Introduction

to

Control Engineering

Definitions

System: Combination of physical components that acts together and performs a certain objective

Control System: Control the output in some prescribed manner by the input through the elements of the control system.

A control system is an interconnection of components or devices so that the output of the overall system will follow as closely as possible a desired signal

Definitions

- Control system is a device or set of devices used to manage, command, direct or regulate the behaviour of other devices or systems.
- This field is wide. It is also applied in economy, finance, political science, physics, mathematics and biological sciences.
- There are three things that define control systems: input, systems and output.

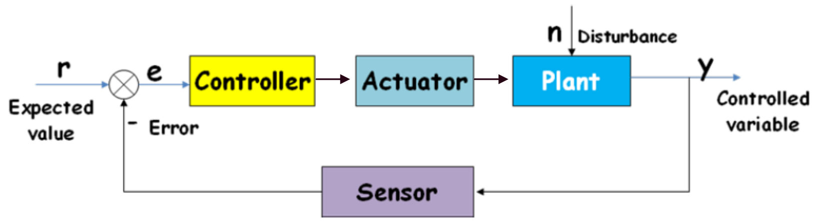
Control System Representation

- We can represent a control system in block diagrams, in terms of input, system and output.

- The output may or may not be equal to the specified response by the input.
- Thus, the purpose of control system is regulate the system to produce the desired output.

Control System Representation

- Expected value : The desired value of controlled variable based on requirement, often it is used as the reference input.
- Output (Controlled variable) – It is the actual response obtained from a control system.
- Actuating error signal – It is the difference between the reference input and feedback signal.



Control System Representation

- Controller: Detects the error signal which is usually at very low level and amplifies it to a sufficiently high level. The output is fed to an actuator.
- Actuator: The device that provide the motive power to the plant. The actuator gets its instruction directly from the controller such as (motor, heater)
- Plant: System to be controlled. It is any physical object to be controlled such as (antenna, heating furnace, robot arm and chemical processes).

Control System Representation

- Sensor: Measure the system output and convert the physical output action into an electric signal which is fed back to the controller. Another name (measuring device) .
- Disturbance: It is a signal that tends to adversely affect the value of the output of a system (undesired input signal) .
- Comparator: Subtracts the feedback signal from the input to determine the error .
- Stability: Means that the system output will be able to follow the input .The system is said to be unstable if its output is out of control.

Types of Control System

- Broadly speaking, there are three major type of control systems:
  - Man made control system
  - Natural control system
  - Mixed (combination) control system

Man Made Control System

- The system (technology) is created by human.
- Example : electrical switch

Natural Control System

- Also called biological control.
- The type of control is available in nature.
- Example : pointing a finger.

- Input : precise direction of the object
- Output : actual pointed direction

Mixed (Combination) Control System

- The system is controlled by nature (human) through man-made technology.
- Example : driving a car

Application Examples

- Home heating or air-conditioning, controlled by a thermostat.
- Home entertainment system with built-in control.
- Cruise (speed) control of an automobile.
- Electronic voltage regulator.
- Automatic bread toaster.
- Photographic automatic focus control.
- Altitude control of space vehicle.
- Automatic washing machine.
- Law and order.





Type of Control System

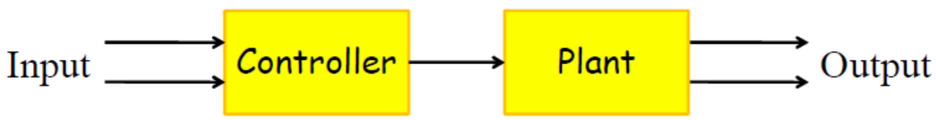
- Two types : open loop and closed loop.
- Open loop : systems that utilizes a device to control the process without using feedback.
- Closed loop : systems that uses a measurement of the output (usually a sensor) and compares it with the desired input.

Open Loop System

- Also known as "feed-forward" control system.
- Characteristics:
  - Simplest (and cheap too!) type of control
  - Contains no feedback
  - The output is not affected by the input
- Application examples:
  - Simple electric switch
  - Kettle or water heating devices
  - Mobile phone
  - Word processor
  - Alarm clock

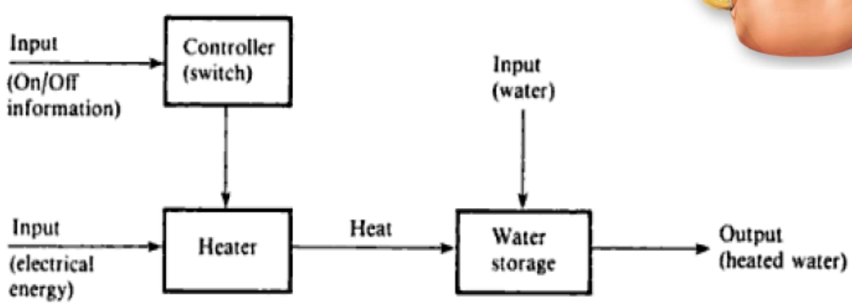
Typical Open Loop Block Diagram

- We can generally design or draw a block diagram for any system provided that we know the input and the output.
- Contains no feedback.
- Sometimes, the input is also called the "desired input" or the "reference input".
- The output is sometimes called the "actual output" or "actual response".



Example 1 : Kettle

- It is a merely an on-off device.
- Block diagram:



Source : Warwick, An Introduction to Control Systems

Example 2 : Mobile Phone

- It is an open loop system.

- Why?
  1. Phone received call/signals.
  2. As the phone is turned on, it will make connecti satellite until the signal (call) is terminated by operator (human).
  3. The phone is **unable to turn itself off** even after a conversation between humans have ended.
  4. Hence, it is an open loop system.

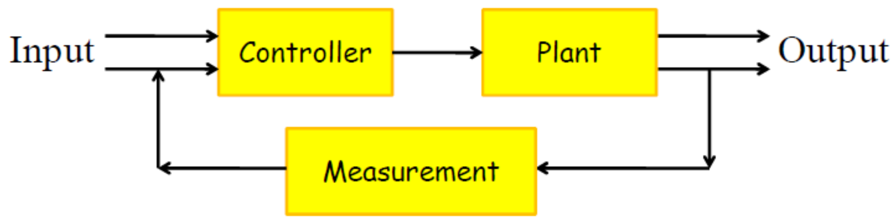
Example 3 : Word Processor

- Control type : open loop system

- Why?
  - The monitor continues to display output characters on the computer monitor if the human give suitable input via keyboard.
  - No input, then no output.

Closed Loop System

- Also known as the feedback system.
- The system uses the measurement of the actual output to compare with the input, hence producing a very effective output.
- The block diagram representation is given as follows:



Example Application of Closed-Loop System

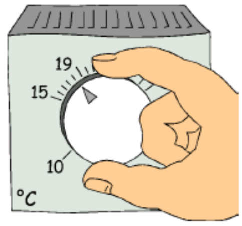
- Example applications:
  - Washing machine
  - Oven
  - Driving an automobile
  - Law and order

- Why are the above example falls in the category of closed-loop system?

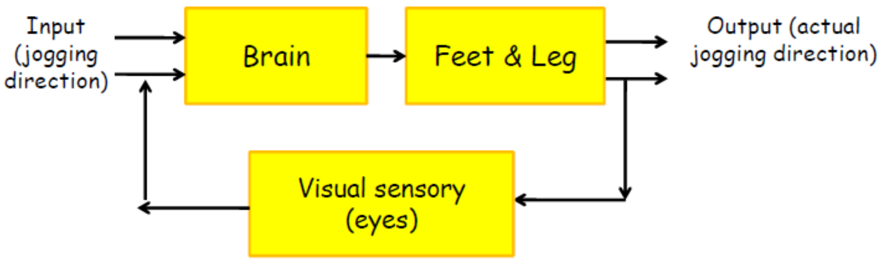
Example 4 : Air Conditioner Control

- Control type : Closed loop.

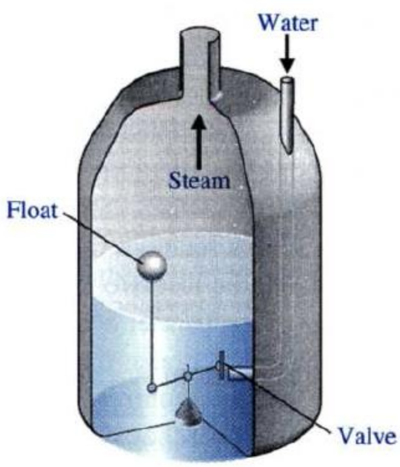
- Why?
  - It is a self-regulating machine performing the operation with and without the need of the human.
  - This machine will keep the surrounding temperature to that of the preset value.
  - Sensor is used to maintain the temperature in which the air-conditioner is placed.

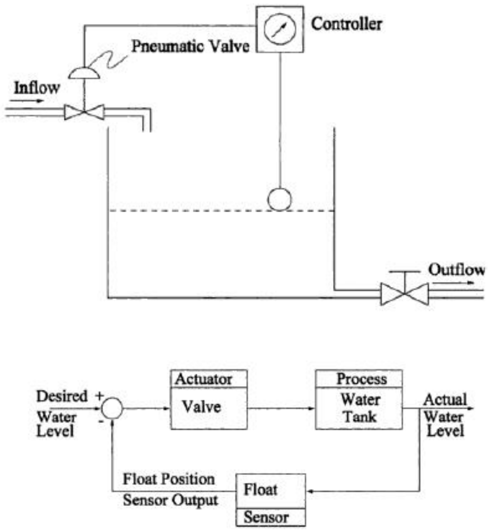


Example 7 : Jogging System



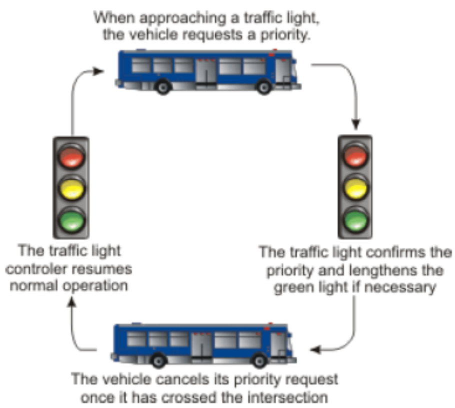
Example 8 : Water Level System





Example 9 : Traffic Light Control System

- The idea is to minimize the waiting time. Furthermore, it is also intended to make the traffic flow smooth.
- Many control techniques can be used: intelligent control system is one of them.



Input and Output System

- Sometimes, we might have one input and one output, but there are cases where we might have multiple input and multiple output.

- The one (single) input and one (single) output is sometimes called the SISO system.

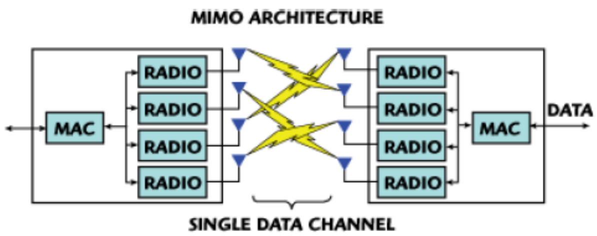
- On the other hand, the multiple input and multiple output is sometimes called the MIMO system.

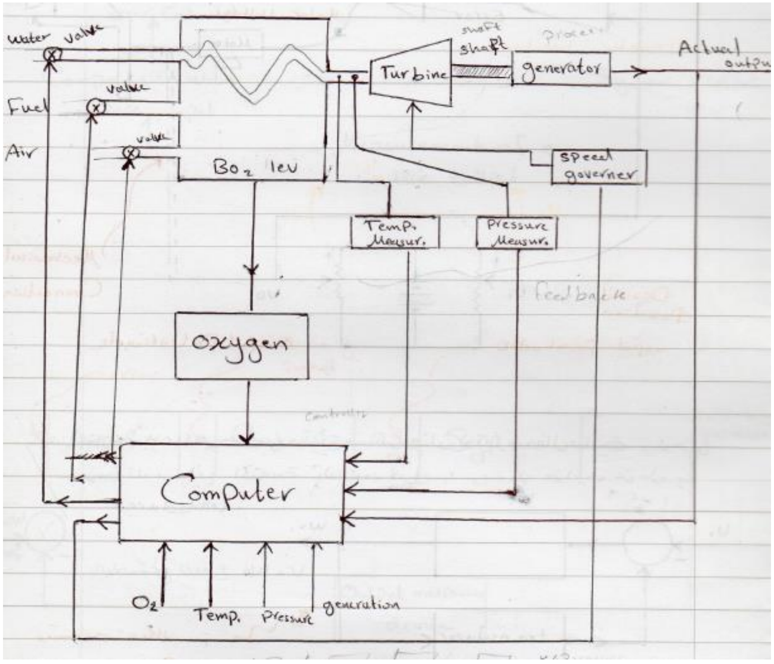
Example 10 : SISO and MIMO system

- The following example illustrates the application in telecommunication engineering.

- SISO system:

- MIMO system:





Classes of Control Systems

- We can also categorize a control system in two (2) classes: **servomechanism** and **regulators**.
- Servomechanism is a power amplifying feedback device in which the controlled variable is a mechanical position or time derivative of position such as velocity or acceleration.
- A regulator is a system where the reference input is constant for a long period of time.

Servomechanism

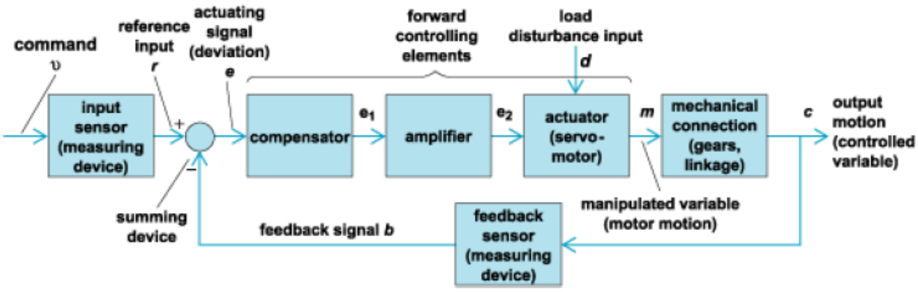
- Usually, we use servo motors for servomechanism applications.
- Characteristics:
  - Closed loop system.
  - The control action is dependent on the desired result.
  - Automatic (intelligent) control.
  - Measures position (displacement), velocity and/or acceleration.
- Application example:
  - (speed) cruise control of cars.
  - Water level system.

Servomechanism

- Purpose of servomechanism:
  - (1) accurate control of motion without the need for human attendants (automatic control);
  - (2) maintenance of accuracy with mechanical load variations, changes in the environment, power supply fluctuations, and aging and deterioration of components (regulation and self-calibration);
  - (3) control of a high-power load from a low-power command signal (power amplification);
  - (4) control of an output from a remotely located input, without the use of mechanical linkages (remote control, shaft repeater).

Servomechanism

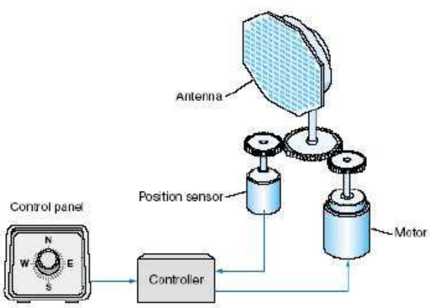
- A servomechanism is typically a feedback system. The following block diagram illustrates the effect of servomechanism.



- The constant speed control system of a DC motor is a servomechanism that monitors any variations in the motor's speed so that it can quickly and automatically return the speed to its correct value. Servomechanisms are also used for the control systems of guided missiles, aircraft, and manufacturing machinery.

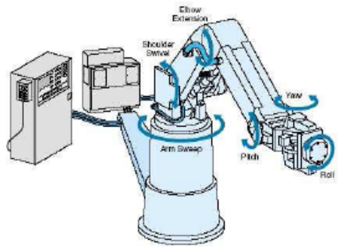
Example 11 : Remote Antenna Positioning System

- One example using a servomechanism is the positioning system for a radar antenna.



- In this case, the controlled variable is the antenna position. The antenna is rotated with an electric motor connected to the controller that is located some distance away. The user selects a direction, and the controller directs the antenna to rotate to a specific position.

Example 12 : Industrial Robot



- Sophisticated robots use closed-loop position systems for all joints. An example is the industrial robot.

- The robot has six independently controlled axes (known as six degrees of freedom) allowing it to get to difficult-to-reach places. The robot comes with and is controlled by a dedicated computer-based controller.

- This unit is also capable of translating human instructions into the robot program during the "teaching" phase. The arm can move from point to point at a specified velocity and arrive within a few thousandths of an inch.

Example 13 : Regulators

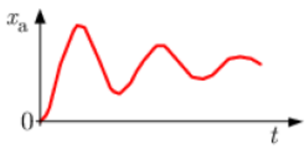
- Characteristics :
  - closed loop system.
  - The input (setpoint) is held constant.

- Application example:
  - Car (power) window.
  - Human body temperature.
  - Automatic temperature regulated over.
  - Human perspiration system.

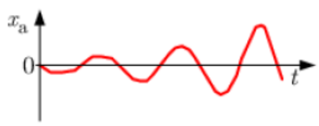
Stable and Unstable System

- If a system is brought to any particular initial condition (or state) and its response decays continuously to zero state, the system is said to be stable of a particular kind called asymptotically stable.

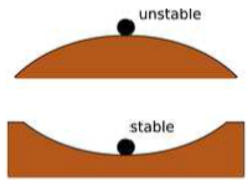
- If a system grows out of bound without any limit, then the system is an unstable system.



A stable system



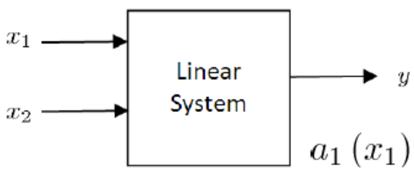
Unstable system



Analogy

Linear vs Non-linear System

- Linear system is a type of system that satisfies the principle of superposition and homogeneity.



$$y = x_1 + x_2$$

$$y = a_1 x_1 + a_2 x_2$$

$$a_1 (x_1) + a_2 (x_2) = a_1 f(x_1) + a_2 f(x_2)$$

- A non-linear system is not a linear system. Mathematically, it is a set of non-linear equations where the variables to be solved cannot be written as a linear combination of independent components.

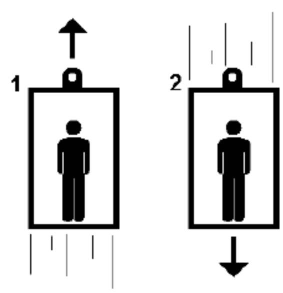
Analysis of Control System

- The main objective of a control system is to produce a desired system, reducing errors and achieving system's stability.

- What do we analyze in control system?
  - Transient (temporary) response
  - Steady-state response
  - Stability

Transient Response

- Also known as the natural response (remember differential equations?) – it is the homogeneous solution.



- Example : consider an elevator going from the first floor to the fourth floor.

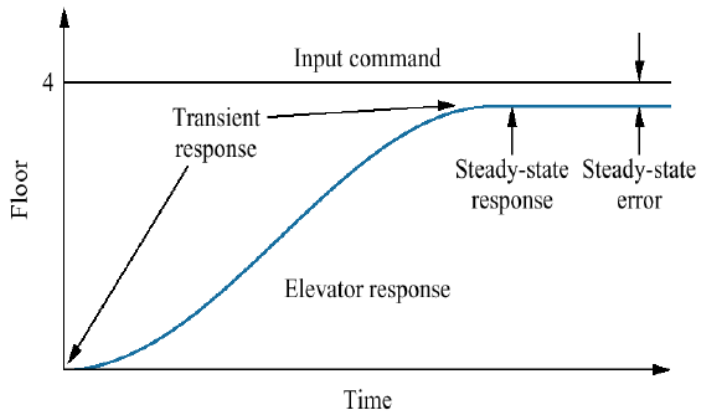
- If a transient response is:
  - Too slow – passenger would be angry
  - Too fast – you would be scared

Steady State Response

- An approximation to the desired response.
- It is also the response that exist for a long time following the given input signal.
- In the previous lift example, the steady state response is when the lift is about to reach the fourth floor.
- We will also examine the steady state error, which is how accurately the system performs.

Output Response of Control System

- Consider an example of an elevator going from the first floor to the fourth floor.
- The output of the elevator can be represented as follows (Nise, 2007)



Stability

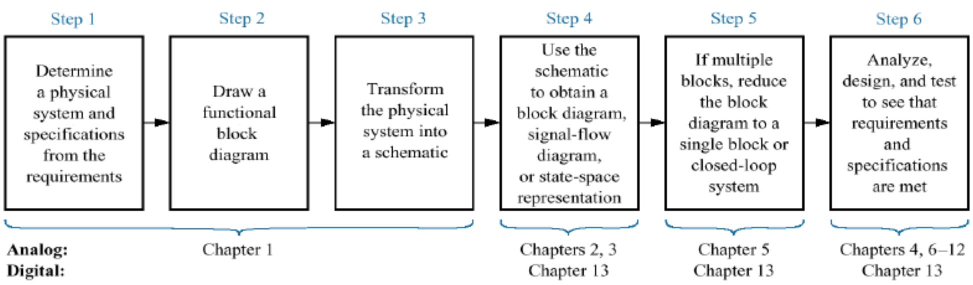
- It is a performance measure of a system.
- If a system is stable, then it should operate properly.
- An unstable system would lead to self-destruction or chaos. For example, in flight control system, if it is unstable, it would crash.
- The total response of the system is given by:

$$x = \text{natural response} + \text{forced response}$$
$$x = x_h + x_p$$

- For a particular control system to be useful, we want the natural response to either approach to zero or oscillate. Sometimes, the natural response will go out of bound, hence the system would be unstable.
- We can use mathematical techniques to analyze and control the stability of a particular control system.

Control System Design Process

- The following are the steps as outlined by Nise (2007) in his book:



Mathematical Modeling

- It uses mathematical language to describe a particular system.

- Why?
  - Important to gain understanding and further insight to the system, hence enabling us to perform analysis.
  - Useful for prediction, formulation and simulation.
  - Useful for estimation and prediction of unforeseeable event that could somehow affect the system.

- Type of mathematical model studied in control engineering:
  - Classical form : representation of n^{th} order differential equations
  - Transfer functions : the ratio between the output to the input, found after taking the Laplace transform of differential equations.
  - State space : a representation of a set of n^{th} order simultaneous first-order differential equations.

Control System Design Example

- Antenna Azimuth Positioning System

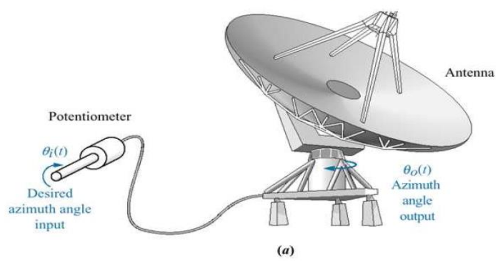


Figure (a) : System Concept [source: Nise, 2007]

Control System Design Example

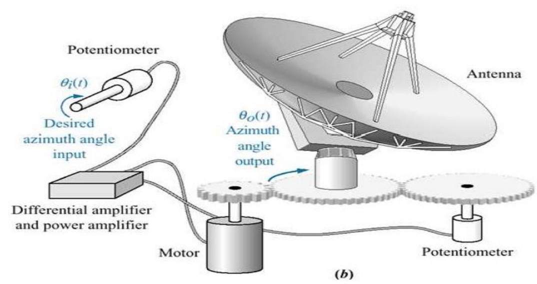


Figure (b) : Detailed System layout [source: Nise, 2007]

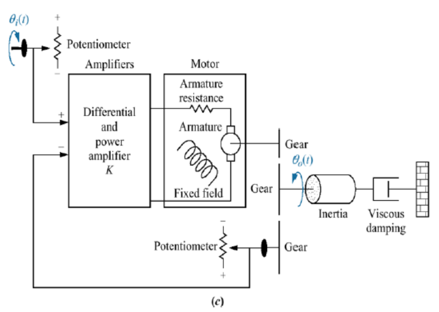


Figure (c) : System Schematic [Source : Nise, 2007]

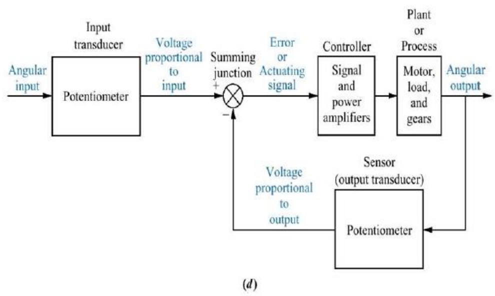


Figure (d) : Functional Block Diagram [source : Nise, 2007]

Transfer Function and Modeling of Mechanical and Electrical Systems

Transfer Functions

- The transfer function is the ratio between the output to the input. It is usually obtained by taking the forward Laplace transform of differential equation, assuming zero initial conditions.

- Definition:

Let $x(t)$ be the input to a general linear time-invariant system, and $y(t)$ be the output, and the bilateral Laplace transform of $x(t)$ and $y(t)$ be

$$X(s) = \mathcal{L}\{x(t)\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$Y(s) = \mathcal{L}\{y(t)\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} y(t)e^{-st}\,dt.$$

Then the output is related to the input by the **transfer function** $G(s)$ as

$$Y(s) = G(s)X(s)$$

and the transfer function itself is therefore

$$G(s) = \frac{Y(s)}{X(s)}.$$

(Source : Wikipedia, 2010)

Transfer Functions

- A transfer function is basically written in the following format:

$$G(s) = \frac{\text{Output}}{\text{Input}} = \frac{C(s)}{R(s)}$$

- Again, the transfer function is found by taking forward Laplace transform of differential equations, assuming zero initial condition.

- We will also come across the terms gain, zeros and poles when dealing with transfer function.

- Another alternative form of a transfer function is:

$$G(s) = \frac{K(s+z)}{s+p}$$

Transfer Functions

- Now, "K" stands for gain, "z" stands for zeros and "p" stand for poles.

- A zero is a number that causes the numerator of the transfer function to be zero.

- A pole is a number that causes the denominator of the transfer function to be zero. Another name for a pole is root.

Transfer Functions

- We can plot the values of the pole(s) and zero(s) on a pole-zero map, also known as the pz-map.

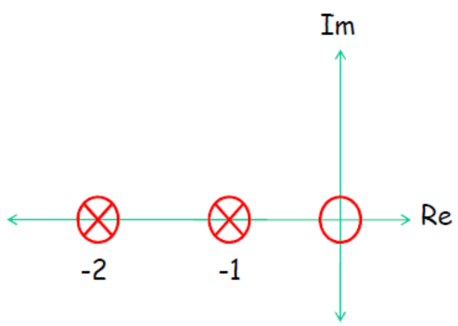
- A pole is marked with the symbol cross (x) while a zero is marked with the symbol empty circle (o).

- Example: Plot the poles and zeros for the following:

$$G(s) = \frac{2s}{(s+1)(s+2)}$$

Transfer Functions

- Solution to the example in the previous slide:



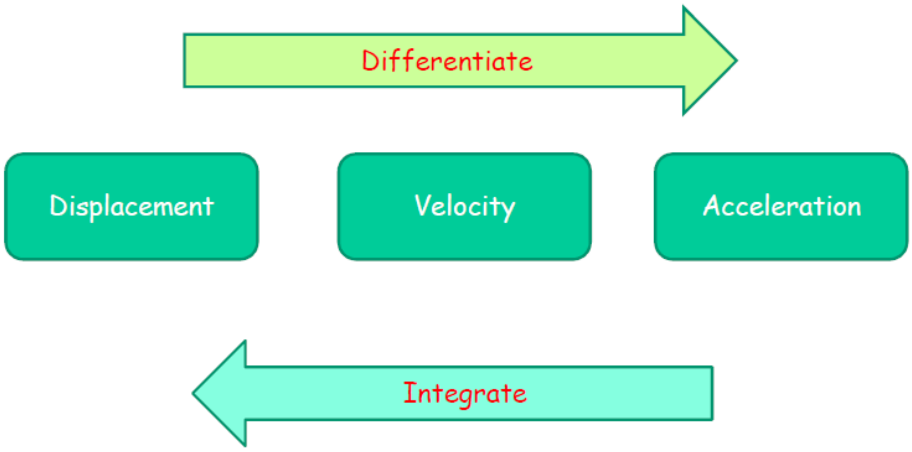
Modeling Mechanical Systems

- A mechanical system is basically a system of elements that interact based on fundamental mechanical principles.
- There are two types of mechanical system:
  - Translational system
    - The motion is generally linear
  - Rotational system
    - The motion is rotational (revolution)
    - We need to consider torque and moments of inertia
- In this lecture, we shall be discussing modeling translational mechanical system. The next lecture will cover modeling of rotational mechanical systems.

Time Derivative Motion

- Time derivative is the rate of change of a function with respect to time.
- Displacement
  - A vector quantity representing the length of separation between two objects or bodies,
- Velocity
  - The rate of change of displacement
- Acceleration
  - The rate of change of velocity

Time Derivative Motion



Newton's Law of Motion

- Law I : Mass (inertia)
  - An object will stay at rest until there is an external force applied to it.
- Law II : Acceleration/gravity
  - $F=ma$
- Law III : Action/Reaction
  - For every action, there will always be an equal and opposite reaction, that is : $F_{action}=F_{reaction}=-F_{action}$.

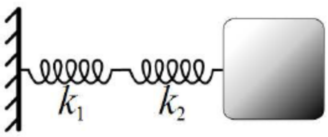
Translational motion Spring

- A mechanical device that stores energy.
- Most spring obeys Hooke's Law, that is, the force with which the spring pushes back is linearly proportional to the distance from its equilibrium length.
- Mathematically:

$$F = -k\,x$$

Hooke's law accurately models the physical properties of common mechanical springs for small changes in length.
(Source: Wikipedia, 2010).

Equivalent System : Series Spring

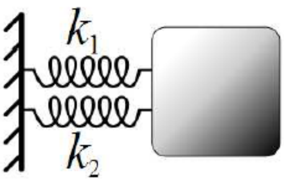


- Spring constant: $$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\frac{1}{k_{eq}} = \sum_{i=1}^{i=n} \frac{1}{k_i}$$

- Spring force: $$F_{eq} = F_1 = F_2$$

Equivalent System : Parallel Spring



In parallel spring, the total force is the sum of the individual force acting on a particular spring. Note that the displacement (or extension or elongation) of the spring is the same.

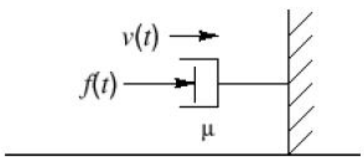
- Spring constant: $k_{eq} = \sum_{i=1}^{i=n} k_i$

- Spring force:

$$F_{eq} = F_1 + F_2$$
$$F_{eq} = k_1 x + k_2 x$$
$$F_{eq} = (k_1 + k_2)x$$
$$F_{eq} = k_{eq}\, x$$

Damper (Dashpot)

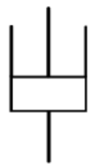
- Also known as dashpot.
- It is a mechanical device that resist the motion via viscous friction.



Schematic diagram

Modeling Dampers

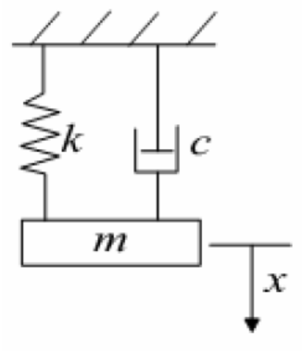
- Damping constant or dashpot impedance.
- Symbol : μ, f_v or b.
- For the purpose of this class, we shall use the symbol b.



$$F_d = b\,v(t) = b\,\dot{x} = b\frac{dx}{dt}$$

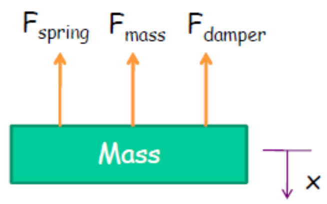
Modeling Example 1

- Find a mathematical model for the following system:



Solution to Modeling Example 1

- Step I : Draw a FBD.



- Step II : Obtain a set of differential equations of motion.

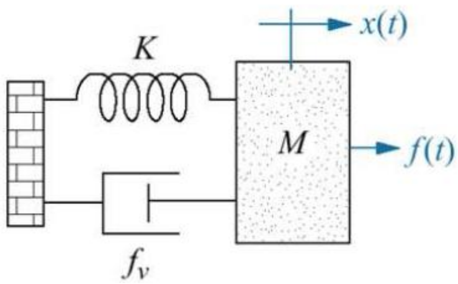
$$\sum F = 0$$

$$F_{mass} + F_{spring} + F_{damper} = 0$$

$$m\ddot{x} + kx + b\dot{x} = 0$$

Modeling Example 2

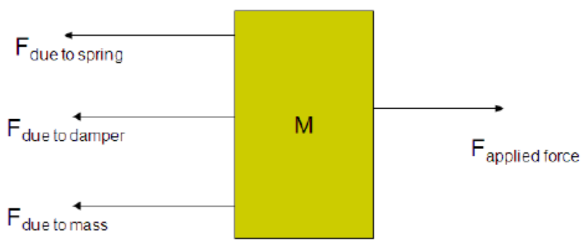
- Find the transfer function and state space representation for the following system :



Source : Nise, 2007

Solution to Modeling Example 2

- Step I : Draw a FBD



$F_{\text{due to spring}} + F_{\text{due to damper}} + F_{\text{due to mass}} = F_{\text{applied force}}$

Solution to Modeling Example 2

- Step II : Obtain a set of differential equation of motion.

$$F_{\text{mass}} + F_{\text{damper}} + F_{\text{spring}} = F_{\text{applied}}$$

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + k\,x(t) = f(t)$$

$$M\,\ddot{x} + f_v\,\dot{x} + k\,x = f(t)$$

- Step III : Determine the transfer function. Take the forward Laplace transform and assume zero initial condition throughout. Remember that the transfer function is the ratio of the output to the input.

$$\mathcal{L}\left\{M\,\ddot{x} + f_v\,\dot{x} + k\,x = f(t)\right\}$$

$$Ms^2X(s) + f_v\,sX(s) + k\,X(s) = F(s)$$

$$\left(Ms^2 + f_v\,s + k\right)X(s) = F(s)$$

Solution to Modeling Example 2

- Next, define the input and the output:

$$X(s) \mapsto output$$
$$F(s) \mapsto input$$

- Then, compute the transfer function:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

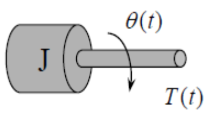
Rotational motion

1. *Inertia*

Newton's Law for rotation motion :

$$\sum T(t) = J\alpha(t) = J\frac{d}{dt}\omega(t) = J\frac{d^2}{dt^2}\theta(t)$$

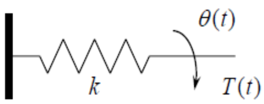
T : Torque J : Inertia α : Angular acceleration
ω : Angular velocity θ : Angular displacement



2. *Torsional Spring*

$$T(t) = k\theta(t)$$

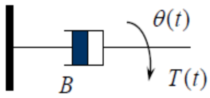
k : Torsional spring constant



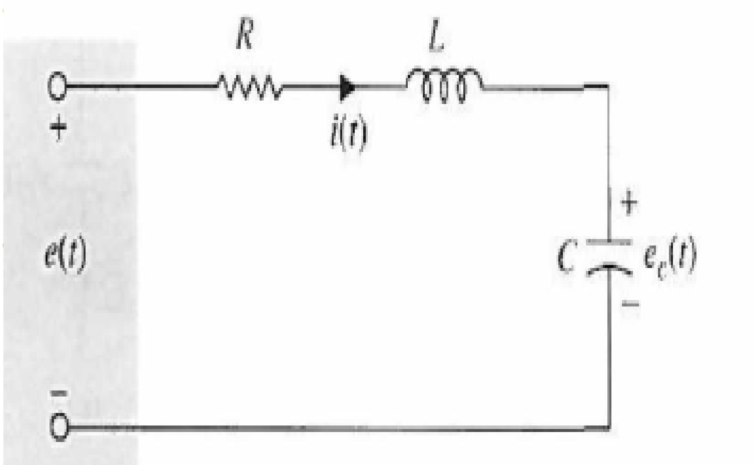
3. *Friction for rotational motion* : *viscous, static, coulomb friction*

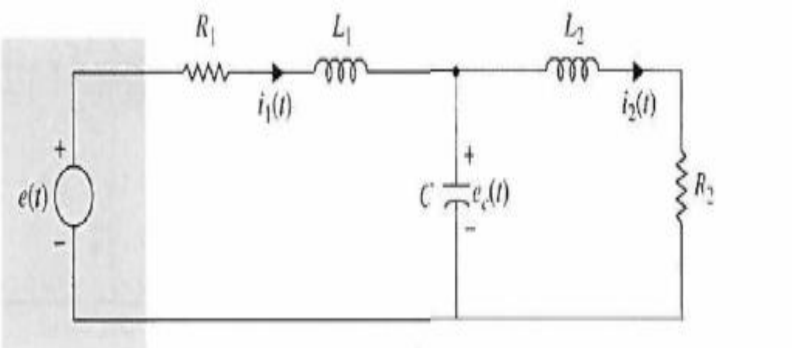
$$T(t) = B\frac{d}{dt}\theta(t)$$

B : Viscous friction coefficient



Electrical and Electronic Networks





(a)

$$L_1 \frac{di_1(t)}{dt} = -R_1 i_1(t) - e_c(t) + e(t)$$

$$L_2 \frac{di_2(t)}{dt} = -R_2 i_2(t) + e_c(t)$$

$$C \frac{de_c(t)}{dt} = i_1(t) - i_2(t)$$

$$\frac{I_1(s)}{E(s)} = \frac{L_2Cs^2 + R_2Cs + 1}{\Delta}$$

$$\frac{I_2(s)}{E(s)} = \frac{1}{\Delta}$$

$$\frac{E_c(s)}{E(s)} = \frac{L_2s + R_2}{\Delta}$$

where

$$\Delta = L_1L_2Cs^3 + (R_1L_2 + R_2L_1)Cs^2 + (L_1 + L_2 + R_1R_2C)s + R_1 + R_2$$

Transformer

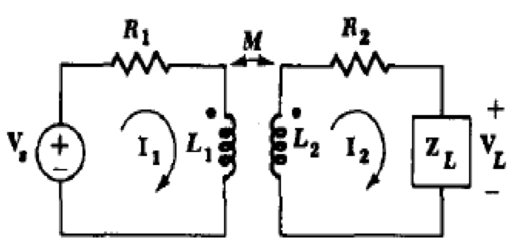
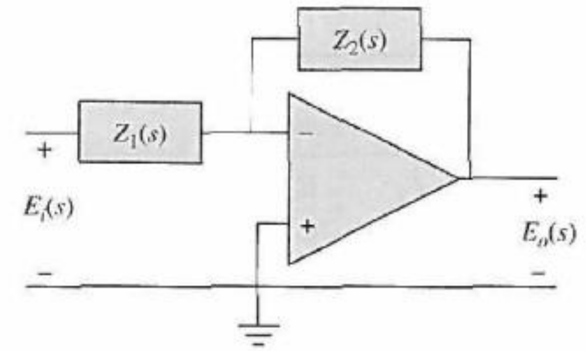


Fig. 15-11 A linear transformer containing a source in the primary circuit and a load in the secondary circuit. Resistance is also included in both the primary and secondary.

$$V_s = I_1(R_1 + sL_1) - I_2sM$$

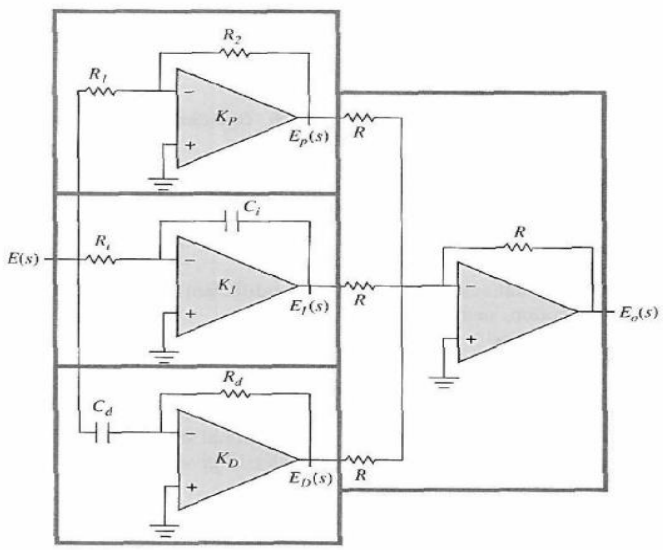
$$0 = -I_1sM + I_2(R_2 + sL_2 + Z_L)$$

Inverting Amplifier



$$G(s) = \frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

PID controller



Proportional: $$\frac{E_P(s)}{E(s)} = -\frac{R_2}{R_1}$$

Integral: $$\frac{E_I(s)}{E(s)} = -\frac{1}{R_iC_is}$$

Derivative: $$\frac{E_D(s)}{E(s)} = -R_dC_ds$$

The output voltage is

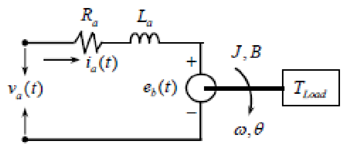
$$E_o(s) = -[E_P(s) + E_I(s) + E_D(s)]$$

Thus, the transfer function of the PID op-amp circuit is

$$G(s) = \frac{E_o(s)}{E(s)} = \frac{R_2}{R_1} + \frac{1}{R_iC_is} + R_dC_ds$$

Electromechanical Systems - Armature controlled DC Motor

Mathematical modeling of PM DC motor:



R_a	Armature resistance	L_a	Armature inductance
$v_a(t)$	Applied voltage	$i_a(t)$	Armature current
$e_b(t)$	Back-emf	k_b	Back-emf constant
k_m	Torque constant	$T_m(t)$	Motor torque
$T(t)$	Load torque	T_d	Disturbance torque
J	Rotor inertia	B	Viscous friction coefficient
$\theta(t)$	Rotor displacement	$\omega(t)$	Rotor angular velocity

The air-gap flux of the dc motor is

$$\phi(t) = k_f i_f(t)$$

and the torque developed by by the dc motor is

$$\begin{aligned} T_m(t) &= k_1\phi(t)i_a(t) = k_1k_fi_f(t)i_a(t) \\ &= k_1k_fI_fi_a(t), \quad i_f(t) = \text{constant} \\ &= k_mi_a(t) \end{aligned}$$

Electromechanical Systems

The cause and effect equation for the motor circuit are

$$v_a(t) = R_a i_a(t) + L_a \frac{d}{dt} i_a(t) + e_b(t)$$

$$e_b(t) = k_b \omega(t) = k_b \dot{\theta}(t)$$

$$T(t) = J \frac{d}{dt}\omega(t) + B\omega(t) + T_d$$

$$= J \frac{d^2}{dt^2}\theta(t) + B \frac{d}{dt}\theta(t) + T_d$$

The motor torque is equal to the torque delivered to the load, that is

$$T_m(t) = T(t)$$

Hence the transfer function of the dc motor, with T_d=0, *is*

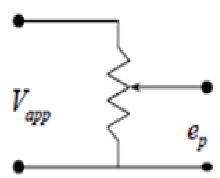
$$\frac{\omega(s)}{V_a(s)} = \frac{k_m}{(R_a + sL_a)(Js + B) + k_b k_m}$$

Electromechanical Systems

Potentiometer: is an electromechanical transducer that covert mechanical energy into electrical energy. Variable resistors can measure both translational and rotational position.

- Potentiometer

$$e_p(t) = k_p \theta(t), \quad k_p = \frac{V_{app}}{2\pi N}$$



Electromechanical Systems

Tachometers: electromechanical devices that convert mechanical energy into electrical energy. The device works as essentially as voltage generator. They can be used as velocity indicators to provide shaft – speed readout, speed control or stabilization. There are many others sensors like: shaft encoder, gyroscope and accelerometer

- Tachometer

$$e_t(t) = k_t\omega(t) = k_t\frac{d}{dt}\theta(t)$$

Block Diagram

Introduction

- A control system consists of a number of components connected to perform a desired function.
- Once a components (system or subsystem) is reduced to a mathematical model, it can be represented as a block with the component operation described by the mathematical function.
- It is convenient and useful to represent the element of a control system by blocks.
- The properties of the block are contained in the transfer function, represented by a Laplace transform.
- Thus, a block diagram is a representation of an entire control system in terms of all the elements and their transfer function.

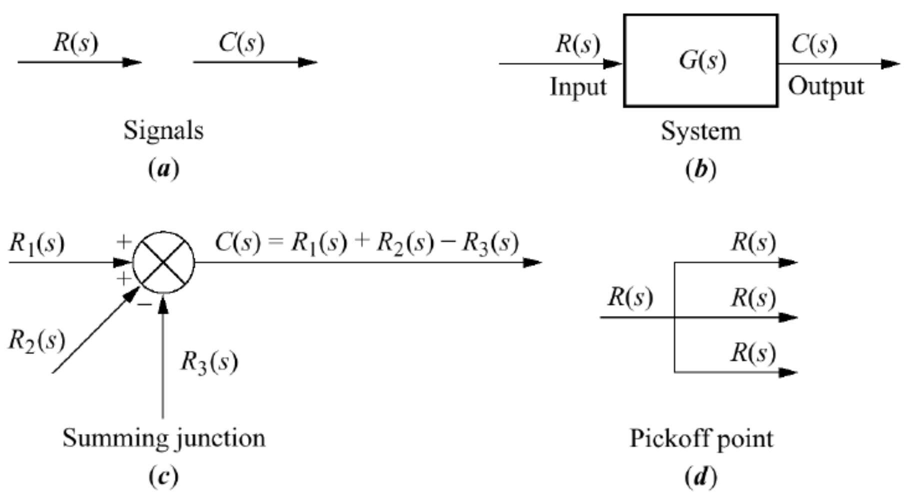
Basic Structure

- Contains input, denoted by R(s) and output, denoted by *C*(s).

Block Diagram Components

- As mentioned, a system is represented by input, output and transfer function.
- Many systems are composed of multiple subsystems.
- When multiple subsystems are interconnected, then, a few more schematic elements must be added to the block diagram.
- The elements are summing and pick-off points.
- The characteristics of summing point is that the input point is the algebraic sum of the many input signals.
- A pick-off point distributed the input signal to several output points.

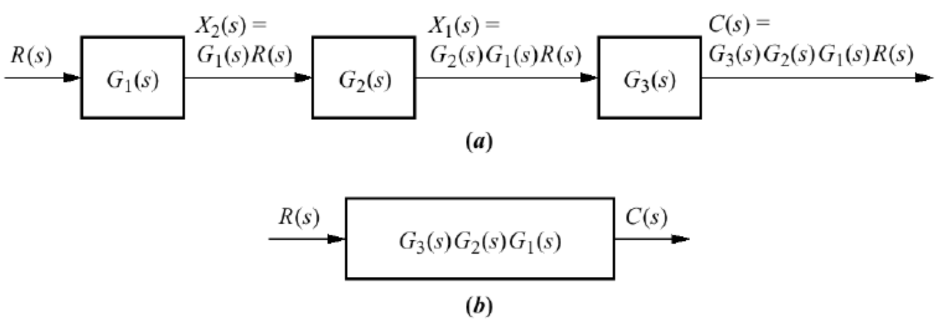
Block Diagram Components



Representation in Block Diagram

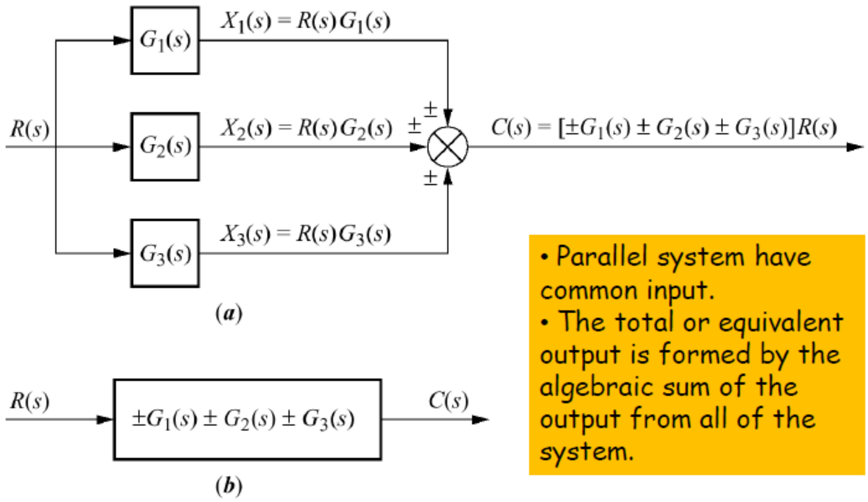
- There are generally three structures of block diagram:
  - Cascade form
  - Parallel form
  - Feedback form

Block Diagram : Cascade Form



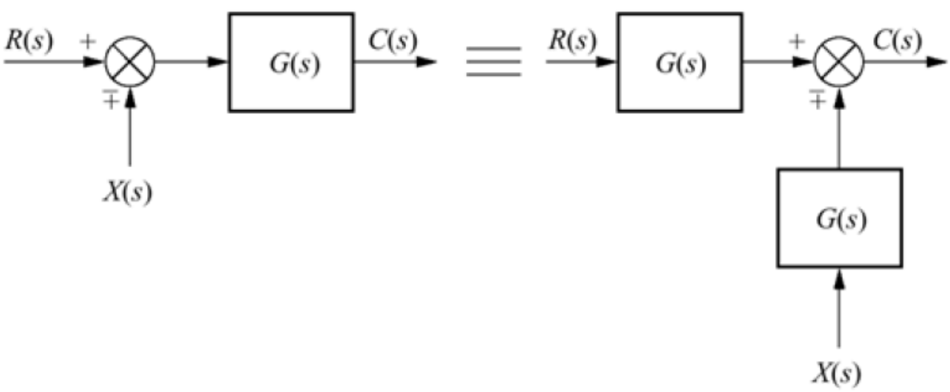
$$G_{eq}(s) = G_3(s)G_2(s)G_1(s)$$

Block Diagram : Parallel Form

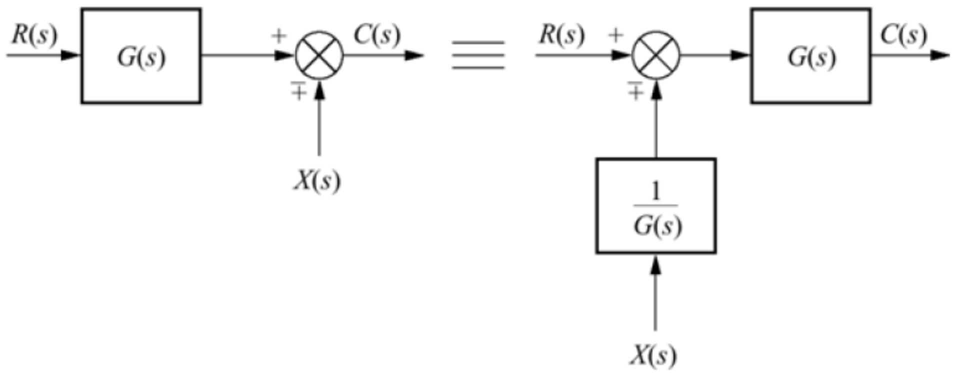


- Parallel system have common input.
- The total or equivalent output is formed by the algebraic sum of the output from all of the system.

Moving a summing point behind a block

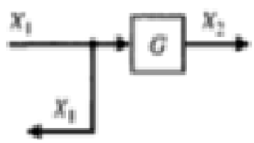


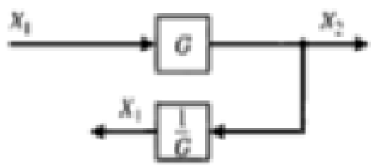
Moving a summing point ahead of a block



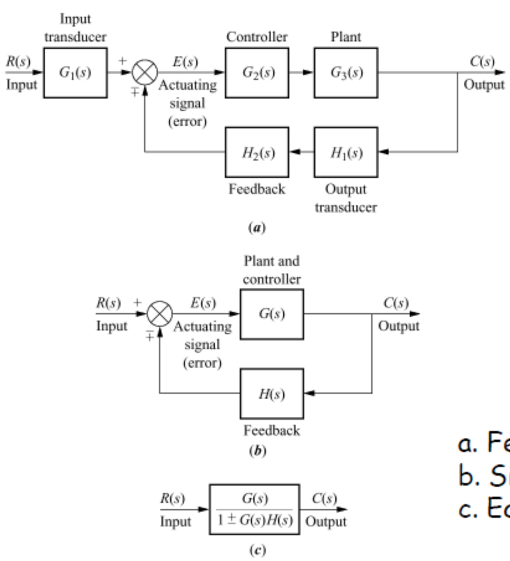
Moving a pick-off point ahead of a block

Moving a pick-off point behind a block





Block Diagram : Feedback Form



a. Feedback control system;
b. Simplified model;
c. Equivalent transfer function

Feedback System : Transfer Function

- Deriving the transfer function for feedback system:

$$E(s) = R(s) \pm C(s)H(s)$$

- Since: $C(s) = E(s)G(s)$

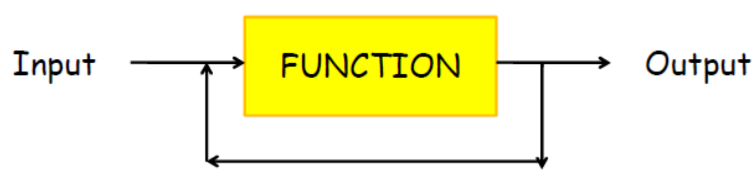
- Then, upon substitution, we see that:

$$G_{eq}(s) = \frac{G(s)}{1 \pm G(s)H(s)}$$

Unity Feedback System

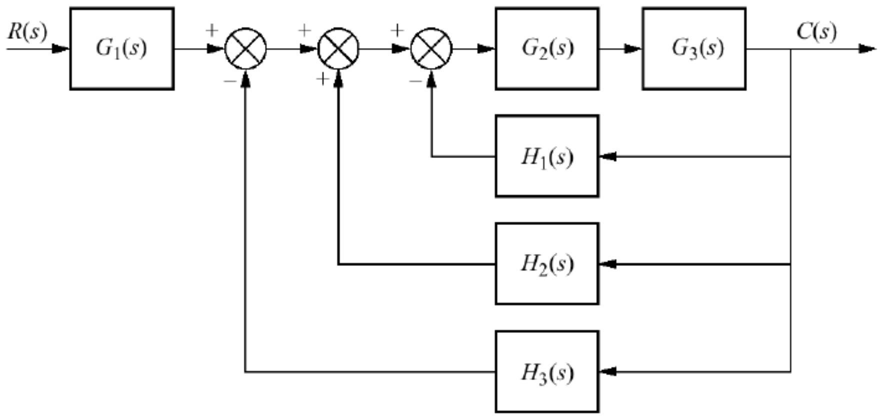
- A unity feedback system is one in which the primary feedback is identically equal to the controlled output.

- The advantage of a unity feedback system is that the reference input and the system output represent the same quantity and it can be compared directly.

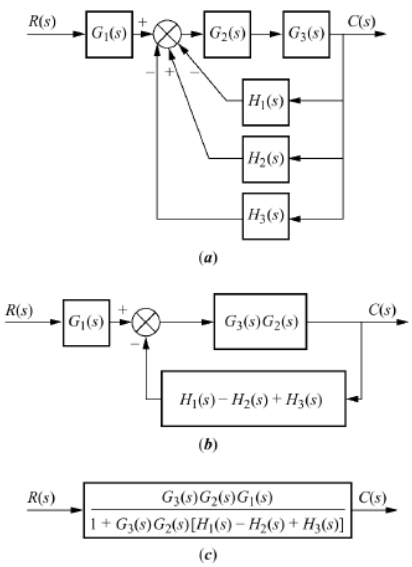


Example 1

- Reduce the following block diagram to a single transfer function.

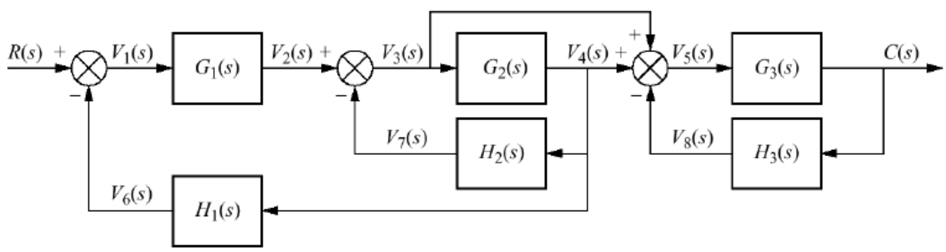


Solution to Example 1

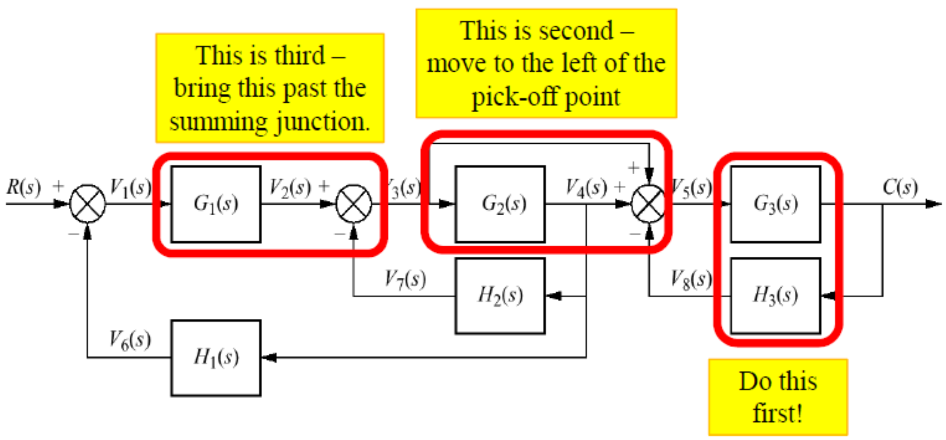


Example 2

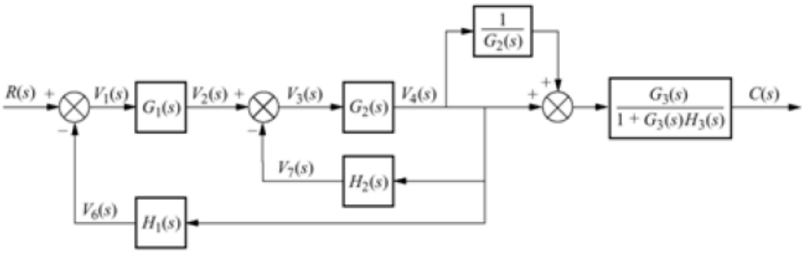
- Reduce the following block diagram to a single transfer function:



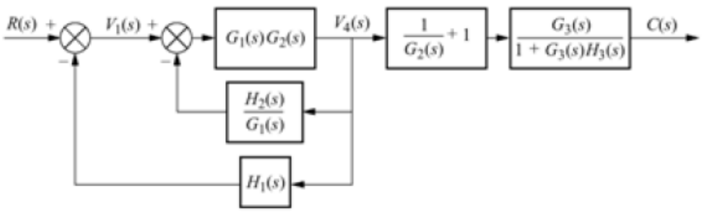
Solution to Example 2



Solution to Example 2

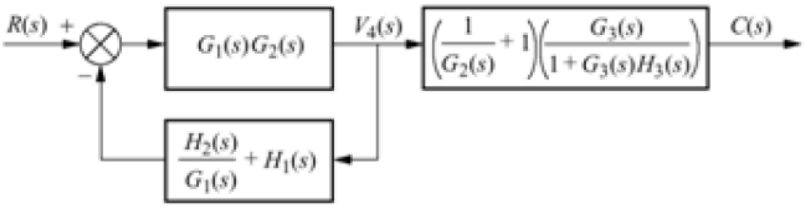


(a)

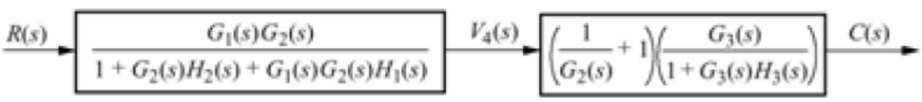


(b)

Solution to Example 2



(c)



(d)

Multiple Input

Sometimes it is necessary to evaluate system performance when several inputs are simultaneously applied at different points of the system.

When multiple inputs are present in a *linear* system, each is treated independently of the others. The output due to all stimuli acting together is found in the following manner. We assume zero initial conditions, as we seek the system response only to inputs.

Step 1: Set all inputs except one equal to zero.

Step 2: Transform the block diagram to canonical form, using the transformations of Section 7.5.

Step 3: Calculate the response due to the chosen input acting alone.

Step 4: Repeat Steps 1 to 3 for each of the remaining inputs.

Step 5: Algebraically add all of the responses (outputs) determined in Steps 1 to 4. This sum is the total output of the system with all inputs acting simultaneously.

We reemphasize here that the above superposition process is dependent on the system being linear.

EXAMPLE We determine the output C due to inputs U and R

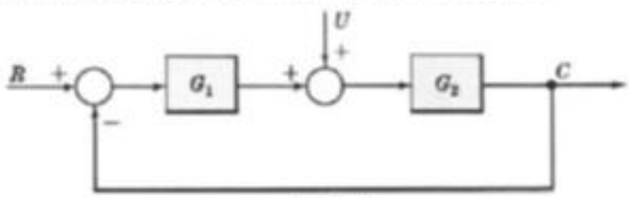
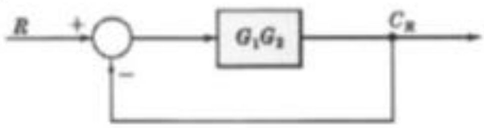


Fig. 7-9

Step 1: Put $U = 0$.

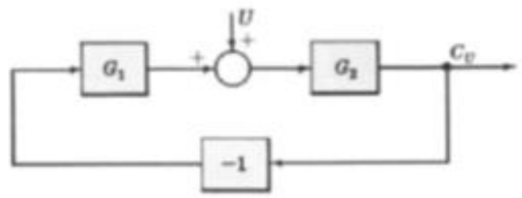
Step 2: The system reduces to



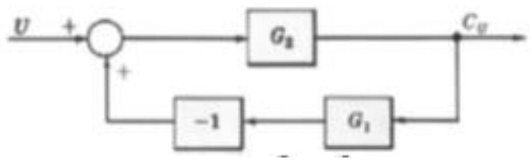
Step 3: By Equation (7.3), the output C_R due to input R is $C_R = [G_1G_2/(1 + G_1G_2)]R$.

Step 4a: Put $R = 0$.

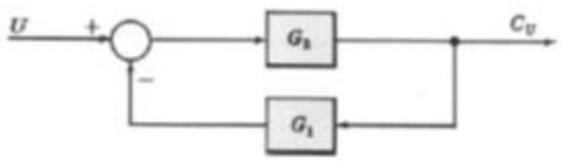
Step 4b: Put -1 into a block, representing the negative feedback effect:



Rearrange the block diagram:



Let the -1 block be absorbed into the summing point:



Step 4c: By Equation (7.3), the output C_U due to input U is $C_U = [G_2/(1 + G_1G_2)]U$.

Step 5: The total output is

$$C = C_R + C_U = \left[\frac{G_1G_2}{1 + G_1G_2}\right]R + \left[\frac{G_2}{1 + G_1G_2}\right]U = \left[\frac{G_2}{1 + G_1G_2}\right][G_1R + U]$$

Multiple inputs multiple outputs

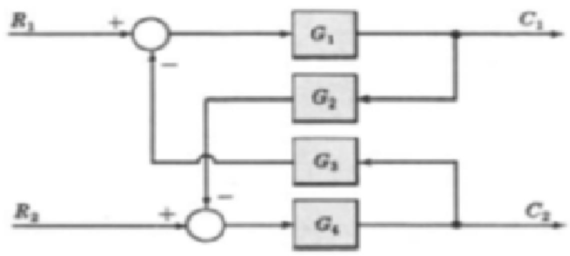
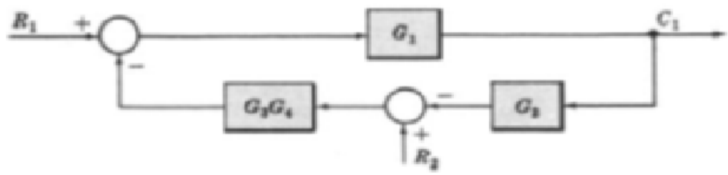


Fig. 7-32

First put the block diagram in the form of Fig. 7-33, ignoring the output C_2.



Letting $R_2 = 0$ and combining the summing points, we get Fig. 7-34.

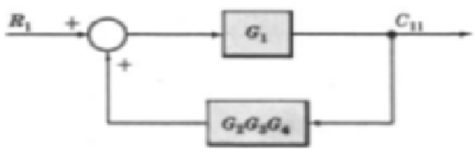


Fig. 7-34

Hence C_{11}, the output at C_1 due to R_1 alone, is $C_{11} = G_1R_1/(1 - G_1G_2G_3G_4)$. For $R_1 = 0$, we have Fig. 7-35.

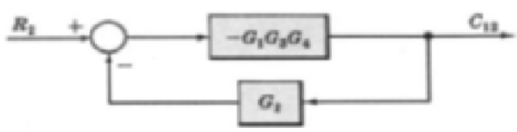


Fig. 7-35

Hence $C_{12} = -G_1G_3G_4R_2/(1 - G_1G_2G_3G_4)$ is the output at C_1 due to R_2 alone. Thus $C_1 = C_{11} + C_{12} = (G_1R_1 - G_1G_3G_4R_2)/(1 - G_1G_2G_3G_4)$.

Now we reduce the original block diagram, ignoring output C_1. First we obtain Fig. 7-36.

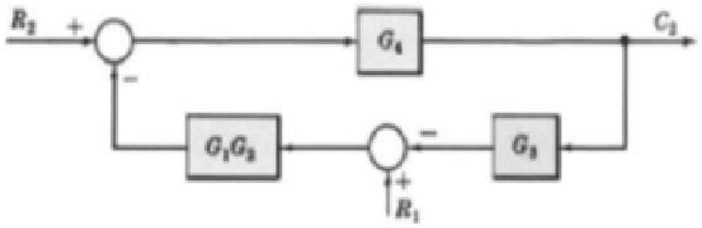


Fig. 7-36

Then we obtain the block diagram given in Fig. 7-37. Hence $C_{22} = G_4R_2/(1 - G_1G_2G_3G_4)$. Next, letting $R_2 = 0$, we obtain Fig. 7-38. Hence $C_{21} = -G_1G_2G_4R_1/(1 - G_1G_2G_3G_4)$. Finally, $C_2 = C_{22} + C_{21} = (G_4R_2 - G_1G_2G_4R_1)/(1 - G_1G_2G_3G_4)$.

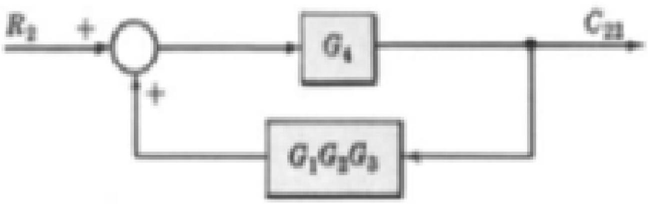
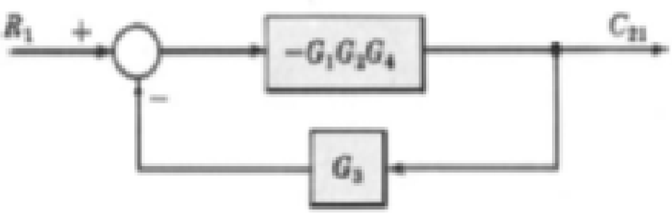
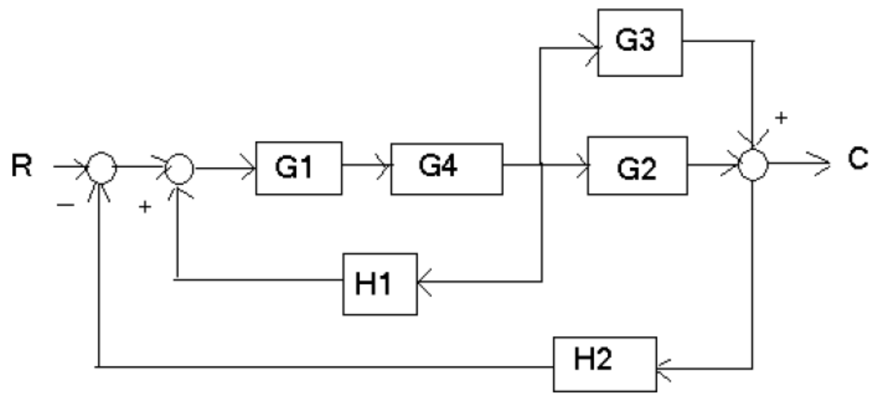


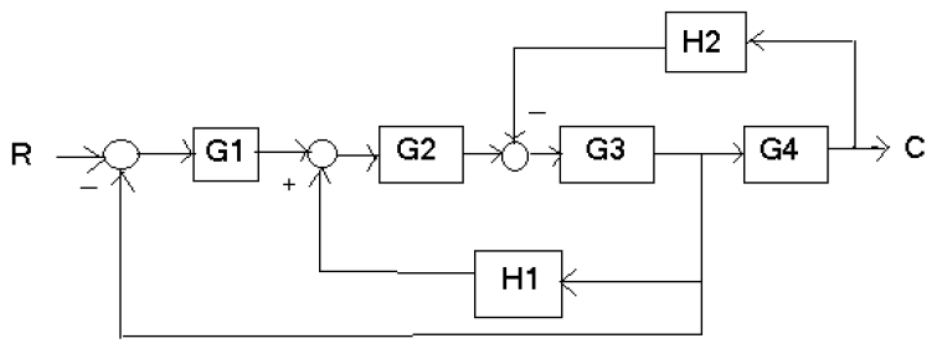
Fig. 7-37



Example:



Example:



Signal Flow Graph

Introduction to **Signal Flow**

- Alternative method to block diagram representation, developed by Samuel Jefferson Mason.

- A signal-flow graph consists of a network in which nodes are connected by directed branches.

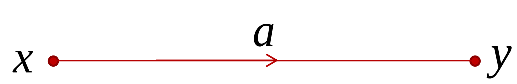
- It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

Fundamentals of Signal Flow Graphs

- Consider a simple equation below and draw its signal flow graph:

$$y = ax$$

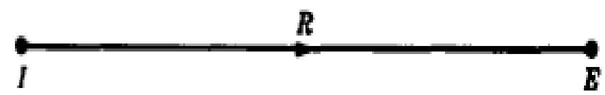
- The signal flow graph of the equation is shown below;



- Every variable in a signal flow graph is designed by a **Node**.
- Every transmission function in a signal flow graph is designed by a **Branch**.
- Branches are always **unidirectional**.
- The arrow in the branch denotes the **direction** of the signal flow.

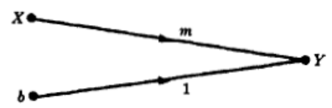
Signal-Flow Graph Models

Example: ohm's law states that E =IR ,where E is voltage ,
I a current , and R a resistance . The S.F.G for this equation.

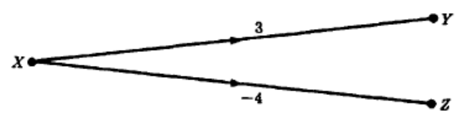


S. F.G algebra:

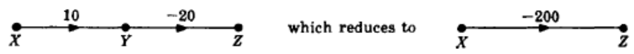
Addition rule: Y = mX +b



Transmission rule: Y = 3X , Z = - 4X

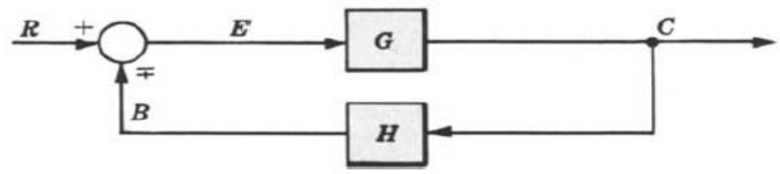


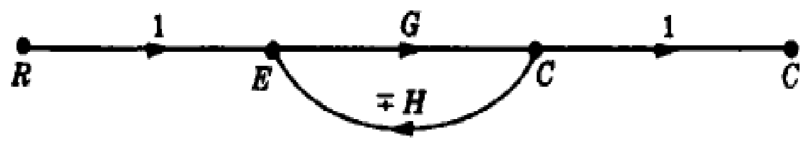
Multiplication rule: Y=10X , Z=-20Y , Z=-200X



Construction of S.F.G:

The SFG of the block diagram of the canonical (standard) feedback control system is as shown below. Each **variable** of the block diagram becomes a **node** and each **block** become a **branch.**

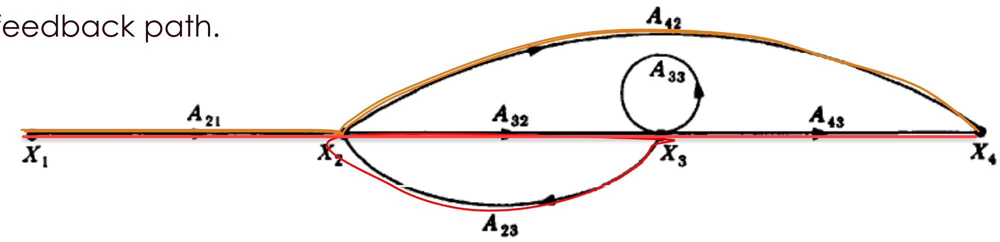




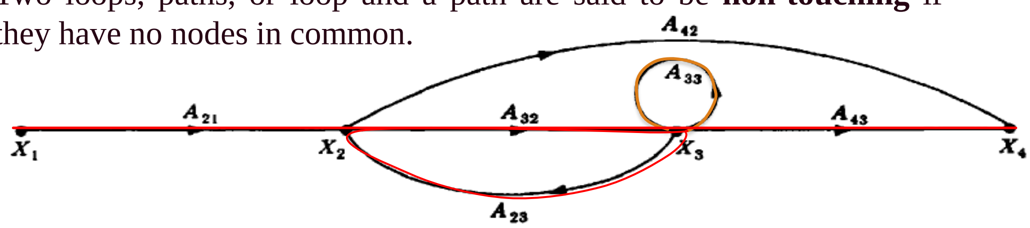
Terminologies

- An **input node** or source contain only the outgoing branches. i.e., X_1
- An **output node** or sink contain only the incoming branches. i.e., X_4
- A **path** is a continuous, unidirectional succession of branches along which no node is passed more than ones. i.e.,

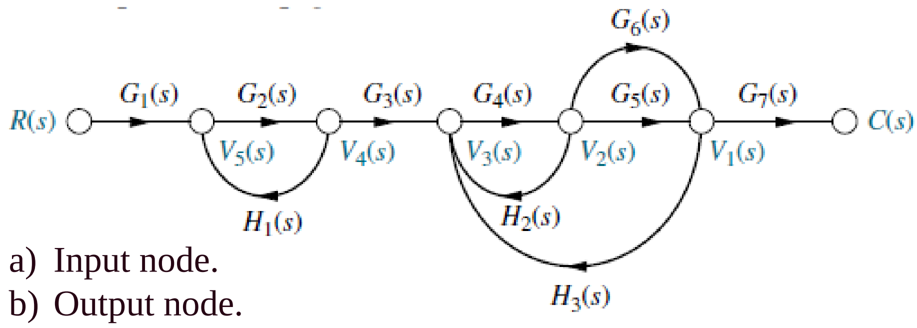
  X_1 *to* X_2 *to* X_3 *to* X_4 X_1 *to* X_2 *to* X_4 X_2 *to* X_3 *to* X_4
- A **forward path** is a path from the input node to the output node. i.e., **X_1 to X_2 to X_3 to X_4**, and **X_1 to X_2 to X_4**, are forward paths.
- A **feedback path** or feedback loop is a path which originates and terminates on the same node. i.e.; **X_2 to X_3 and back to X_2** is a feedback path.



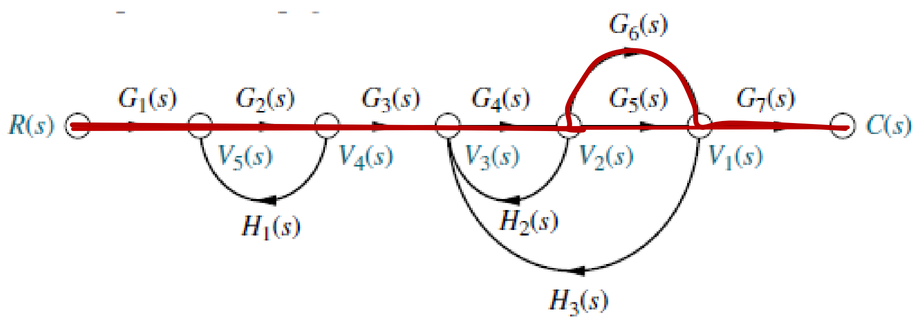
- A **self-loop** is a feedback loop consisting of a single branch. i.e.; A_{33} is a self loop.
- The **gain** of a branch is the transmission function of that branch.
- The **path gain** is the product of branch gains encountered in traversing a path. i.e. the gain of forwards path X_1 *to* X_2 *to* X_3 *to* X_4 is $A_{21}A_{32}A_{43}$
- The **loop gain** is the product of the branch gains of the loop. i.e., the loop gain of the feedback loop from X_2 to X_3 and back to X_2 is $A_{32}A_{23}$.
- Two loops, paths, or loop and a path are said to be **non-touching** if they have no nodes in common.



Consider the signal flow graph below and identify the following



a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths (loops).
e) Determine the loop gains of the feedback loops.
f) Determine the path gains of the forward paths.
g) Non-touching loops

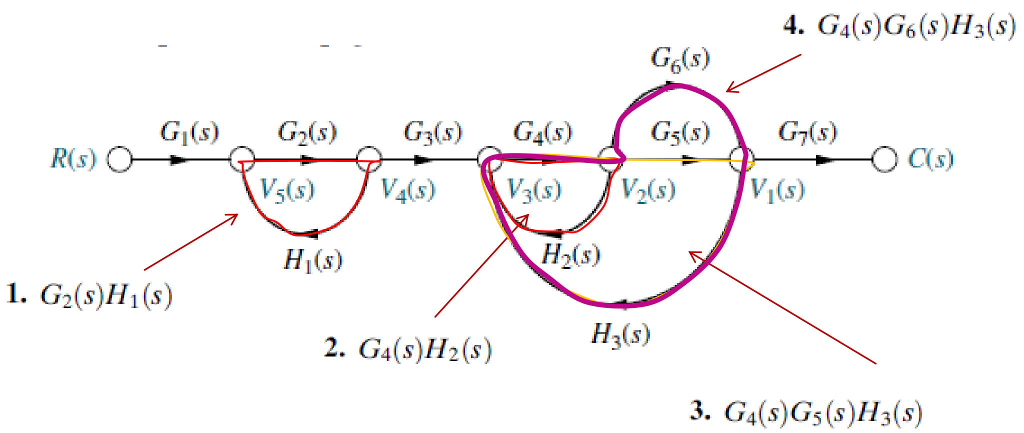


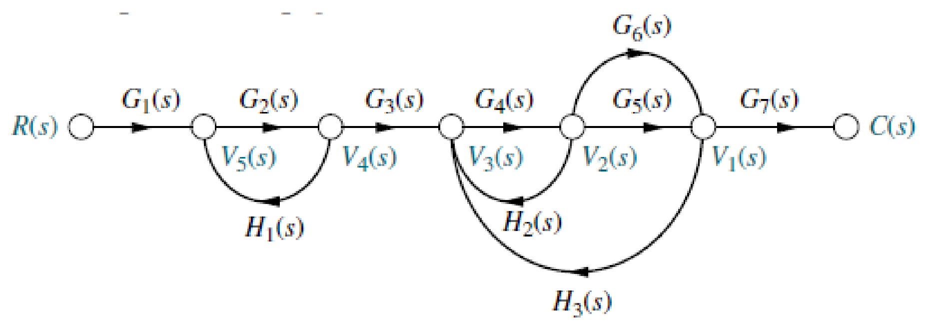
- There are two forward path gains;

1. $G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s)$

2. $G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s)$

- There are four loops

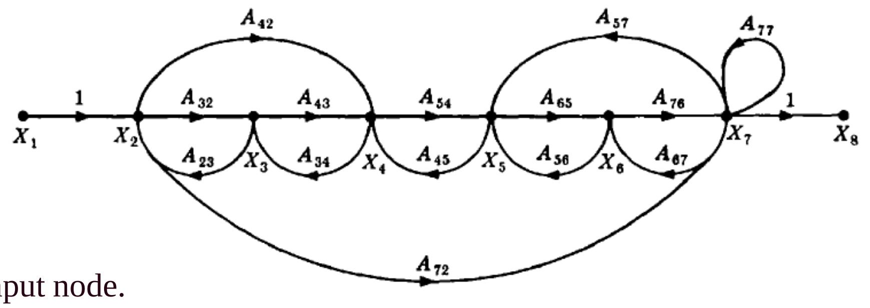




- Nontouching loop gains;

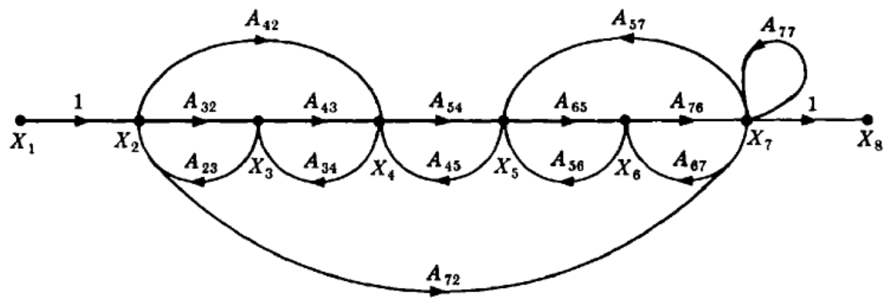
1. $[G_2(s)H_1(s)][G_4(s)H_2(s)]$
2. $[G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]$
3. $[G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]$

Consider the signal flow graph below and identify the following



a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths.
e) Self loop.
f) Determine the loop gains of the feedback loops.
g) Determine the path gains of the forward paths.

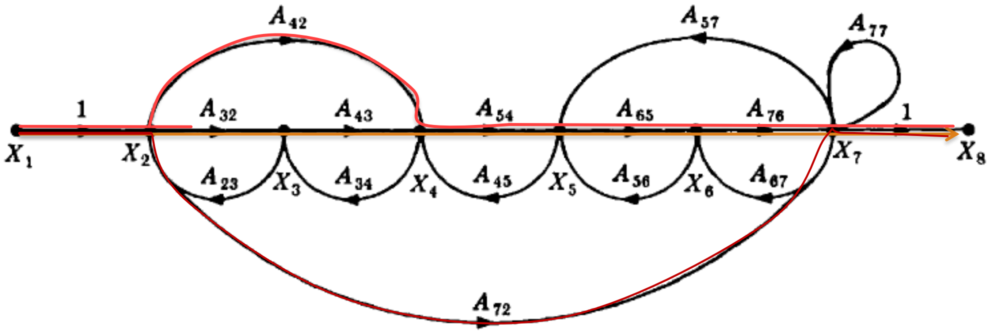
Input and output Nodes



a) Input nod X_1

b) Output node X_8

(c) Forward Paths



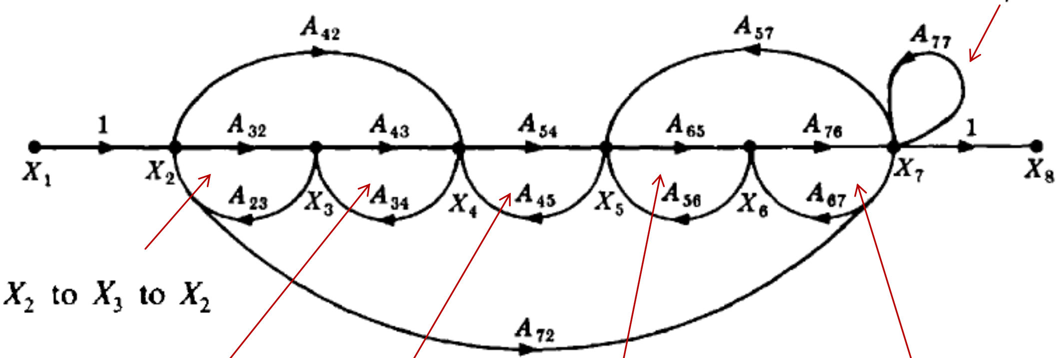
X_1 to X_2 to X_3 to X_4 to X_5 to X_6 to X_7 to X_8

X_1 to X_2 to X_7 to X_8

X_1 to X_2 to X_4 to X_5 to X_6 to X_7 to X_8

(d) Feedback Paths or Loops

X_7 to X_7



X_2 to X_3 to X_2

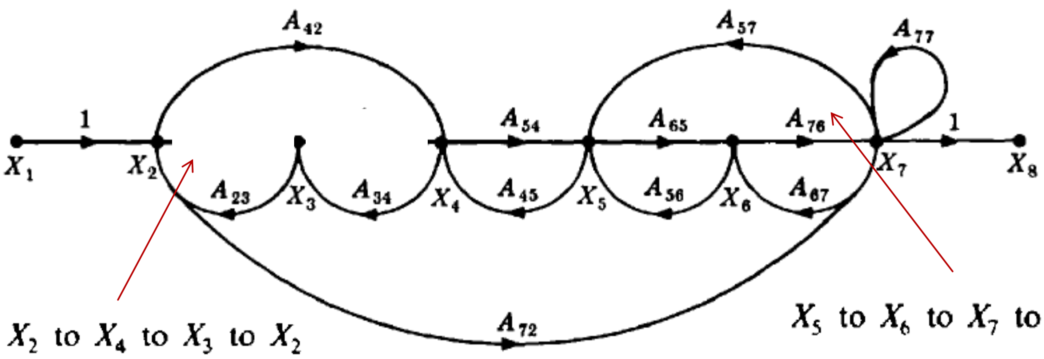
X_3 to X_4 to X_3

X_4 to X_5 to X_4

X_5 to X_6 to X_5

X_6 to X_7 to X_6;

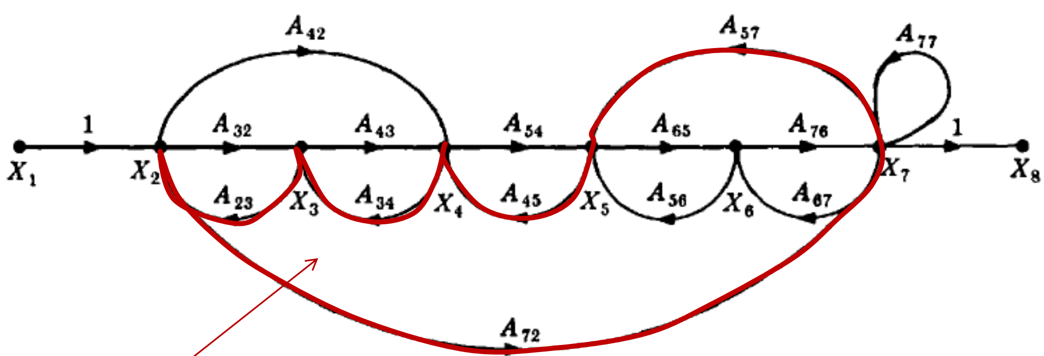
(d) Feedback Paths or Loops



X_2 to X_4 to X_3 to X_2

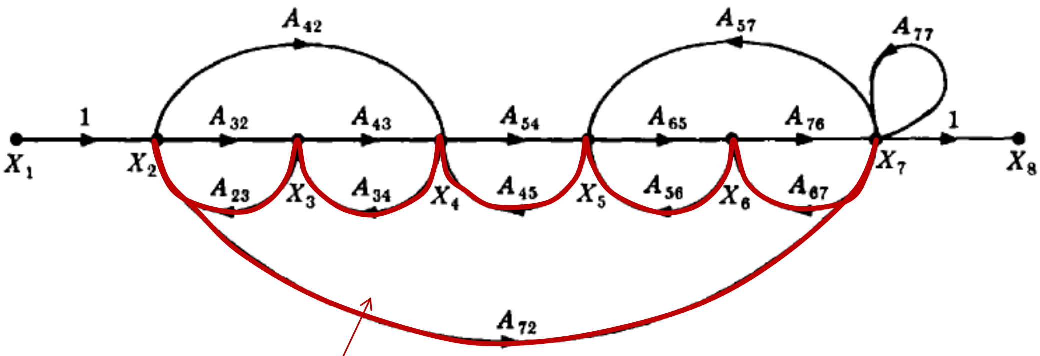
X_5 to X_6 to X_7 to X_5

(d) Feedback Paths or Loops



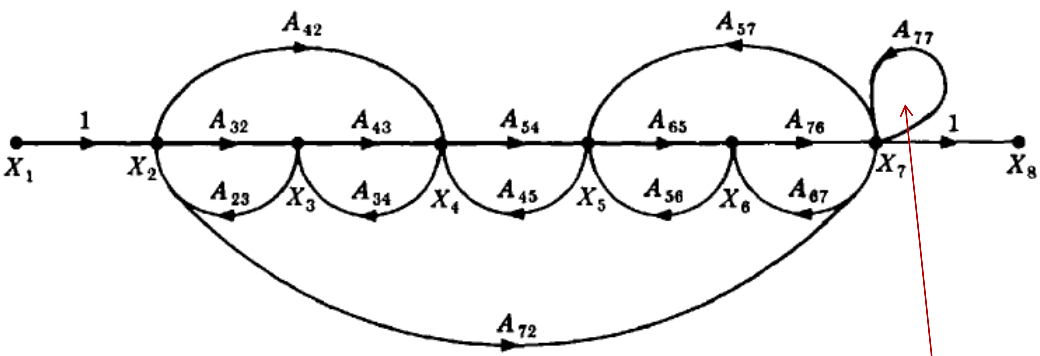
X_2 to X_7 to X_5 to X_4 to X_3 to X_2

(d) Feedback Paths or Loops



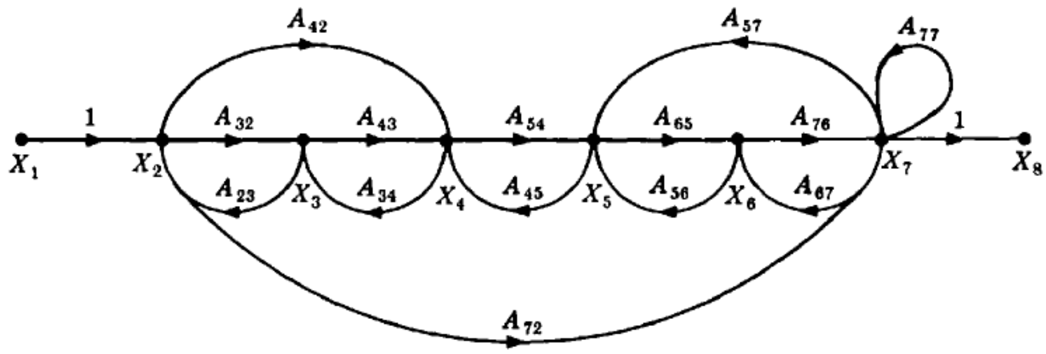
X_2 to X_7 to X_6 to X_5 to X_4 to X_3 to X_2

(e) Self Loop(s)



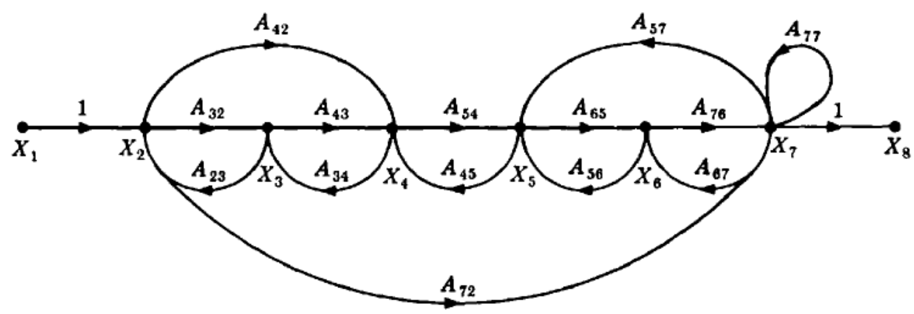
X_7 to X_7

(f) Loop Gains of the Feedback Loops



$A_{32}A_{23}$	$A_{76}A_{67}$	$A_{72}A_{57}A_{45}A_{34}A_{23}$
$A_{43}A_{34}$	$A_{65}A_{76}A_{57}$	$A_{72}A_{67}A_{56}A_{45}A_{34}A_{23}$
$A_{54}A_{45}$	A_{77}	
$A_{65}A_{56}$	$A_{42}A_{34}A_{23}$	

(g) Path Gains of the Forward Paths



$A_{32}A_{43}A_{54}A_{65}A_{76}$

A_{72}

$A_{42}A_{54}A_{65}A_{76}$

Mason's Rule (Mason, 1953)

- The block diagram reduction technique requires successive application of fundamental relationships in order to arrive at the system transfer function.
- On the other hand, Mason's rule for reducing a signal-flow graph to a single transfer function requires the application of one formula.
- The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph.

Mason's Rule:

- The transfer function, ***C(s)/R(s)***, of a system represented by a signal-flow graph is;

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

Where

n = number of forward paths.
P_i = the i^{th} forward-path gain.
Δ = Determinant of the system
Δ_i = Determinant of the i^{th} forward path

- Δ is called the signal flow graph determinant or characteristic function. Since $\Delta=0$ is the system characteristic equation.

Mason's Rule

$$\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}$$

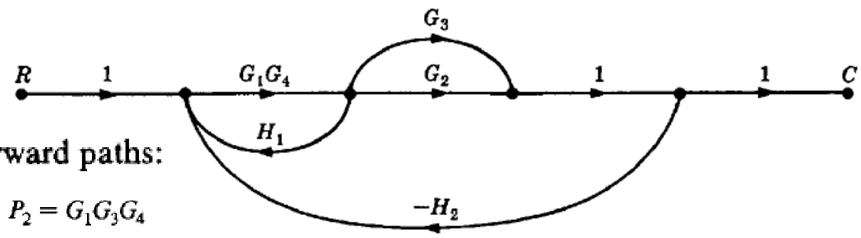
Δ = 1- (sum of all individual loop gains) + (sum of the products of the gains of all possible two loops that do not touch each other) – (sum of the products of the gains of all possible three loops that do not touch each other) + … and so forth with sums of higher number of non-touching loop gains

Δ_i = value of Δ for the part of the block diagram that does not touch the i-th forward path (Δ_i = 1 if there are no non-touching loops to the i-th path.)

Systematic approach

1. Calculate forward path gain P_i for each forward path *i*.
2. Calculate all loop transfer functions
3. Consider non-touching loops 2 at a time
4. Consider non-touching loops 3 at a time
5. etc
6. Calculate Δ from steps 2,3,4 and 5
7. Calculate Δ_i as portion of Δ not touching forward path *i*

Example1: Apply Mason's Rule to calculate the transfer function of the system represented by following Signal Flow Graph



There are two forward paths:

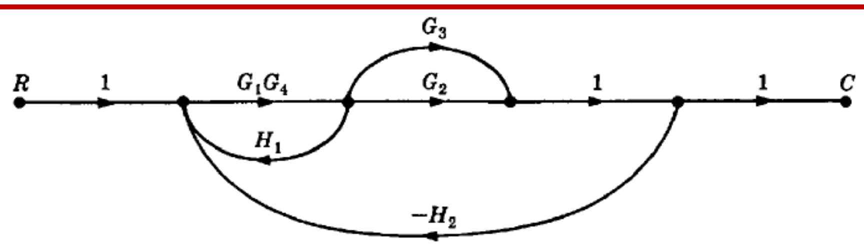
$P_1 = G_1G_2G_4$ $\quad$ $P_2 = G_1G_3G_4$

Therefore,

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

There are three feedback loops

$$L_1 = G_1G_4H_1, \quad L_2 = -G_1G_2G_4H_2, \quad L_3 = -G_1G_3G_4H_2$$

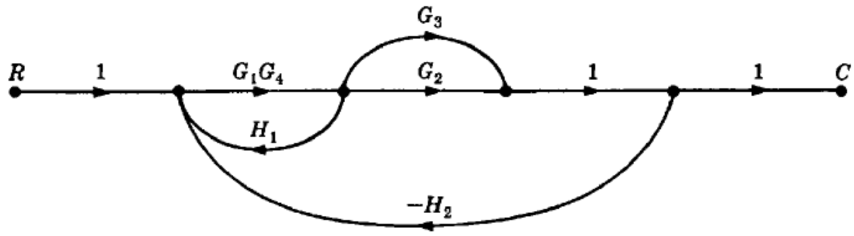


There are no non-touching loops, therefore

Δ = 1- (sum of all individual loop gains)

$$\Delta = 1 - (L_1 + L_2 + L_3)$$

$$\Delta = 1 - (G_1G_4H_1 - G_1G_2G_4H_2 - G_1G_3G_4H_2)$$



Eliminate forward path-1

Δ_1 = 1- (sum of all individual loop gains)+...

$\Delta_1 = 1$

Eliminate forward path-2

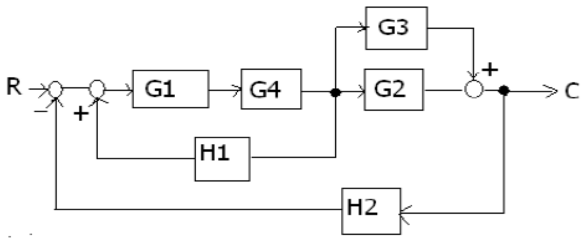
Δ_2 = 1- (sum of all individual loop gains)+...

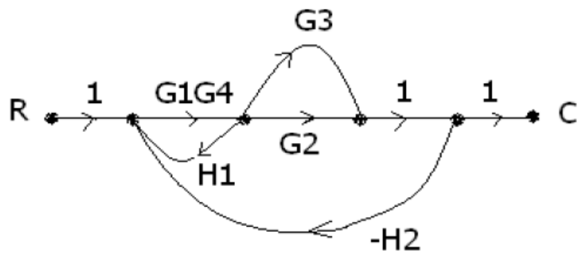
$\Delta_2 = 1$

$$\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

$$= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}$$

Determine the transfer function $C(s)/R(s)$ of the following system:-





<u>Note</u> :The takeoff point and the summing point must be separated by a unity gain branch.

There are <u>two</u> forward paths through the system : $T_1 = G_1G_4G_2$ and $T_2 = G_1G_4G_3$

The system has <u>three</u> feedback loops :-

$L_1 = G_1G_4H_1$ $\quad L_2 = -G_1G_4G_2H_2$ $\quad$ and $\quad L_3 = -G_1G_4G_3H_2$

There are no non-touching loops … so,

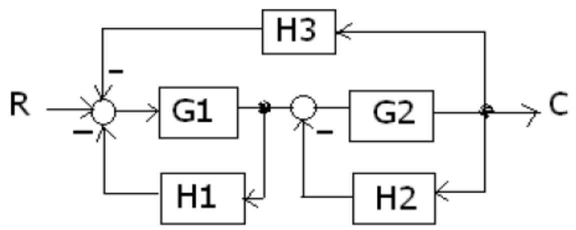
$\Delta = 1-(L_1+L_2+L_3)$

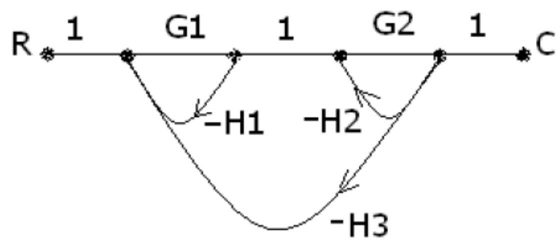
Δ_1 is obtained from Δ by removing the loops that touch T_1 .Similarly $\Delta_2 = 1$

Since all the loops touch both the forward paths. Therefore,

$$T = \frac{C}{R} = \frac{T_1\Delta_1 + T_2\Delta_2}{\Delta} = \frac{G_1G_4G_2 + G_1G_4G_3}{1 - G_1G_4H_1 + G_1G_4G_2H_2 + G_1G_4G_3H_2}$$

Example-2 : Find the C(S) / R (s) transfer function for ;





There is one forward path $T_1 = G_1G_2$

There are three loops $L_1 = -G_1H_1$, $L_2 = -G_2H_2$, and $L_3 = -G_1G_2H_3$

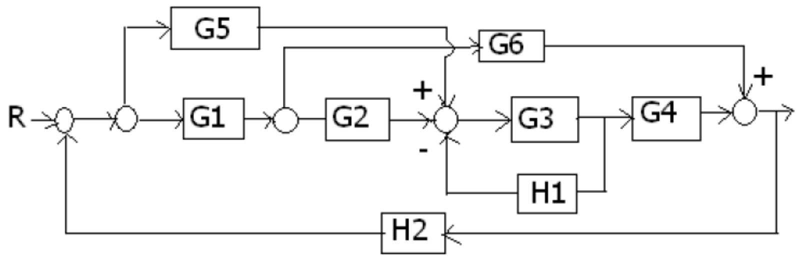
There are two non-touching loops L_1 and L_2

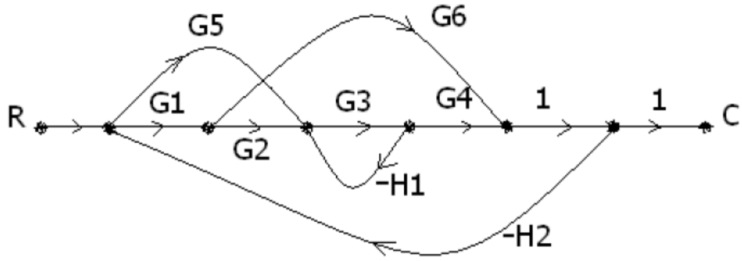
$\therefore \quad \Delta = 1 - (L_1 + L_2 + L_3) + L_1L_2$

$\Delta_1 = 1$

$$\therefore \; T = \frac{G_1G_2}{\Delta}$$

Example 3 : Consider the following system, find C(s)/R(s) Using S.F.G.





There are three distinct forward paths

$T_1 = G_1G_2G_3G_4$, $T_2 = G_5G_3G_4$, and $T_3 = G_1G_6$

The system has four loops

$L_1 = -G_3H_1$, $L_2 = -G_1G_6H_2$, $L_3 = -G_5G_3G_4H_2$, $L_4 = -G_1G_2G_3G_4H_2$

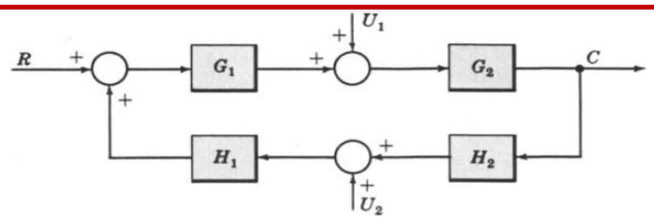
There is only L_1 which does not touch L_2

$\therefore \quad \Delta = 1 - (L_1 + L_2 + L_3 + L_4) + L_1L_2$

$\Delta_1 = 1$, $\Delta_2 = 1$, and $\Delta_3 = 1 - L_1$

$$T = \frac{T_1\Delta_1 + T_2\Delta_2 + T_3X_3}{\Delta}$$

Example: determine the output C for the following system using SFG



With $U_1 = U_2 = 0$, we have Fig. 8-48. Then $P_1 = G_1G_2$ and $P_{11} = G_1G_2H_1H_2$. Hence $\Delta = 1 - P_{11} = 1 - G_1G_2H_1H_2$, $\Delta_1 = 1$, and

$$C_R = TR = \frac{P_1\Delta_1 R}{\Delta} = \frac{G_1G_2R}{1 - G_1G_2H_1H_2}$$

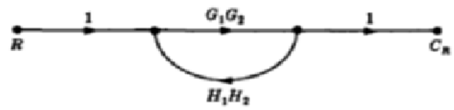
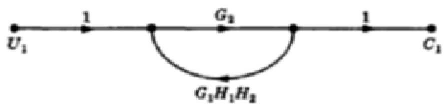


Fig. 8-48

Now put $U_2 = R = 0$ (Fig. 8-49).



Then $P_1 = G_2$, $P_{11} = G_1G_2H_1H_2$, $\Delta = 1 - G_1G_2H_1H_2$, $\Delta_1 = 1$, and

$$C_1 = TU_1 = \frac{G_2U_1}{1 - G_1G_2H_1H_2}$$

Now put $R = U_1 = 0$ (Fig. 8-50).

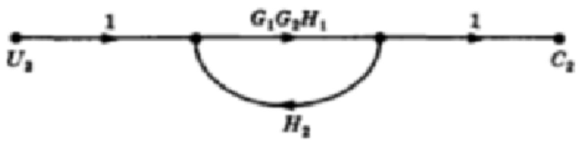


Fig. 8-50

Then $P_1 = G_1G_2H_1$, $P_{11} = G_1G_2H_1H_2$, $\Delta = 1 - G_1G_2H_1H_2$, $\Delta_1 = 1$, and

$$C_2 = TU_2 = \frac{P_1\Delta_1U_2}{\Delta} = \frac{G_1G_2H_1U_2}{1 - G_1G_2H_1H_2}$$

Finally, we have

$$C = C_R + C_1 + C_2 = \frac{G_1G_2R + G_2U_1 + G_1G_2H_1U_2}{1 - G_1G_2H_1H_2}$$

Example:

Draw a signal flow graph for the resistance network shown in Fig. 8-41 in which $v_2(0) = v_3(0) = 0$. v_2 is the voltage across C_1.

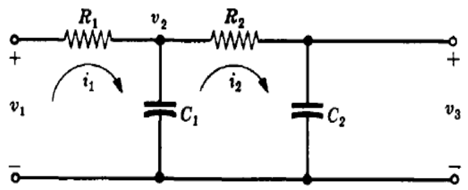


Fig. 8-41

The five variables are v_1, v_2, v_3, i_1, and i_2; and v_1 is the input. The four independent equations derived from Kirchhoff's voltage and current laws are

$$i_1 = \left(\frac{1}{R_1}\right)v_1 - \left(\frac{1}{R_1}\right)v_2 \qquad v_2 = \frac{1}{C_1}\int_0^t i_1\,dt - \frac{1}{C_1}\int_0^t i_2\,dt$$

$$i_2 = \left(\frac{1}{R_2}\right)v_2 - \left(\frac{1}{R_2}\right)v_3 \qquad v_3 = \frac{1}{C_2}\int_0^t i_2\,dt$$

The signal flow graph can be drawn directly from these equations (Fig. 8-42).

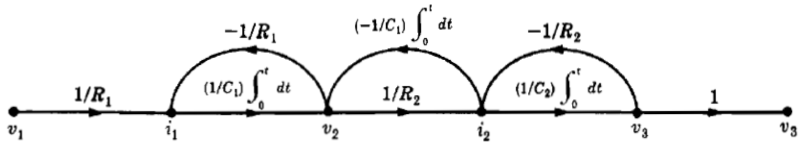


Fig. 8-42

In Laplace transform notation, the signal flow graph is given in Fig. 8-43.

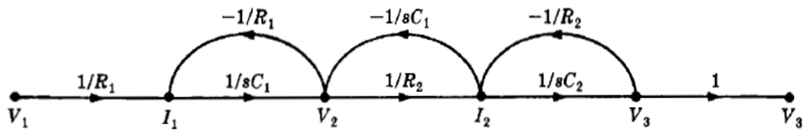
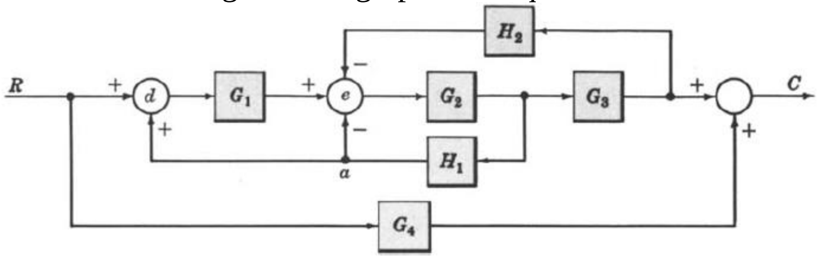
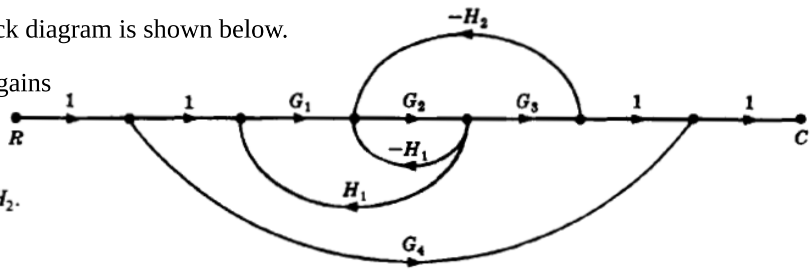


Fig. 8-43

Example-: Determine the transfer function C/R for the block diagram below by signal flow graph techniques.



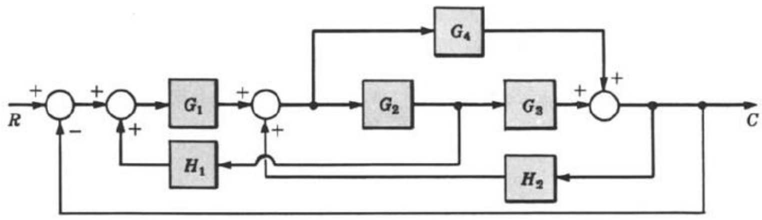
- The signal flow graph of the above block diagram is shown below.



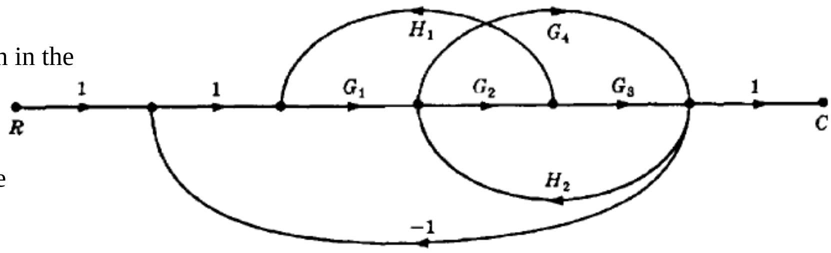
- There are two forward paths. The path gains $P_1 = G_1G_2G_3$ and $P_2 = G_4$
- The three feedback loop gains $P_{11} = -G_2H_1$, $P_{21} = G_1G_2H_1$, $P_{31} = -G_2G_3H_2$.
- No loops are non-touching, $\Delta = 1 - (P_{11} + P_{21} + P_{31})$
- Because the loops touch the nodes of P1, hence $\Delta_1 = 1$
- Since no loops touch the nodes of P2, therefore $\Delta_2 = \Delta$.
- Hence the control ratio T = C/R is

$$T = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_4 + G_2G_4H_1 - G_1G_2G_4H_1 + G_2G_3G_4H_2}{1 + G_2H_1 - G_1G_2H_1 + G_2G_3H_2}$$

Example: Find the control ratio C/R for the system given below.



- The signal flow graph is shown in the figure.



- The two forward path gains are $P_1 = G_1G_2G_3$ and $P_2 = G_1G_4$
- The five feedback loop gains are $P_{11} = G_1G_2H_1$, $P_{21} = G_2G_3H_2$, $P_{31} = -G_1G_2G_3$, $P_{41} = G_4H_2$, and $P_{51} = -G_1G_4$.
- All feedback loops touches the two forward path: $\Delta_1 = \Delta_2 = 1$
- Hence the control ratio $T = \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4}$
- There are no non-touching loops, hence

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51})$$
$$= 1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4$$

Time Domain Analysis

Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.
- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.
- Usually, the input signals to control systems are not known fully ahead of time.
- It is therefore difficult to express the actual input signals mathematically by simple equations.

Standard Test Signals

- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.

- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.

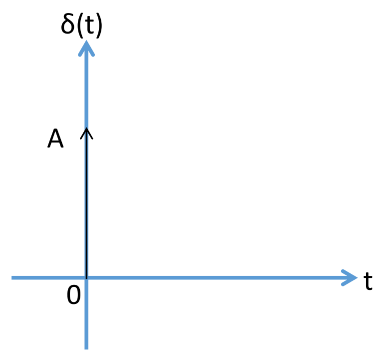
- The other standard signal of great importance is a sinusoidal signal.

Standard Test Signals

- Impulse signal
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

  - If A=1, the impulse signal is called unit impulse signal.

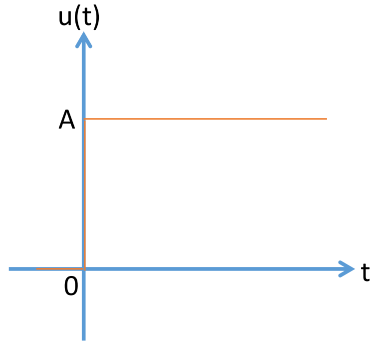


Standard Test Signals

- Step signal
  - The step signal imitate the sudden change characteristic of actual input signal.

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If A=1, the step signal is called unit step signal

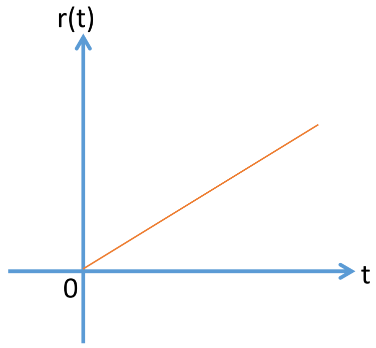


Standard Test Signals

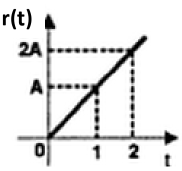
- Ramp signal
  - The ramp signal imitate the constant velocity characteristic of actual input signal.

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

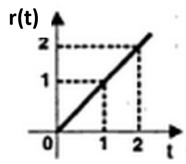
  - If *A=1*, the ramp signal is called unit ramp signal



ramp signal with slope A



unit ramp signal

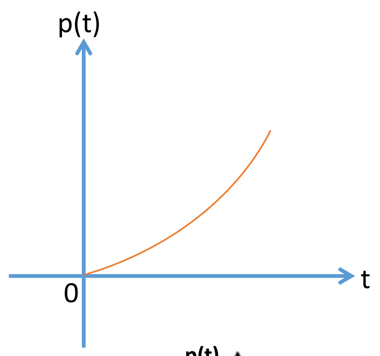


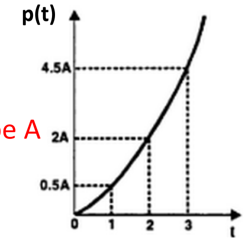
Standard Test Signals

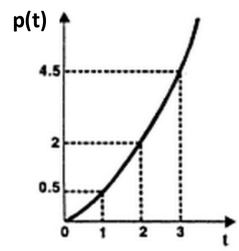
- Parabolic signal
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If *A=1*, the parabolic signal is called unit parabolic signal.







Relation between standard Test Signals

- Impulse $\quad \delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$
- Step $\quad u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$
- Ramp $\quad r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$
- Parabolic $\quad p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$

$\int$ $\quad \int$ $\quad \int$

$\dfrac{d}{dt}$ $\quad \dfrac{d}{dt}$ $\quad \dfrac{d}{dt}$

Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

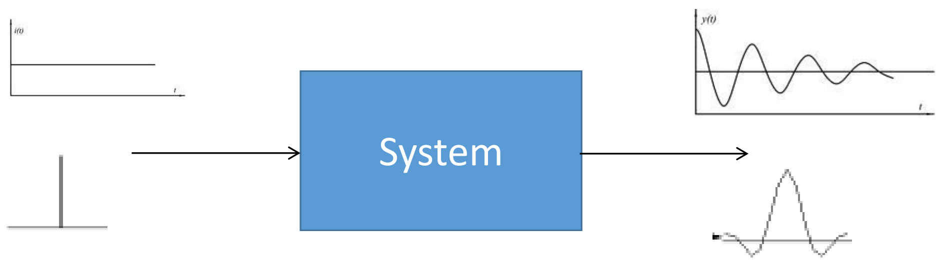
- Parabolic

$$p(t) = \begin{cases} \frac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

Time Response of Control Systems

- Time response of a dynamic system response to an input expressed as a function of time.



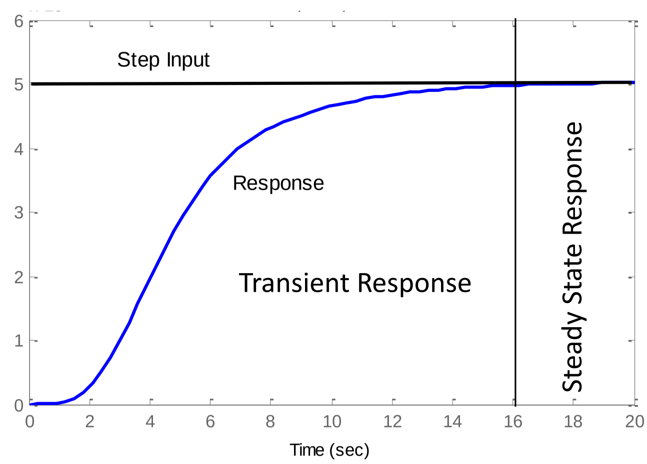
- The time response of any system has two components
    - Transient response
    - Steady-state response.

Time Response of Control Systems

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.



Time Response of Control Systems

- Transient response depend upon the system poles only and not on the type of input.

- It is therefore sufficient to analyze the transient response using a step input.

- The steady-state response depends on system dynamics and the input quantity.

- It is then examined using different test signals by final value theorem.

Introduction

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1^{st} order system responds to a unit step input.

- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Introduction

- The first order system given below.

$$G(s) = \frac{10}{3s+1}$$

- D.C gain is 10 and time constant is 3 seconds.

- For the following system

$$G(s) = \frac{3}{s+5} = \frac{3/5}{1/5s+1}$$

- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

Steady State Error

- If the output of a control system at steady state does not exactly match with the input, the system is said to have steady state error

- Any physical control system inherently suffers steady-state error in response to certain types of inputs.

- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.

Classification of Control Systems

- Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.

- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Classification of Control Systems

- Consider the unity-feedback control system with the following open-loop transfer function

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

- It involves the term s^N in the denominator, representing N poles at the origin.

- A system is called type 0, type 1, type 2, ... , if N=0, N=1, N=2, ... , respectively.

Classification of Control Systems

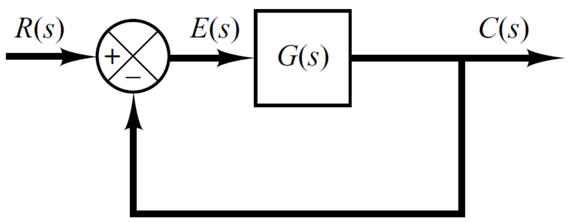
- As the type number is increased, accuracy is improved.

- However, increasing the type number aggravates the stability problem.

- A compromise between steady-state accuracy and relative stability is always necessary.

Steady State Error of Unity Feedback Systems

- Consider the system shown in following figure.



- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \qquad G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

Steady State Error of Unity Feedback Systems

- Steady state error is defined as the error between the input signal and the output signal when $t \to \infty$.

- The transfer function between the error signal E(s) and the input signal R(s) is $\dfrac{E(s)}{R(s)} = \dfrac{1}{1+G(s)}$
- The final-value theorem provides a convenient way to find the steady-state performance of a stable system.
- Since E(s) is $E(s) = \dfrac{1}{1 + G(s)} R(s)$
- The steady state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Static Error Constants

- The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.

- In a given system, the output may be the position, velocity, pressure, temperature, or the like.

- Therefore, in what follows, we shall call the output “position,” the rate of change of the output “velocity,” and so on.

- This means that in a temperature control system “position” represents the output temperature, “velocity” represents the rate of change of the output temperature, and so on.

Static Position Error Constant (K_p)

- The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s\to 0} \frac{\cancel{s}}{1 + G(s)} \frac{1}{\cancel{s}} = \frac{1}{1 + G(0)}$$

- The static position error constant K_p is defined by

$$K_p = \lim_{s\to 0} G(s) = G(0)$$

- Thus, the steady-state error in terms of the static position error constant K_p is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

Static Position Error Constant (K_p)

- For a Type 0 system

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For Type 1 or higher order systems

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

- For a unit step input the steady state error **e_{ss}** is

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

Static Velocity Error Constant (K_v)

- The steady-state error of the system for a unit-ramp input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2} = \lim_{s \to 0} \frac{1}{sG(s)}$$

- The static velocity error constant K_v is defined by

$$K_v = \lim_{s \to 0} sG(s)$$

- Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

Static Velocity Error Constant (K_v)

- For a Type 0 system

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 2 or higher order systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

Static Velocity Error Constant (K_v)

- For a ramp input the steady state error e_{ss} is

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

Static Acceleration Error Constant (K_a)

- The steady-state error of the system for parabolic input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

- The static acceleration error constant K_a is defined by

$$K_a = \lim_{s \to 0} s^2 G(s)$$

- Thus, the steady-state error in terms of the static acceleration error constant K_a is given by

$$e_{ss} = \frac{1}{K_a}$$

Static Acceleration Error Constant (K_a)

- For a Type 0 system

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For type 2 systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 3 or higher order systems

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

Static Acceleration Error Constant (K_a)

- For a parabolic input the steady state error e_{ss} is

$$e_{ss} = \infty, \qquad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \qquad \text{for type 2 systems}$$

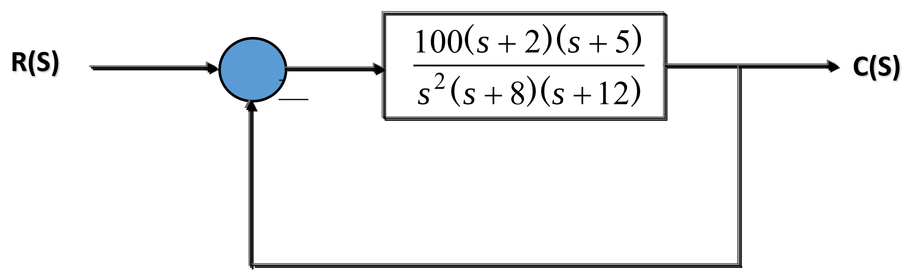
$$e_{ss} = 0, \qquad \text{for type 3 or higher systems}$$

Summary

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Example

- For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



Example

$$G(s) = \frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p = \lim_{s\to 0} G(s)$$

$$K_p = \lim_{s\to 0}\left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_p = \infty$$

$$K_v = \lim_{s\to 0} sG(s)$$

$$K_v = \lim_{s\to 0}\left(\frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_v = \infty$$

$$K_a = \lim_{s\to 0} s^2G(s)$$

$$K_a = \lim_{s\to 0}\left(\frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_a = \left(\frac{100(0+2)(0+5)}{(0+8)(0+12)}\right) = 10.4$$

Example

$$K_p = \infty \qquad K_v = \infty \qquad K_a = 10.4$$

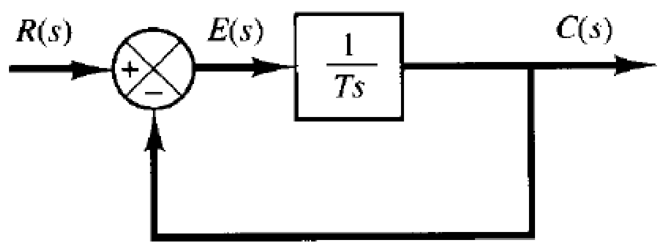
$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$

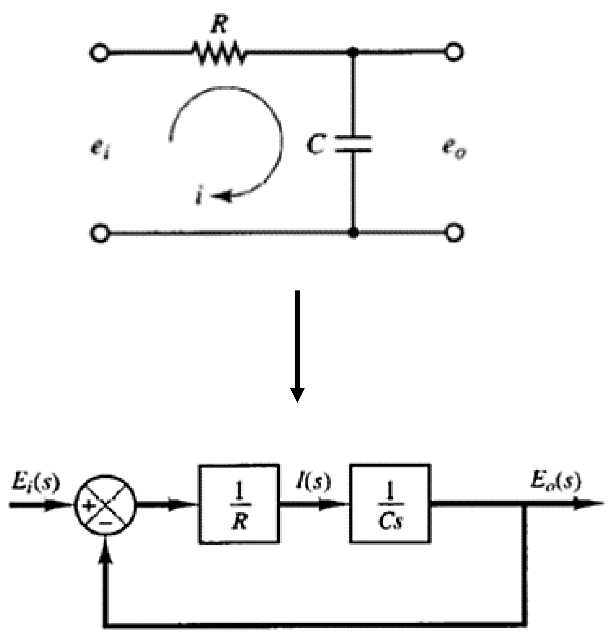
First – order systems

Consider the first –order system shown in figure below.
Physically, this system may be represent an RC electric circuit.



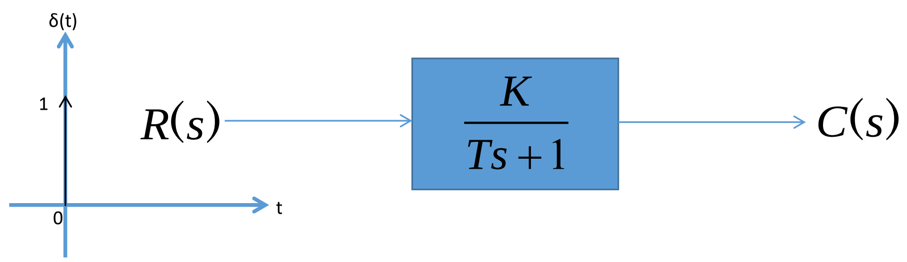
The closed – loop T.F is :
We shall analyze the system responses to such inputs as the unit-step, unit – ramp, and unit – impulse functions. The initial conditions are zero .

$$\frac{C(s)}{R(s)} = \frac{1}{Ts + 1}$$



Impulse Response of 1st Order System

- Consider the following 1st order system



$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts+1}$$

- Re-arrange following equation as

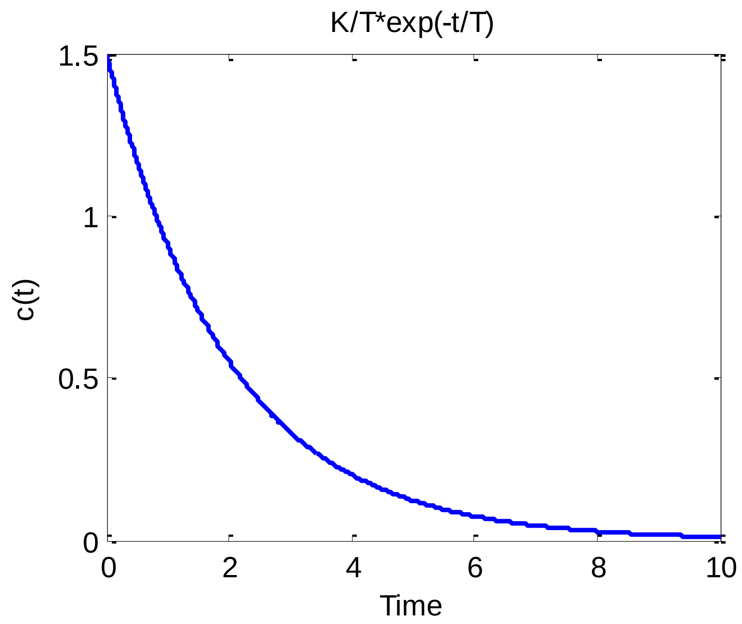
$$C(s) = \frac{K/T}{s+1/T}$$

- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at} \qquad c(t) = \frac{K}{T}e^{-t/T} \quad \text{for } t \geq 0$$

Impulse Response of 1st Order System

- If K=3 and T=2s then $c(t) = \frac{K}{T} e^{-t/T}$



Step Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion.

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

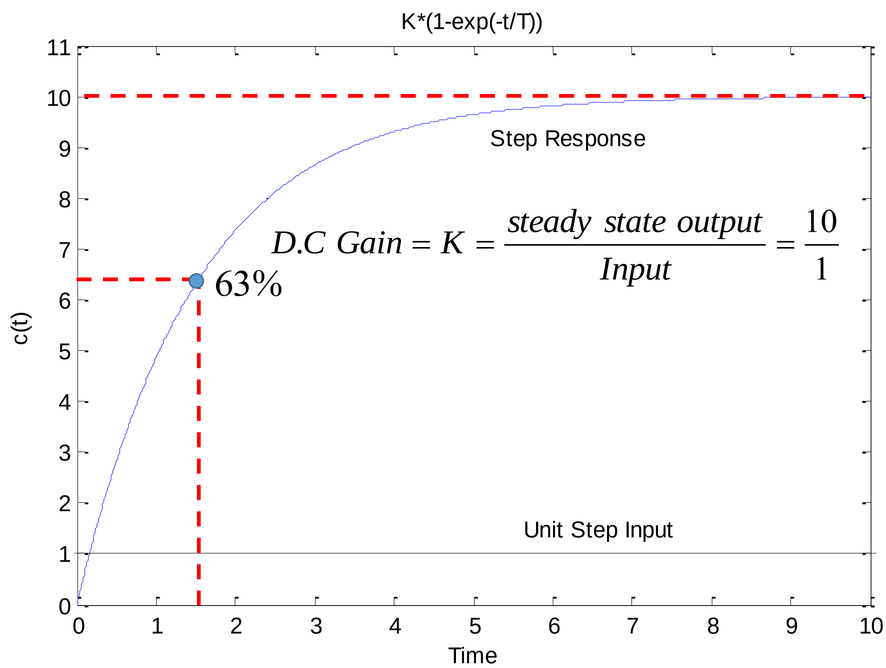
$$c(t) = K\left(1 - e^{-t/T}\right)$$

- The above equation states that initially the output c(t) is zero and finally it becomes unity .
- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

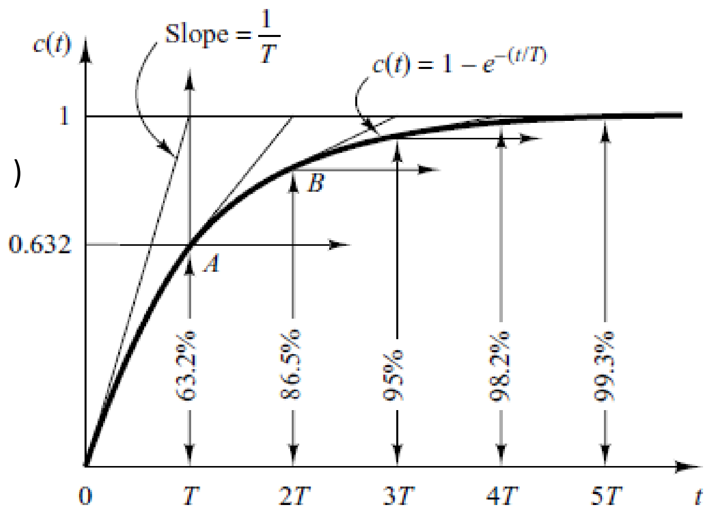
Step Response of 1st Order System

- If K=10 and T=1.5s then $c(t) = K\left(1 - e^{-t/T}\right)$



Step Response of 1st order System

- System takes five time constants to reach its final value.



e(t) = r(t) – c(t) = 1- $(1 - e^{-(t/T)})$

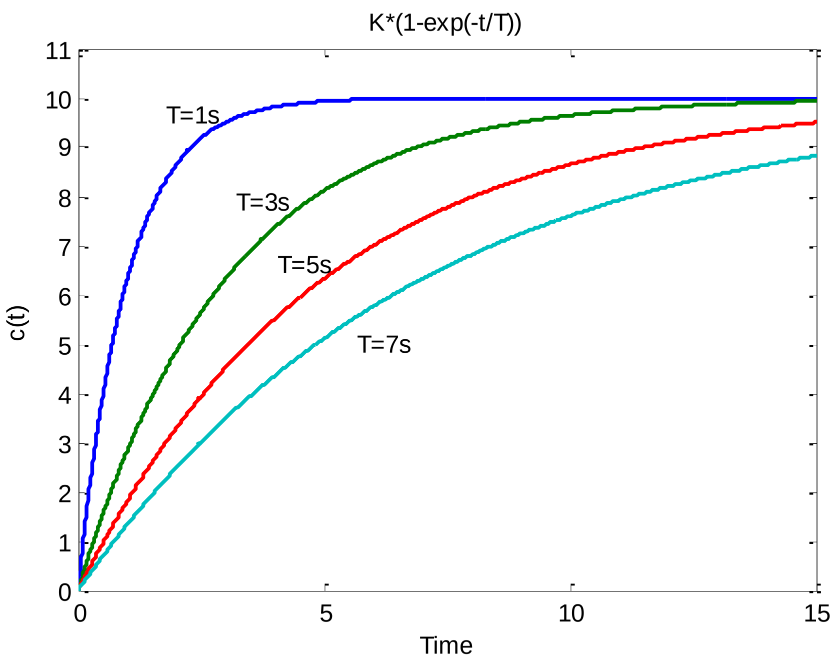
$$e_{ss} = \lim_{t\to\infty} e(t)$$

$$e_{ss} = \lim_{s\to 0} s \cdot \frac{1}{1+G(s)} \cdot \frac{A}{s}$$

$e_{ss} = 0$

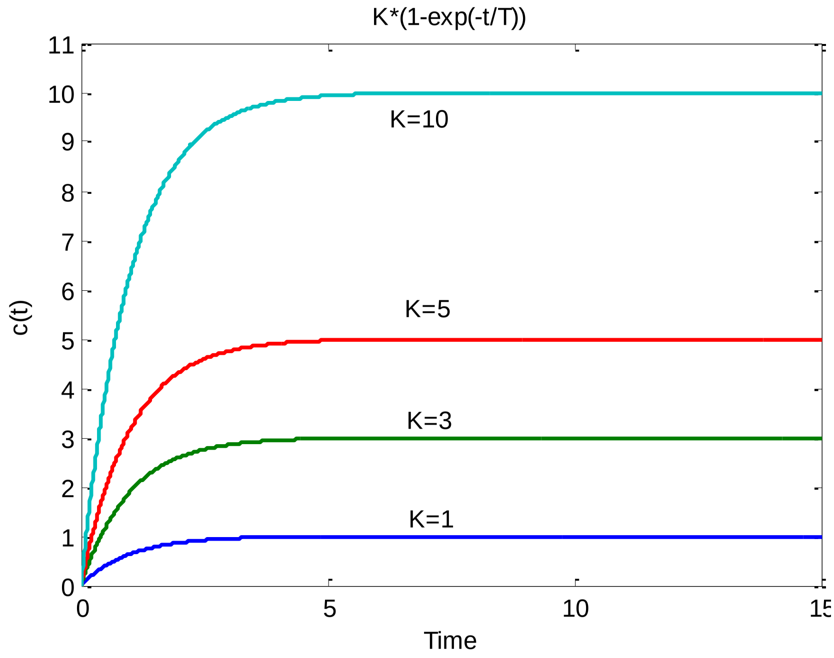
Step Response of 1st Order System

- If K=10 and T=1, 3, 5, 7 $\quad c(t) = K\left(1 - e^{-t/T}\right)$



Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $c(t) = K\left(1 - e^{-t/T}\right)$



Relation Between Step and impulse response

- The step response of the first order system is

$$c(t) = K\left(1 - e^{-t/T}\right) = K - Ke^{-t/T}$$

- Differentiating c(t) with respect to t yields

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

$$\frac{dc(t)}{dt} = \frac{K}{T}e^{-t/T}$$

Ramp Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{K}{s^2(Ts+1)}$$

- The ramp response is given as

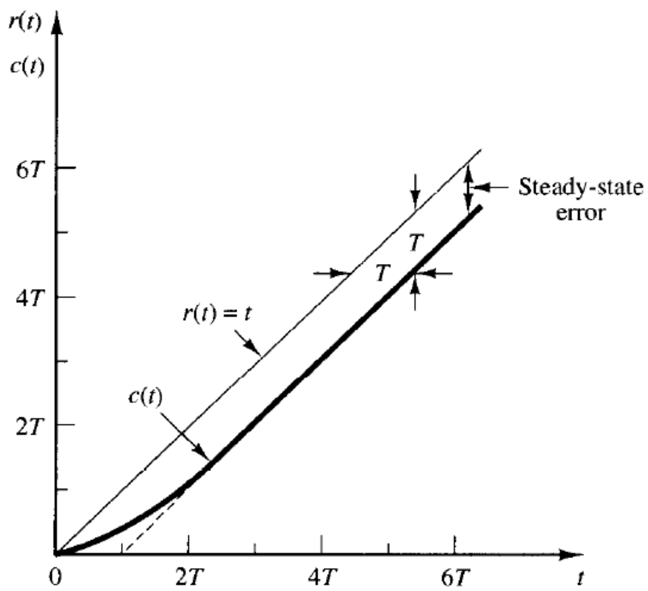
$$c(t) = K\left(t - T + Te^{-t/T}\right) \qquad \text{for } t \geq 0$$

The error signal $e(t)$ is then

$$e(t) = r(t) - c(t)$$
$$= T(1 - e^{-t/T})$$

As t approaches infinity, $e^{-t/T}$ approaches zero, and thus the error signal $e(t)$ approaches T or

The unit- ramp input and the system output are shown in figure. The error in following the unit- ramp input is equal to T for sufficiently large t. The smaller the time constant T, the smaller the ess .



Parabolic Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^3} \quad \text{Therefore,} \quad C(s) = \frac{K}{s^3(Ts+1)}$$

Second Order System

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.

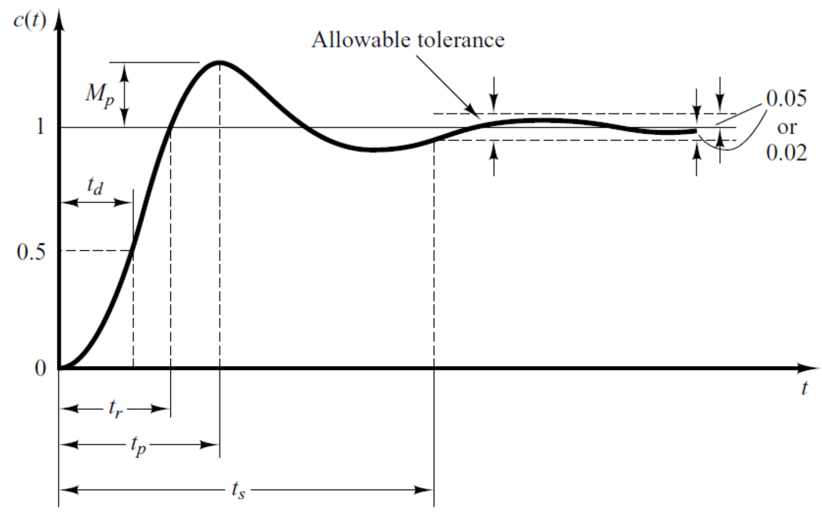
- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response

- Whereas, changes in the parameters of a second-order system can change the *form of* the response.

- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or pure oscillations for its *transient response*.

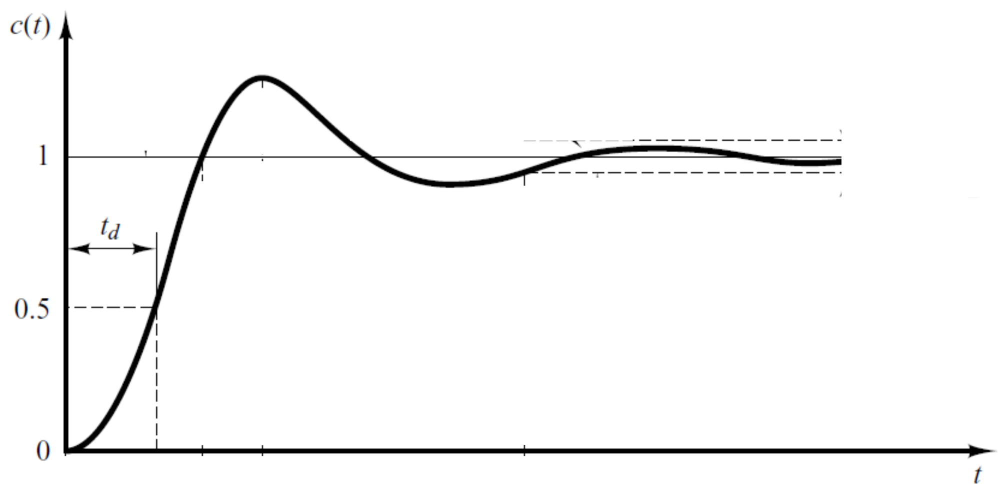
For the 2nd order system's response due to a unit step input is as follows.

Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



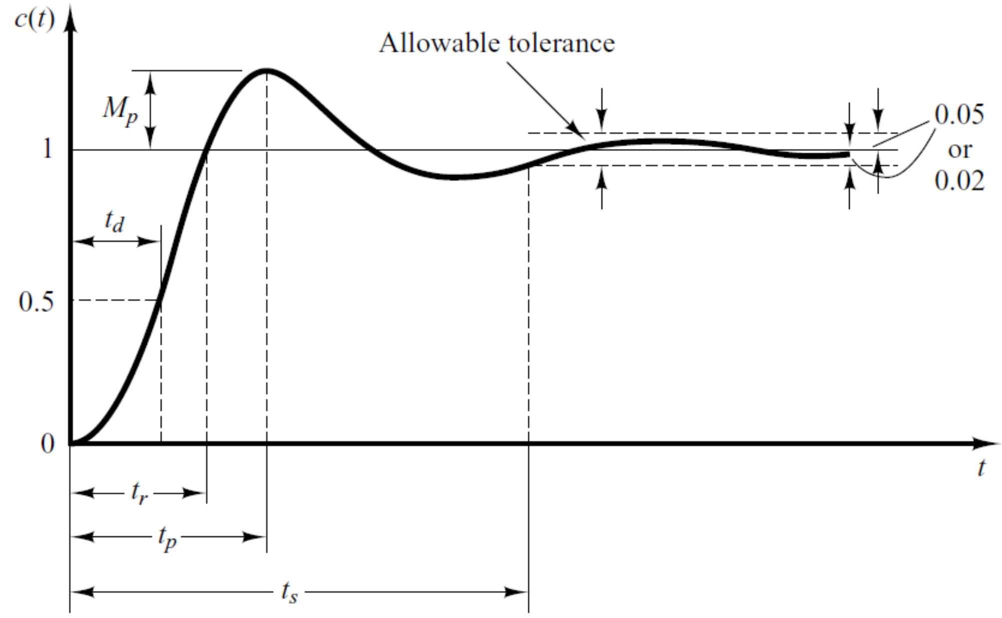
Delay Time

- The delay (t_d) time is the time required for the response to reach half the final value the very first time.



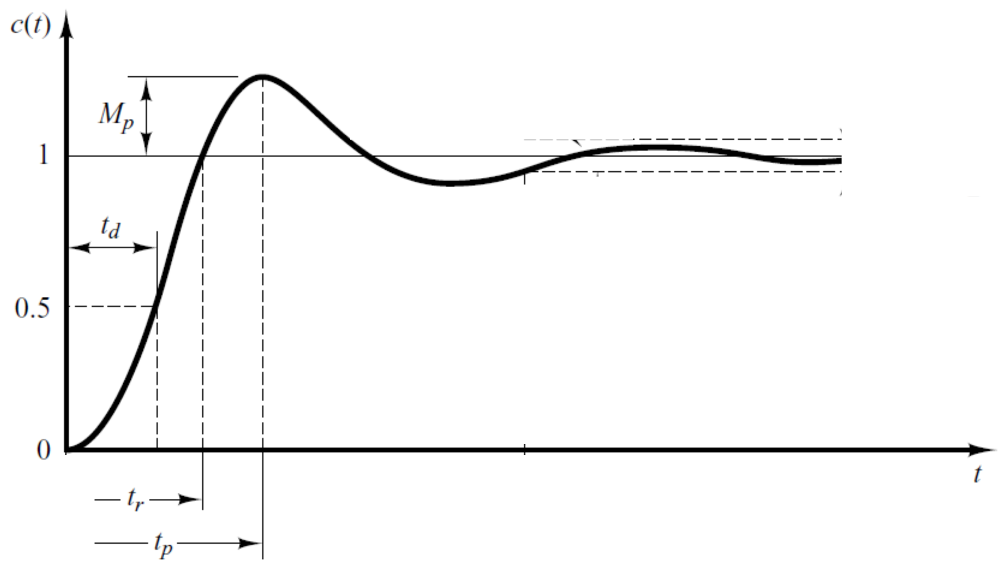
Rise Time

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



Peak Time

- The peak time is the time required for the response to reach the first peak of the overshoot.



Maximum Overshoot

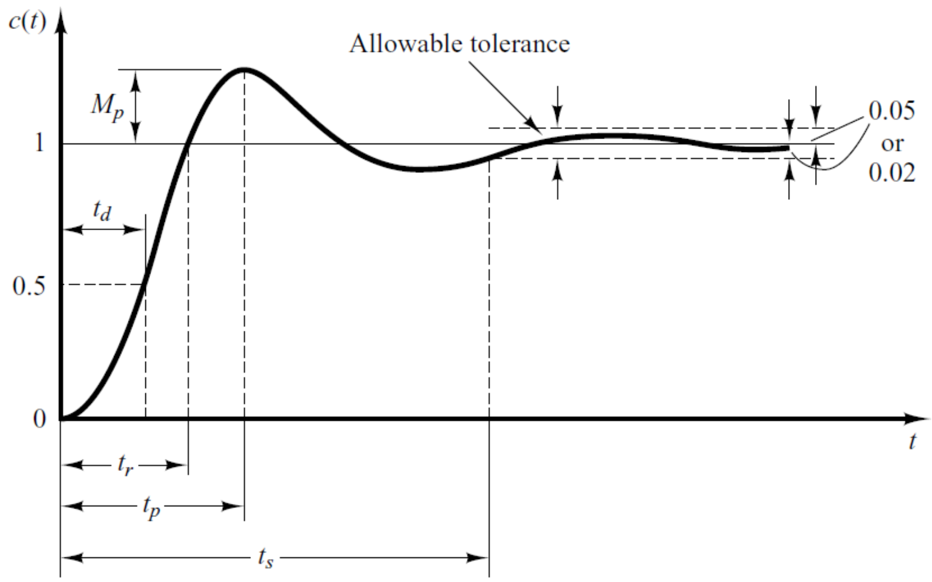
The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Settling Time

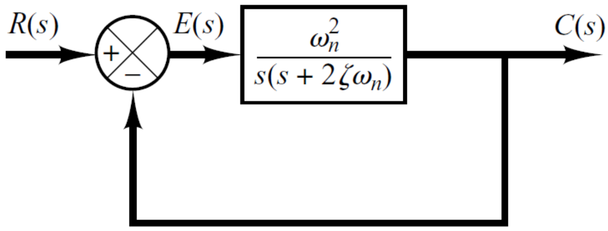
- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



Introduction

- A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$\omega_n \longrightarrow$ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

$\zeta \longrightarrow$ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

Example 2

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n^2 = 4 \qquad \Rightarrow \omega_n = 2$

$$\cancel{s^2} + 2\zeta\omega_n s + \cancel{\omega_n^2} = \cancel{s^2} + 2s + \cancel{4}$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

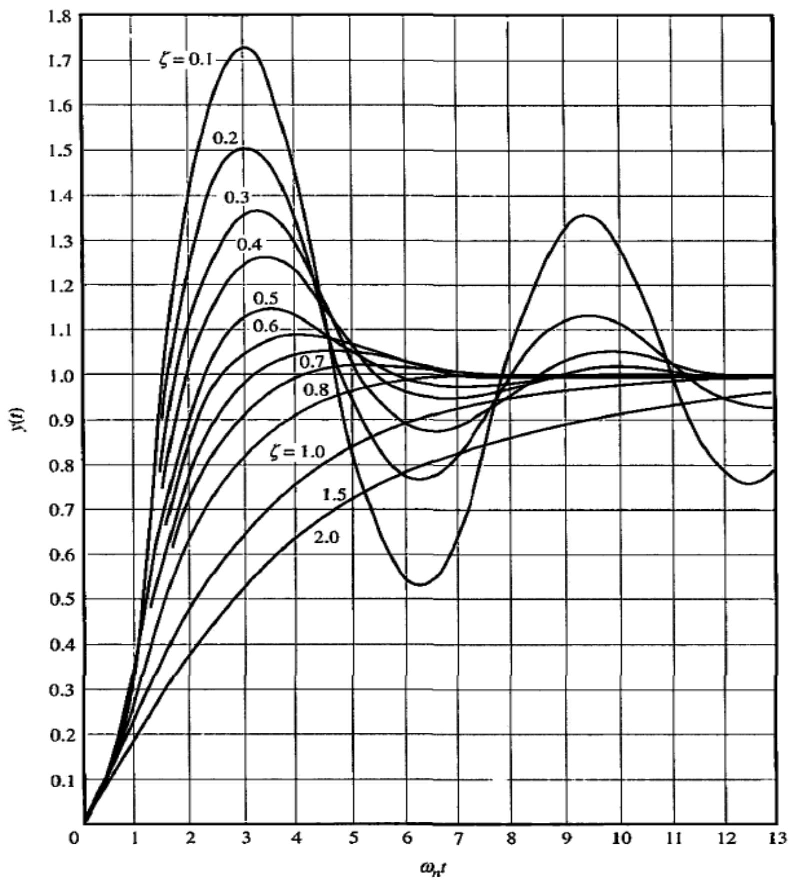
$$\Rightarrow \zeta = 0.5$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

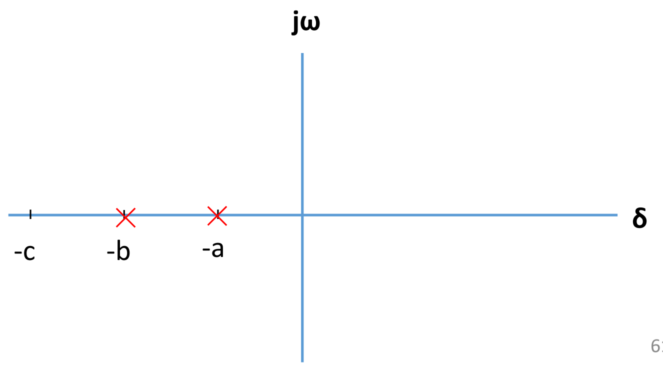


Introduction

$$-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}$$
$$-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}$$

- According the value of ζ, a second-order system can be set into one of the four categories (page 169 in the textbook):

1. Overdamped - when the system has two real distinct poles (ζ >1).

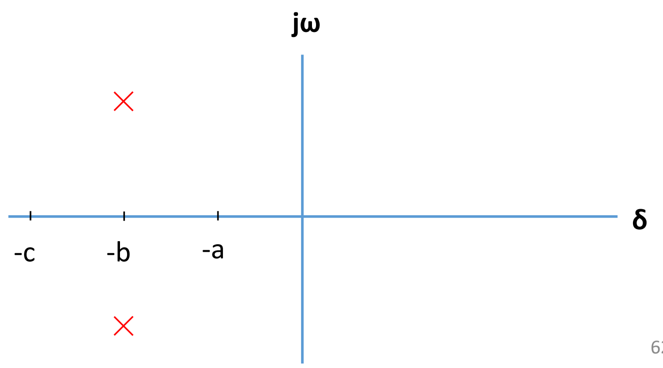


Introduction

$$-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}$$
$$-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}$$

- According the value of ζ, a second-order system can be set into one of the four categories (page 169 in the textbook):

2. *Underdamped* - when the system has two complex conjugate poles (0 <ζ <1)

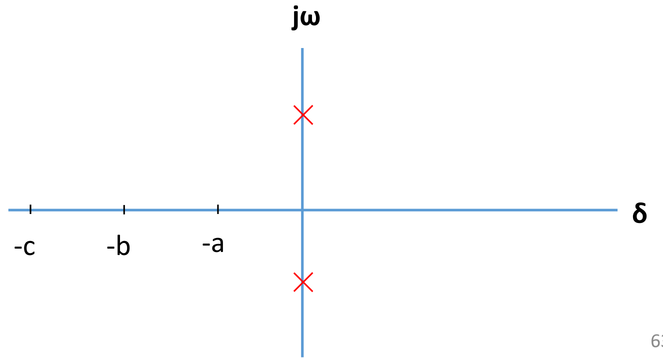


Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories (page 169 in the textbook):

3. *Undamped* - when the system has two imaginary poles (ζ = 0).

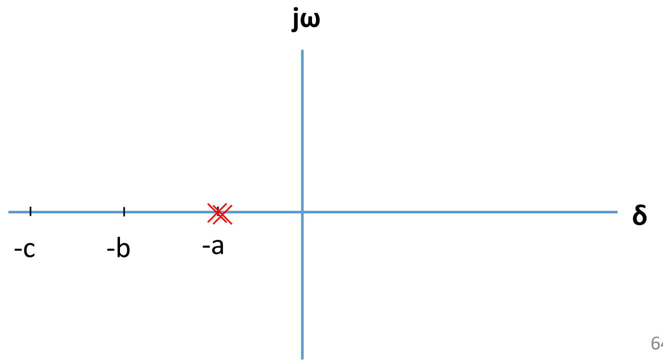


Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

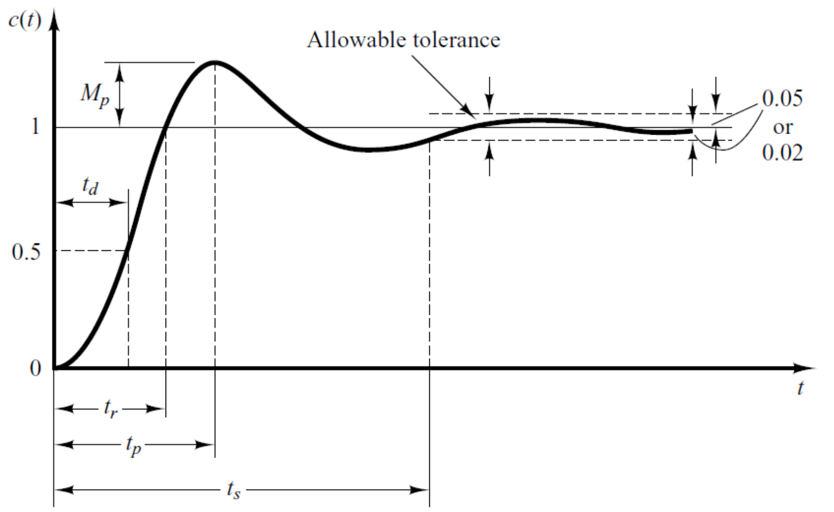
- According the value of ζ , a second-order system can be set into one of the four categories (page 169 in the textbook):

4. *Critically damped* - when the system has two real but equal poles (ζ = 1).



Underdamped System

For $0<\zeta<1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input is as follows.
Important timing characteristics: delay time, rise time, peak time, maximum overshoot, and settling time.



Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\omega_n^2\left(1-\zeta^2\right)$

$(s + 2\zeta\omega_n)^2$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2\left(1-\zeta^2\right)}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

- Above equation can be written as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

- Where $\omega_d = \omega_n\sqrt{1-\zeta^2}$, is the frequency of transient oscillations and is called damped natural frequency.

- The inverse Laplace transform of above equation can be obtained easily if C(s) is written in the following form:

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

Step Response of underdamped System

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

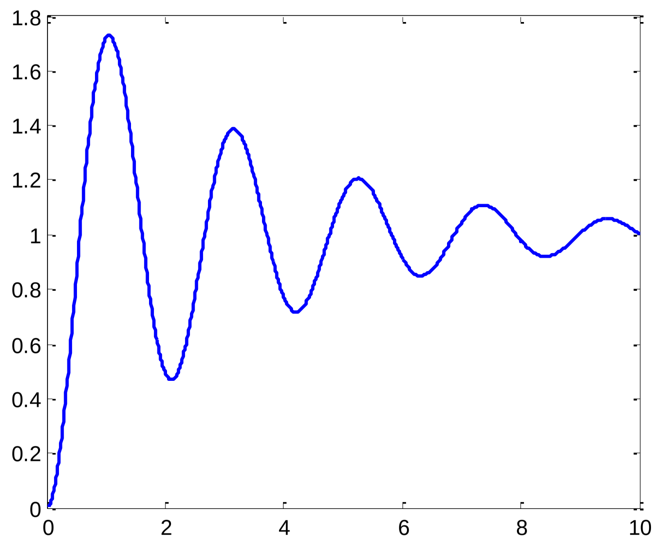
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

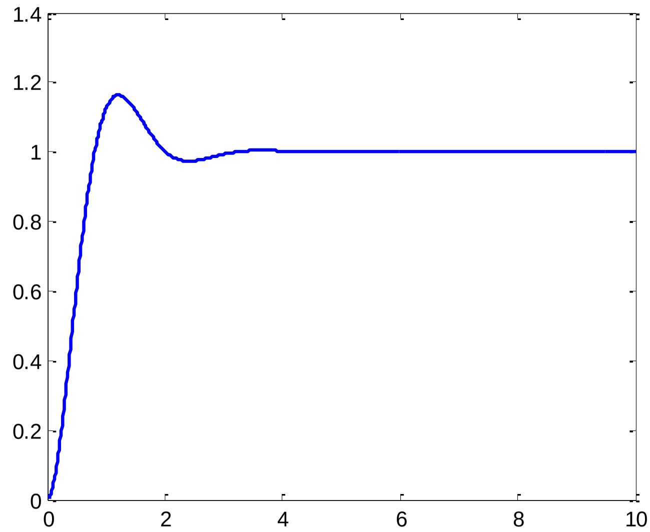
if $\zeta = 0.1$ and $\omega_n = 3$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

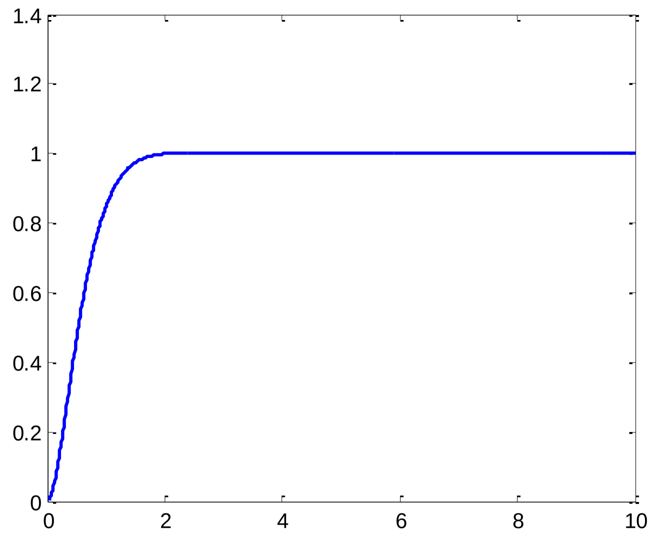
if $\zeta = 0.5$ and $\omega_n = 3$



Step Response of underdamped System

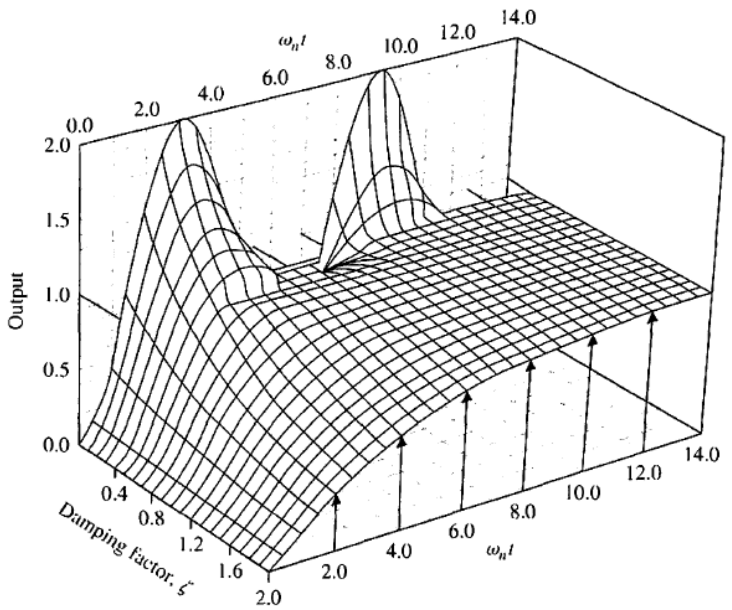
$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.9$ and $\omega_n = 3$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$



Damping ratio and damping factor:

The effects of the system parameters ζ and ω_n on the step response y(t) of the prototype second- order system can be studied by referring to the roots of the characteristic equation.

The two roots can be expressed as

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$
$$= -\alpha \pm j\omega$$

where

$$\alpha = \zeta\omega_n$$

and

$$\omega = \omega_n\sqrt{1-\zeta^2}$$

S1, s2 : complex frequencies. **ζ** : damping ratio **α** : damping factor or damping constant (neper frequency) . **1/ α = T:** time constant . **ω_n** :undamped natural **frequency (resonant frequency) . ω : damped or conditional frequency.**

Natural frequency:

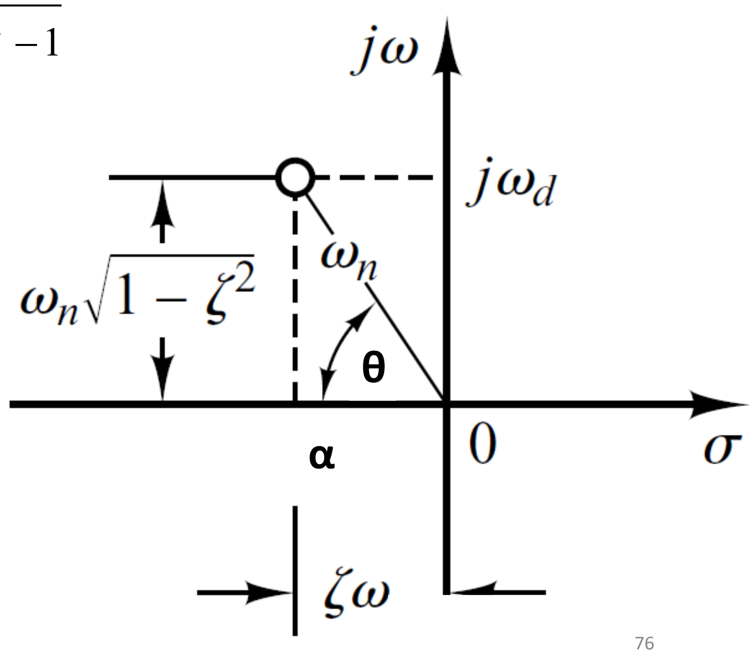
Figure below illustrates the relationships between the locations of the characteristic roots and α, ζ, ω_n, and ω. For the complex-conjugate roots shown,

- ω_n is the radial distance from the roots to the origin of the s-plane.
- α is the real part of the roots.
- ω is the imaginary part of the roots.
- ζ is the cosine of the angle between the radial line to the roots and the negative axis when the roots are in the left-half s-plane, or

S-Plane (Underdamped System)

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

Since $\omega^2\zeta^2 - \omega^2(\zeta^2 - 1) = \omega^2$, the distance from the pole to the origin is ω and $\zeta = cos\theta$



- The left-half s-plane corresponds to positive damping; that is, the damping factor or damping ratio is positive. Positive damping causes the unit-step response to settle to a constant final value in the steady state due to the negative exponent of $\exp(-\zeta\omega_n t)$. The system is stable.
- The right-half s-plane corresponds to negative damping. Negative damping gives a response that grows in magnitude without bound, and the system is unstable.
- The imaginary axis corresponds to zero damping ($\alpha = 0$ or $\zeta = 0$). Zero damping results in a sustained oscillation response, and the system is marginally stable or marginally unstable.

The following classification of the system with respect to the value of ζ is made.

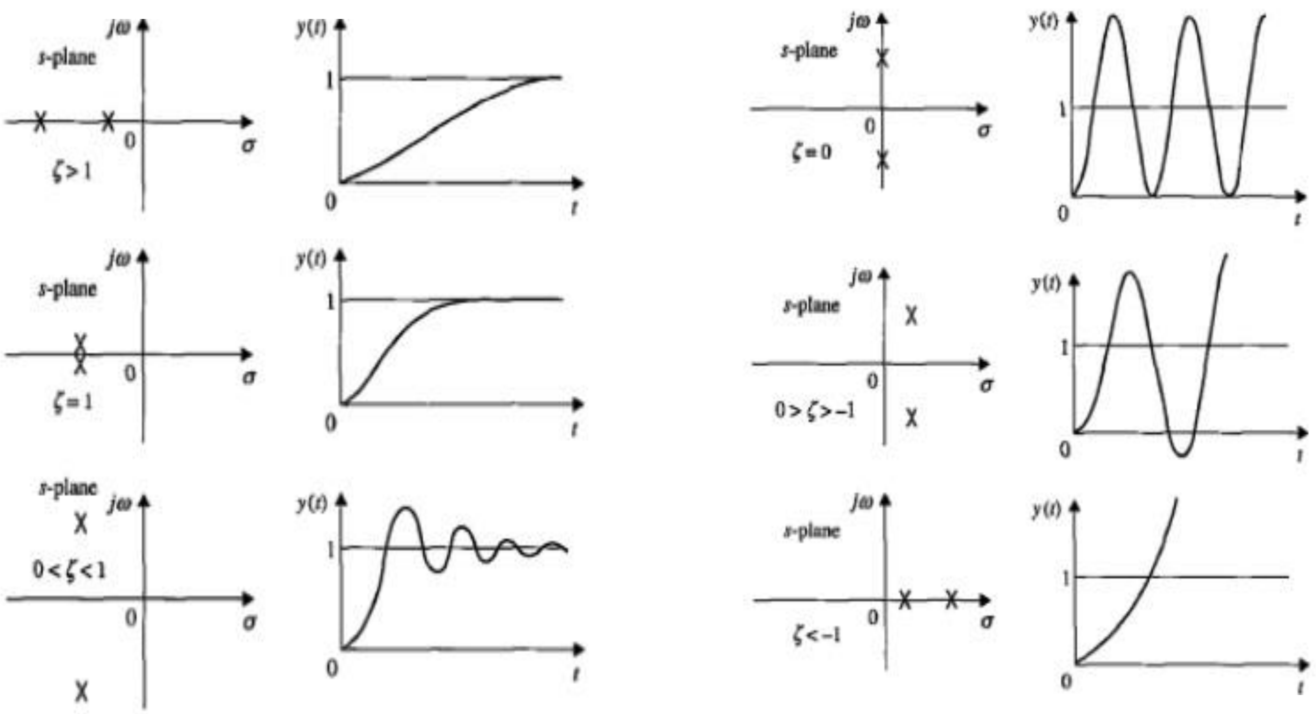
$0<\zeta<1$: $s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ $(-\zeta\omega_n < 0)$ *underdamped*

$\zeta = 1$: $s_1, s_2 = -\omega_n$ *critically damped*

$\zeta > 1$: $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$ *overdamped*

$\zeta = 0$: $s_1, s_2 = \pm j\omega_n$ *undamped*

$\zeta < 0$: $s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ $(-\zeta\omega_n > 0)$ *negatively damped*



Maximum overshoot (M_p) and peak time (t_p):

M_p is defined only for step function input. **M_p** depend on damping ratio (ζ). The relationship between ζ and Mp can be obtained by taking the derivative of output response **y(t)** with respect to **t** and setting the result to **zero**

$$\frac{dy(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t \quad t \geq 0$$

Setting $dy(t)/dt$ to zero, we have the solutions: $t = \infty$ and

$$\omega_n \sqrt{1-\zeta^2}\, t = n\pi \quad n = 0, 1, 2, \ldots$$

Peak time (t_p)= π/ω = $t_{\max} = \dfrac{\pi}{\omega_n \sqrt{1-\zeta^2}}$

M_p= $y(t_p) - 1$

M_p= $e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

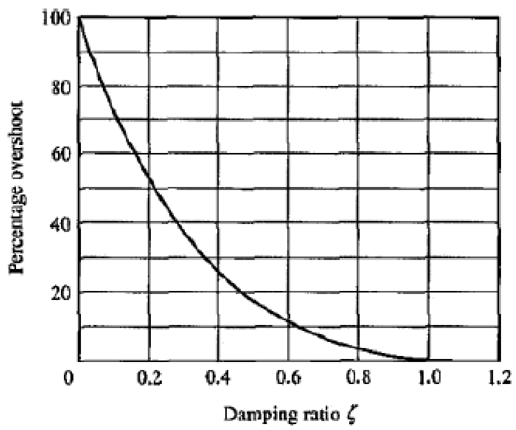


Figure Percent overshoot as a function of damping ratio for the step response of the prototype second-order system.

$$t_d \cong \frac{1.1 + 0.125\zeta + 0.469\zeta^2}{\omega_n} \quad 0 < \zeta < 1.0$$

Rise time (t_r) and settling time (t_s):

For rise time, setting y(t) = 1 , $t_r = \pi - \theta / \omega$

$$t_s = \frac{4}{\zeta\omega_n} \qquad \text{(2\% criterion)}$$

$$t_s = \frac{3}{\zeta\omega_n} \qquad \text{(5\% criterion)}$$

***It is desirable that the transient response be sufficiently fast and be sufficiently damped. Thus $0.4 \leq \zeta \leq 0.8$ or $2.5\% \leq M_p \leq 25\%$**

***For rapid response ω_n must be large.**

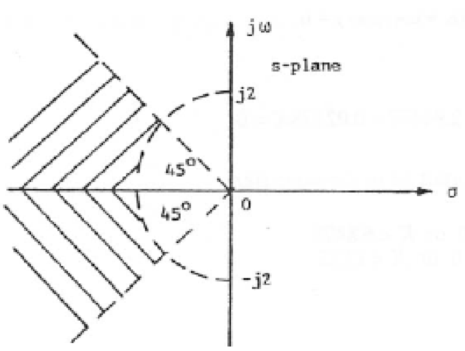
Example:

5-1. A pair of complex-conjugate poles in the s-plane is required to meet the various specifications that follow. For each specification, sketch the region in the s-plane in which the poles should be located.

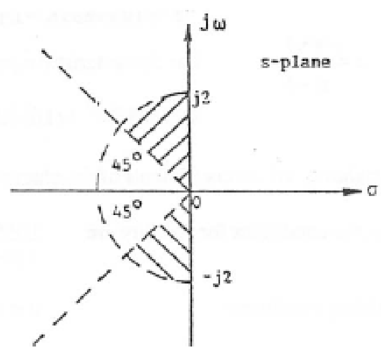
(a) $\zeta \geq 0.707$ $\omega_n \geq 2$ rad/sec (positive damping)
(b) $0 \leq \zeta \leq 0.707$ $\omega_n \leq 2$ rad/sec (positive damping)

Solution:

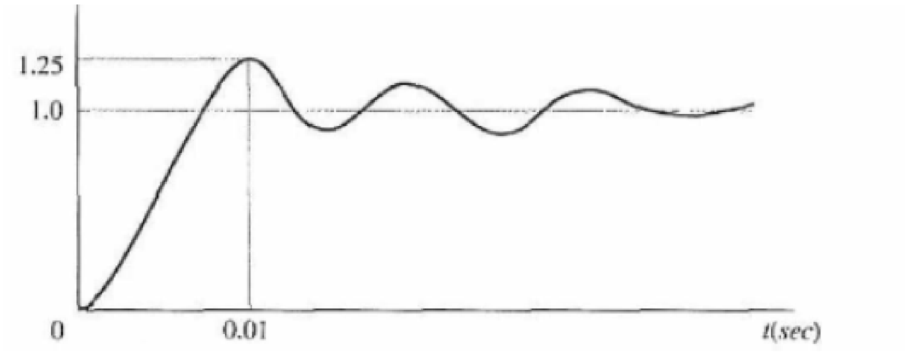
(a) $\zeta \geq 0.707$ $\omega_n \geq 2$ rad / sec



(b) $0 \leq \zeta \leq 0.707$ $\omega_n \leq 2$ rad / sec



Example: The unit step response of a linear control system is shown in figure below. Find the T.F of a second order prototype system to model the system.



Solution:

$$\text{Percent maximum overshoot} = 0.25 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Thus

$$\pi\zeta\sqrt{1-\zeta^2} = -\ln 0.25 = 1.386 \qquad \pi^2\zeta^2 = 1.922\left(1-\zeta^2\right)$$

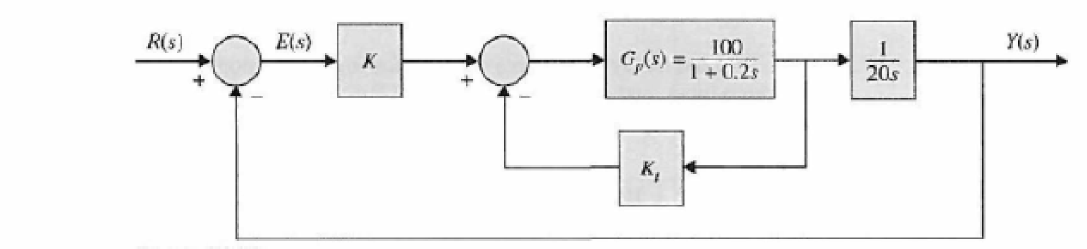
Solving for ζ from the last equation, we have $\zeta = 0.404$.

$$\text{Peak Time} \quad t_{\max} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.01 \quad \text{sec.} \quad \text{Thus,} \quad \omega_n = \frac{\pi}{0.01\sqrt{1-(0.404)^2}} = 343.4 \quad \text{rad / sec}$$

Transfer Function of the Second-order Prototype System:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{117916}{s^2 + 277.3s + 117916}$$

Example: for the control system shown below . Find the values of **k** and **kt** so that the $\zeta = 0.6$ and **ts = 0.1sec** of unit step response .



Solution:

Closed –loop T.F= $Y(S) / R(S) = 25K/ S^2 + (5+500K_t) S +25K$

Compare with prototype second- order system

$25K = \omega_n^2$, $2 \zeta \omega_n = 5 +500k_t = 1.2 \omega_n$, for 5% $t_s = 3/ \zeta \omega_n = 0.1$

$\omega_n = 50$ rad/sec $k_t = 0.11$, $k = 100$

System Response With Zeros

Consider closed loop T.F $= s+ a / s^2+2s+ 9$, a =3 , 5 , 10

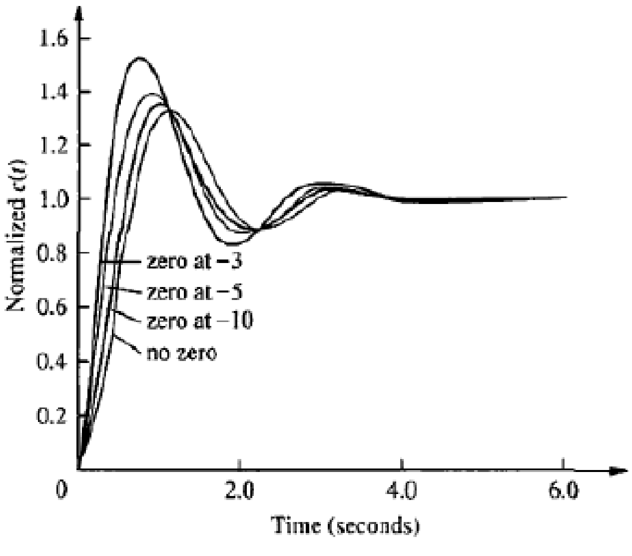


FIGURE 4.25 Effect of adding a zero to a two-pole system

DOMINANT POLES AND ZEROS OF TRANSFER FUNCTIONS

Because most control systems in practice are of orders higher than two, it would be useful to establish guidelines on the approximation of high-order systems by lower-order ones insofar as the transient response is concerned. In design, we can use the dominant poles to control the dynamic performance of the system, whereas the insignificant poles are used for the purpose of ensuring that the controller transfer function can be realized by physical components.

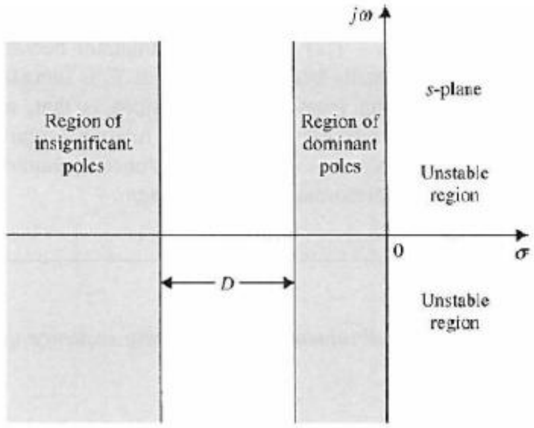


Figure 5-40 Regions of dominant and insignificant poles in the s-plane.

the real part of a pole is at least 5 to 10 times that of a dominant pole or a pair of complex

Example:

$$M(s) = \frac{Y(s)}{R(s)} = \frac{20}{(s+10)(s^2+2s+2)}$$

The pole at $s = -10$ is 10 times the real part of the complex conjugate poles, which are at $-1 \pm j1$. We can refer to the relative damping ratio of the system as 0.707.

$$M(s) \cong \frac{20}{10(s^2+2s+2)}$$

This way, the steady-state performance of the third-order system will not be affected by the approximation.

PROBLEM: Determine the validity of a second-order approximation for each of these two transfer functions:

a. $G(s) = \dfrac{700}{(s+15)(s^2+4s+100)}$

b. $G(s) = \dfrac{360}{(s+4)(s^2+2s+90)}$

Solution:

a : valid , b: not valid

Evaluating Pole-Zero Cancellation Using Residues

When a pole and zero of a system transfer function nearly cancel each other, the portion of the system response associated with the pole will have a small magnitude.

PROBLEM: For each of the response functions in Eqs. (4.86) and (4.87), determine whether there is cancellation between the zero and the pole closest to the zero. For any function for which pole-zero cancellation is valid, find the approximate response.

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

SOLUTION: The partial-fraction expansion of Eq. (4.86) is

$$C_1(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5}$$

The residue of the pole at -3.5, which is closest to the zero at -4, is equal to 1 and is not negligible compared to the other residues. Thus, a second-order step response approximation cannot be made for $C_1(s)$. The partial-fraction expansion for $C_2(s)$ is

$$C_2(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01}$$

The residue of the pole at -4.01, which is closest to the zero at -4, is equal to 0.033, about two orders of magnitude below any of the other residues. Hence, we make a second-order approximation by neglecting the response generated by the pole at -4.01:

$$C_2(s) \approx \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6}$$

and the response $c_2(t)$ is approximately

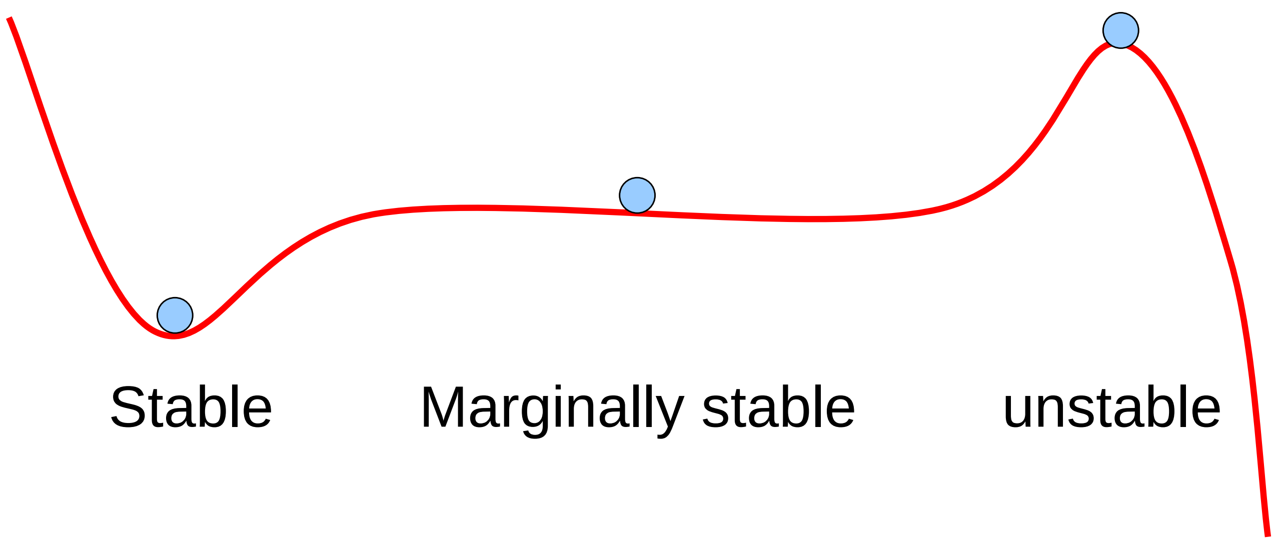
$$c_2(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$$

Stability

Learning Objective

- To be able to
  - Define BIBO stability
  - Explain the connection between stability and pole locations
  - Recognize the importance of stability
  - Classify a system as being stable, unstable or marginally stable

Stability Concept



Introduction

- Stability is the most important issue for control systems
- If a system is unstable then it is not useful (in general)
- If a system is unstable, our first concern in controller design is to stabilize it.

Introduction

- **Absolute stability**: Refers to the condition whether the system stable or unstable
- **Relative stability**: Degree of stability is a measure of relative stability
- **Zero – state response** : is due to the input only ; all the initial condition zeros
- **Zero – input response**: is due to the initial condition only ; all the input zeros

Introduction

- **Zero- input stability**: refers to the stability condition when the input is zero , and the system is driven only by its initial condition. Zero – input stability also depends on the roots of the characteristic equation
- **Total response = zero – state response + zero – input response**

Introduction

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0} = \frac{p(s)}{q(s)}$$

For physical systems $m \le n$

$q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0$ *(characteristic polynomial)*

and, $q(s) = 0$ *(characteristic equation)*

The roots of the characteristic equation are the system poles.

BIBO stability

A system is **BIBO** stable if

The response to any bounded input is bounded

(**B**ounded **I**nput **B**ounded **O**utput stability)

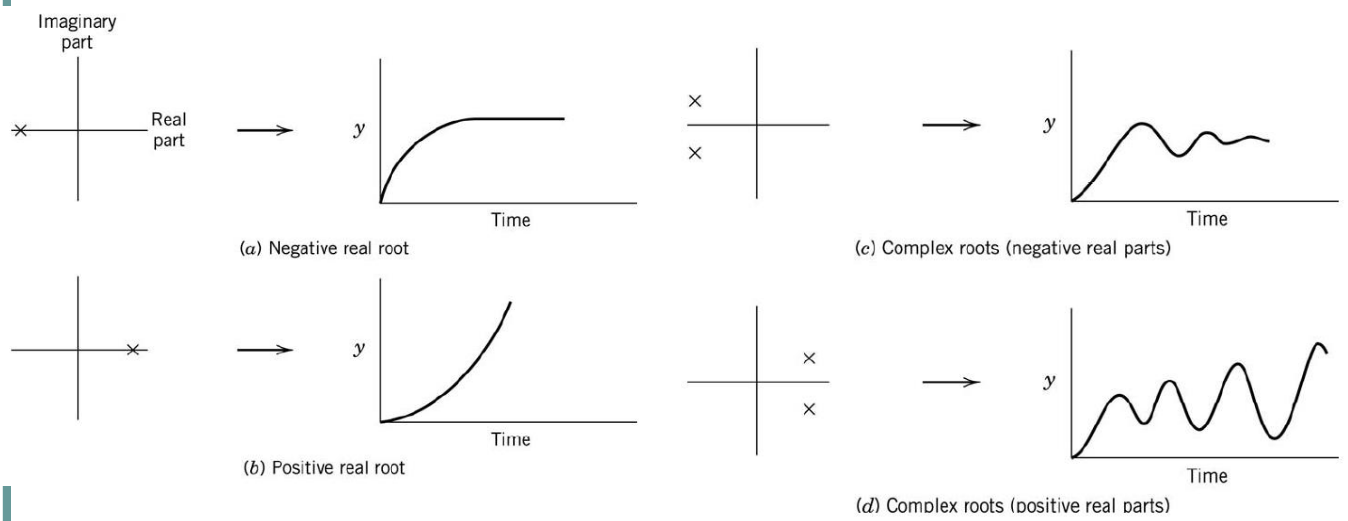
BIBO stability of Linear Systems

For linear systems, **BIBO** stability depends on the location of the **poles** of the system's transfer function

Linear Systems

If ALL poles of the transfer function of a system have negative real parts then the system is BIBO stable.

Poles Location and the response



Systems in (a) and (c) are stable, however systems in (b) and (d) are unstable.

A stable System

r (t) → $G(s) = \dfrac{1}{s^2 + 4s + 3}$ → y (t)

Ch. Eqn is: (s+1)(s+3)=0, So s = -1 and s = -3

This is a **BIBO** stable system

If the input r(t) is bounded then y(t) is bounded

This is true for all bounded inputs r(t)

Unit step → $G(s)$ → $y(t) = \dfrac{1}{3} - \dfrac{1}{2}e^{-t} + \dfrac{1}{6}e^{-3t}$

Unstable System

r (t) → $G(s)=\dfrac{1}{s^2-3s-4}$ → y (t)

Ch. Eqn is: (s+1)(s-4)=0, So s = -1 and s = 4

This is an unstable system

There are some bounded functions that results in unbounded outputs

Unit step → $G(s)$ → $y(t)=\dfrac{-1}{4}+\dfrac{1}{5}e^{-t}+\dfrac{1}{20}e^{4t}$

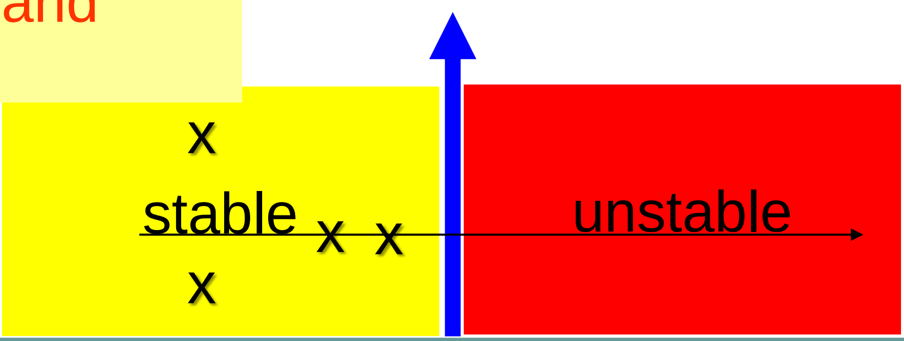
Stable Systems

G(s) is BIBO stable if **<u>ALL</u>** poles have negative real part, i.e **<u>ALL</u>** poles are in **<u>LEFT</u>** hand side.

$$G_1(s)=\frac{s+1}{(s+2)(s+3)} \qquad poles\ -2,-3$$

$$G_2(s)=\frac{s-1}{(s+2)(s+3)} \qquad poles\ -2,-3$$

$$G_3(s)=\frac{2}{(s+2+3j)(s+2-3j)} \qquad poles\ -2\pm 3j$$



Marginal Stability

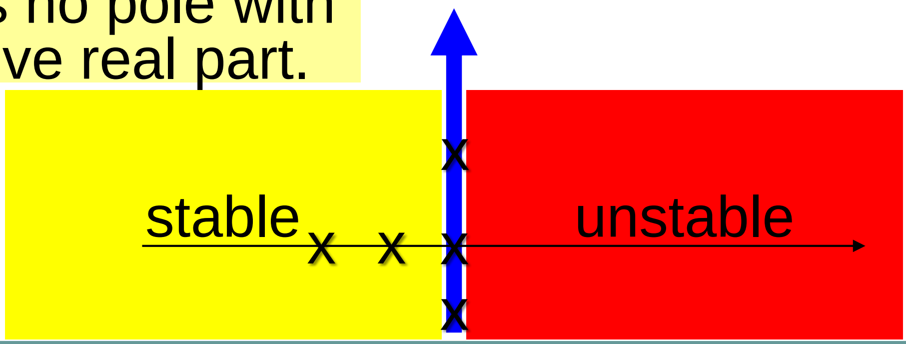
G(s) is marginally stable if

- It has distinct poles with zero real part i.e the poles are on the imaginary axis
- It has no pole with positive real part.

$$G_1(s) = \frac{s+1}{s(s+2)} \qquad poles\ 0,-2$$

$$G_2(s) = \frac{s+5}{(s+1)(s+2j)(s-2j)} \qquad poles\ -1, 0 \pm 2j$$

$$G_3(s) = \frac{2}{s(s+2+3j)(s+2-3j)} \qquad poles\ 0, -2 \pm 3j$$



Unstable Systems

G(s) is unstable if one of the following is true:

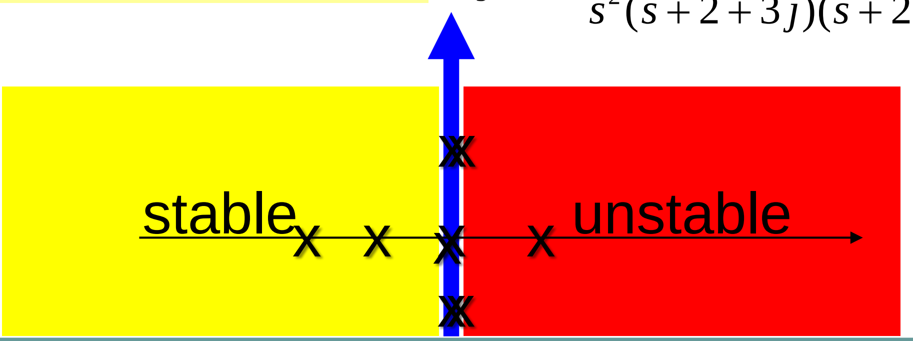
- It has at **least one pole with positive real part.**
- It has **repeated poles with zero real part**.

$$G_1(s) = \frac{s+1}{(s-3)(s-2)}$$

$$G_1(s) = \frac{s+1}{(s+4)(s-2)}$$

$$G_2(s) = \frac{s+5}{(s+1)(s+2j)^2(s-2j)^2}$$

$$G_3(s) = \frac{2}{s^2(s+2+3j)(s+2-3j)}$$



Are these system stable?

$$G_1(s)=\frac{1}{s^2+5s+6}$$

$$G_2(s)=\frac{1}{(s-2)(s+2)}$$

$$G_3(s)=\frac{(s-4)}{(s+1)(s+3)}$$

$$G_4(s)=\frac{1}{s(s^2+5s+4)}$$

$$G_5(s)=\frac{1}{(s-3+2j)(s-3-2j)(s+2)}$$

$$G_6(s)=\frac{(s+2)}{(s+2+j)(s+2-j)(s+0.12)}$$

- Stable
- Unstable
- Marginally stable

Are these system stable?

$$G_1(s)=\frac{1}{s^2+5s+6}$$ $poles: -2,-3$ Stable

$$G_2(s)=\frac{1}{(s-2)(s+2)}$$

$$G_3(s)=\frac{(s-4)}{(s+1)(s+3)}$$ $poles: -1,-3$ Stable

$$G_4(s)=\frac{1}{s(s^2+5s+4)}$$

$$G_5(s)=\frac{1}{(s-3+2j)(s-3-2j)(s+2)}$$

$$G_6(s)=\frac{(s+2)}{(s+2+j)(s+2-j)(s+0.12)}$$ $, poles: -2\pm j,-0.12$ Stable

Are these system stable?

$$G_1(s) = \frac{1}{s^2 + 5s + 6}$$

$$G_2(s) = \frac{1}{(s-2)(s+2)}, \quad poles: 2, -2$$ Unstable

$$G_3(s) = \frac{(s-4)}{(s+1)(s+3)}$$

$$G_4(s) = \frac{1}{s(s^2 + 5s + 4)}, \quad poles: 0, -1, -4$$ Marginally Stable

$$G_5(s) = \frac{1}{(s-3+2j)(s-3-2j)(s+2)}, \quad poles: 3 \pm 2j, -2$$ Unstable

$$G_6(s) = \frac{(s+2)}{(s+2+j)(s+2-j)(s+0.12)}$$

Example:

The impulse responses of several linear continuous systems are given below. For each case determine if the impulse response represents a stable or an unstable system.

(a) $h(t) = e^{-t}$, (b) $h(t) = te^{-t}$, (c) $h(t) = 1$, (d) $h(t) = e^{-t}\sin 3t$, (e) $h(t) = \sin \omega t$.

If the impulse response decays to zero as time approaches infinity, the system is stable. As can be seen in Fig. 5-1, the impulse responses (a), (b), and (d) decay to zero as time approaches infinity and therefore

Example:

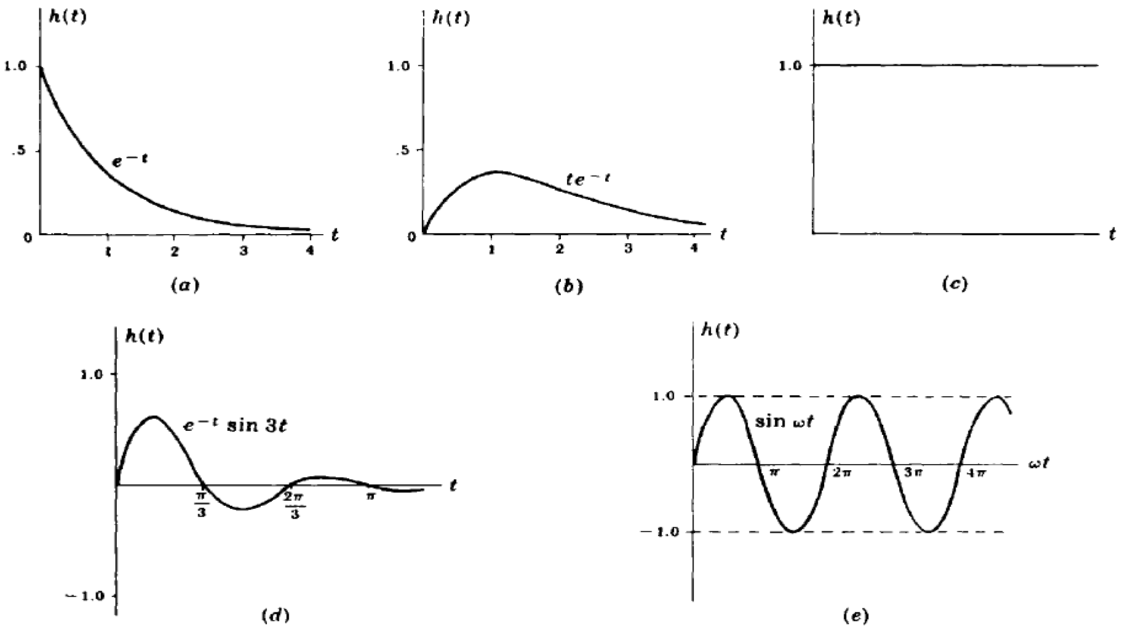


Fig. 5-1

represent *stable* systems. Since the impulse responses (c) and (e) do not approach zero, they represent *unstable* systems.

Example:

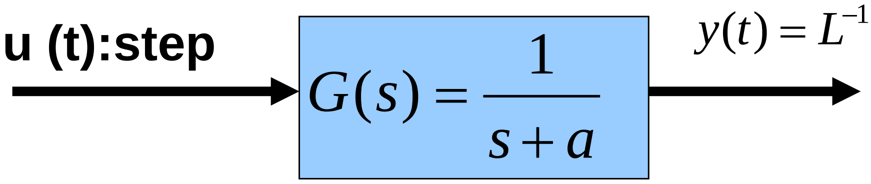
A system has poles at -1 and -5 and zeros at 1 and -2. Is the system stable?

The system is stable since the poles are the roots of the system characteristic equation (Chapter 3) which have negative real parts. The fact that the system has a zero with a positive real part does not affect its stability.

Why do we study stability?

- In general, systems that are unstable are not useful
- Many physical systems are unstable without feedback
- We need to design controllers to stabilize them.

Open loop System



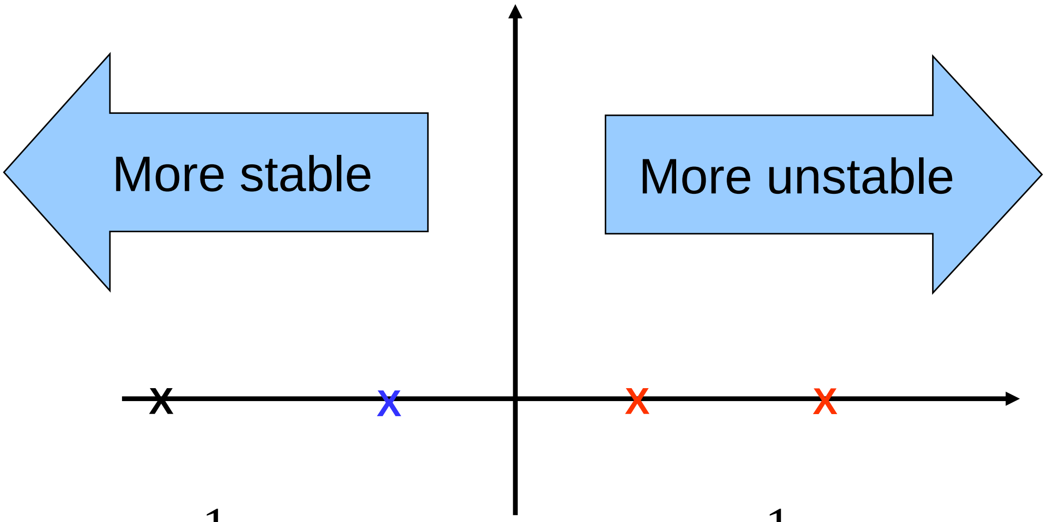
$$y(t) = L^{-1}\left\{\frac{1}{s(s+a)}\right\} = \frac{1}{a}(1-e^{-at})$$

$$Case1: \quad a > 0 \quad |y(t)| \le \infty \quad \textit{stable system}$$

$$Case2: \quad a = 0 \quad |y(\infty)| \to \infty \quad \text{marginally stable system}$$

$$Case3: \quad a < 0 \quad |y(\infty)| \to \infty \quad \textit{unstable system}$$

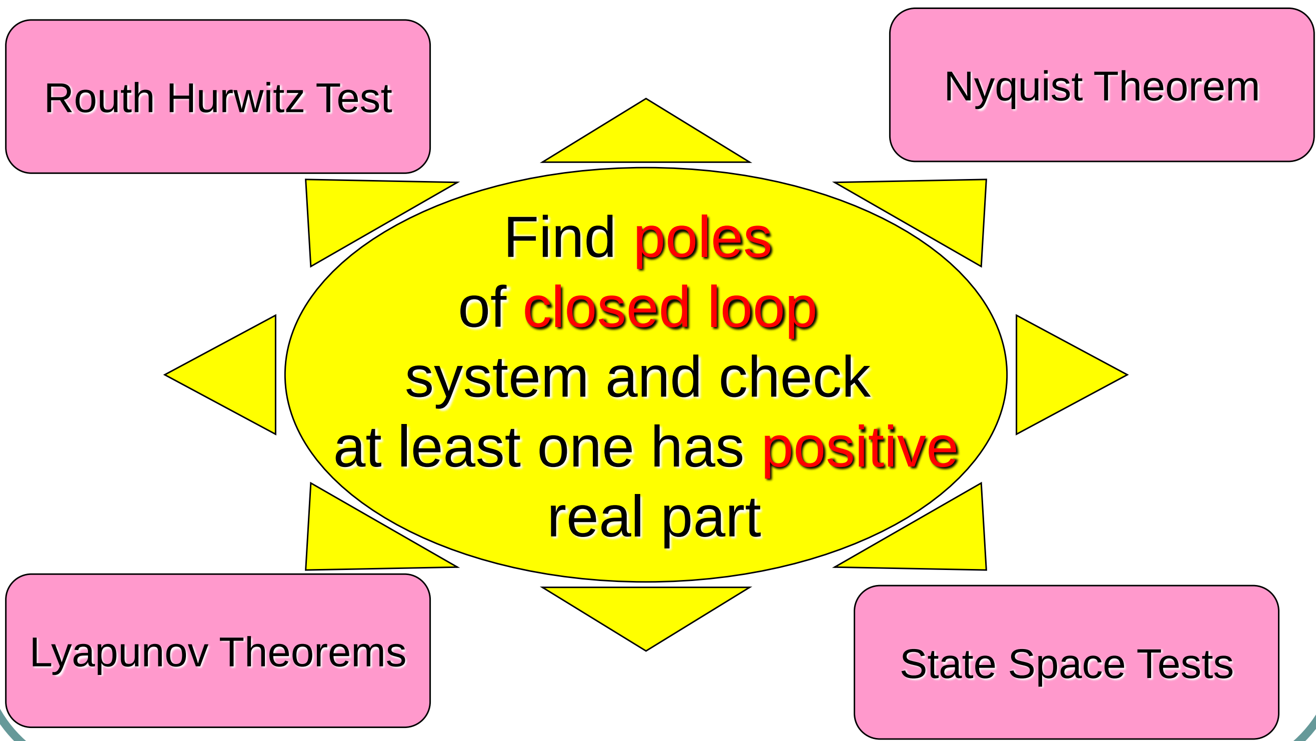
Relative stability_ single pole case



$$\frac{1}{s+3} \text{ is more stable than } \frac{1}{s+1}$$

$$\frac{1}{s-3} \text{ is more unstable than } \frac{1}{s-1}$$

How Do We Check Stability?



Stability Test

- Poles of a system are the roots of the characteristic equation.
- If any coefficient of the characteristic polynomial is –ve then there is at least one root with (positive) real part, and then the system is **unstable**.
- If all the pole are in the LHS of the S-plane, then all the coefficients must be > 0.

Routh Hurwitz Stability Test

- For higher order systems, it is too difficult to solve the characteristic equation.
- Some tricks allows us to determine stability without determining the values of the poles.
- Determining stability is reduced to determining the signs of the roots of the characteristic polynomial.
- Routh-Hurwitz stability test is used to count the RHS poles, without solving the Ch. Eqn.

Routh Array (Routh-Hurwitz table)

Problem : Determine whether all roots of

$a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0$ have positive real part (or zero)

Solution :

$$\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\cdots & \cdots & & & \\
s^0 & & & &
\end{array}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-3} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$c_{n-1} = \frac{b_{n-1}a_{n-3} - a_{n-1}b_{n-3}}{b_{n-1}}$$

Stability test

$$\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\cdots & \cdots & & & \\
s^0 & & & &
\end{array}$$

The number of positive roots = The number of <u>sign changes</u> in the first column

Example of Stability test

How many roots of $s^4 + 4s^3 + 6s^2 + 3s + 3 = 0$ *have positive real part?*

$+s^4$	1	6	3
$+s^3$	4	3	0
$+s^2$	$\frac{21}{4}$	3	
$+s^1$	$\frac{5}{7}$	0	
$+s^0$	3		

First two rows are generated from the polynomial coefficients

$$\frac{(4\times6)-(1\times3)}{4} = \frac{21}{4}, \quad \frac{(4\times3)-(1\times0)}{4} = 3$$

$$\frac{(\frac{21}{4}\times3)-(4\times3)}{\frac{21}{4}} = \frac{5}{7}$$

No sign changes, so the system is STABLE

Example of Stability test

How many roots of $s^5 + 8s^4 + 2s^3 + 4s^2 + 2s + 4 = 0$ *have positive real part?*

$+s^5$	1	2	2	
$+s^4$	8	4	4	$(Divide\ by\ +4)$
$+s^3$	$\frac{3}{2}$	$\frac{3}{2}$		$(Divide\ by\ +\frac{3}{2})$
$-s^2$	-1	1		
$+s^1$	2			
$+s^0$	1			

First two rows are generated from the polynomial coefficients

2 sign changes → 2 roots with positive real part. ∴ UNSTABLE system

Example of Stability test

How many roots of $a_2 s^2 + a_1 s + a_0 = 0$ *have positive real part*?

$$\begin{array}{c|cc} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & a_0 & \end{array}$$

The second order polynomial has no roots with positive real part if and only if all coefficients have same sign.

Special cases (a row begins with zero element)

- <u>A row begins with zero element and some of the elements in that row are not zero</u>.
  - A zero in the first column indicates that there is a pair of purely imaginary roots.
  - Replace zero by ε (a small positive number)
  - Continue computation of the table by taking the limit as ε = 0.

Special cases (a row begins with zero element)

How many roots of $s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$ *have positive real part*?

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^2			
s^1			
s^0			

First element in a row is zero but some other elements in the row are nonzero.

Special case

Replace 0 in the first column by a positive number ε and take limit as ε approaches zero.

s^5	1	2	11
s^4	2	4	10
s^3	ε	6	0
s^2	c	10	
s^1	d		
s^0	10		

$$c = \frac{4\varepsilon - 12}{\varepsilon} = negative$$

$$d = \frac{6c - 10\varepsilon}{c} = positive$$

Two sign changes, So two poles are in the right half plane, hence the system is UNSTABLE.

Special case (All-zero row)

How many roots of $s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63 = 0$ *have positive real part*?

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	0
s^2	21	63	0
s^1	0	0	0
s^0			

Row of zeros

Special cases

- <u>All-zero row</u>
  - Obtain the auxiliary equation and use its derivative to replace the all-zero row.
  - Continue computation of the table

Special case_ zero row

How many roots of $s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63 = 0$ *have positive real part?*

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	0
s^2	21	63	0
s^1	0	0	0
s^0			

Auxiliary equation

$$21s^2 + 63 = 0$$

or

$$s^2 + 3 = 0$$

All-zero row

Auxiliary Equations

How many roots of $s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63 = 0$ *have positive real part?*

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	0
s^2	21	63	0
s^1	0	0	0
s^0			

Auxiliary polynomial

$$21s^2 + 63 = 0$$

The order of the auxiliary polynomial is equal to the number of roots symmetrically located about the origin

Auxiliary Polynomial

How many roots of $s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63 = 0$ *have positive real part?*

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	0
s^2	21	63	0
s^1	0	0	0
s^1	42	0	0
s^0	63	0	0

Auxiliary polynomial

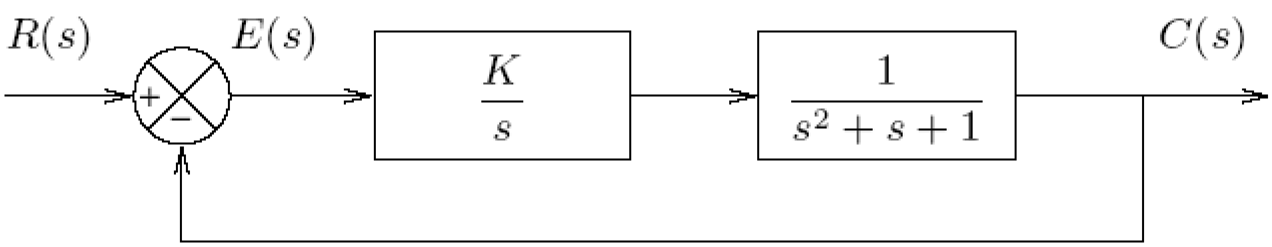
$21s^2 + 63 = 0$

Replace the zero row by the derivative of the auxiliary polynomial

$42\,s$

Designing for Stability

- Consider the following feedback control system:



- What are the values of K that produce stable systems?

Designing for Stability

- **Solution**
  - Compute the transfer function

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s^2+s+1)}}{1+\frac{K}{s(s^2+s+1)}} = \frac{K}{s^3+s^2+s+K}$$

Designing for Stability

- Apply Routh-Hurwitz test to

$$s^3+s^2+s+K$$

for all positive coefcients: K >0

- Routh Array:

$$\begin{array}{lcc} s^3: & 1 & 1 \\ s^2: & 1 & K \\ s: & 1-K & \\ 1: & K & \end{array}$$

Designing for Stability

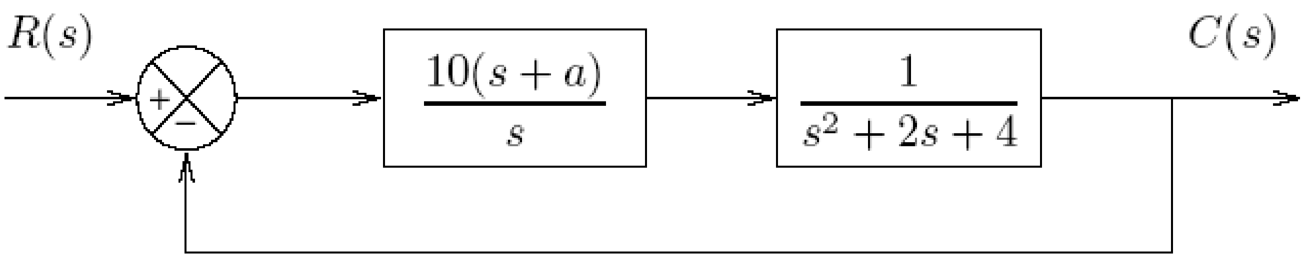
- Routh Array:

$$\begin{array}{rcc} s^3: & 1 & 1 \\ s^2: & 1 & K \\ s: & 1-K & \\ 1: & K & \end{array}$$

- To have all positive elements in column 1, we must have 0 < K < 1.

Stability and Zero Position

- For what values of a is this system stable?



Stability and Zero Position

- **Solution**
  - Compute the transfer function

$$\frac{C(s)}{R(s)}=\frac{10(s+a)/(s^3+2s^2+4s)}{1+10(s+a)/(s^3+2s^2+4s)}$$

$$=\frac{10(s+a)}{(s^3+2s^2+14s+10a)}$$

Stability and Zero Position

- Apply Routh-Hurwitz test to

$$s^3+2s^2+14s+10a$$

All coefficients must be positive, so a > 0.

- Routh Array:

$$\begin{array}{rcc} s^3: & 1 & 14 \\ s^2: & 2 & 10a \\ s: & 14-5a & \\ 1: & 10a & \end{array}$$

Stability and Zero Position

- Routh Array:

$$\begin{array}{lcc} s^3: & 1 & 14 \\ s^2: & 2 & 10a \\ s: & 14-5a & \\ 1: & 10a & \end{array}$$

- Thus, if 0 < a < $\frac{14}{5}$, then all the entries in the first column will be positive, indicating that the system is stable.