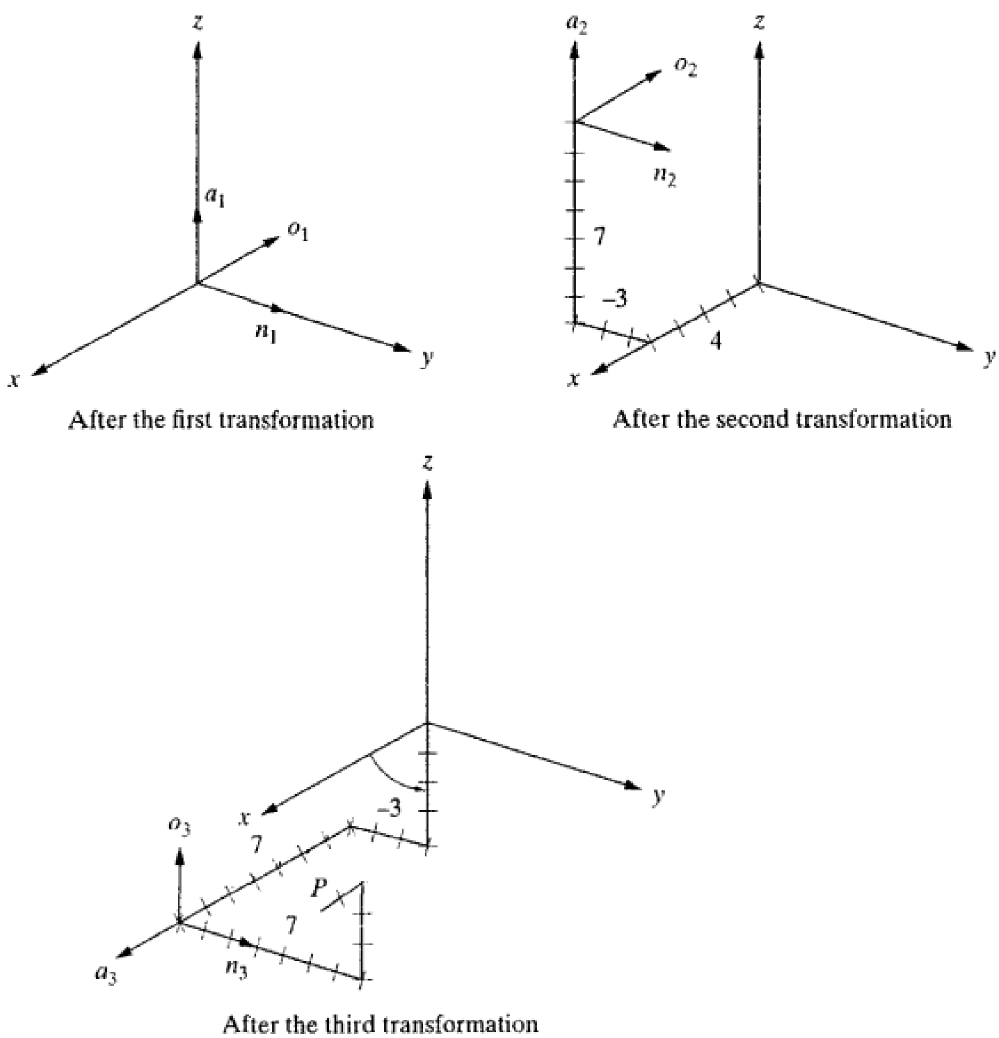



**Figure 2.14** Changing the order of transformations will change the final result.

6) [2] Page (47) **Assume that the same point as in (5) is now subjected to the same transformations, but all relative to the current moving frame, as listed next. Find the coordinates of the point relative to the reference frame after transformations are completed:**

**a) A rotation of 90° about the $\bar{a}$-axis,**

**b) Then a translation of [4, -3,7] along $\bar{n}, \bar{o}, \bar{a}$**

**c) Followed by a rotation of 90° about the $\bar{o}$-axis.**

Solution:

$$P_{xyz} = \mathrm{Rot}(\bar{a}, 90)\ \mathrm{Trans}(4, -3, 7)\ \mathrm{Rot}(\bar{O}, 90) P_{noa} =$$

$$P_{xyz} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

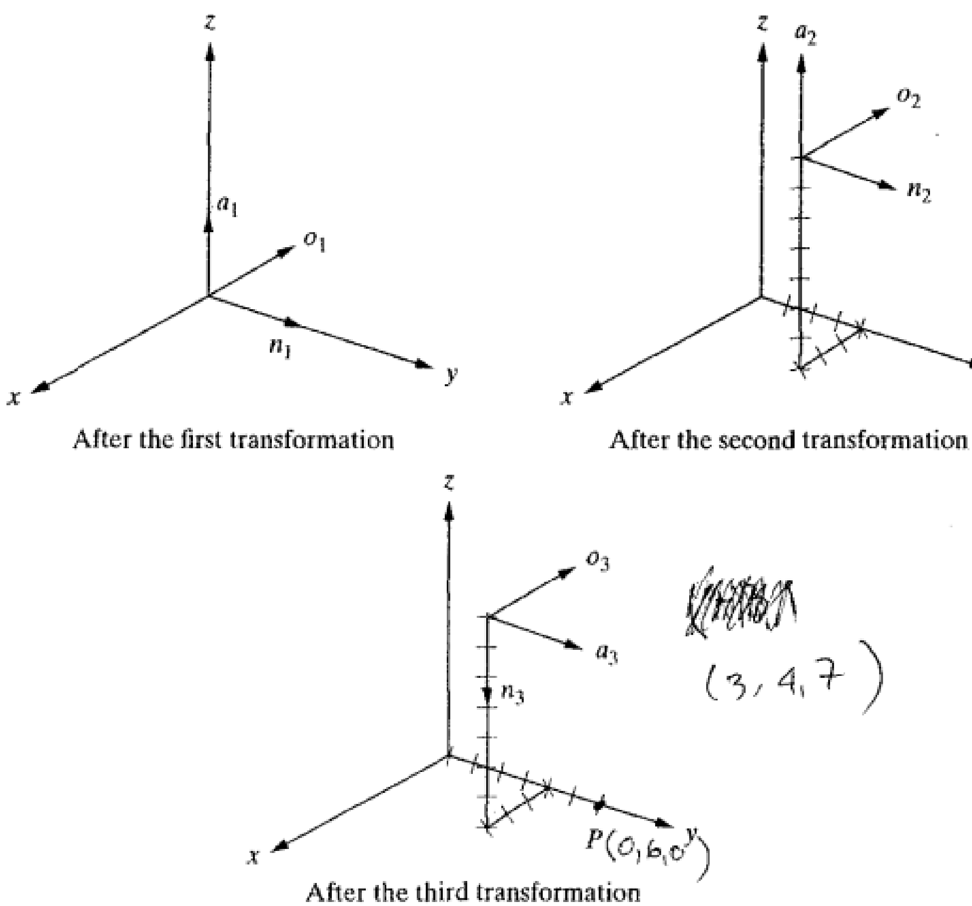



**Figure 2.15** Transformations relative to the current frames.

**7) [2] Page (47) A frame B was rotated about the x-axis 90°; it was then translated about the current a-axis 3 inches before being rotated about the z-axis 90°. Finally, it was translated about current o-axis 5 inches.**

**(a) Write an equation describing the motions.**

**(b) Find the final location of a point P(1,5,4) attached to the frame relative to the reference frame.**

Solution:

In this case, motions alternate relative to the reference frame and current frame.

$$ {}^{U}P_B = \mathrm{Rot}(z\,,90)\ \mathrm{Rot}(x,90)\mathrm{Trans}(0,0,3)\ \mathrm{Trans}(0,5,0) = $$

$$ {}^{U}P = {}^{U}T_B \ X\ {}^{B}P $$

$$ {}^{U}P = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 \\ 5 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 10 \\ 1 \end{bmatrix} $$

**8) [2] Page (51) Calculate the matrix representing $\mathbf{Rot(x, 40°)^{-1}}$**

Solution:

The matrix representing a 40° rotation about the x-axis is

$$ \mathrm{R}(x, 40^o) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & -0.643 & 0 \\ 0 & 0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The inverse of this matrix is

$$ \mathrm{Rot}(x, 40°)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & 0.643 & 0 \\ 0 & -0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

**9) [2] Page (52) Calculate the inverse of the following transformation matrix:**

$$\mathbf{T} = \begin{bmatrix} \mathbf{0.5} & \mathbf{0} & \mathbf{0.866} & \mathbf{3} \\ \mathbf{0.866} & \mathbf{0} & \mathbf{-5} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{5} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution:

$${}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & & & -{}^{A}_{B}R^{T}\ {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.5 & 0.866 & 0 & -(3*0.5+2*0.866+5*0) \\ 0 & 0 & 1 & -(3*0+2*0+5*1) \\ 0.866 & -0.5 & 0 & -(3*0.866+2*-0.5+5*0) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.5 & 0.866 & 0 & -3.23 \\ 0 & 0 & 1 & -5 \\ 0.866 & -0.5 & 0 & -1.598 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: You may want to verify that $T * T^{-1}$will be an identity matrix.

**10) [2] Page (52) In a robotic setup, a camera is attached to the fifth link of a robot with six degrees of freedom. The camera observes an object and determines its frame relative to the camera's frame. Using the following information, determine the necessary motion the end effector has to make to get to the object:**

$$
{}^{5}T_{cam} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{5}T_{H} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{cam}T_{obj} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{H}T_{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Solution:

$$
{}^{E}T_{obj} = {}^{E}T_{H}\ {}^{H}T_{5}\ {}^{5}T_{cam}\ {}^{cam}T_{obj} = {}^{H}T_{E}^{-1}\ {}^{5}T_{H}^{-1}\ {}^{5}T_{cam}\ {}^{cam}T_{obj}
$$

$$
{}^{H}T_{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{5}T_{H}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{E}T_{obj} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{E}T_{obj} = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

**11) [9] (Page 38) From the figure 2.4 the frame $x_1y_1z_1$ is rotated through an angle θ about the $z_0-$ axis, and it is desired to find the resulting transformation matrix $R_1^0$.**

**Note that by convention the positive sense for the angle θ is given by the right-hand rule; that is, a positive rotation of θ degrees about the z-axis would advance a right-hand threaded screw along the positive z-axis.**

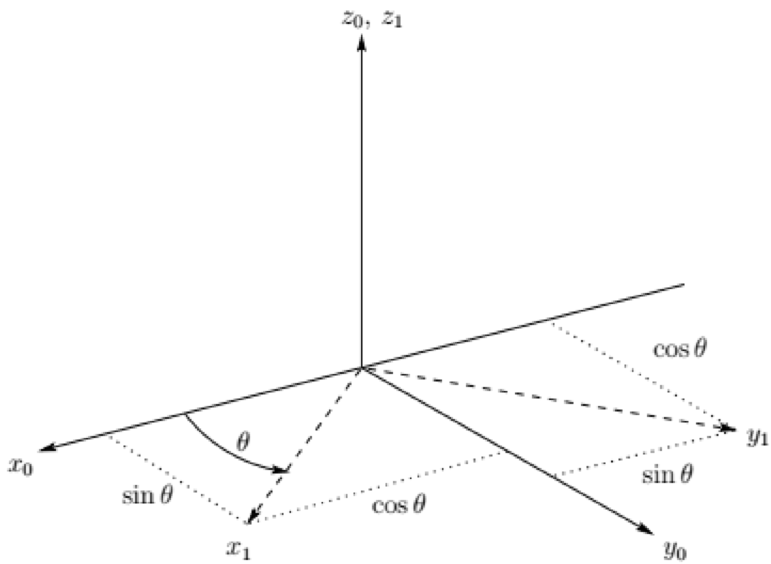



Figure 2.4: Rotation about $z_0$.

Solution:

$$R_1^0 = \begin{bmatrix} x_1.x_0 & y_1.x_0 & z_1.x_0 \\ x_1.y_0 & y_1.y_0 & z_1.y_0 \\ x_1.z_0 & y_1.z_0 & z_1.z_0 \end{bmatrix}$$

$x_1.x_0 = \cos\theta \qquad y_1.x_0 = -\sin\theta$

$x_1.y_0 = \sin\theta \qquad y_1.y_0 = \cos\theta \qquad z_1.z_0 = 1$

From Figure 2.4 we see that and all other dot products are zero. Thus the transformation $R_1^0$ has a particularly simple form in this case, namely

$$R(z_0, \theta) = R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**12) [9] (Page 44) The vector $\vec{v}$ with coordinates $v^0 = (0,1,1)^T$ is rotated about $y_0$ by $\frac{\pi}{2}$ as shown in Figure 2.8. Find the resulting vector.**

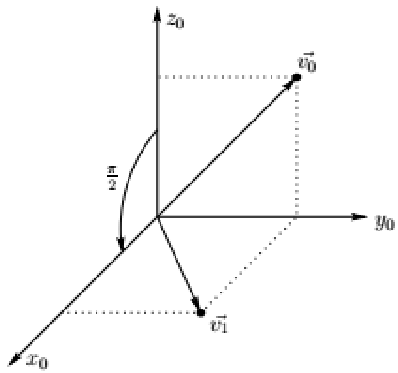



Figure 2.8: Rotating a vector about axis $y_0$.

Solution:

$$v_1^0 = R\left(y, \frac{\pi}{2}\right) v^0 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

**13) [9] (Page 45) Suppose a rotation matrix R represents a rotation of $\varphi$ degrees about the current $y-axis$ followed by a rotation of $\theta$ degrees about the current $z-axis$. Find the matrix R.**

Solution:

$$R = R(y, \varphi)\, R(z, \theta)$$

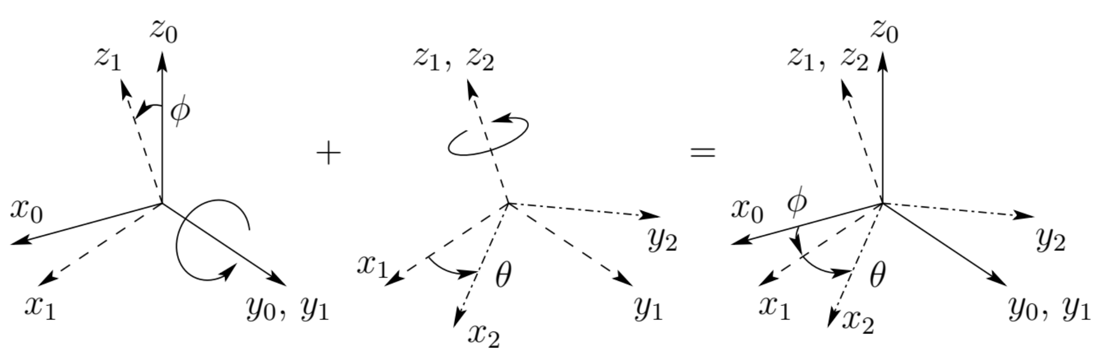



Figure 2.9: Composition of rotations about current axes.

**14) [9] (Page 45) Suppose that the above rotations are performed in the reverse order, that is, first a rotation about the current z-axis followed by a rotation about the current y-axis. Find the resulting rotation matrix.**

Solution:

$$R \quad = R\,(z, \varphi)\; R\,(y, \theta)$$

**15) [9] (page 47) Suppose that a rotation matrix R represents a rotation of $\varphi$ degrees about $y_0$ followed by a rotation of $\theta$ about the fixed $z_0$. Find R.**

Solution:

$$R \quad = R\,(z_0, \varphi)\; R\,(y_0, \theta)$$

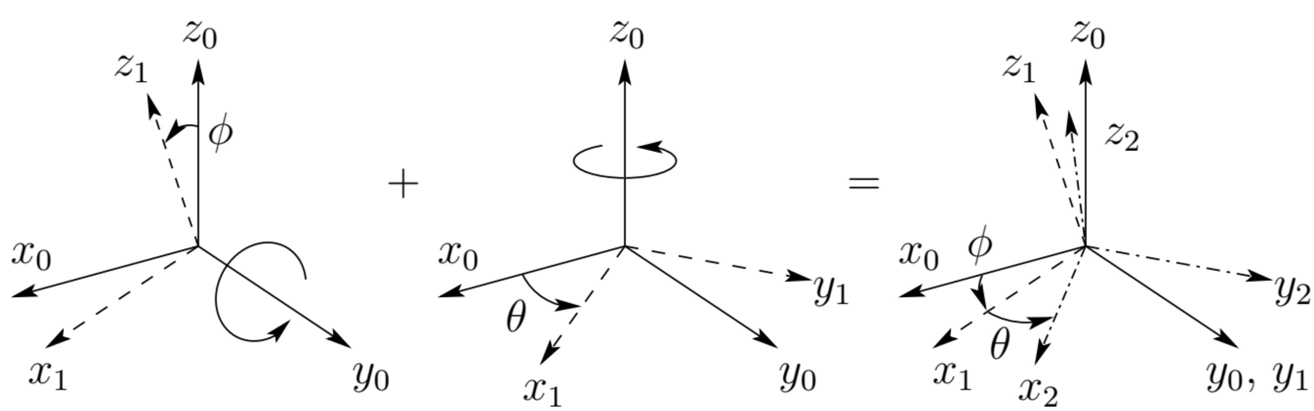



Figure 2.10: Composition of rotations about fixed axes.

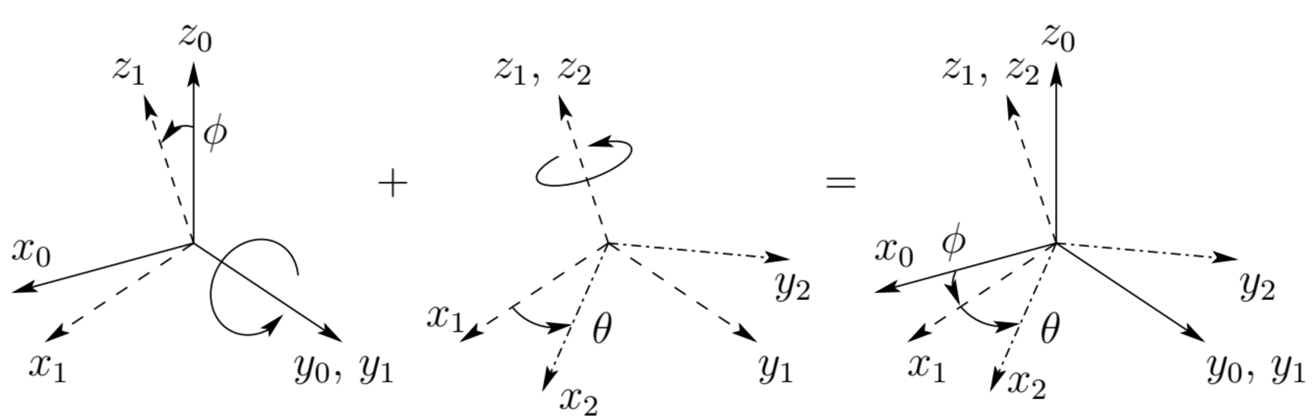



Figure 2.9: Composition of rotations about current axes.

**Summary [9] (Page 48)**

**We can summarize the rule of composition of rotational transformations by the following recipe. Given a fixed frame $x_0y_0z_0$ a current frame $x_1y_1z_1$, together with rotation matrix $R_1^0$ relating them, if a third frame $x_2y_2z_2$ is obtained by a rotation R performed relative to the current frame then postmultiply $R_1^0$ *by* $R = R_2^1$ to obtain**

$$R_2^0 = R_1^0 R_2^1 \text{-------} \quad (2.47)$$

**If the second rotation is to be performed relative to the fixed frame then it is both confusing and inappropriate to use the notation $R_2^1$ to represent this rotation. Therefore, if we represent the rotation by R, we premultiply $R_1^0$ *by* $R$ to obtain**

$$R_2^0 = RR_1^0 \text{-------} \quad (2.48)$$

**In each case $R_2^0$ represents the transformation between the frames $x_0y_0z_0$ and $x_2y_2z_2$. The frame $x_2y_2z_2$ that results in (2.47) will be different from that resulting from (2.48).**

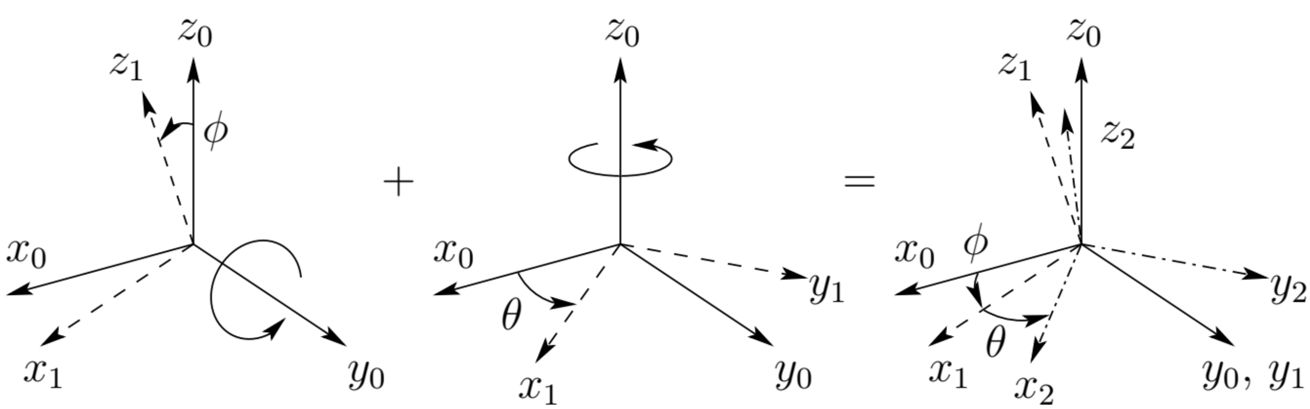



Figure 2.10: Composition of rotations about fixed axes.

16) [9] (Page 53) **Suppose R is generated by a rotation of 90◦ about $z_0$ followed by a rotation of 30◦ about $y_0$ followed by a rotation of 60◦ about $x_0$. Find the rotation equation.**

Solution:

$$R = R(x_0, 60)R(y_0, 30)R(z_0, 90)$$

**17) [9] (Page 58) Find the homogeneous transformation matrix H that represents a rotation of α degrees about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation of θ degrees about the current z-axis.**

Solution:

$$H = Rot(x, \alpha)\ Trans(b, 0, 0)\ Trans(0, 0, d)\ Rot(z, \theta)$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Denavit-Hartenberg Representation of Forward Kinematic Equations

$\boldsymbol{a_i}$ = the distance from $\boldsymbol{\hat{Z}_i}$ to $\boldsymbol{\hat{Z}_{i+1}}$ measured along $\boldsymbol{\hat{X}_i}$;

$\boldsymbol{\alpha_i}$ = the angle from $\boldsymbol{\hat{Z}_i}$ ***to*** $\boldsymbol{\hat{Z}_{i+1}}$ measured about $\boldsymbol{\hat{X}_i}$;

$\boldsymbol{d_i}$ = the distance from $\boldsymbol{\hat{X}_{i-1}}$ ***to*** $\boldsymbol{\hat{X}_i}$ measured along $\boldsymbol{\hat{Z}_i}$; and

$\boldsymbol{\theta_i}$ = the angle from $\boldsymbol{\hat{X}_{i-1}}$ ***to*** $\boldsymbol{\hat{X}_i}$ measured about $\boldsymbol{\hat{Z}_i}$.

$a_i = the\ distance\ from\ \hat{Z}_i\ to\ \hat{Z}_{i+1}\ measured\ along\ \hat{X}_i;$

$\alpha_i = the\ angle\ from\ \hat{Z}_i\ to\ \hat{Z}_{i+1}\ measured\ about\ \hat{X}_i;$

$d_i = the\ distance\ from\ \hat{X}_{i-1}\ to\ \hat{X}_i\ measured\ along\ \hat{Z}_i;$ and

$\theta_i = the\ angle\ from\ \hat{X}_{i-1}\ to\ \hat{X}_i\ measured\ about\ \hat{Z}_i.$

**[1] (page 69)**

$$ {}^{i-1}_{\ \ i}T = R(X, \alpha_{(i-1)})\, T(X, a_{(i-1)})\, R(Z, \theta_i)\, T(Z, d_{(i)}) $$

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{(i-1)} \\ S\theta_i\, C\alpha_{(i-1)} & C\theta_i\, C\alpha_{(i-1)} & -s\alpha_{(i-1)} & -s\alpha_{(i-1)}\, d_i \\ S\theta_i\, S\alpha_{(i-1)} & C\theta_i\, S\alpha_{(i-1)} & C\alpha_{(i-1)} & C\alpha_{(i-1)}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

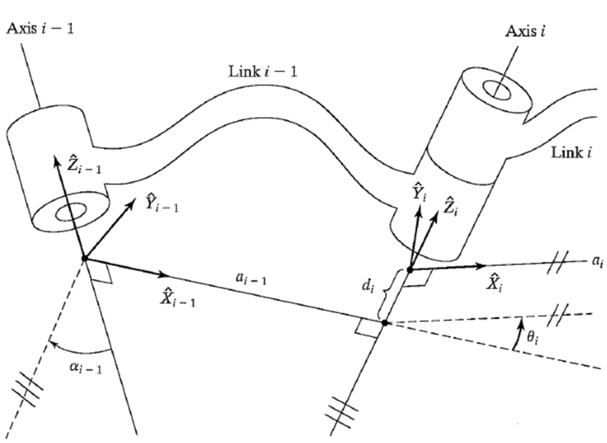
$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (3.6) $$

**[1] (page 75)**

$$ {}^{0}_{N}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T\ \dots\ {}^{N-1}_{\ \ N}T $$

$$ {}^{0}_{N}T = {}^{0}_{1}T\, {}^{1}_{2}T\, {}^{2}_{3}T \dots {}^{N-1}_{\ \ N}T. \qquad (3.8) $$

**[1] (page 76)**



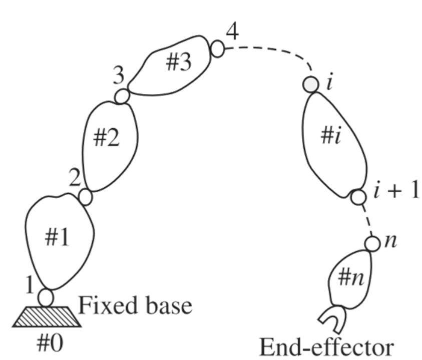




**Fig. 5.27** Serial manipulator

1) [1] (Page 69) **Figure 3.6(a) shows a three-link planar arm. Because all three joints are revolute, this manipulator is sometimes called an RRR (or 3R) mechanism. Fig. 3.6(b) is a schematic representation of the same manipulator.**

**Note the double hash marks indicated on each of the three axes, which indicate that these axes are parallel. Assign link frames to the mechanism and give the Denavit-Hartenberg parameters.**

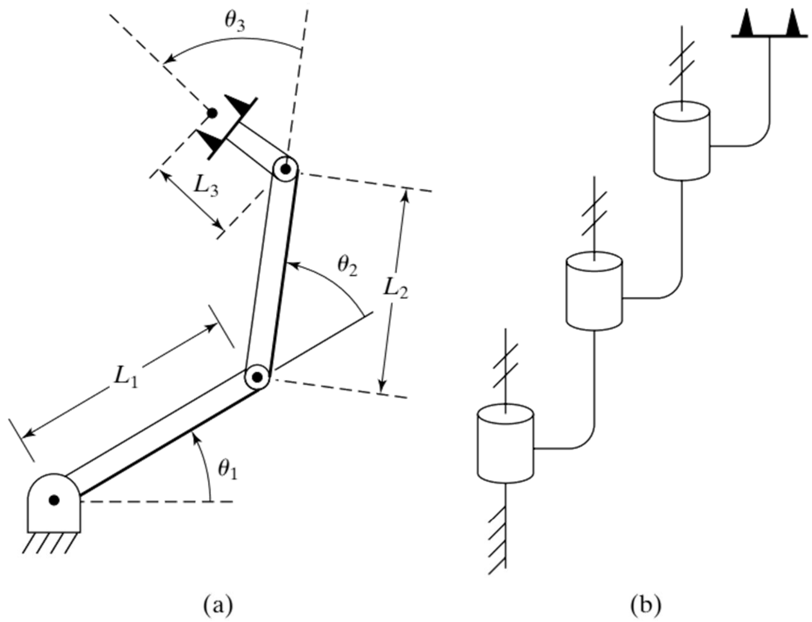



FIGURE 3.6: A three-link planar arm. On the right, we show the same manipulator by means of a simple schematic notation. Hash marks on the axes indicate that they are mutually parallel.

Solution:

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | L1 | 0 | 0 | $\theta_2$ |
| 3 | L2 | 0 | 0 | $\theta_3$ |

$a_i$ = *the distance from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured along* $\hat{X}_i$;

$\alpha_i$ = *the angle from* $\hat{Z}_i$ *to* $\hat{Z}_{i+1}$ *measured about* $\hat{X}_i$;

$d_i$ = *the distance from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured along* $\hat{Z}_i$; and

$\theta_i$ = *the angle from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured about* $\hat{Z}_i$.

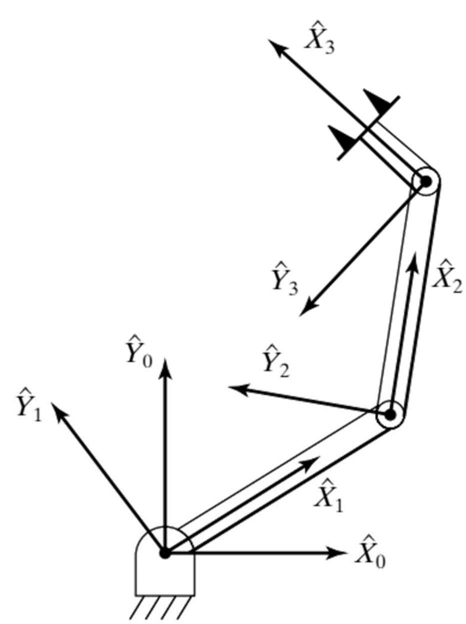



FIGURE 3.7: Link-frame assignments.

2) [1] (Page 71) **Figure 3.9(a) shows a robot having three degrees of freedom and one prismatic joint. This manipulator can be called an "RPR mechanism," in a notation that specifies the type and order of the joints. It is a "cylindrical" robot whose first two joints are analogous to polar coordinates when viewed from above. The last joint (joint 3) provides "roll" for the hand. Figure 3.9(b) shows the same manipulator in schematic form.**

**Note the symbol used to represent prismatic joints, and note that a "dot" is used to indicate the point at which two adjacent axes intersect. Also, the fact that axes 1 and 2 are orthogonal has been indicated.**

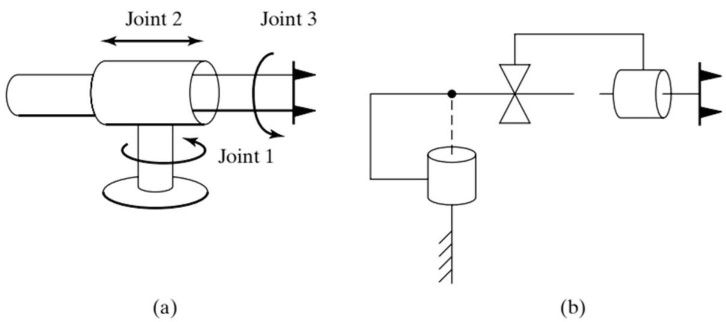



FIGURE 3.9: Manipulator having three degrees of freedom and one prismatic joint.

Solution:

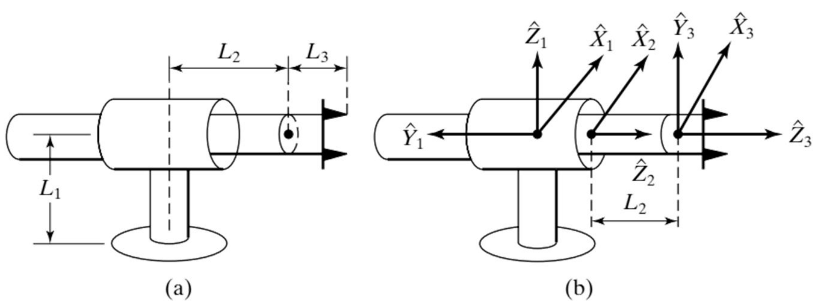



FIGURE 3.10: Link-frame assignments.

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | 0 | 90 | $d_2$ | 0 |
| 3 | 0 | 0 | $L_2$ | $\theta_3$ |

$a_i = the\ distance\ from\ \hat{Z}_i\ to\ \hat{Z}_{i+1}\ measured\ along\ \hat{X}_i$;

$\alpha_i = the\ angle\ from\ \hat{Z}_i\ to\ \hat{Z}_{i+1}\ measured\ about\ \hat{X}_i$;

$d_i = the\ distance\ from\ \hat{X}_{i-1}\ to\ \hat{X}_i\ measured\ along\ \hat{Z}_i$; and

$\theta_i = the\ angle\ from\ \hat{X}_{i-1}\ to\ \hat{X}_i\ measured\ about\ \hat{Z}_i$.

3) [1] (Page 72) **Figure 3.12(a) shows a three-link, 3R manipulator for which joint axes 1 and 2 intersect and axes 2 and 3 are parallel. Figure 3.12(b) shows the kinematic schematic of the manipulator.**

**Note that the schematic includes annotations indicating that the first two axes are orthogonal and that the last two are parallel. Demonstrate the nonuniqueness of frame assignments and of the Denavit–Hartenberg parameters by showing several possible correct assignments of frames {1} and {2}.**

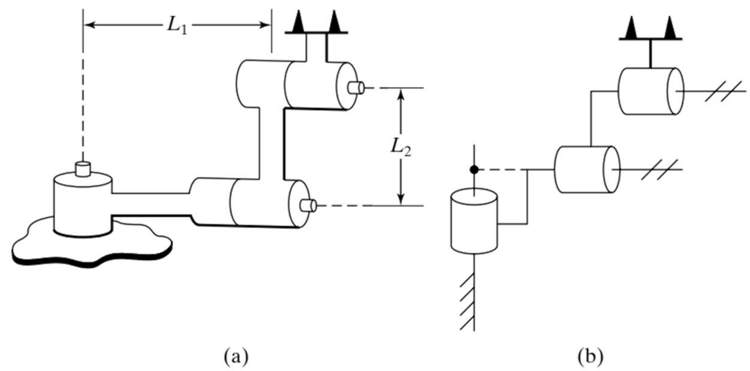



FIGURE 3.12: Three-link, nonplanar manipulator.

Solution:

Figure 3.13 shows two possible frame assignments and corresponding parameters for the two possible choices of direction of $\hat{Z}_2$. In general, when $\hat{Z}_i$ and $\hat{Z}_{i+1}$ intersect, there are two choices for $\hat{X}_i$. In this example, joint axes 1 and 2 intersect, so there are two choices for the direction of $\hat{X}_i$.

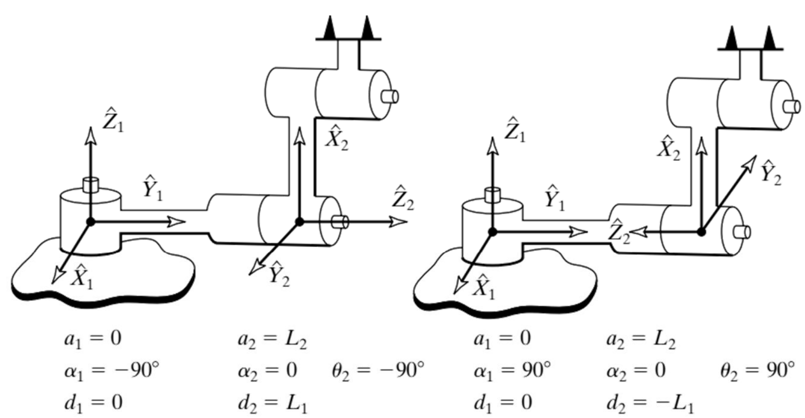



FIGURE 3.13: Two possible frame assignments.

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | -90 | 0 | |
| 2 | $L_2$ | 0 | $L_1$ | -90 |
| 3 | | | | |

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 90 | 0 | |
| 2 | $L_2$ | 0 | $-L_1$ | 90 |
| 3 | | | | |

Figure 3.14 shows two more possible frame assignments, corresponding to the second choice of $\hat{X}_i$.

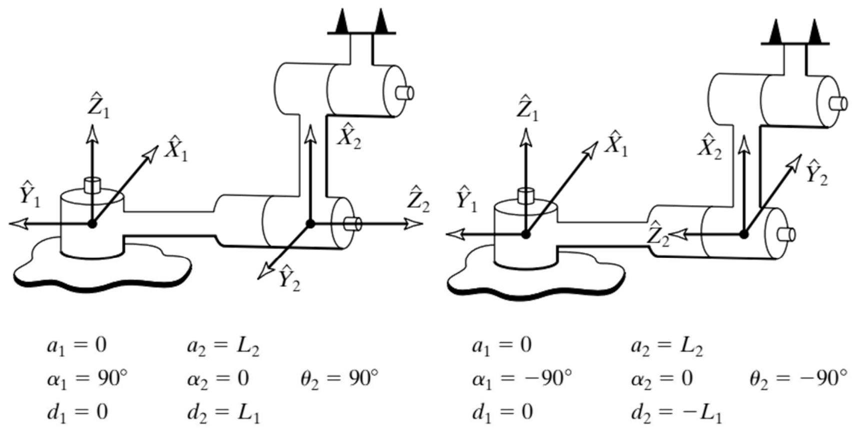



FIGURE 3.14: Two more possible frame assignments.

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 90 | 0 | |
| 2 | $L_2$ | 0 | $L_1$ | 90 |
| 3 | | | | |

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | -90 | 0 | |
| 2 | $L_2$ | 0 | $-L_1$ | -90 |
| 3 | | | | |

In fact, there are four more possibilities, corresponding to the preceding four choices, but with $\hat{Z}_1$pointing downward.

$a_i = the\ distance\ from\ \hat{Z}_i\ to\ \hat{Z}_{i+1}\ measured\ along\ \hat{X}_i;$

$\alpha_i = the\ angle\ from\ \hat{Z}_i\ to\ \hat{Z}_{i+1}\ measured\ about\ \hat{X}_i;$

$d_i = the\ distance\ from\ \hat{X}_{i-1}\ to\ \hat{X}_i\ measured\ along\ \hat{Z}_i;$ and

$\theta_i = the\ angle\ from\ \hat{X}_{i-1}\ to\ \hat{X}_i\ measured\ about\ \hat{Z}_i.$

4) [1] (Page 92) (3.3) **The arm with three degrees of freedom shown in Fig. 3.29. Joint l's axis is not parallel to the other two. Instead, there is a twist of 90 degrees in magnitude between axes 1 and 2. Derive link parameters and the kinematic equations for $^{B}_{W}T$ ($^{0}_{3}T$ ). Note that no $I_3$ need be defined.**

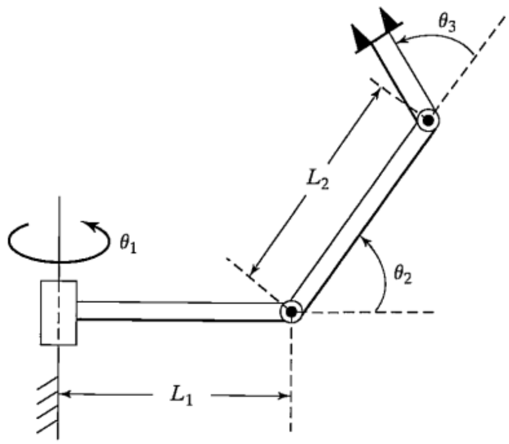
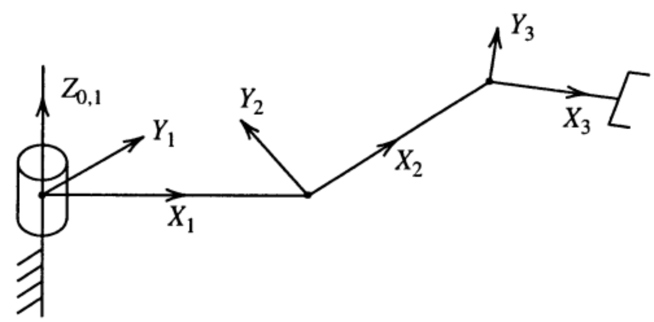



FIGURE 3.29: The 3*R* nonplanar arm (Exercise 3.3).

Solution:

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | $L_1$ | 90 | 0 | $\theta_2$ |
| 3 | $L_2$ | 0 | 0 | $\theta_3$ |




$$^{i-1}_{\ \ i}T = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{(i-1)} \\ S\theta_i\,C\alpha_{(i-1)} & C\theta_i\,C\alpha_{(i-1)} & -s\alpha_{(i-1)} & -s\alpha_{(i-1)}\,d_i \\ S\theta_i\,S\alpha_{(i-1)} & C\theta_i\,S\alpha_{(i-1)} & C\alpha_{(i-1)} & C\alpha_{(i-1)}\,d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}_{1}T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^{1}_{2}T = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ^{2}_{3}T = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & L_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{B}_{W}T = {}^{0}_{3}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T$$

5) **[1] (Page 95) (3.16) Assign link frames to the RPR planar robot shown in Fig. 3.36, and give the linkage parameters.**

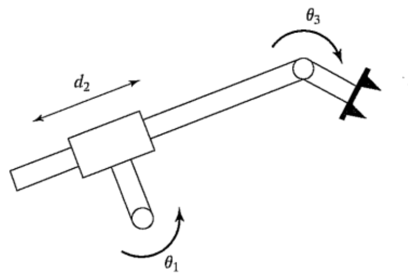
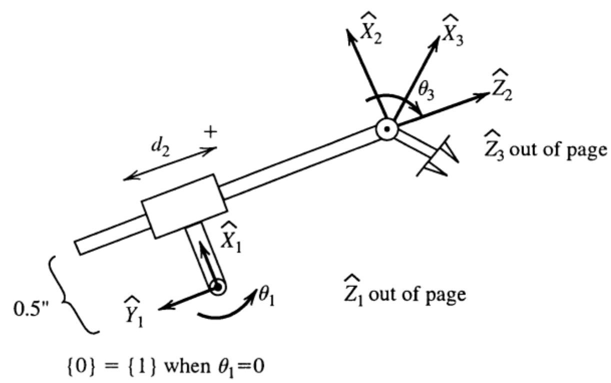



FIGURE 3.36: *RPR* planar robot (Exercise 3.16).

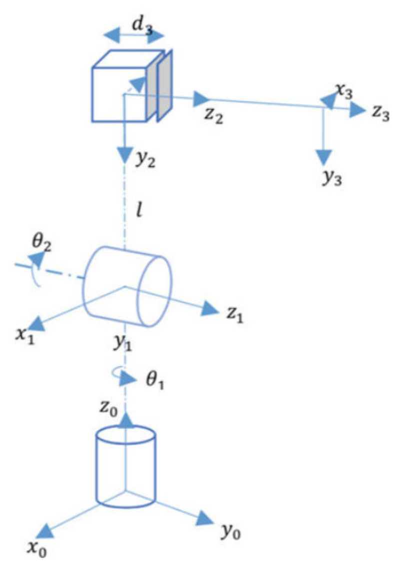
Solution:

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $d_1$ | 0 |
| 2 | $a_1$ | 90 | 0 | $\theta_2$ |
| 3 | 0 | -90 | $d_3$ | 0 |




6) **[15] (Page 7) The robotic device has three active DOFs arranged in a revolute–revolute-prismatic (R–R–P) configuration as shown in Figure below.**

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | -90 | 0 | $\theta_1$ |
| 2 | 1 | 90 | 0 | $\theta_2$ |
| 3 | 0 | 0 | $d_3$ | 0 |



7) [12] (page 40) **Let us assign the link frames to the PUMA-type manipulator shown in figure 2.23 and obtain ${}^{0}T_6$ .**

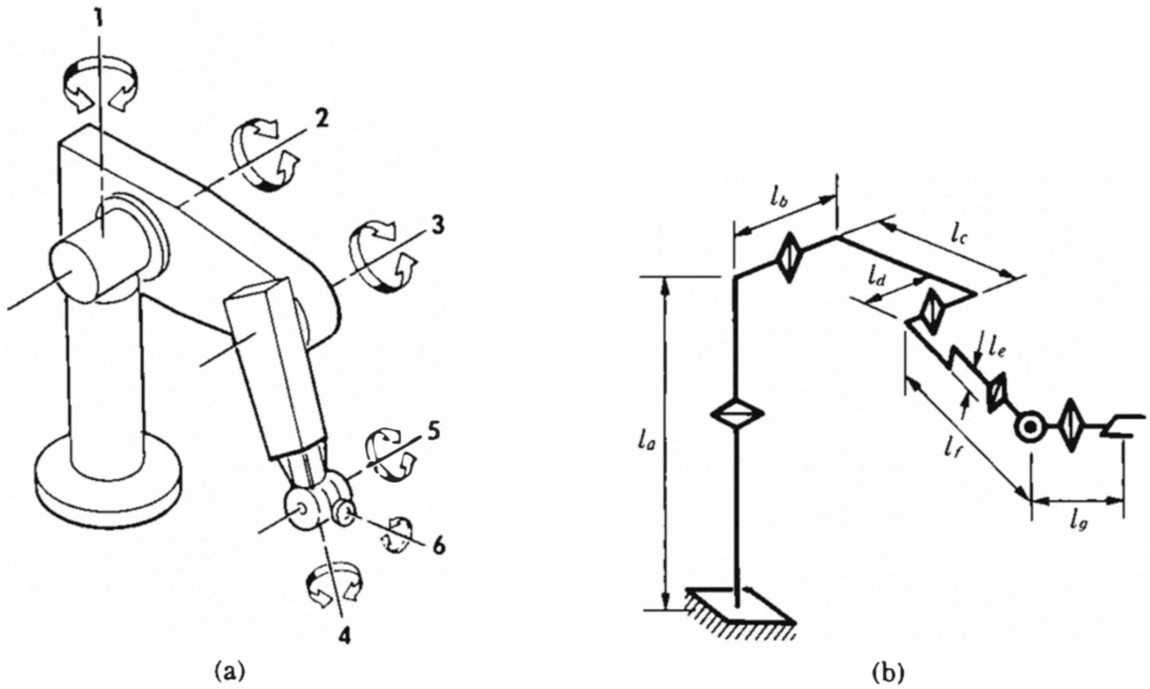



**Figure 2.23**
PUMA-type manipulator. (a) PUMA robot (courtesy of Westinghouse Automation Division/Unimation Inc.). (b) Link structure.

Solution:

Following the scheme of the previous subsection, we obtain the link frames shown in figure 2.24 and the link parameters in table below.

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| **1** | 0 | 0 | 0 | $\theta_1$ |
| **2** | 0 | -90 | $l_b - l_d$ | $\theta_2$ |
| **3** | $l_c$ | 0 | 0 | $\theta_3$ |
| **4** | $l_e$ | -90 | $l_f$ | $\theta_4$ |
| **5** | 0 | 90 | 0 | $\theta_5$ |
| **6** | 0 | -90 | 0 | $\theta_6$ |

$$
{}^{i-1}_{\ \ i}T = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{(i-1)} \\ S\theta_i\, C\alpha_{(i-1)} & C\theta_i\, C\alpha_{(i-1)} & -s\alpha_{(i-1)} & -s\alpha_{(i-1)}\, d_i \\ S\theta_i\, S\alpha_{(i-1)} & C\theta_i\, S\alpha_{(i-1)} & C\alpha_{(i-1)} & C\alpha_{(i-1)}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

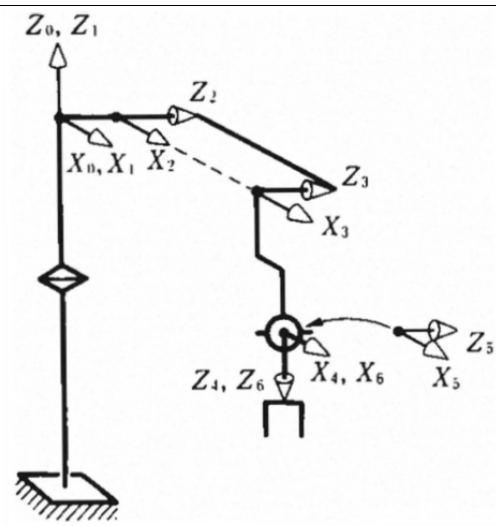



**Figure 2.24**
Link frames for PUMA robot.

8) [12] (Page 36) **The Stanford manipulator, developed mainly for research purposes, has the mechanism shown in figure 2.20.**

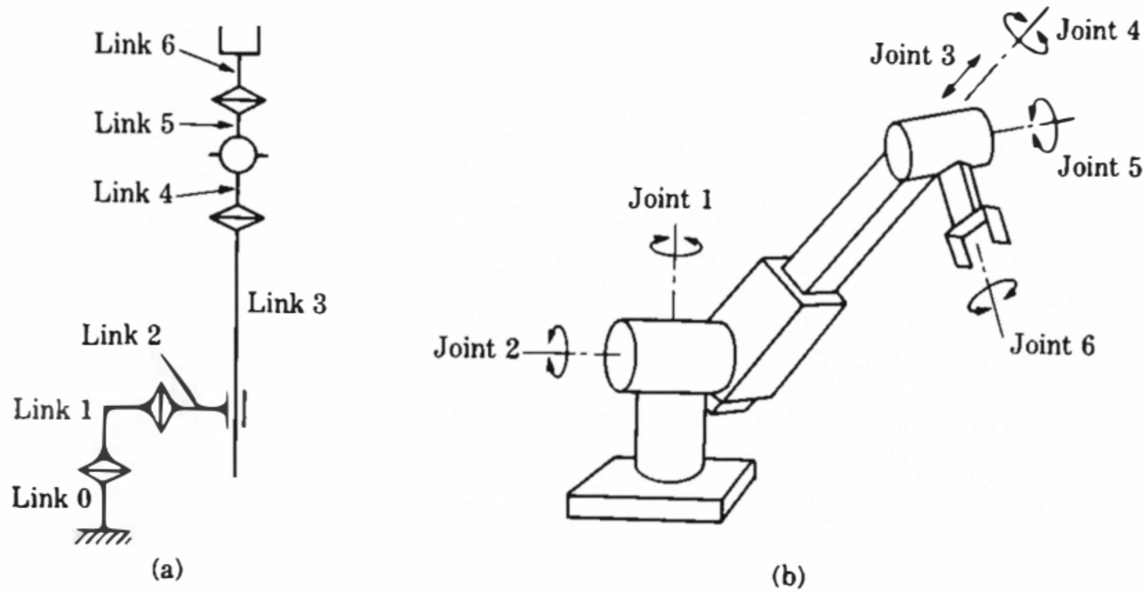



**Figure 2.20**
**Stanford manipulator. (a) Link structure. (b) Appearance.**

Solution:

The link frames determined by the above procedure for this manipulator are shown in figure 2.2 1; the link parameters are given in table 2.1.

Note that figure 2.21 shows a reference configuration for which $\theta_i = 0$ ($i = 1,2,4,5,6$) and all the $X_i$ axes are in the same direction. Also note that $d_3$ is not taken to be zero in the figure, because a configuration with $d_3 = 0$ is unattainable.

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | 0 | -90 | $d_2$ | $\theta_2$ |
| 3 | 0 | 90 | $d_3$ | 0 |
| 4 | 0 | 0 | 0 | $\theta_4$ |
| 5 | 0 | -90 | 0 | $\theta_5$ |
| 6 | 0 | 90 | 0 | $\theta_6$ |

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{(i-1)} \\ S\theta_i\, C\alpha_{(i-1)} & C\theta_i\, C\alpha_{(i-1)} & -s\alpha_{(i-1)} & -s\alpha_{(i-1)}\, d_i \\ S\theta_i\, S\alpha_{(i-1)} & C\theta_i\, S\alpha_{(i-1)} & C\alpha_{(i-1)} & C\alpha_{(i-1)}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

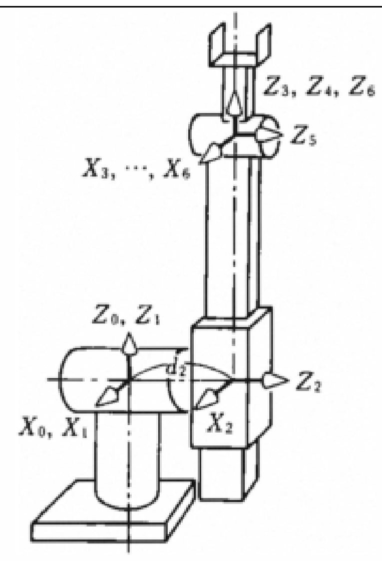



**Figure 2.21**
**Link frames for Stanford manipulator.**

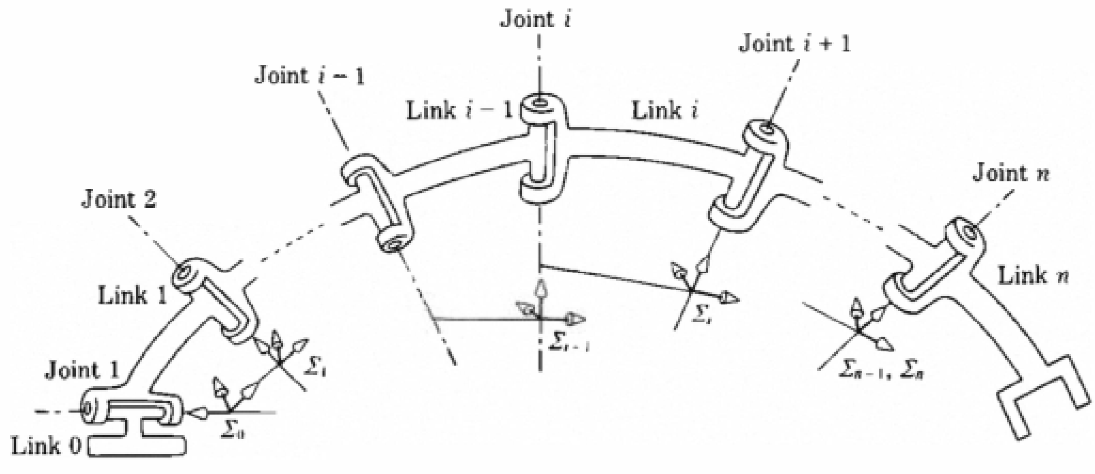



**Figure 2.22**
Another assignment of link frames.

9) [13] (Page 121) **As an example, consider a 6-DOF manipulator (Stanford Manipulator) whose rigid body and coordinate frame assignment are illustrated in Figure 3. Note that the manipulator has an Euler wrist whose three axes intersect at a common point. The first (RRP) and last three (RRR) joints are spherical in shape. P and R denote prismatic and revolute joints, respectively. Find the DH parameters corresponding to this manipulator.**

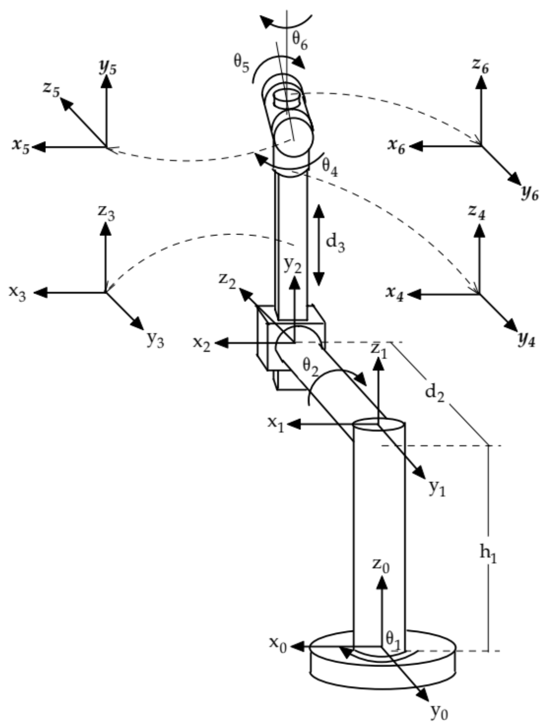



Figure 3. Rigid body and coordinate frame assignment for the Stanford Manipulator.

**Solution:**

| | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | $h_1$ | $\theta_1$ |
| 2 | 0 | 90 | $d_2$ | $\theta_2$ |
| 3 | 0 | -90 | $d_3$ | 0 |
| 4 | 0 | 0 | 0 | $\theta_4$ |
| 5 | 0 | 90 | 0 | $\theta_5$ |
| 6 | 0 | -90 | 0 | $\theta_6$ |

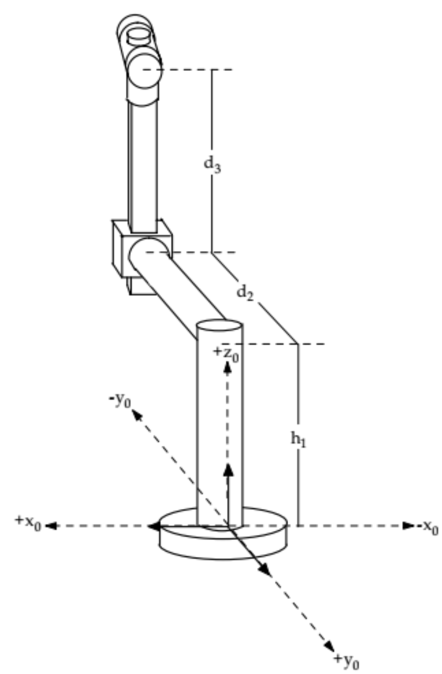



Figure 4. Zero position for the Stanford Manipulator.


# Introduction to Inverse Kinematics

Lectur-8-

1

# Introduction

## Forward Kinematics:

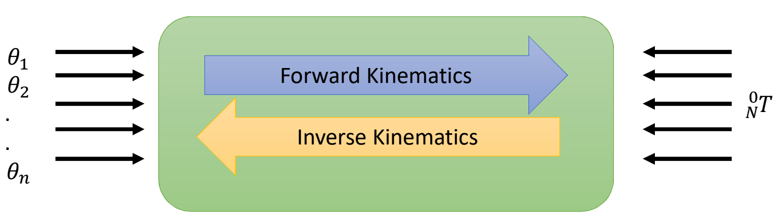
➢Joint variables are given ( $\theta$ $or$ $d$) depending whether ( R or P Joints)

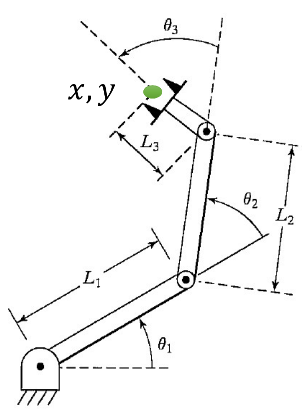
➢Calculate the location of the end-effectors location and Orientation

## Inverse Kinematics:

➢Given end-effector position (X,Y and Z)

➢Find Joint variables ( $\theta_1, \theta_2$ $or$ $d_1, d_2$)

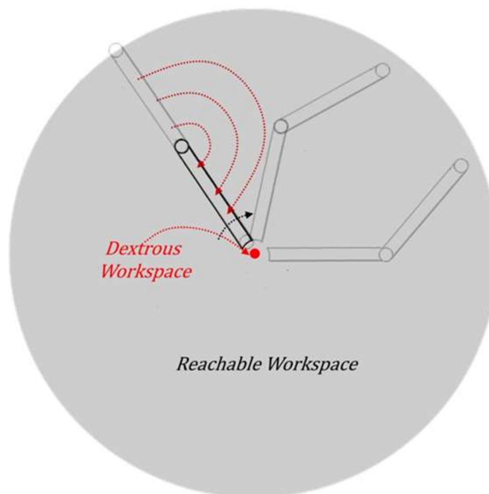



2

# REACHABLE And dextrous workspaces

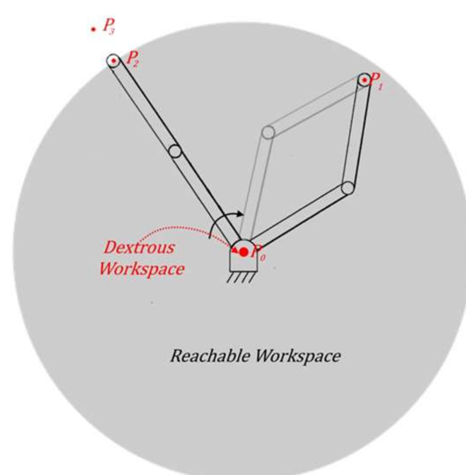
- **Reachable workspace: Set of end frames reachable in at least one orientation**
  - Always includes the edge of the workspace
- **Dextrous workspace: Set of end frames reachable in any orientation**
  - Never includes the edge of the workspace




3

# Existence and Uniqueness

- In linear algebra: linear equations always have one and only one solution (y=mx+c;y=0)
- Nonlinear equations can have none or many ($Y=ax^2+bx+c;Y=0$)
- Same in higher dimensions, e.g. kinematics of robotics
- Solutions may not exist – lie outside of workspace
- Solutions may not be unique – more than one set of joint angles achieves the goal




4

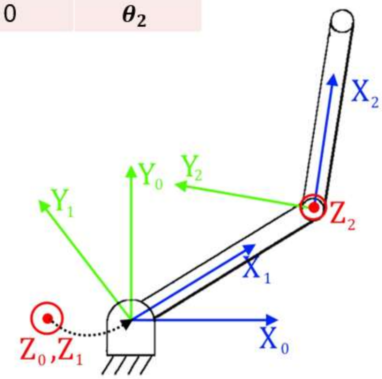
# 2 LINK PLANAR RR: RECAP

| $i$ | $a_{i-1}$ | $\alpha_{i-1}$ | $d_i$ | $\theta_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | $\theta_1$ |
| 2 | $L_1$ | 0 | 0 | $\theta_2$ |

- What is the position of the end point of Link 2 in the GCS?

$$^{0}P_{End2} = {}^{0}_{2}T\ {}^{2}P_{End2}$$

$$= \begin{pmatrix} L_2 c_{\theta_1+\theta_2} + L_1 c_{\theta_1} \\ L_2 s_{\theta_1+\theta_2} + L_1 s_{\theta_1} \\ 0 \\ 1 \end{pmatrix}$$




- Worked example two weeks ago – see also Craig, Ch. 3.

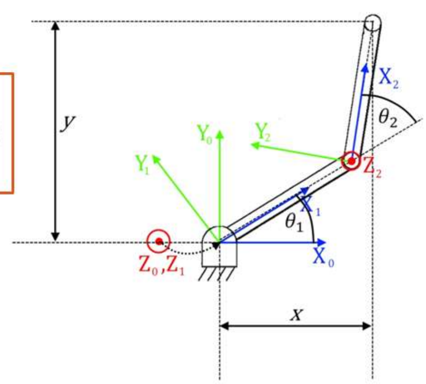


# 2 LINK PLANAR RR INVERSE KINEMATICS

- Given a specified X and Y position in the GCS for the end of the second link, what are the joint angles which achieve it?

$$\begin{pmatrix} L_2 c_{\theta_1+\theta_2} + L_1 c_{\theta_1} \\ L_2 s_{\theta_1+\theta_2} + L_1 s_{\theta_1} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \\ 1 \end{pmatrix} \quad \textbf{(1)}$$

$$\theta_1 = ?$$
$$\theta_2 = ?$$



# Inverse Kinematics RP Example :

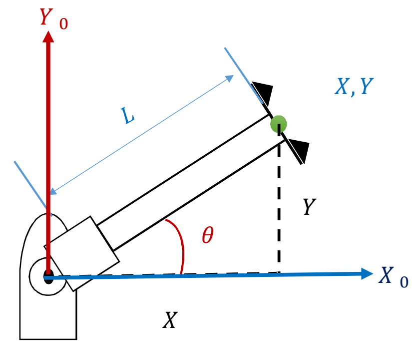
- **Find the inverse Kinematics for the RP Robot ?**

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

- **Find L ?**

$$L = \sqrt{x^2 + y^2}$$




7

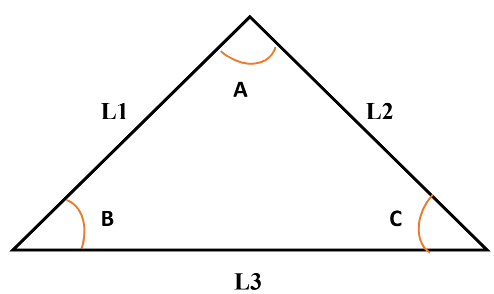
# The Laws of Cosines and Sines

Sin Law

$$\frac{\sin A}{L3} = \frac{\sin B}{L2} = \frac{\sin C}{L1}$$

Cos Law

$$L_1{}^2 = L_2{}^2 + L_3{}^2 - 2L_2L_3\cos(C)$$




8

# Inverse kinematics for RR

- RR robot planar robot find its Inverse Kinematics
- Find $\theta_2$

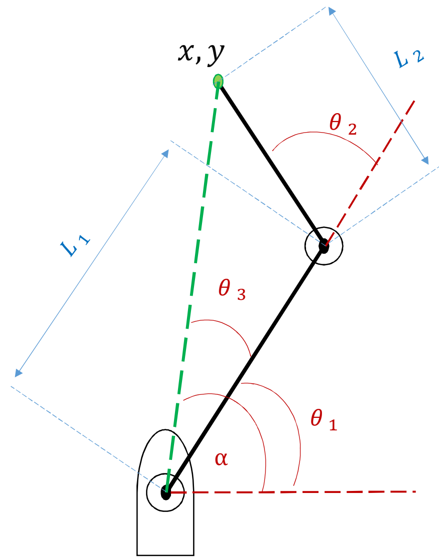
Applying Cosine Law

$c^2=a^2+b^2-2ab\cos(angle\ opposite\ (c))$

$x^2+y^2=L_1{}^2+L_2{}^2-2L_1L_2\cos(180-\theta_2)$

$\cos(180-\theta_2)= - \cos(\theta_2)$

$\theta_2=\cos^{-1}\left(\frac{x^2+y^2-L_1{}^2-L_2{}^2}{2L_1L_2}\right)$






# Inverse kinematics for RR

- Find $\theta_1$

From geometry

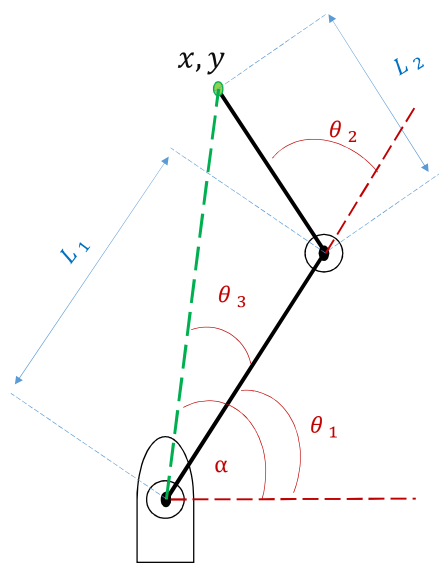
$\theta_1=\alpha-\theta_3$---------- $\alpha=\tan^{-1}\frac{y}{x}$

Using Sine Law $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

$a, b\ and\ c: opposite\ members\ for\ each\ angle$

$\frac{\sin\theta_3}{L_2}=\frac{\sin(180-\theta_2)}{\sqrt{x^2+y^2}}=\frac{\sin(\theta_2)}{\sqrt{x^2+y^2}},$

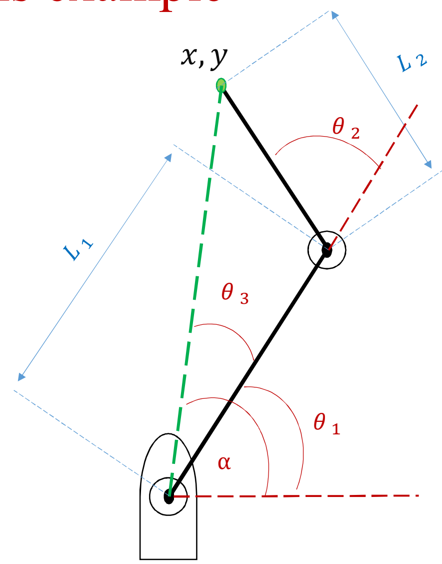
$\theta_3=\sin^{-1}\frac{L_2\sin(\theta_2)}{\sqrt{x^2+y^2}}$



# Inverse kinematics for RR for this example

$$\theta_2 = \cos^{-1}\left(\frac{x^2 + y^2 - L_1{}^2 - L_2{}^2}{2L_1 L_2}\right)$$

$$\theta_1 = \tan^{-1}\frac{y}{x} - \sin^{-1}\left(\frac{L_2 \sin(\theta_2)}{\sqrt{x^2 + y^2}}\right)$$



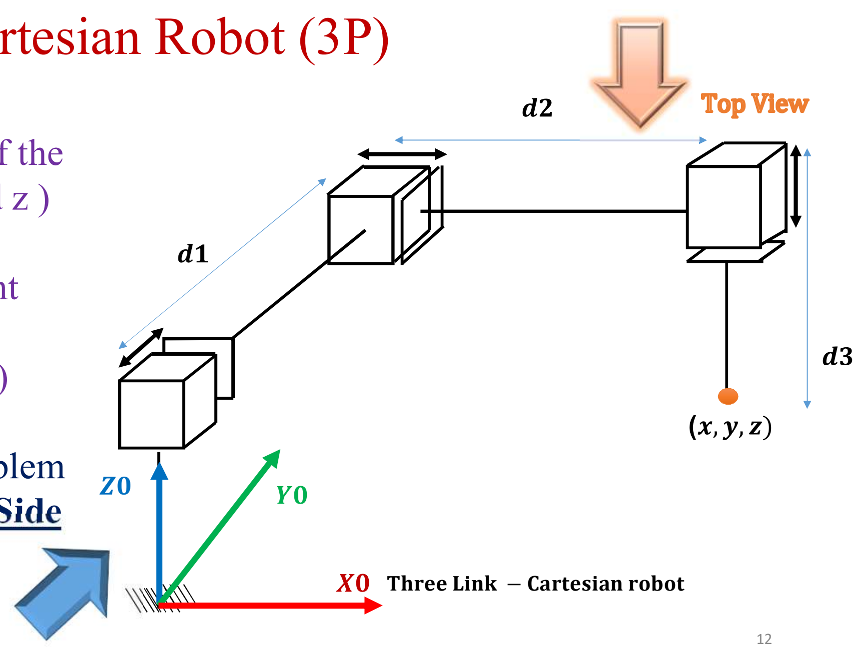

11

# Three Link Cartesian Robot (3P)

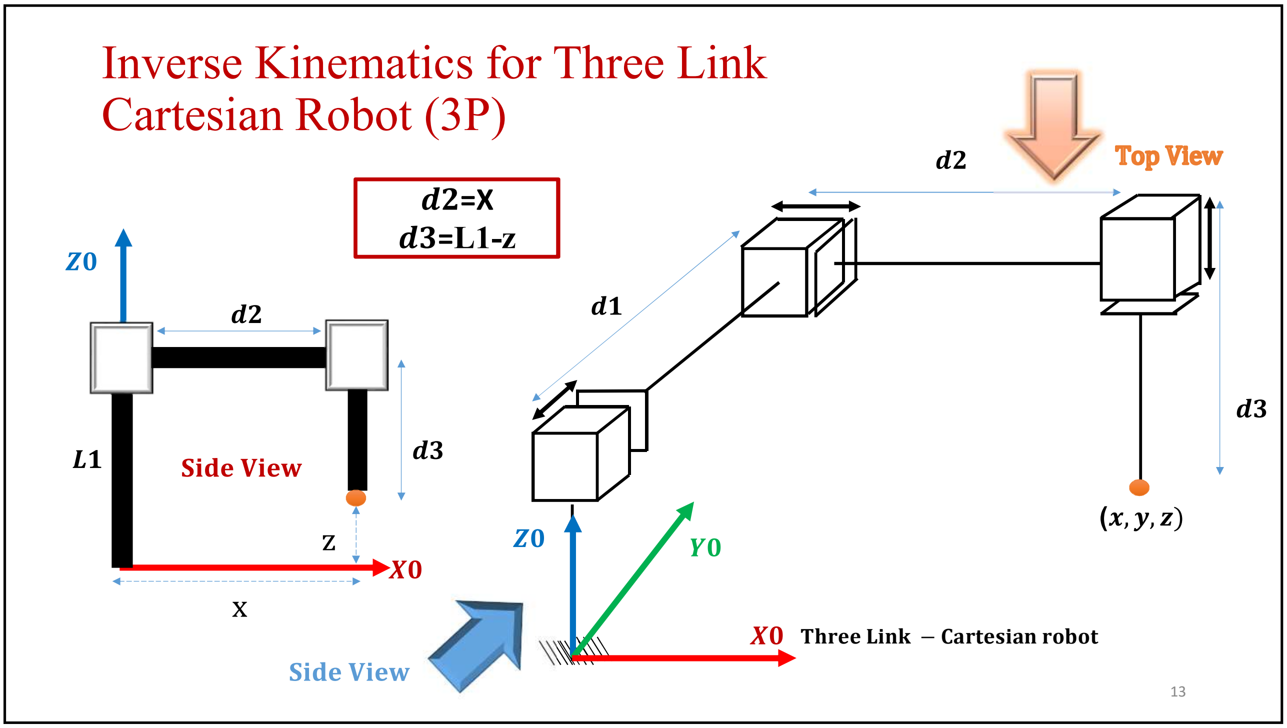
Given the coordinate of the end –effector ( x, y and z )

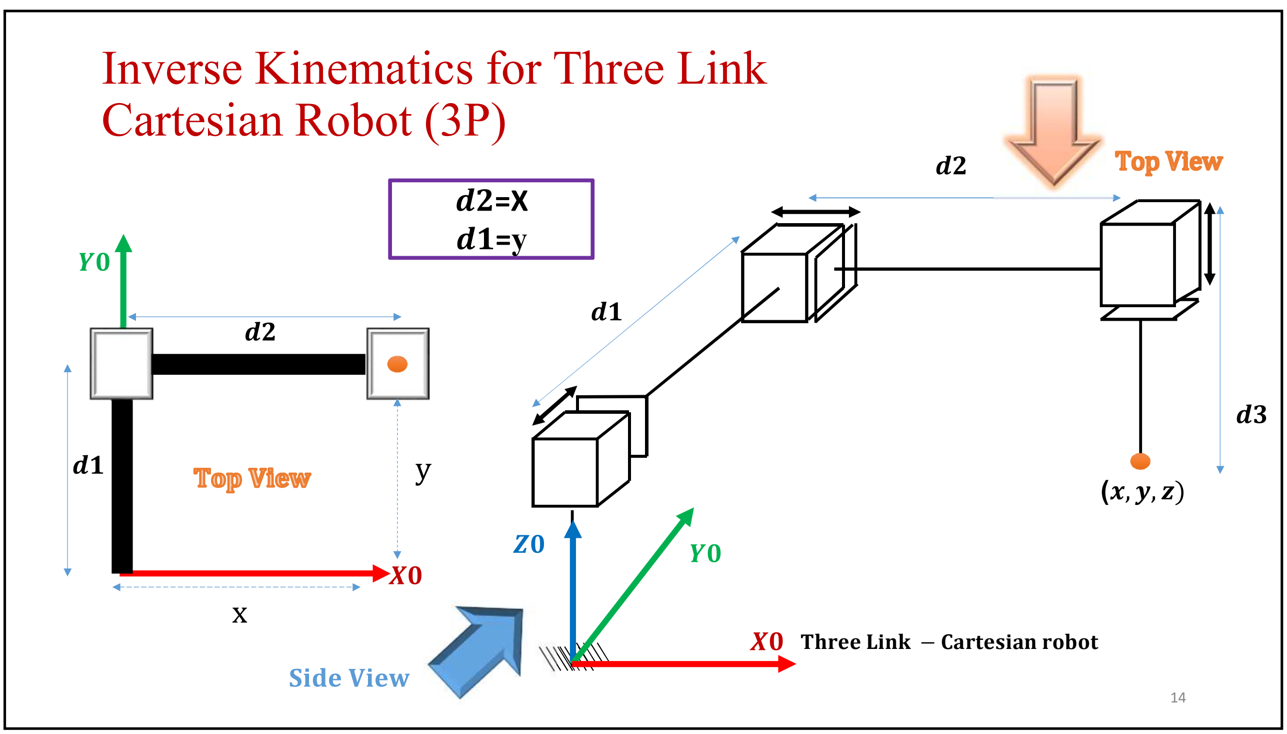
Required : find the Joint variables d1,d2,d3 as a function of (x , y and z)

- To simplify the problem we will Look from **Side and Top views**



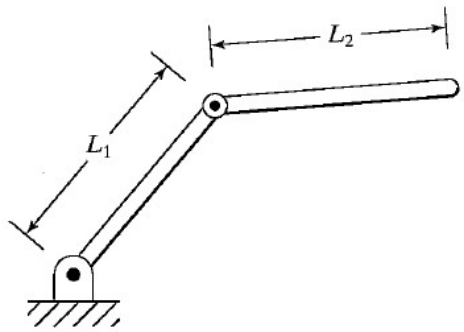

12


Inverse Kinematics for Three Link Cartesian Robot (3P)
d2=X
d3=L1-z
Z0
d2
L1
Side View
d3
z
X0
x
Side View
d1
d2
Top View
d3
(x, y, z)
Z0
Y0
X0 Three Link – Cartesian robot
13


Inverse Kinematics for Three Link Cartesian Robot (3P)
d2=X
d1=y
Y0
d2
d1
Top View
y
X0
x
Side View
d1
d2
Top View
d3
(x, y, z)
Z0
Y0
X0 Three Link – Cartesian robot
14

## Two-link manipulator with link lengths $L_1$ and $L_2$.

- If $L_1 = L_2$ reachable workspace consists of a disc of radius $2L_1$.
- The dextrous workspace has single point which is the origin.
- If $L_1 \neq L_2$ , there is no dextrous workspace and the reachable workspace is realised in two cases a ring of outer radius $L_1 + L_2$ and ring with inner radius $|L_1 - L_2|$.
- Inside the reachable workspace there are two possible orientations of the end-effector while in on the boundaries only one possible solution can be existed.




## Multiple solutions

- In the absence of the obstacle, the upper dashed configuration in Fig. 10.3 would be chosen.
- Weights might be applied in the calculation of which solution is "closer". That means the movement of the smaller Joints are easier than the larger ones that carry links

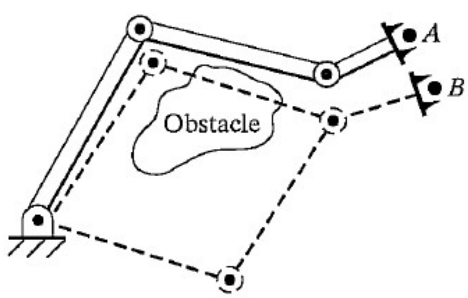



Fig10.3: One of the two possible solutions to reach point B causes a collision.

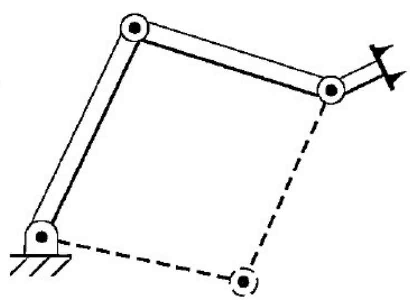


Fig10.2: Three-link manipulator. Dashed lines indicate a second solution.

# Multiple solutions

Number of solution depends on:

1- Number of joints as a function of Link parameters (a ,d , alpha and theta).

2- Allowable ranges of motion of the joints.

e.g. PUMA 560 can reach certain goals with eight different solutions. Fig 10.4 shows four possible solution for the same XYZ coordinate. In addition each one of these four position can be reached by oriented the Joints 4,5 and 6 as follow:

$$\theta_4' = \theta_4 + 180°,$$

$$\theta_5' = -\theta_5,$$

$$\theta_6' = \theta_6 + 180°.$$

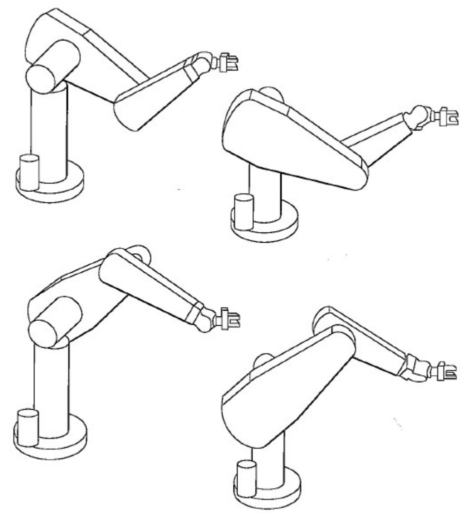


Fig10.4: Four solutions of the PUMA 560

# Number of solutions vs. nonzero $a_1$

| $a_i$ | *Number of Solution* |
|---|---|
| $a_2 = a_3 = a_5 = 0$ | $\leq 4$ |
| $a_3 = a_5 = 0$ | $\leq 8$ |
| $a_3 = 0$ | $\leq 16$ |
| $All\, a_i \neq 0$ | $\leq 16$ |

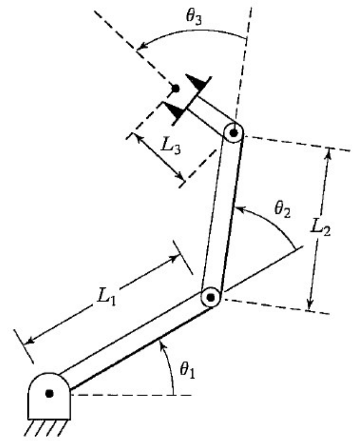
## Ex: Give a description of the subspace of ${}^{B}_{W}T$ for the three-link manipulator

The subspace of ${}^{B}_{W}T$ is given by:

$$ {}^{B}_{W}T = \begin{bmatrix} C_{\emptyset} & S_{\emptyset} & 0 & x \\ -S_{\emptyset} & C_{\emptyset} & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Link lengths and joint limits restrict the workspace of the manipulator to be a subset of this subspace.

Workspace ⊂ subspace ⊂ space




## Algebraic solution for RRR

$$ {}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0.0 & l_1 s_1 + l_2 s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The subspace can therefore be given as

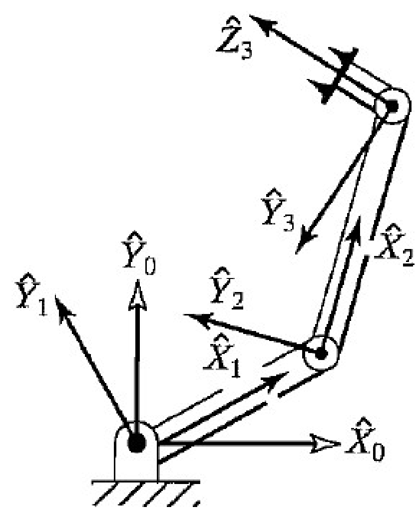
$$ {}^{B}_{W}T = \begin{bmatrix} c_{\emptyset} & -s_{\emptyset} & 0.0 & x \\ s_{\emptyset} & c_{\emptyset} & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$c_{\emptyset} = c_{123}$,........ (1)

$s_{\emptyset} = s_{123}$,....... (2)

$x = l_1 c_1 + l_2 c_{12}$,........(3)

$y = l_1 s_1 + l_2 s_{12}$, .........(4)



## Algebraic solution

*We use the trigonometric formula*

$c_{12}=c_1 c_2 - s_1 s_2,$

$s_{12}=c_1 s_2 + s_1 c_2,$

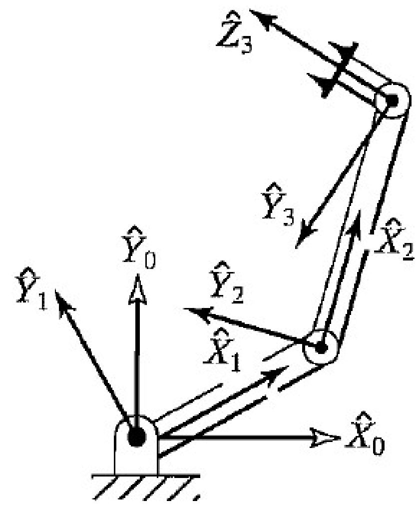
*Then*

$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2,$

$c_2 = \frac{x^2+y^2-l_1^2-l_2^2}{2 l_1 l_2}$

$s_2 = \sqrt{1 - c_2^2}$

$\theta_2 = \tan^{-1} \frac{s_2}{c_2}$




## Algebraic solution

*We attempt to find* $\theta_1$

$x = k_1 c_1 - k_2 s_1$,.........(5)

$y = k_1 s_1 - k_2 c_1$,........(6)

*From eqn. (3) and (4)*

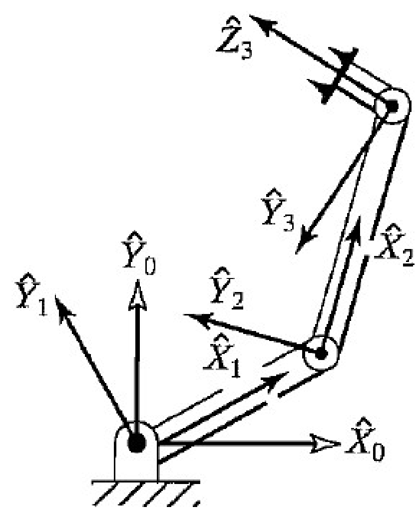
$k_1 = l_1 + l_2 c_2,$

$k_2 = l_2 s_2,$

*Find the radius*

$r = +\sqrt{k_1^2 + k_2^2}$

$\gamma = \tan^{-1} \frac{k_2}{k_1}$



## Algebraic solution

$$k_1 = r\cos\gamma$$
$$k_2 = r\sin\gamma$$

Eqn. (4) and (5) can be written now:

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$$

$$\frac{y}{r} = \cos\gamma\sin\theta_1 - \sin\gamma\cos\theta_1$$

$$\cos(\gamma + \theta_1) = \frac{x}{r}$$

$$\sin(\gamma + \theta_1) = \frac{x}{r}$$

Using the two-argument arctangent, we get

$$\theta_1 = Atan2(y, x)) - Atan2(k_2, k_1)$$