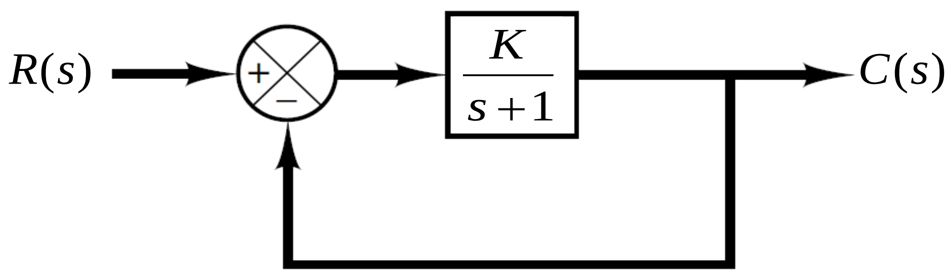
Control Systems

Root Locus

Introduction

- Consider a unity feedback control system shown below.



- The open loop transfer function *G(s)* of the system is $G(s)=\frac{K}{s+1}$
- And the closed transfer function is

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{K}{s+1+K}$$

Introduction

- The open loop stability does not depend upon gain K.

$$G(s) = \frac{K}{s+1}$$

- Whereas, the location of closed loop poles vary with the variation in gain.

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

Introduction

Importance of poles and zeros of the closed loop transfer function of a linear control system:

- The behavior of the roots of the characteristic equation **(poles of the closed loop transfer function)** affect the system **stability**

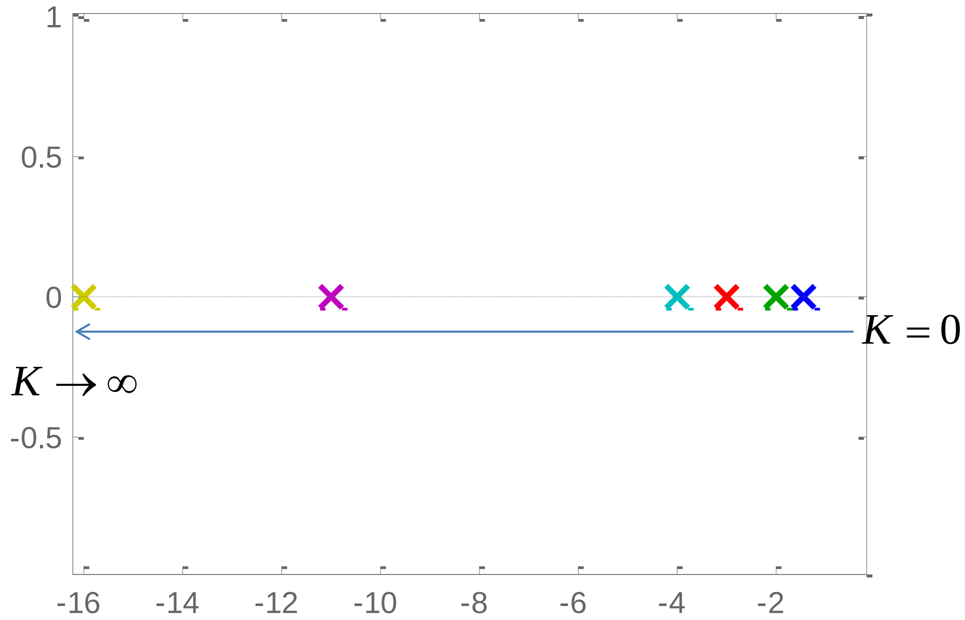
- The **transient** behavior of the system is governed by the poles and zeros of the closed loop transfer function

Introduction

- Location of closed loop Pole for different values of K (remember K>0).

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16



What is Root Locus?

- The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter varies from zero to infinity.

- *Root locus* graphically shows how poles of CL system varies as K varies from 0 to infinity.

How to Sketch root locus?

- One way is to compute the roots of the characteristic equation for all possible values of K.

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16

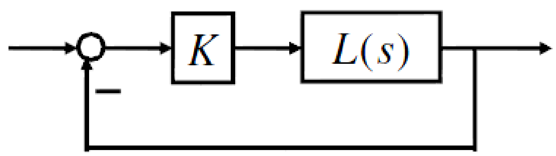
How to Sketch root locus?

- Computing the roots for all values of K might be tedious for higher order systems.

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+10)(s+20)+K}$$

K	Pole
0.5	?
1	?
2	?
3	?
5	?
10	?
15	?

Root Locus – A Simple Example

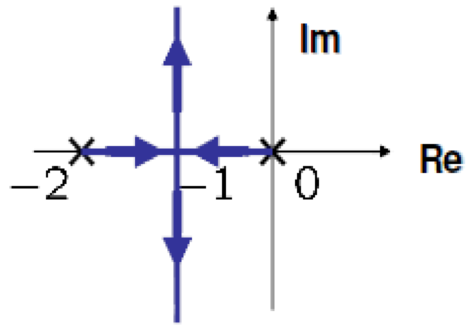


$$L(s) = \frac{1}{s(s+2)}$$

Characteristic eq. $$1 + K\frac{1}{s(s+2)} = 0$$

➡ $s^2 + 2s + K = 0$ ➡ $s = -1 \pm \sqrt{1-K}$

$K = 0$: $s = 0,-2$
$K = 1$: $s = -1, -1$
$K > 1$: complex numbers



Construction of Root Loci

- Finding the roots of the characteristic equation of degree higher than 3 is laborious and will need computer solution.

- A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering.

- This method, called the *root-locus method,* is one in which the roots of the characteristic equation are plotted for all values of a system parameter.

Construction of Root Loci

- The roots corresponding to a particular value of this parameter can then be located on the resulting graph.

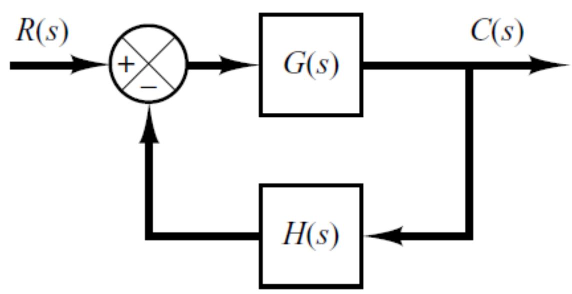
- Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used.

- By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros.

Angle & Magnitude Conditions

- In constructing the root loci angle and magnitude conditions are important.

- Consider the system shown in following figure.



- The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Construction of Root Loci

- The characteristic equation is obtained by setting the denominator polynomial equal to zero.

$$1+G(s)H(s)=0$$

- Or

$$G(s)H(s)=-1$$

- Where *G(s)H(s)* is a ratio of polynomial in s.

- Since *G(s)H(s)* is a complex quantity it can be split into angle and magnitude part.

Angle & Magnitude Conditions

- The angle of *G(s)H(s)=-1* is

$$\angle G(s)H(s)=\angle -1$$

$$\angle G(s)H(s)=\pm 180^{\circ}(2k+1)$$

- Where k=1,2,3…

- The magnitude of *G(s)H(s)=-1* is

$$|G(s)H(s)|=|-1|$$

$$|G(s)H(s)|=1$$

Angle & Magnitude Conditions

- Angle Condition

$$\angle G(s)H(s) = \pm 180^\circ (2k+1) \qquad (k = 1,2,3\ldots)$$

- Magnitude Condition

$$|G(s)H(s)| = 1$$

- The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

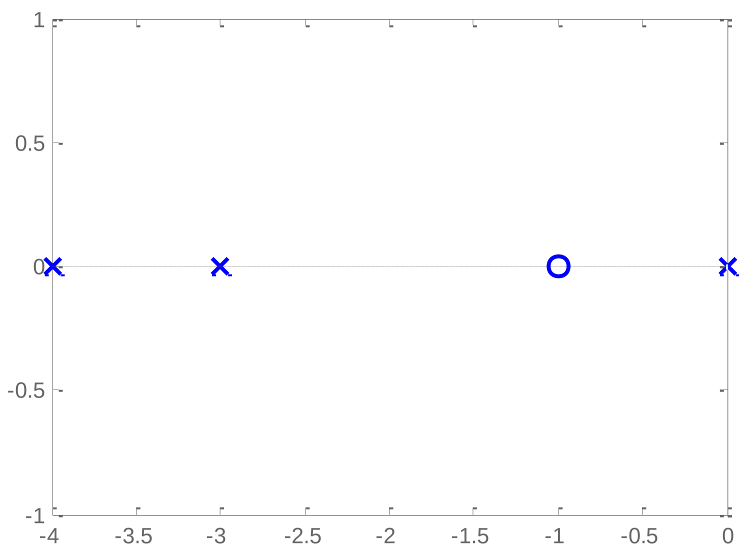
- A locus of the points in the complex plane satisfying the angle condition alone is the root locus.

Angle and Magnitude Conditions (Graphically)

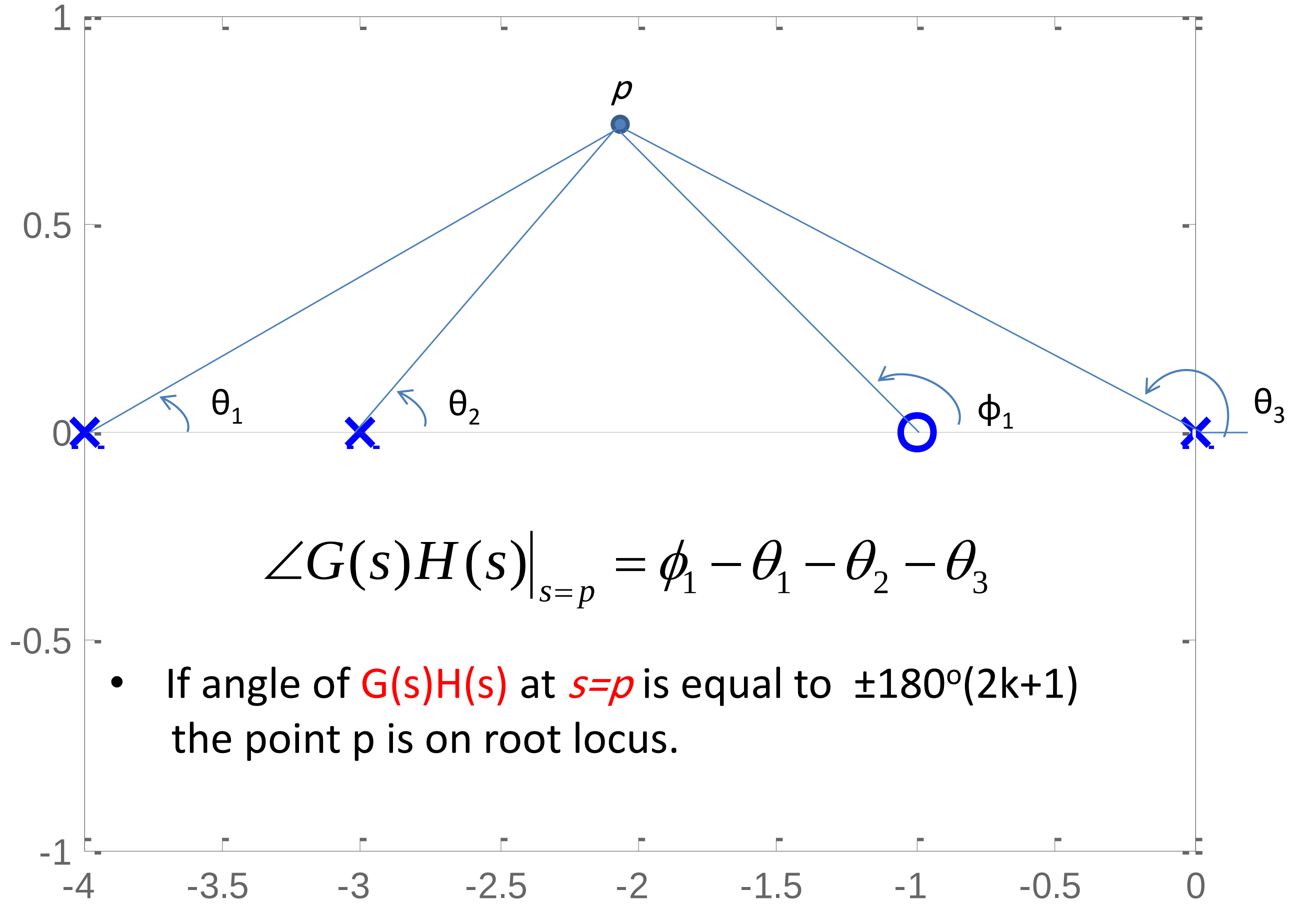
- To apply Angle and magnitude conditions graphically we must first draw the poles and zeros of G(s)H(s) in s-plane.

- For example if G(s)H(s) is given by

$$G(s)H(s) = \frac{s+1}{s(s+3)(s+4)}$$



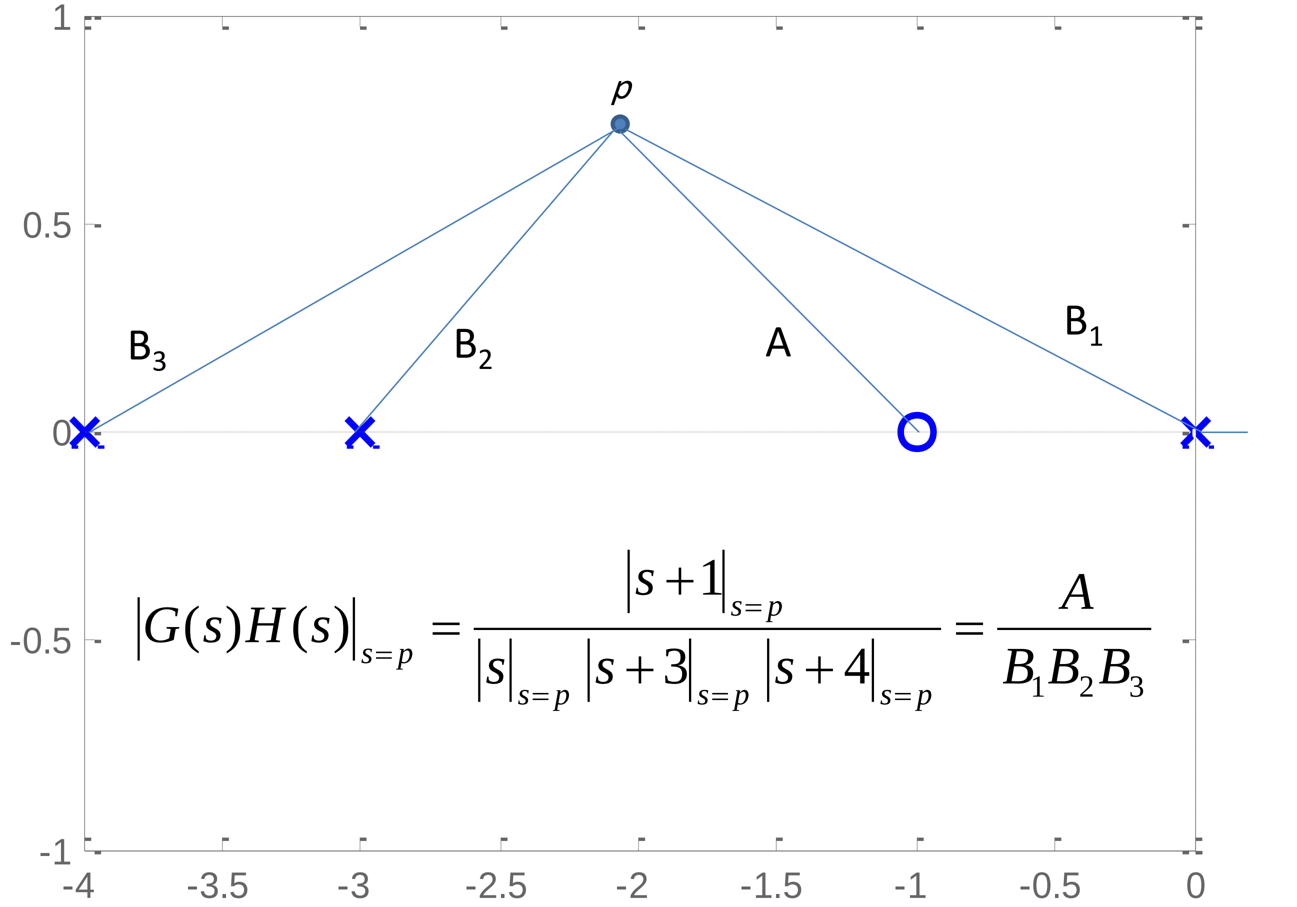
Angle and Magnitude Conditions (Graphically)



$$\angle G(s)H(s)\big|_{s=p} = \phi_1 - \theta_1 - \theta_2 - \theta_3$$

- If angle of G(s)H(s) at *s=p* is equal to ±180°(2k+1) the point p is on root locus.

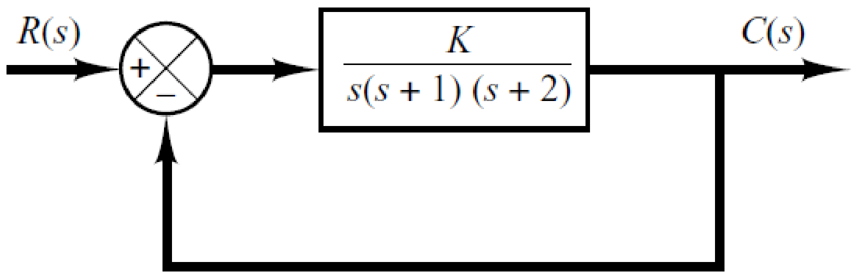
Angle and Magnitude Conditions graphically



$$\left|G(s)H(s)\right|_{s=p} = \frac{|s+1|_{s=p}}{|s|_{s=p}\,|s+3|_{s=p}\,|s+4|_{s=p}} = \frac{A}{B_1B_2B_3}$$

Example1

- Apply angle and magnitude conditions (Analytically as well as graphically) on following unity feedback system at s=-0.25.



Example1

- Here $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

- For the given system the angle condition becomes

$$\angle G(s)H(s) = \angle \frac{K}{s(s+1)(s+2)}$$

$$\angle G(s)H(s) = \angle K - \angle s - \angle(s+1) - \angle(s+2)$$

$$\angle K - \angle s - \angle(s+1) - \angle(s+2) = \pm 180°(2k+1)$$

Example1

$$\angle G(s)H(s)\big|_{s=-0.25} = \angle K\big|_{s=-0.25} - \angle s\big|_{s=-0.25} - \angle(s+1)\big|_{s=-0.25} - \angle(s+2)\big|_{s=-0.25}$$

$$\angle G(s)H(s)\big|_{s=-0.25} = -\angle(-0.25) - \angle(-0.75) - \angle(1.75)$$

$$\angle G(s)H(s)\big|_{s=-0.25} = -180^\circ - 0^\circ - 0^\circ$$

$$\angle G(s)H(s)\big|_{s=-0.25} = \pm 180^\circ(2k+1)$$

Example1

- Here $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$
- And the Magnitude condition becomes

$$|G(s)H(s)| = \left|\frac{K}{s(s+1)(s+2)}\right| = 1$$

Example1

- Now we know from angle condition that the point s=-0.25 is on the rot locus. But we do not know the value of gain K at that specific point.

- We can use magnitude condition to determine the value of gain at any point on the root locus.

$$\left|\frac{K}{s(s+1)(s+2)}\right|_{s=-0.25} = 1$$

$$\left|\frac{K}{(-0.25)(-0.25+1)(-0.25+2)}\right|_{s=-0.25} = 1$$

Example1

$$\left|\frac{K}{(-0.25)(-0.25+1)(-0.25+2)}\right|_{s=-0.25} = 1$$

$$\left|\frac{K}{(-0.25)(0.75)(1.75)}\right| = 1$$

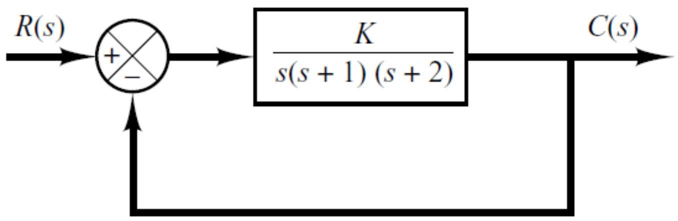
$$\left|\frac{K}{-0.3285}\right| = 1$$

$$\frac{K}{0.328} = 1$$

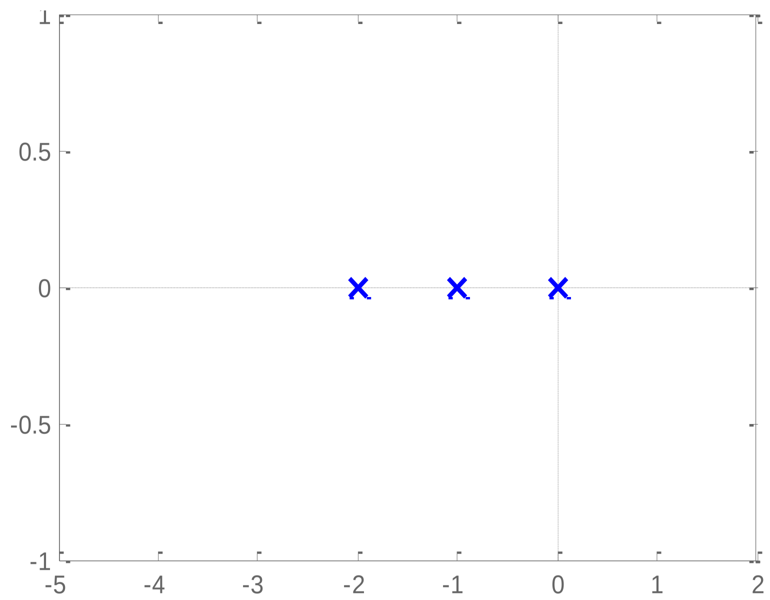
$$K = 0.328$$

Construction of root loci

- Step-1: The first step in constructing a root-locus plot is to locate the open-loop poles and zeros in s-plane.



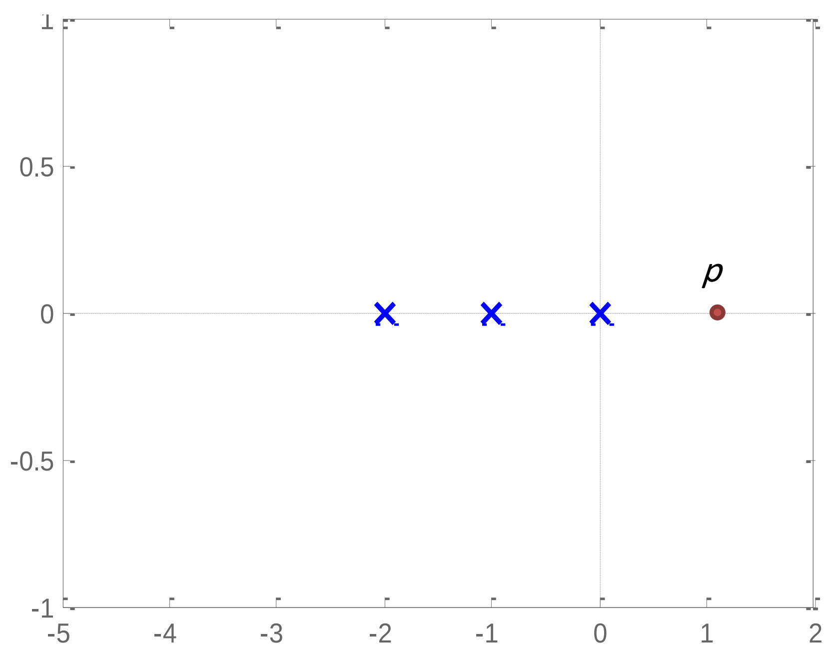
$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$



Construction of root loci

- Step-2: Determine the root loci on the real axis.

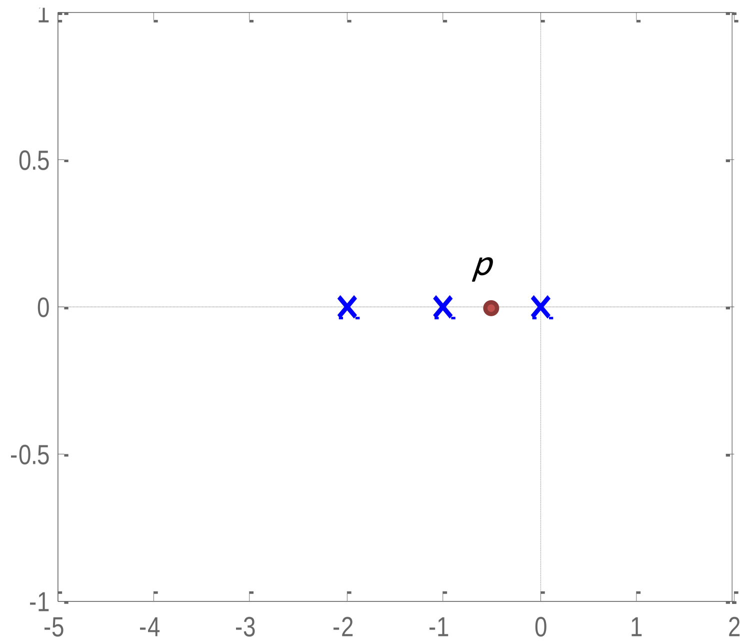
- Hence, there is no root locus on the positive real axis.



Construction of root loci

- Step-2: Determine the root loci on the real axis.

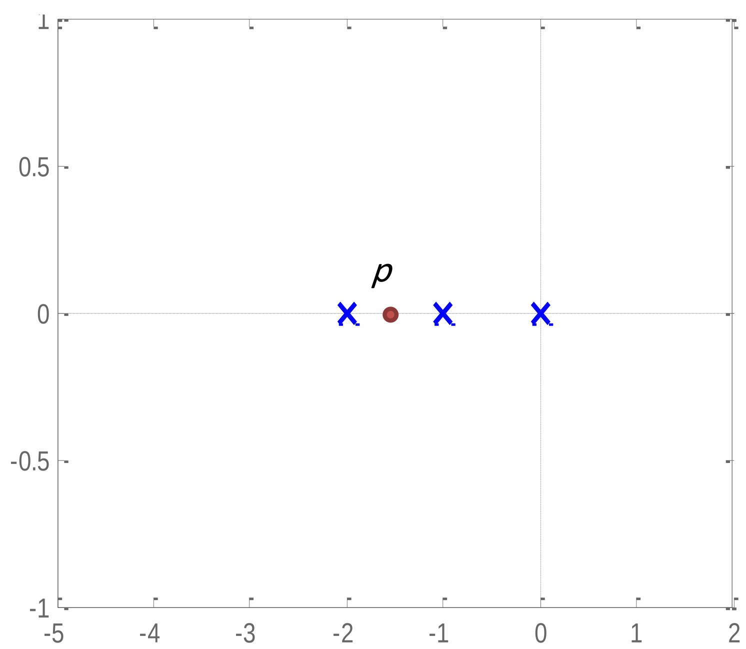
- Next, select a test point on the negative real axis between 0 and –1.
- Therefore, the portion of the negative real axis between 0 and –1 forms a portion of the root locus.



Construction of root loci

- Step-2: Determine the root loci on the real axis.

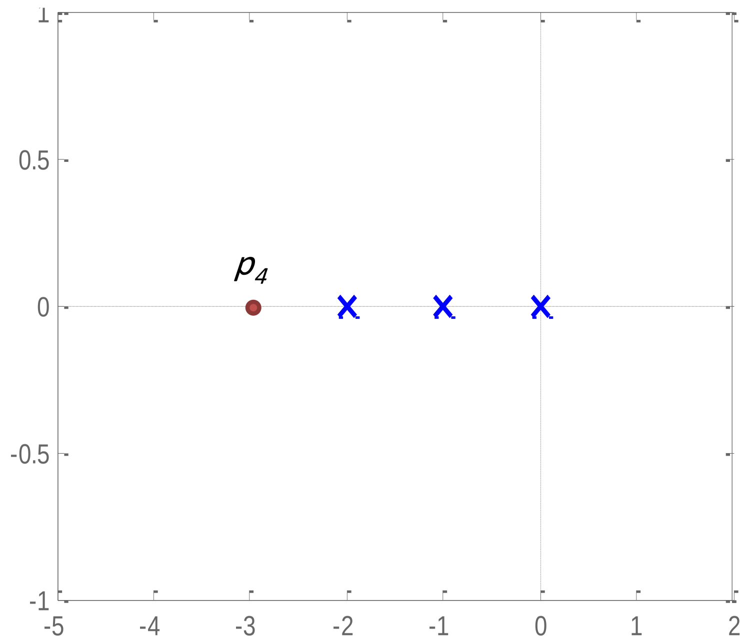
- Now, select a test point on the negative real axis between -1 and –2.
- Therefore, the negative real axis between -1 and –2 is not a part of the root locus.



Construction of root loci

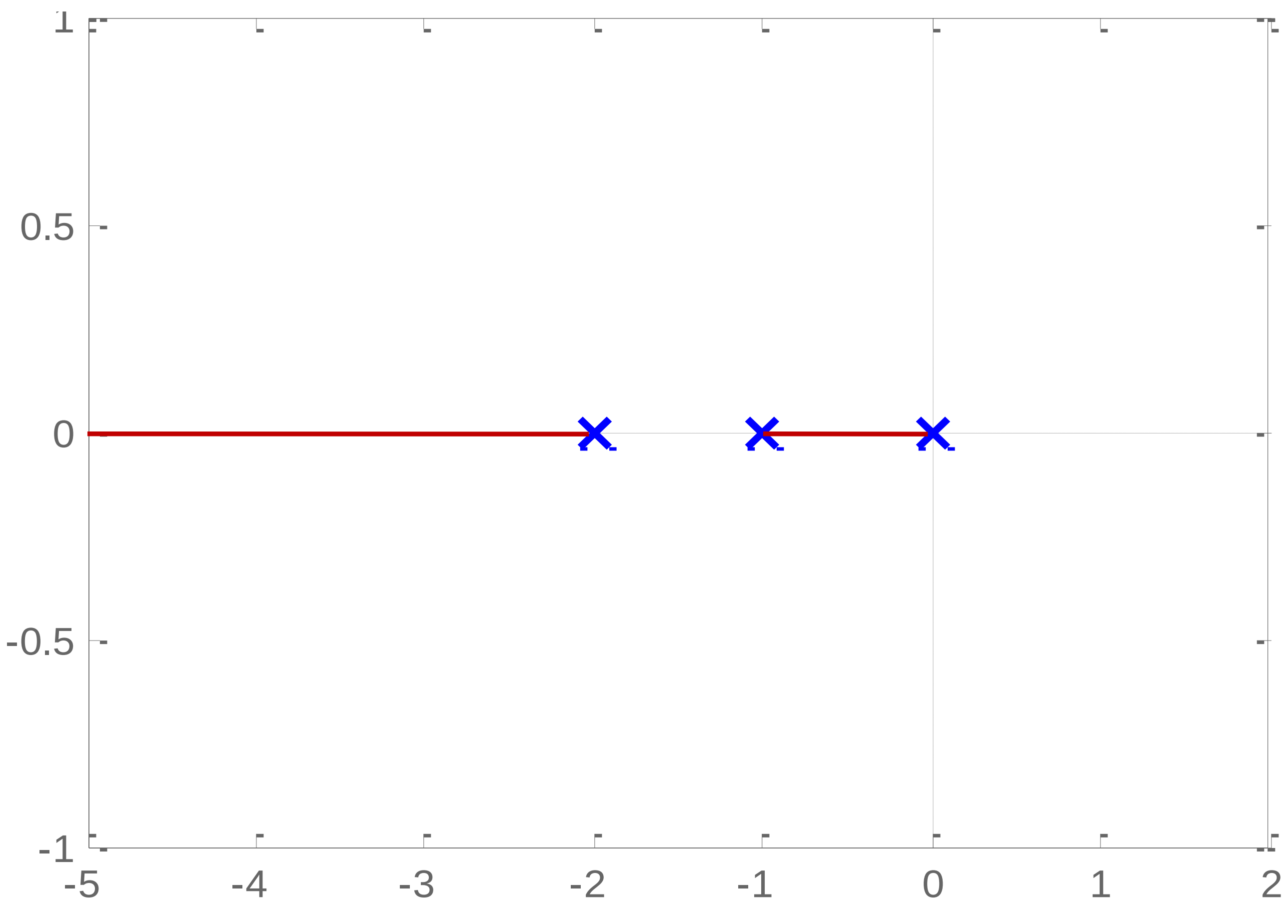
- Step-2: Determine the root loci on the real axis.

- Similarly, test point on the negative real axis between -2 and $-\infty$.
- Therefore, the negative real axis between -2 and $-\infty$ is part of the root locus.



Construction of root loci

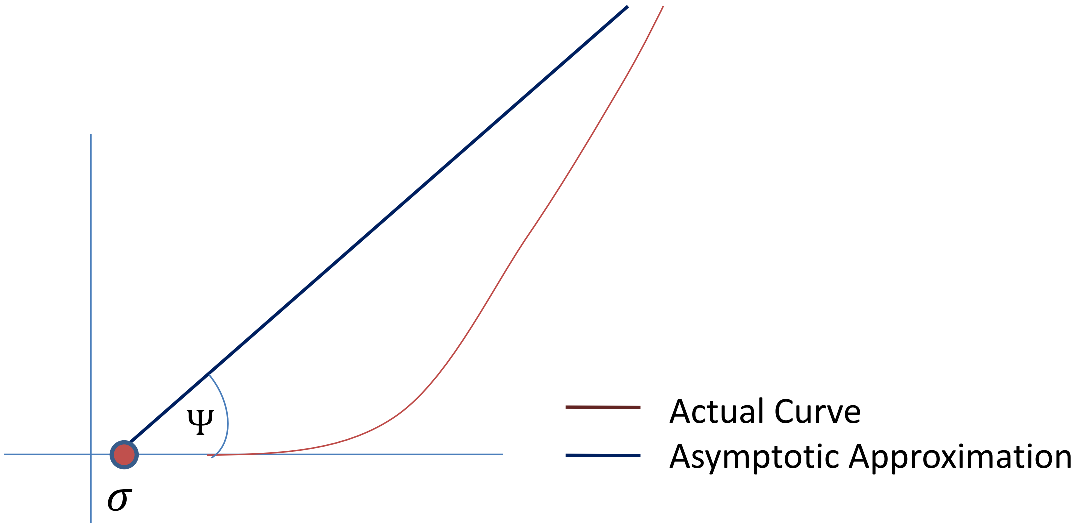
- Step-2: Determine the root loci on the real axis.



Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

Asymptote is the straight line approximation of a curve



$\sigma \longrightarrow$ *Centroid of Asymptotes*

$\Psi \longrightarrow$ *Angle of Asymptotes*

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

$$Angle\ of\ asymptotes = \psi = \frac{\pm 180°(2k+1)}{n-m}$$

- where
- n-----> number of poles
- m-----> number of zeros

- For this Transfer Function $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

$$\psi = \frac{\pm 180°(2k+1)}{3-0}$$

K=0,1,2,3,...

K=n-m-1

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

$$\psi = \frac{\pm 180°(2k+1)}{3-0}$$

K=0,1,2

$$\begin{aligned} \psi &= \pm 60° && \text{when } k=0 \\ &= \pm 180° && \text{when } k=1 \\ &= \pm 300° && \text{when } k=2 \\ &= \pm 420° && \text{when } k=3 \end{aligned}$$

- Since the angle repeats itself as k is varied, the distinct angles for the asymptotes are determined as 60°, –60°, -180°and 180°.
- Thus, there are three asymptotes having angles 60°, –60°, 180°.

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

- Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis.

- Point of intersection of asymptotes on real axis (or centroid of asymptotes) can be find as out

$$\sigma = \frac{\sum poles - \sum zeros}{n-m}$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

- For $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

$$\sigma = \frac{(0-1-2)-0}{3-0}$$

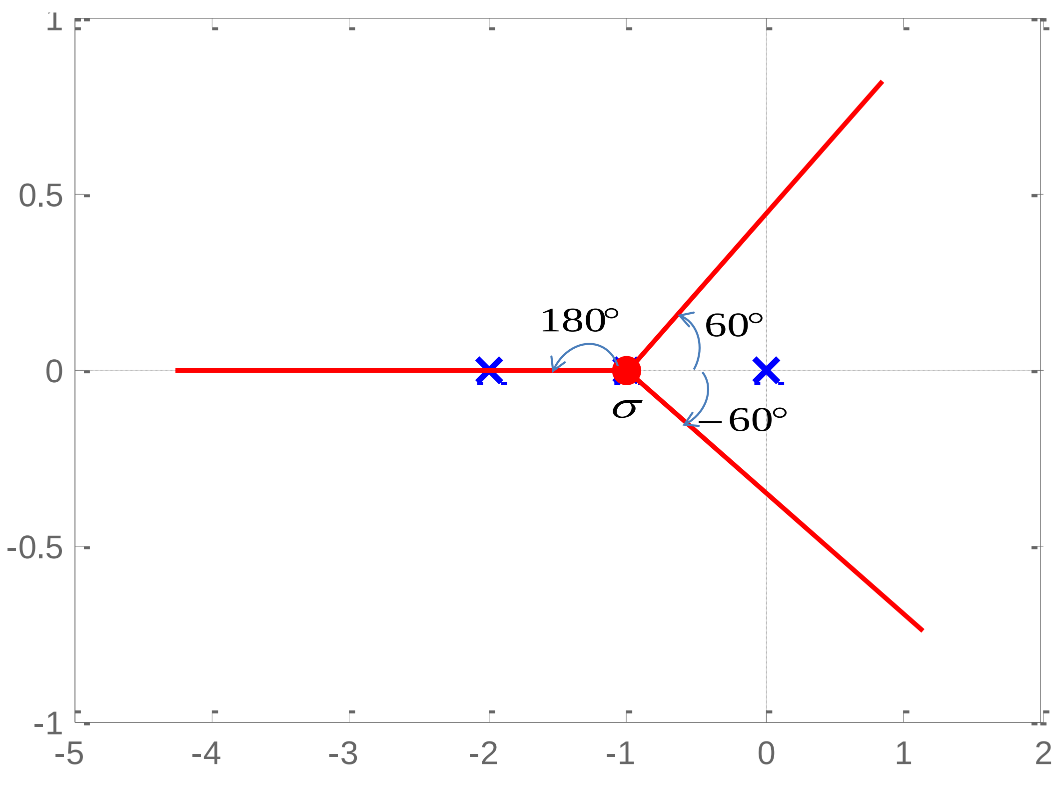
$$\sigma = \frac{-3}{3} = -1$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

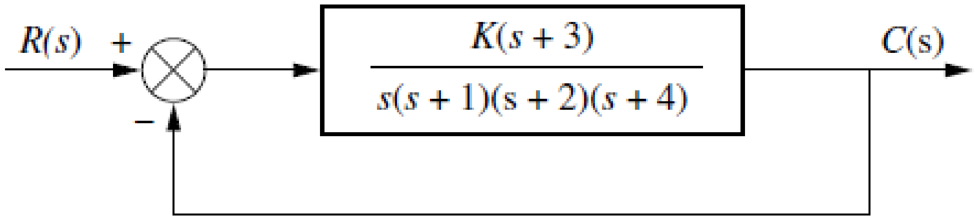
$\psi = 60^\circ, -60^\circ, 180^\circ$

$$\sigma = -1$$



Home Work

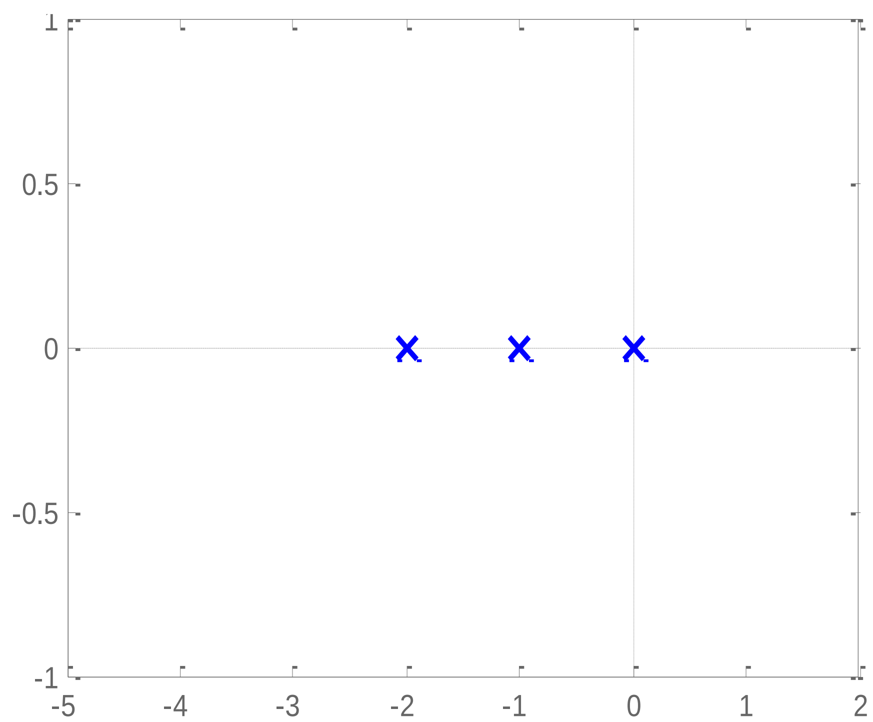
- Consider following unity feedback system.



- Determine
  - Root loci on real axis
  - Angle of asymptotes
  - Centroid of asymptotes

Construction of root loci

- Step-4: Determine the *breakaway point* & *break in point*
  - The breakaway point corresponds to a point in the s plane where multiple roots of the characteristic equation occur.
  - It is the point from which the root locus branches leaves real axis and enter in complex plane.



Construction of root loci

- Step-4: Determine the *breakaway point* & *break in point*

  - The breakaway can be determined from the roots of

$$\frac{dK}{ds} = 0$$

  - It should be noted that not all the solutions of dK/ds=0 correspond to actual breakaway points.

  - If a point at which dK/ds=0 is on a root locus, it is an actual breakaway.

Construction of root loci

- Step-4: Determine the *breakaway point* & *break in point*

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

- The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$\frac{K}{s(s+1)(s+2)} = -1$$

$$K = -[s(s+1)(s+2)]$$

- The breakaway point can now be determined as

$$\frac{dK}{ds} = -\frac{d}{ds}[s(s+1)(s+2)]$$

Construction of root loci

- Step-4: Determine the *breakaway point* & *break in point*

$$\frac{dK}{ds} = -\frac{d}{ds}\left[s(s+1)(s+2)\right]$$

$$\frac{dK}{ds} = -\frac{d}{ds}\left[s^3 + 3s^2 + 2s\right]$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2$$

- Set *dK/ds=0* in order to determine breakaway point.

$$-3s^2 - 6s - 2 = 0$$

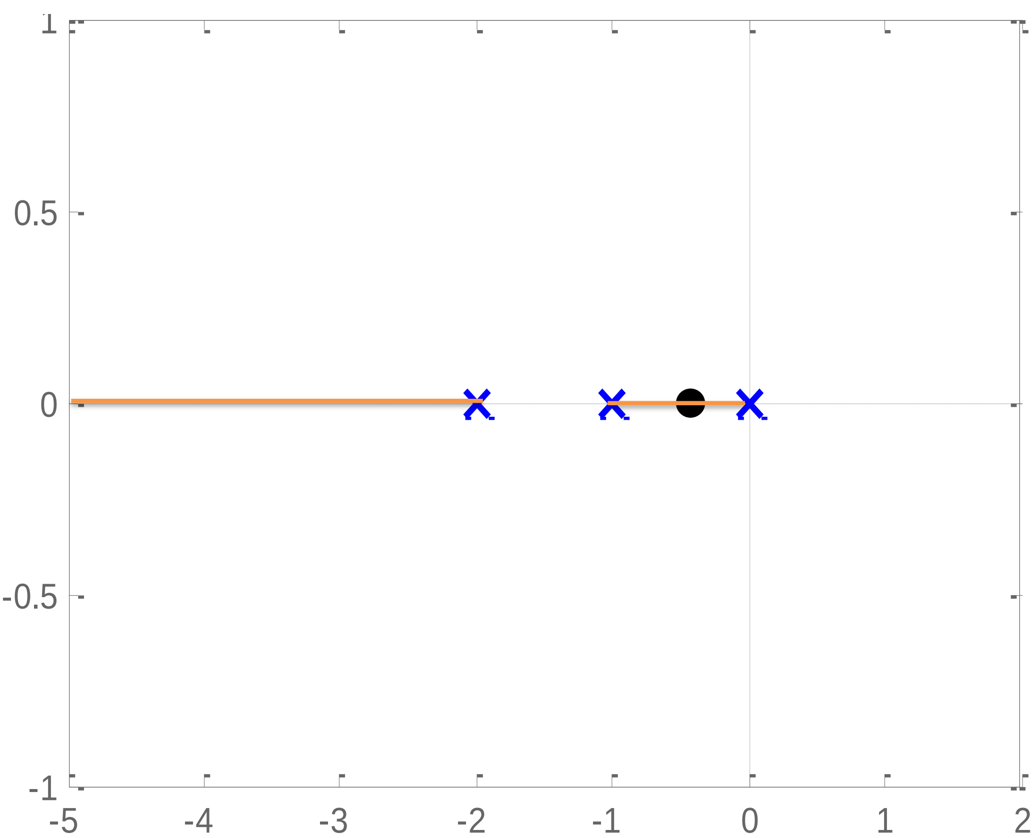
$$3s^2 + 6s + 2 = 0$$

$$s = -0.4226$$
$$= -1.5774$$

Construction of root loci

- Step-4: Determine the *breakaway point* & *break in point*

$$s = -0.4226$$



Construction of root loci

- Step-4: Determine the *breakaway point* & *break in point*

$$s = -0.4226$$
$$= -1.5774$$

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

- Since the breakaway point must lie on a root locus between 0 and –1, it is clear that *s=–0.4226* corresponds to the actual breakaway point.

- Point s=–1.5774 is not on the root locus. Hence, this point is not an actual breakaway.

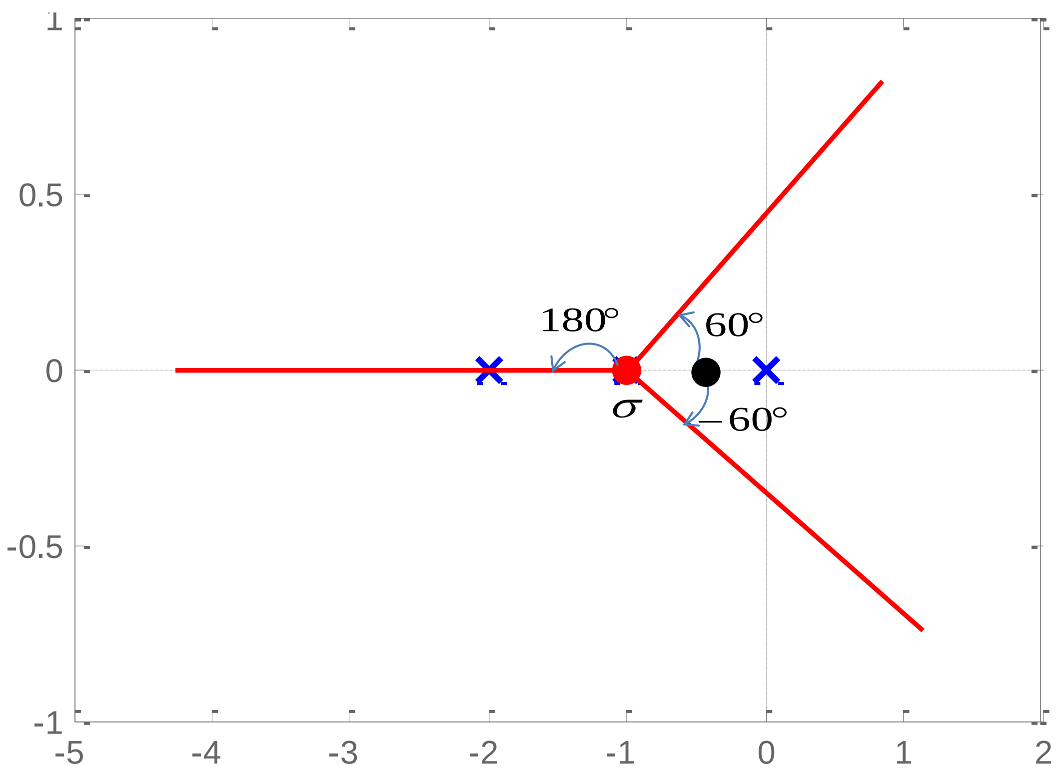
- In fact, evaluation of the values of K corresponding to s=–0.4226 and s=–1.5774 yields

$$K = 0.3849, \quad \text{for } s = -0.4226$$
$$K = -0.3849, \quad \text{for } s = -1.5774$$

Construction of root loci

- Step-4: Determine the *breakaway point* & *break in point*

$$s = -0.4226$$



Home Work

- Determine the Breakaway and break in points

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

Solution

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)}$$

$$\frac{K(s^2-8s+15)}{s^2+3s+2} = -1$$

$$K = -\frac{(s^2+3s+2)}{(s^2-8s+15)}$$

- Differentiating ***K*** with respect to **s** and setting the derivative equal to zero yields;

$$\frac{dK}{ds} = -\frac{[(s^2-8s+15)(2s+3)-(s^2+3s+2)(2s-8)]}{(s^2-8s+15)^2} = 0$$

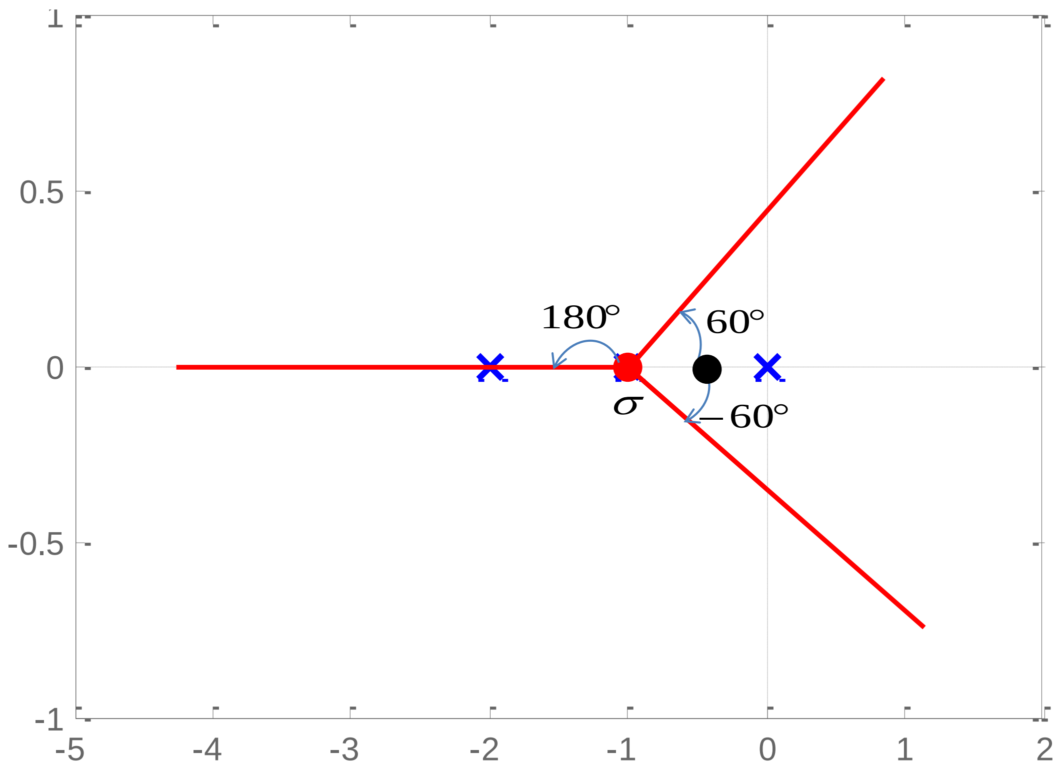
$$11s^2 - 26s - 61 = 0$$

Hence, solving for s, we find the break-away and break-in points;

$$\boxed{s = -1.45 \text{ and } 3.82}$$

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.



Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.
  - These points can be found by use of Routh's stability criterion.
  - Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

  - The Routh Array Becomes

$$\begin{array}{lcc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.

- The value(s) of K that makes the system marginally stable is 6.

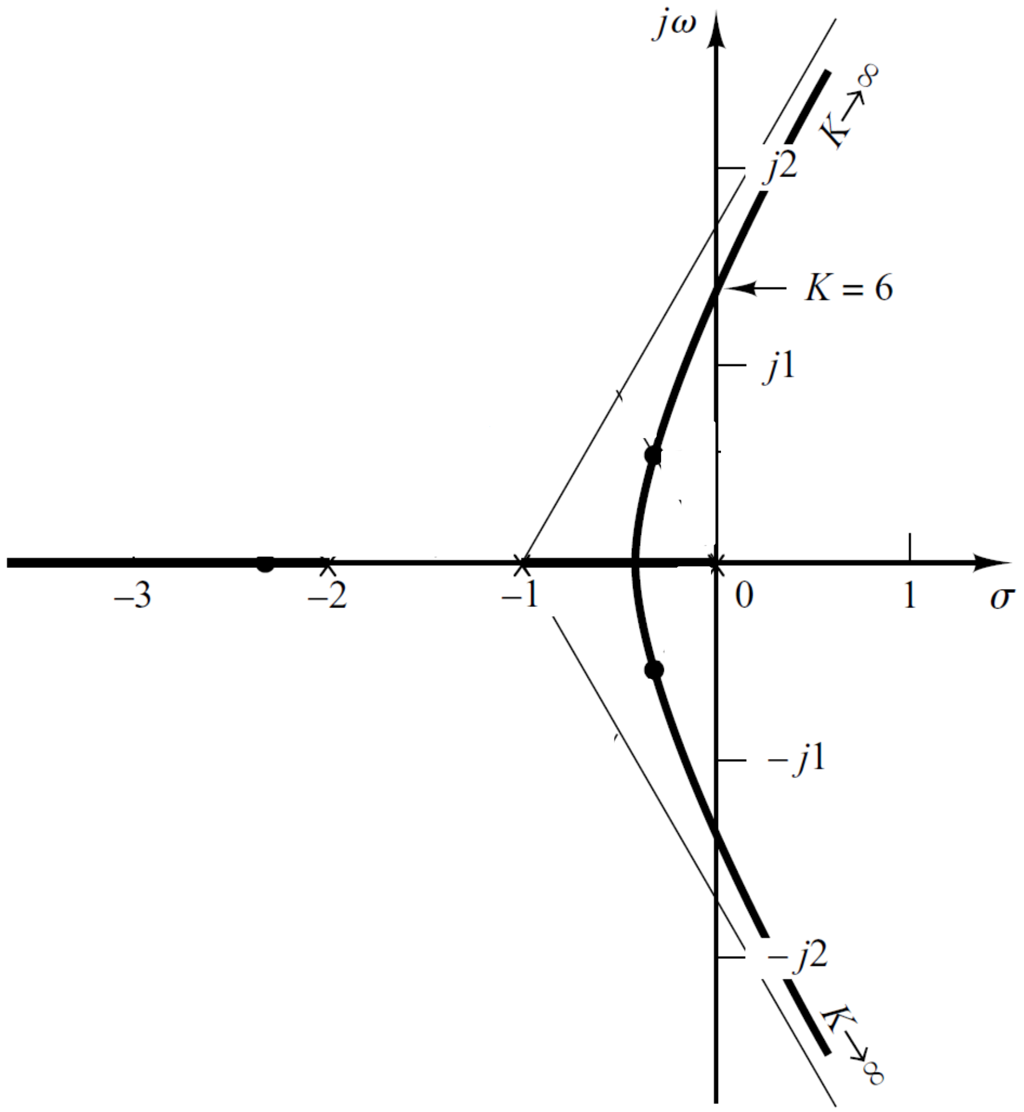
- The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row, that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

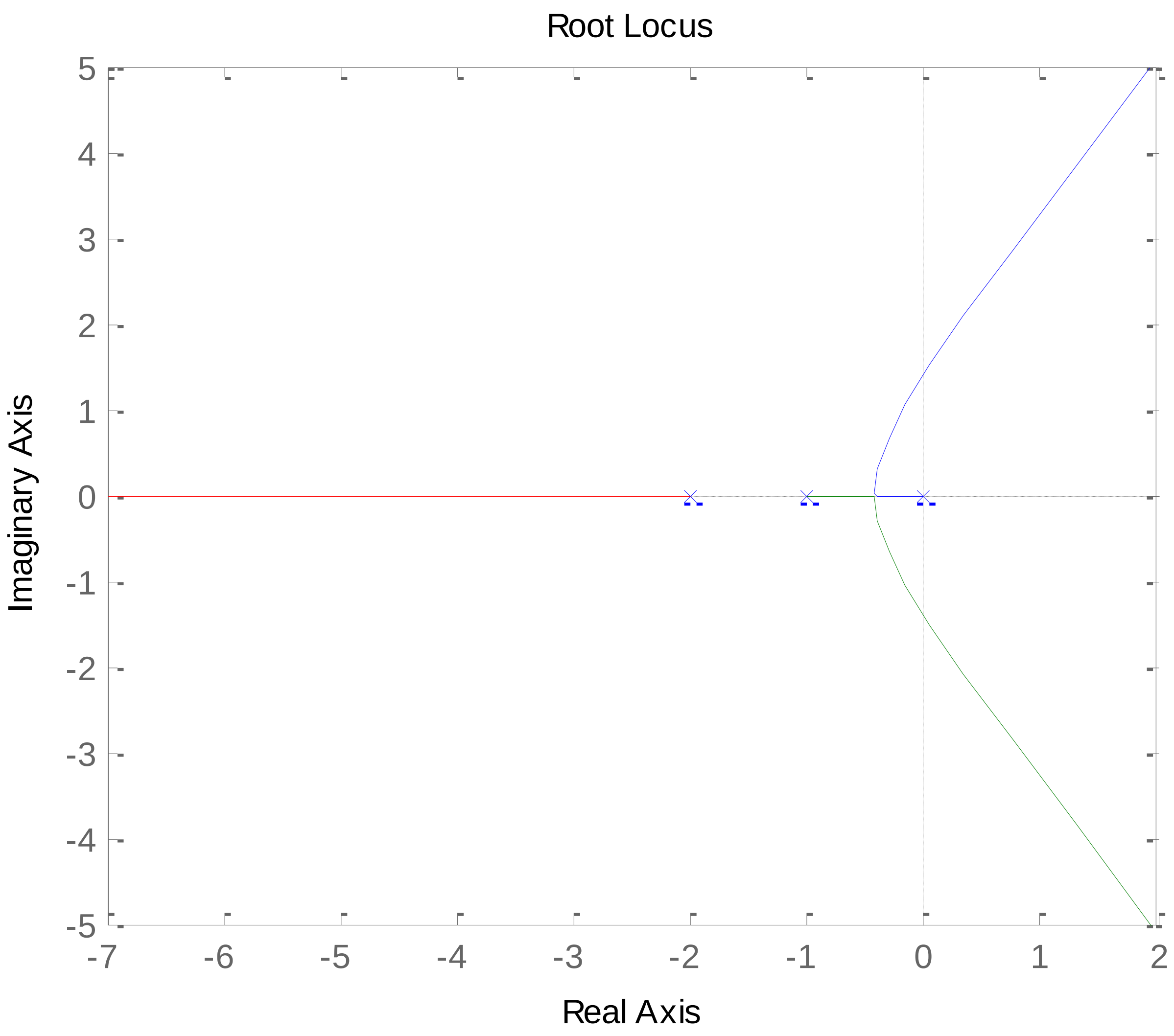
s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

- Which yields

$$s = \pm j\sqrt{2}$$



```
num=[1]
den=[1 3 2 0]
sys=tf(num,den)
rlocus(sys)
```



Summary of Root Locus

Root Locus Basic Steps:

1. **The Loci starts (K=0) at the poles of G(s) and terminates (K=∞) at the zeros of G(s) at infinity**

2. **The Loci exits on the real axis only to the left of an odd number of real poles and zeros of G(s)**

3. **If the loci leaves the real axis, it must be symmetrical about this axis because complex poles appear in complex conjugate pairs**

4. **Loci which terminates at ∞ must follow asymptotes which make an angle θ with the real axis where :**
P=number of poles , Z: number of finite zeros $l=0,1,2,\ldots$

$$\theta = \pm \frac{180^{\circ}(2l+1)}{P-Z}$$

Summary of Root Locus Steps

5. **The asymptotes intersect at point σ on the real axis given by:**

$$\sigma = \frac{\sum P - \sum Z}{P - Z}$$

∑P: Sum of the poles of G(s)
∑Z: Sum of the zeros of G(s)
P, Z: The number of poles and zeros of G(s)

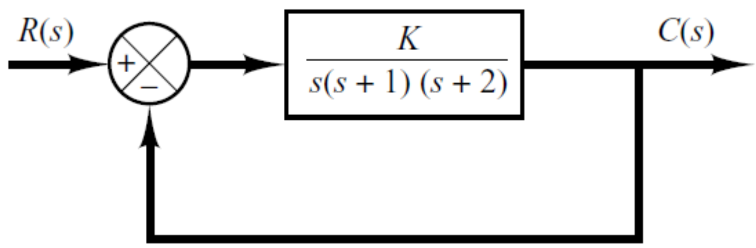
6. **The point of Breakaway from the real axis can be determined by solving the equation** (where 1+G(s)=0 must be satisfied).

$$\frac{dK}{ds} = 0$$

7. **The point at which the loci cross the imaginary axis can be determined by letting s=jω in the characteristic equation.**

Example2

- Consider following unity feedback system.



- Determine the value of K such that the damping ratio of a pair of dominant complex-conjugate closed-loop poles is 0.5.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Example2

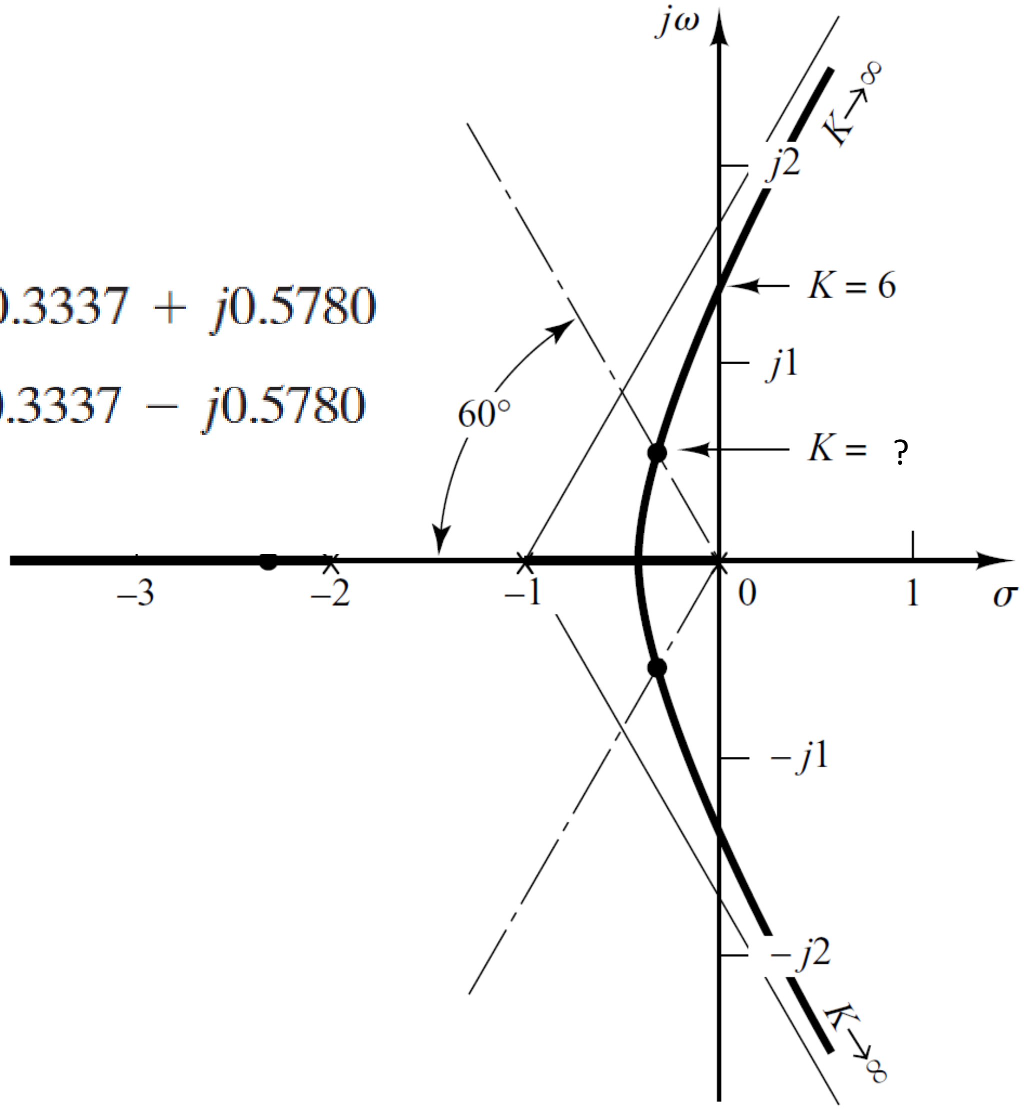
- The damping ratio of 0.5 corresponds to

$$\zeta = \cos\theta$$

$$\theta = \cos^{-1}\zeta$$

$$\theta = \cos^{-1}(0.5) = 60°$$

$$s_1 = -0.3337 + j0.5780$$
$$s_2 = -0.3337 - j0.5780$$

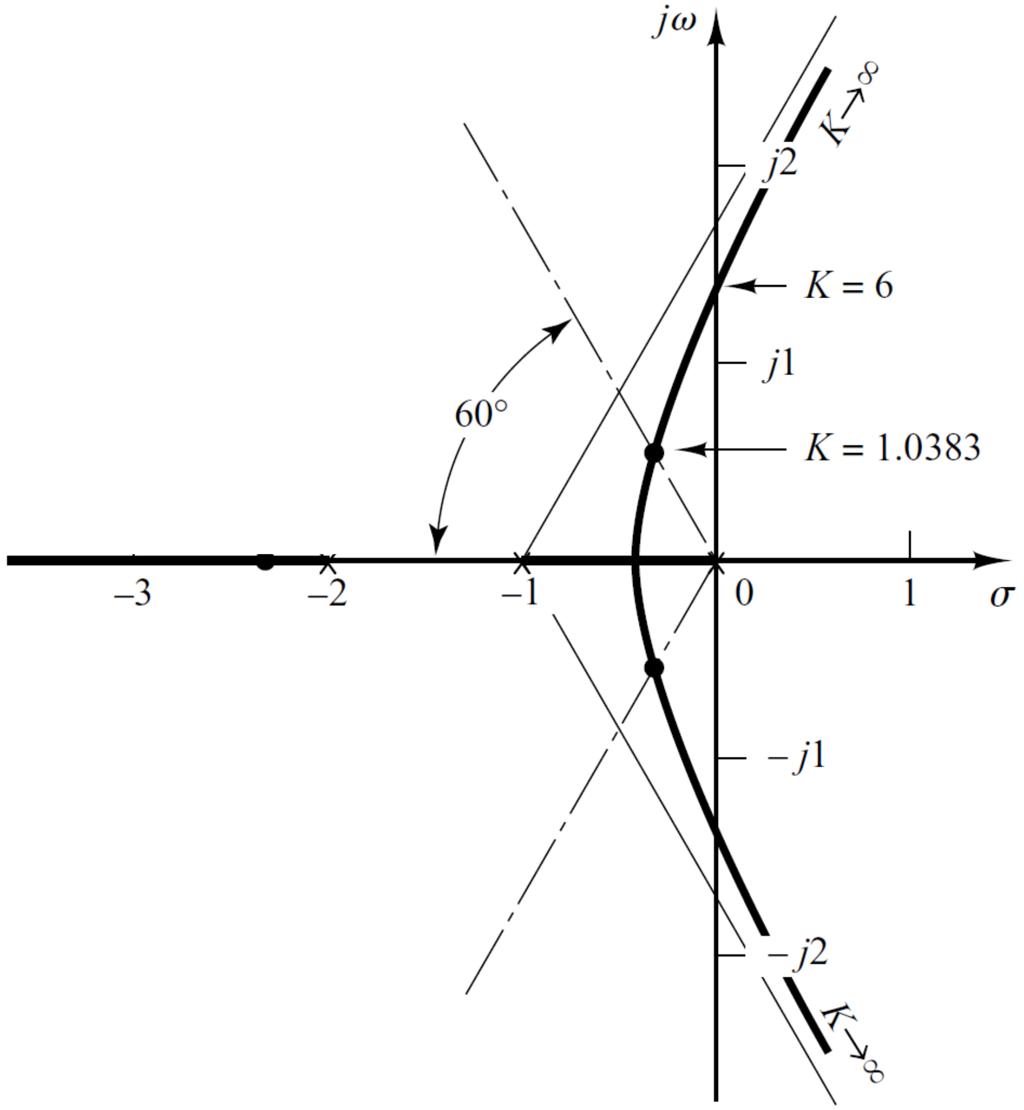


Example2

- The value of K that yields such poles is found from the magnitude condition

$$\left|\frac{K}{s(s+1)(s+2)}\right|_{s=-0.3337+j0.5780} = 1$$

$$K = |s(s+1)(s+2)|_{s=-0.3337+j0.5780}$$
$$= 1.0383$$



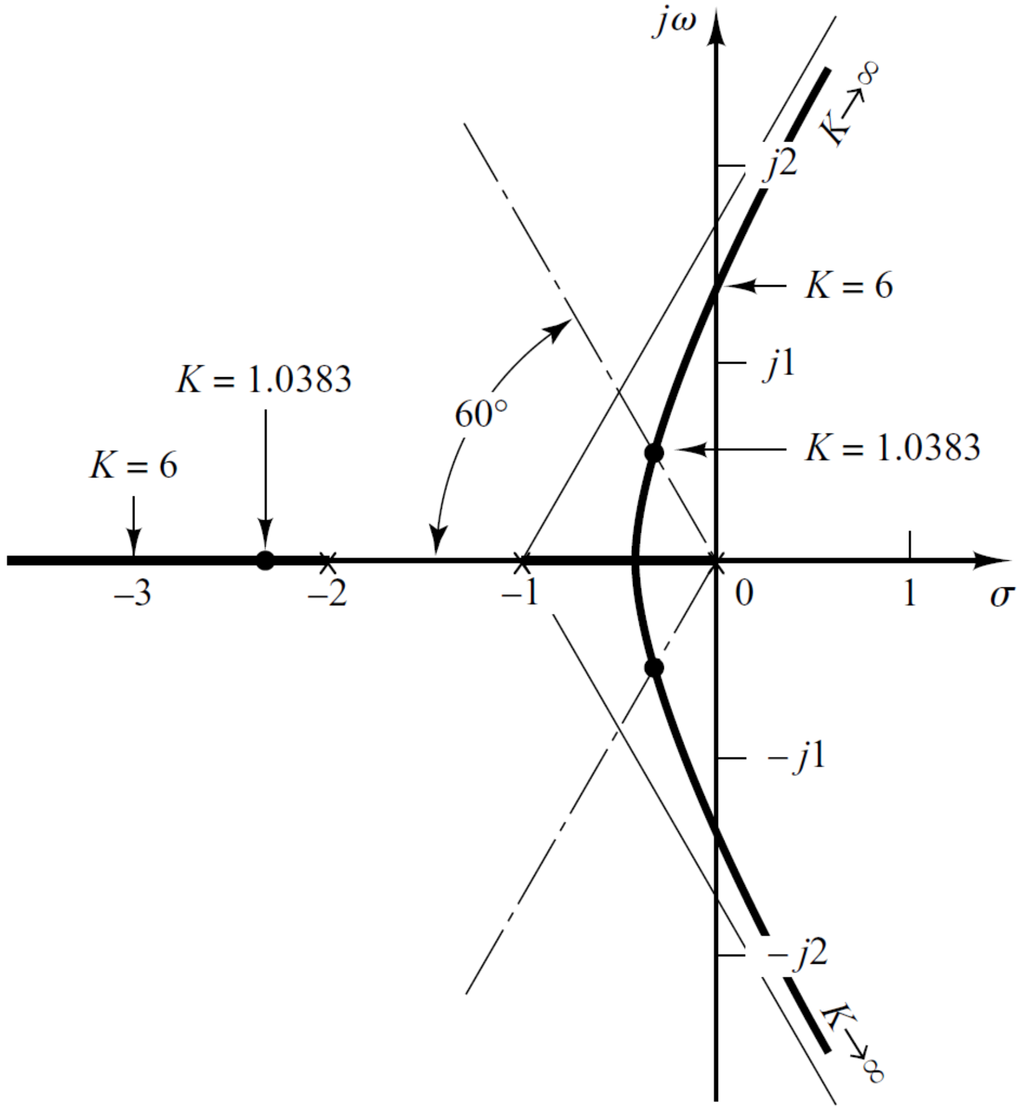
Example2

- The third closed loop pole at K=1.0383 can be obtained as

$$1+G(s)H(s)=1+\frac{K}{s(s+1)(s+2)}=0$$

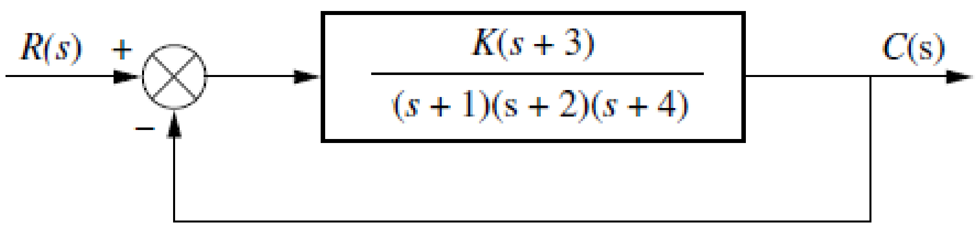
$$1+\frac{1.0383}{s(s+1)(s+2)}=0$$

$$s(s+1)(s+2)+1.0383=0$$



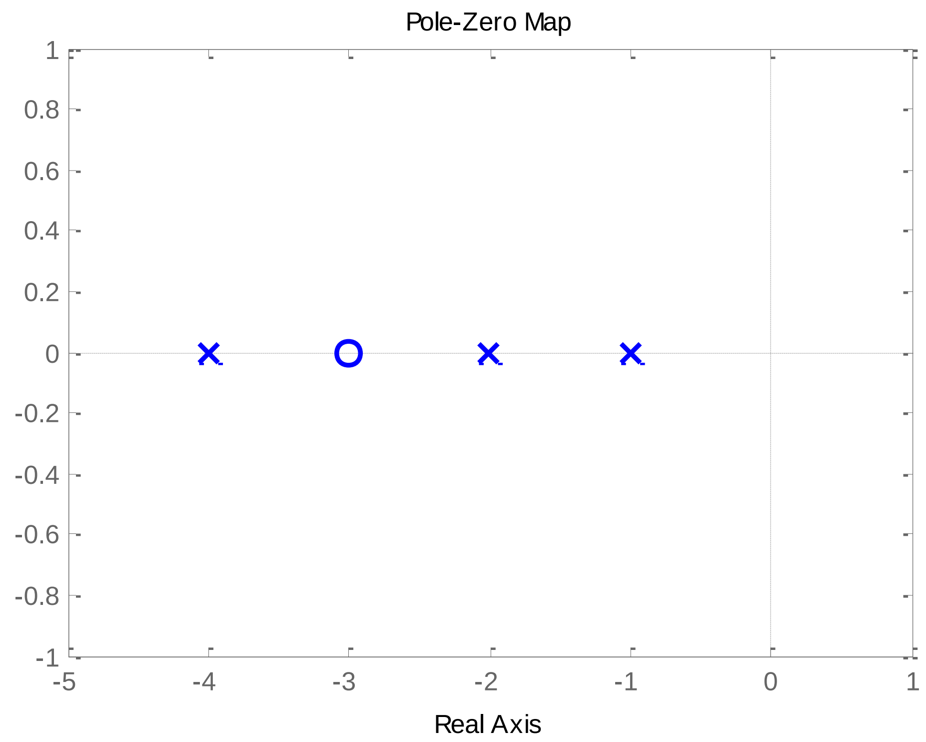
Example3

- Sketch the root locus of following system and determine the location of dominant closed loop poles to yield maximum overshoot in the step response less than 30%.



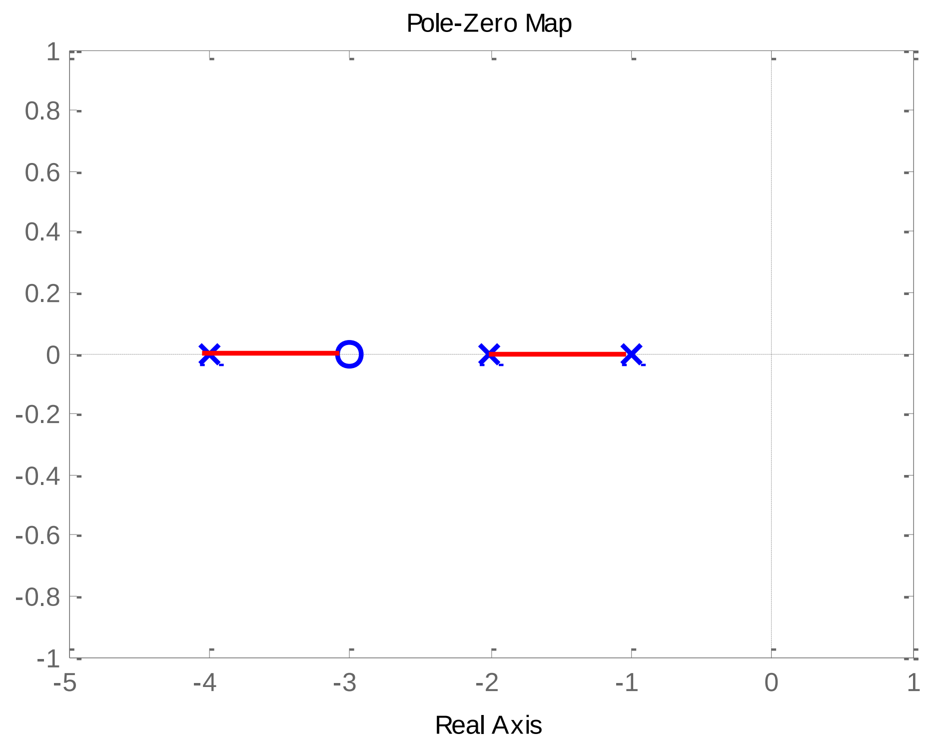
Example3

- Step-1: Pole-Zero Map



Example3

- Step-2: Root Loci on Real axis

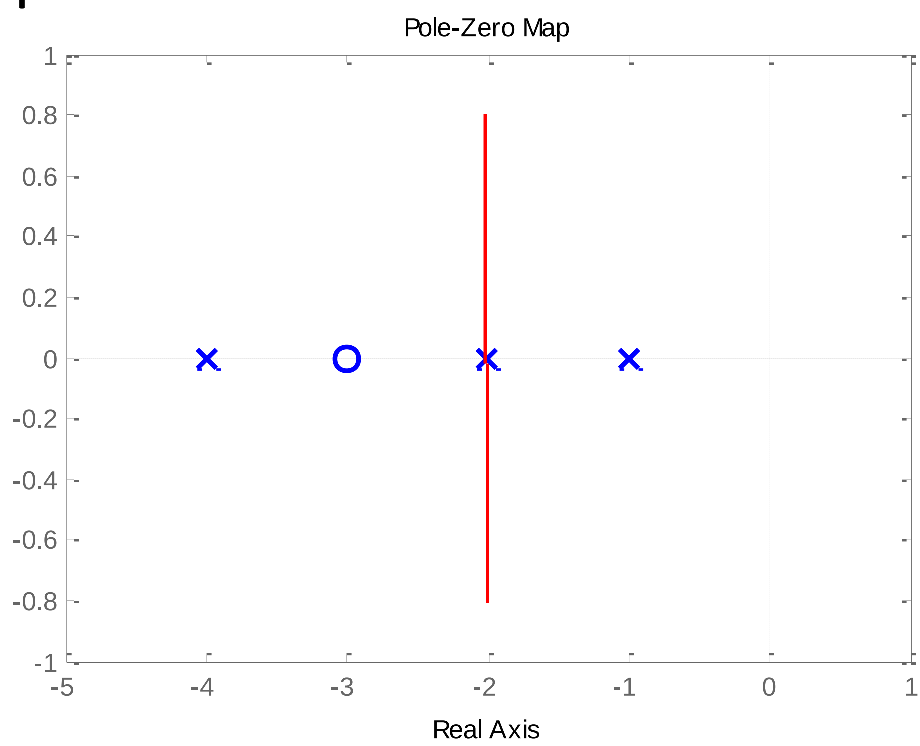


Example3

- Step-3: Asymptotes

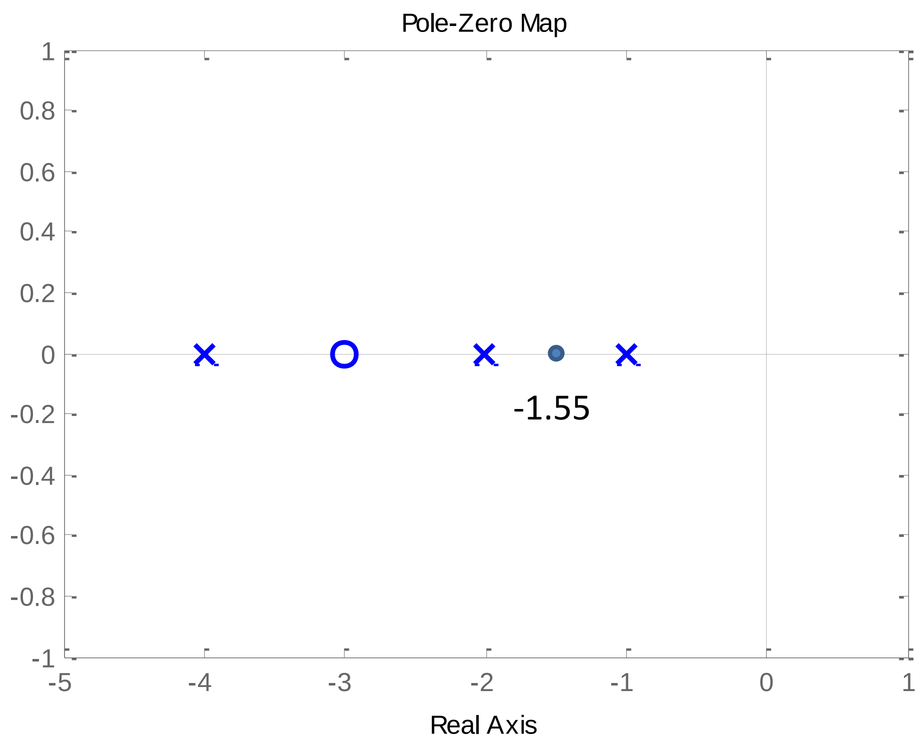
$$\psi = \pm 90^\circ$$

$$\sigma = -2$$

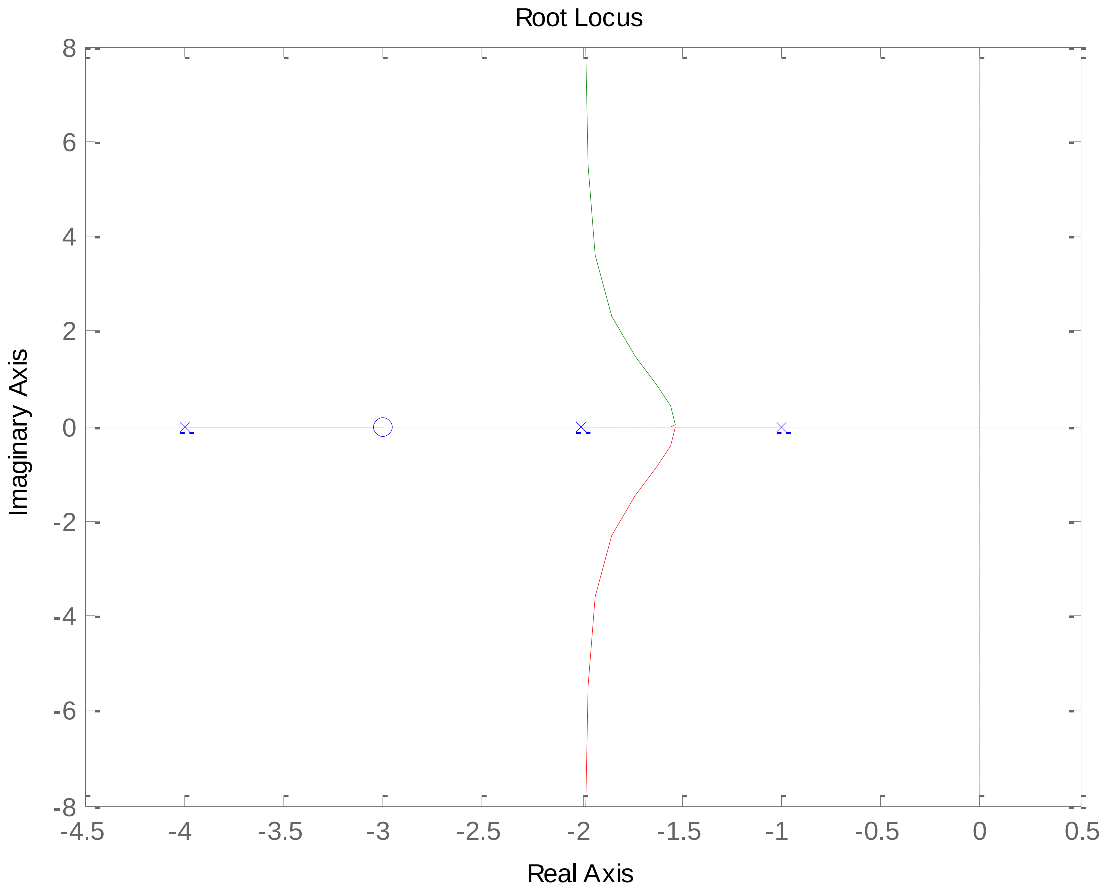


Example3

- Step-4: breakaway point



Example3



Example3

- Mp<30% corresponds to

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

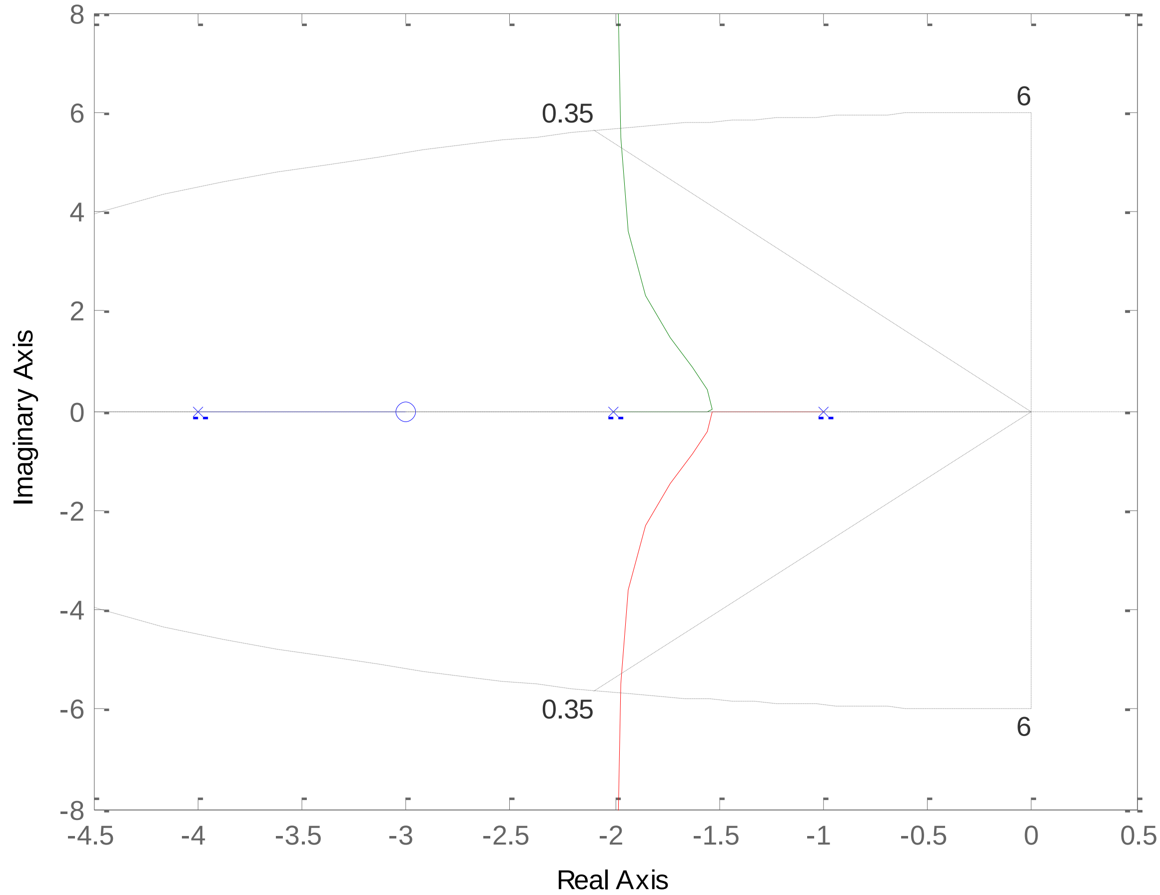
$$30\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$\zeta > 0.35$$

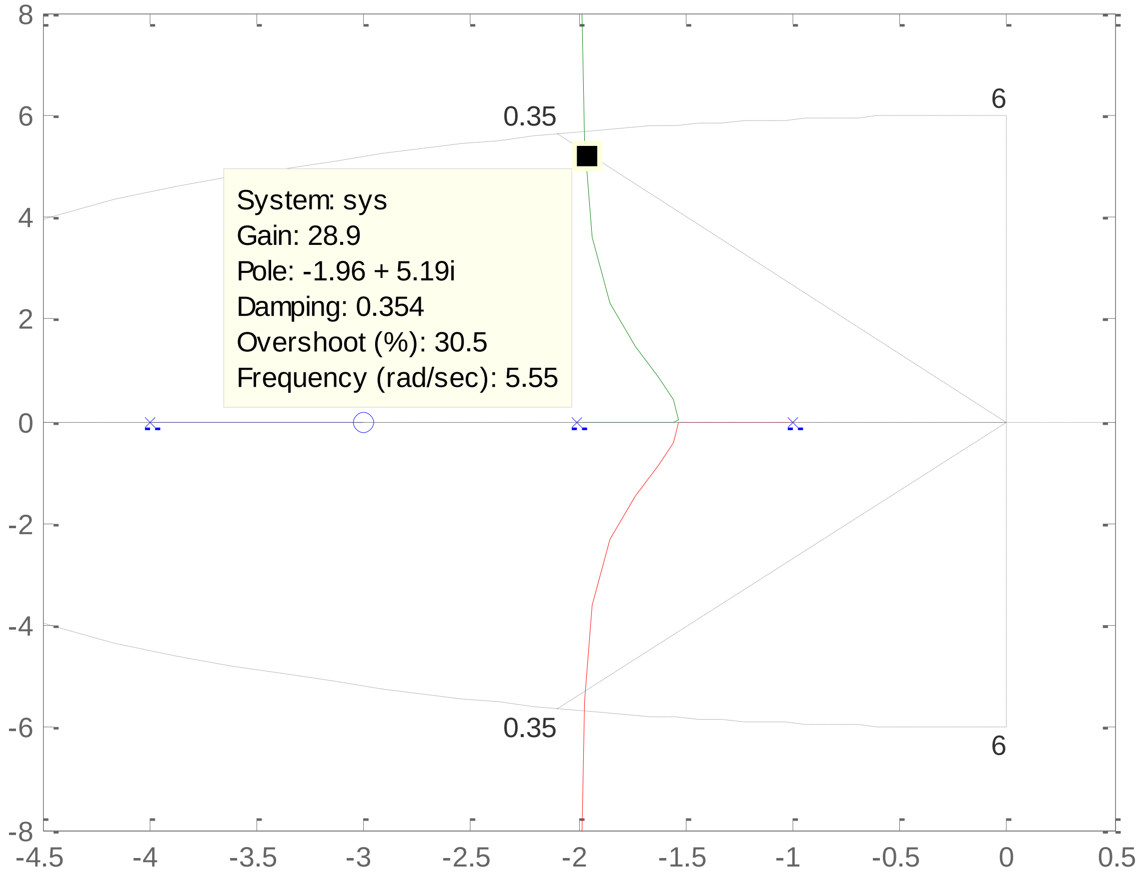
$$\theta < \cos^{-1}(0.35)$$

$$\theta < 69.5°$$

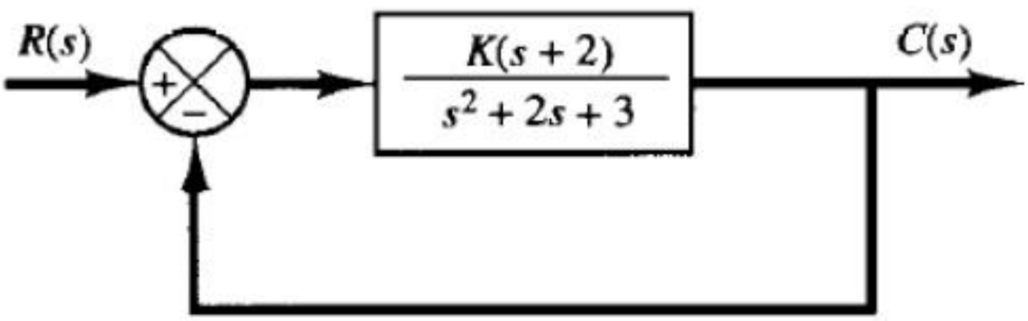
Example3



Example3



Example :Consider the control system shown in figure below. Sketch the root – locus plot and determine the approximate damping ratio for a value of K = 1.33

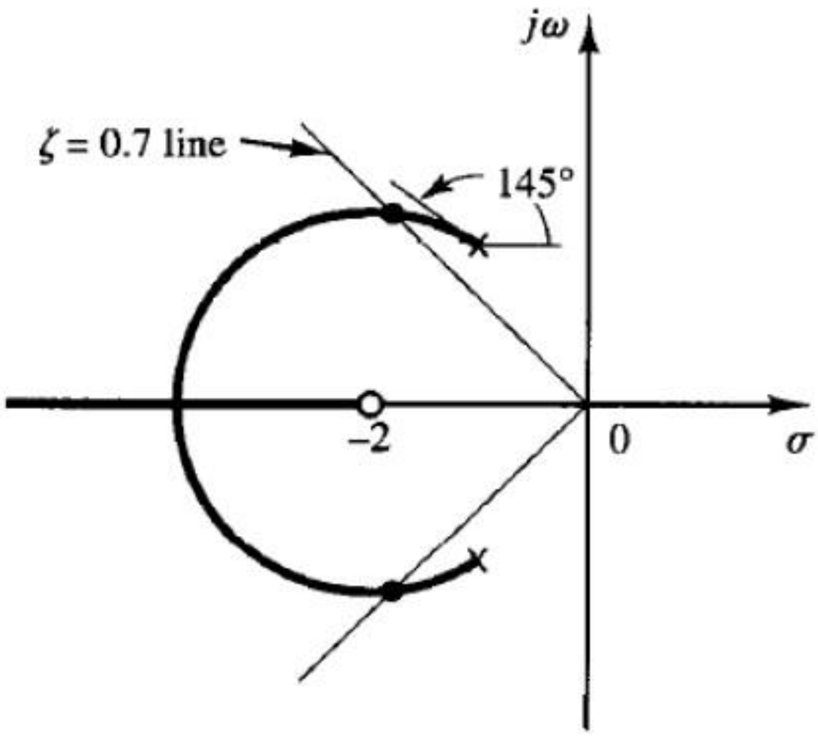


Solution:

n=2 , m = 1 , 2nd order system , 2 loci , 1 asymptotes , 1 angle ,Θ_0=180

It is seen that $G(s)$ has a pair of complex-conjugate poles at

$$s = -1 + j\sqrt{2}, \qquad s = \;\; 1 \quad j\sqrt{2}$$



The angle of departure is

$$180° - 90° + 55° = 145°$$

For this problem, the break-in point can be found as follows: Since

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

$$s^2 + 4s + 1 = 0$$

$$s = -3.7320 \quad \text{or} \quad s = -0.2680$$

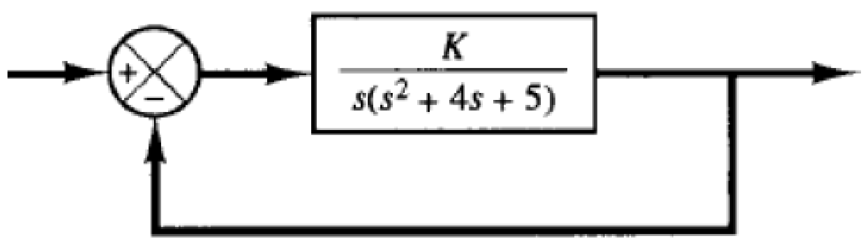
The characteristic equation is:

$S^2 + (2 + K)\,S + 3 + 2K = 0$, for $K = 1.33$

$S^2 + 3.33\,S + 5.66 = 0$, $S_1 = -1.665 + J\,1.699$, $\omega = 1.699$, $\alpha = 1.665$

$\theta = \tan^{-1} = \omega/\alpha = 45^0$, $\zeta = 0.707$

Example: Sketch the root – locus plot for the system shown in figure below



Solution:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, -180°$$

The intersection of the asymptotes and the real axis is located on the real axis at

$$\sigma_a = -\frac{0 + 2 + 2}{3} = -1.3333$$

The angle of departure from a complex pole in the upper half s plane is obtained from

$$\theta = 180° - 153.43° - 90°$$
$$\theta = -63.43°$$

The breakaway and break-in points are found from $dK/ds = 0$. Since the characteristic equation is

$$s^3 + 4s^2 + 5s + K = 0$$

we have

$$K = -(s^3 + 4s^2 + 5s)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 8s + 5) = 0 \qquad s = -1, \qquad s = -1.6667$$

Next we determine the points where root-locus branches cross the imaginary axis. By substituting $s = j\omega$ into the characteristic equation, we have

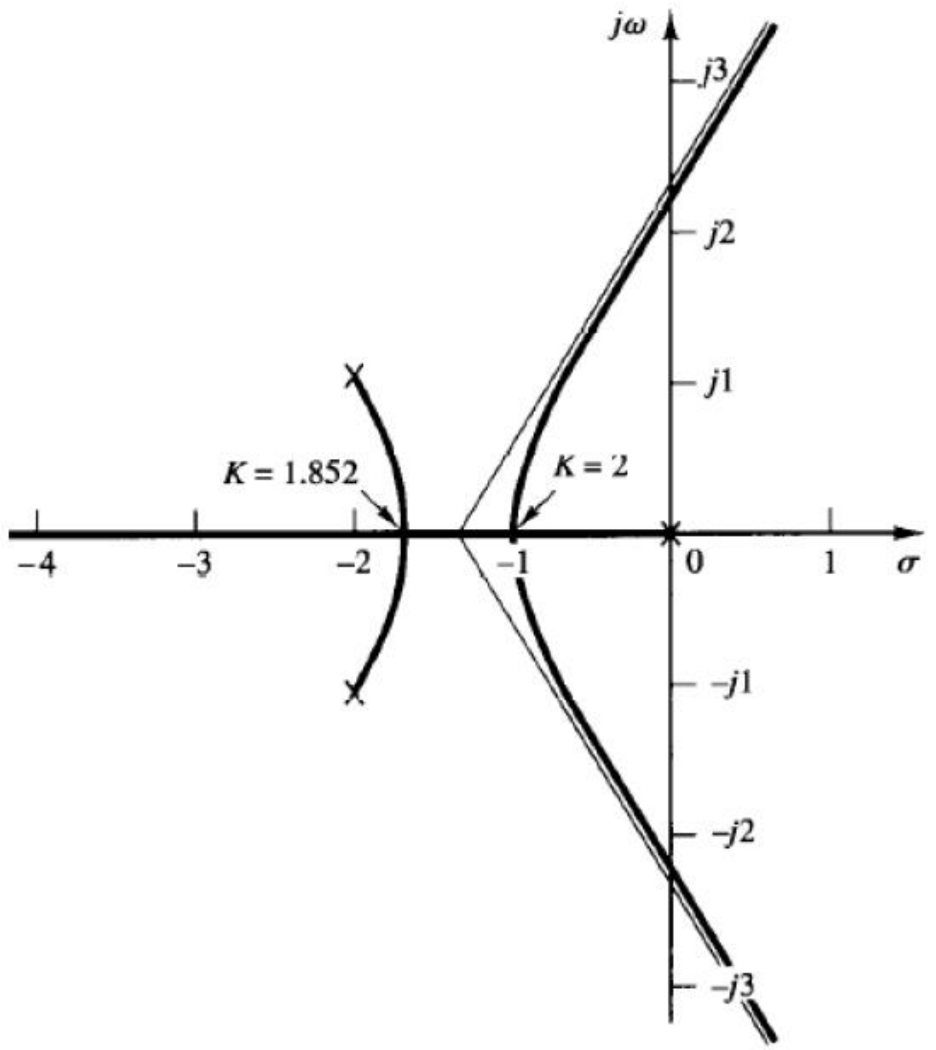
$$(j\omega)^3 + 4(j\omega)^2 + 5(j\omega) + K = 0$$

or

$$(K - 4\omega^2) + j\omega(5 - \omega^2) = 0$$

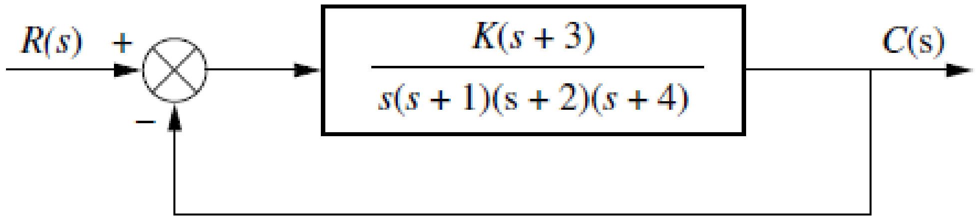
$$\omega = \pm\sqrt{5}, \qquad K = 20 \quad \text{or} \quad \omega = 0, \qquad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = \sqrt{5}$ and $\omega = -\sqrt{5}$. The root-locus branch on the real axis touches the $j\omega$ axis at $\omega = 0$. A sketch of the root loci for the system is shown in

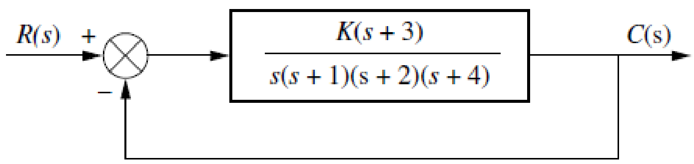


Root Locus of Higher Order System

- Sketch the Root Loci of following unity feedback system



$$G(s)H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$



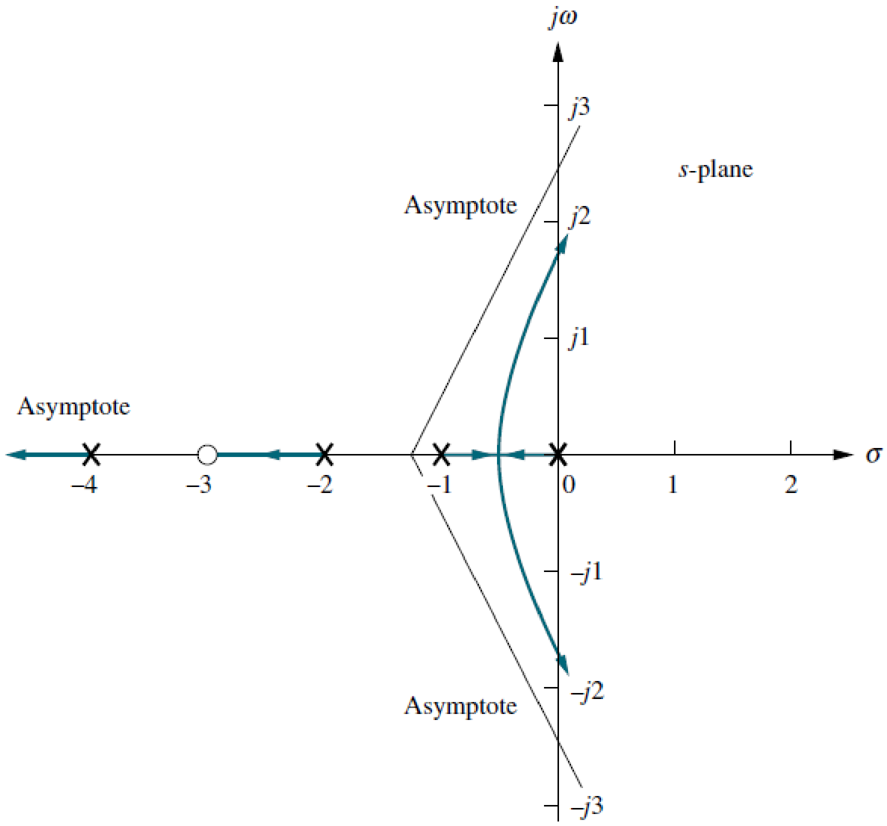
- Let us begin by calculating the asymptotes. The real-axis intercept is evaluated as;

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

- The angles of the lines that intersect at - 4/3, given by

$$\begin{aligned}\theta_a &= \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}\\ &= \pi/3 \qquad \text{for } k = 0\\ &= \pi \qquad \text{for } k = 1\\ &= 5\pi/3 \qquad \text{for } k = 2\end{aligned}$$

- The Figure shows the complete root locus as well as the asymptotes that were just calculated.



Example: Sketch the root locus for the system with the characteristic equation of;

$$1 + GH(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

a) Number of finite poles = n = 4.
b) Number of finite zeros = m = 1.
c) Number of asymptotes = n - m = 3.
d) Number of branches or loci equals to the number of finite poles (n) = 4.
e) The portion of the real-axis between, 0 and -2, and between, -4 and -∞, lie on the root locus for $K > 0$.

- Using Eq. (v), the real-axis asymptotes intercept is evaluated as;

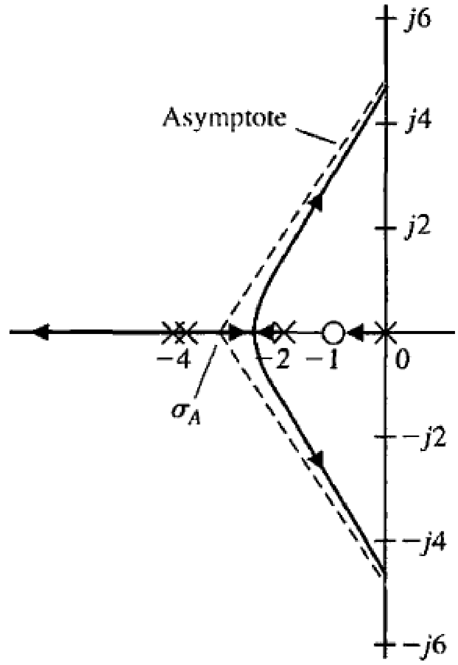
$$\sigma_a = \frac{(-2) + 2(-4) - (-1)}{n - m} = \frac{-10 + 1}{4 - 1} = -3$$

- The angles of the asymptotes that intersect at - 3, given by Eq. (vi), are;

$$\theta_a = \frac{(2k+1)\pi}{n - m} = \frac{(2k+1)\pi}{4 - 1}$$

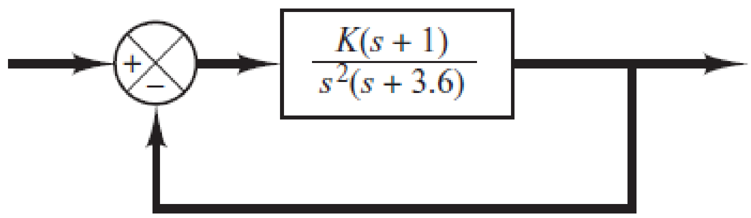
For $K = 0$, $\theta a = 60^{o}$
For $K = 1$, $\theta a = 180^{o}$
For $K = 2$, $\theta a = 300^{o}$

- The root-locus plot of the system is shown in the figure below.
- It is noted that there are three asymptotes. Since n – m = 3.
- The root loci must begin at the poles; two loci (or branches) must leave the double pole at s = -4.
- the breakaway point, σ, can be determine as;



- The solution of the above equation is $\sigma = -2.59$.

Example: Sketch the root loci for the system.



- A root locus exists on the real axis between points ***s* = –1** and ***s* = –3.6**.
- The intersection of the asymptotes and the real axis is determined as,

$$\sigma_a = \frac{0+0+3.6-1}{n-m} = \frac{2.6}{3-1} = -1.3$$

- The angles of the asymptotes that intersect at – 1.3, given by Eq. (vi), are;

$$\theta_a = \frac{(2k+1)\pi}{n-m} = \frac{(2k+1)\pi}{3-1}$$

For $K = 0$, $\theta a = 90^o$
For $K = 1$, $\theta a = -90^o$ or 270^o

- Since the characteristic equation is $s^3 + 3.6s^2 + K(s+1) = 0$
- We have $K = -\frac{s^3 + 3.6s^2}{s+1} \longrightarrow$ (a)

- The breakaway and break-in points are found from Eq. (a) as,

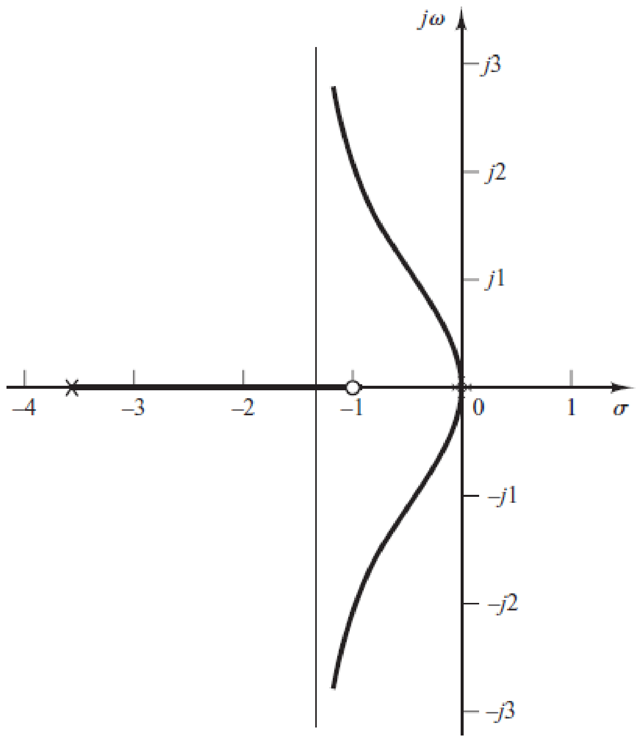
$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 1) - (s^3 + 3.6s^2)}{(s + 1)^2} = 0$$

$$\text{or} \qquad s^3 + 3.3s^2 + 3.6s = 0$$

From which we get,

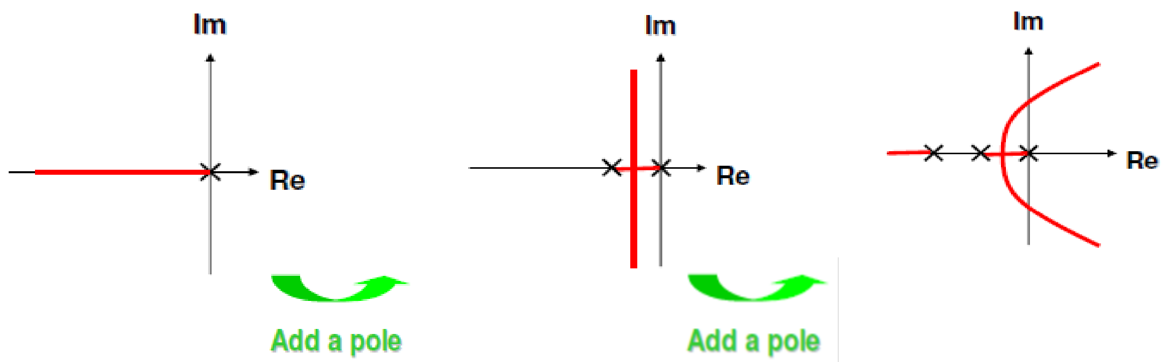
$$s = 0, \qquad s = -1.65 + j0.9367, \qquad s = -1.65 - j0.9367$$

- Point $s = 0$ corresponds to the actual breakaway point. But points $s = 1.65 \pm j0.9367$ neither breakaway nor break-in points, because the corresponding gain values K become complex quantities.



Root Locus – Effect of Adding Poles

Pulling root locus to the RIGHT
– Less stable
– Slow down the settling



Root Locus – Effect of Adding Zeros

Pulling root locus to the LEFT
– More stable
– Speed up the settling

Add a zero

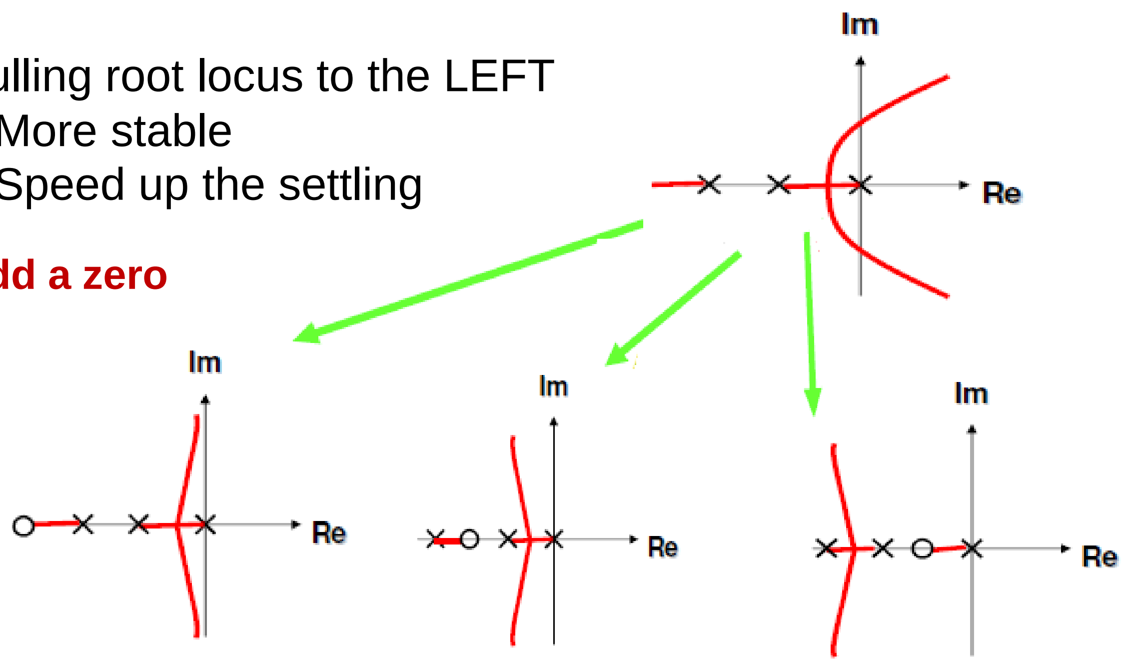
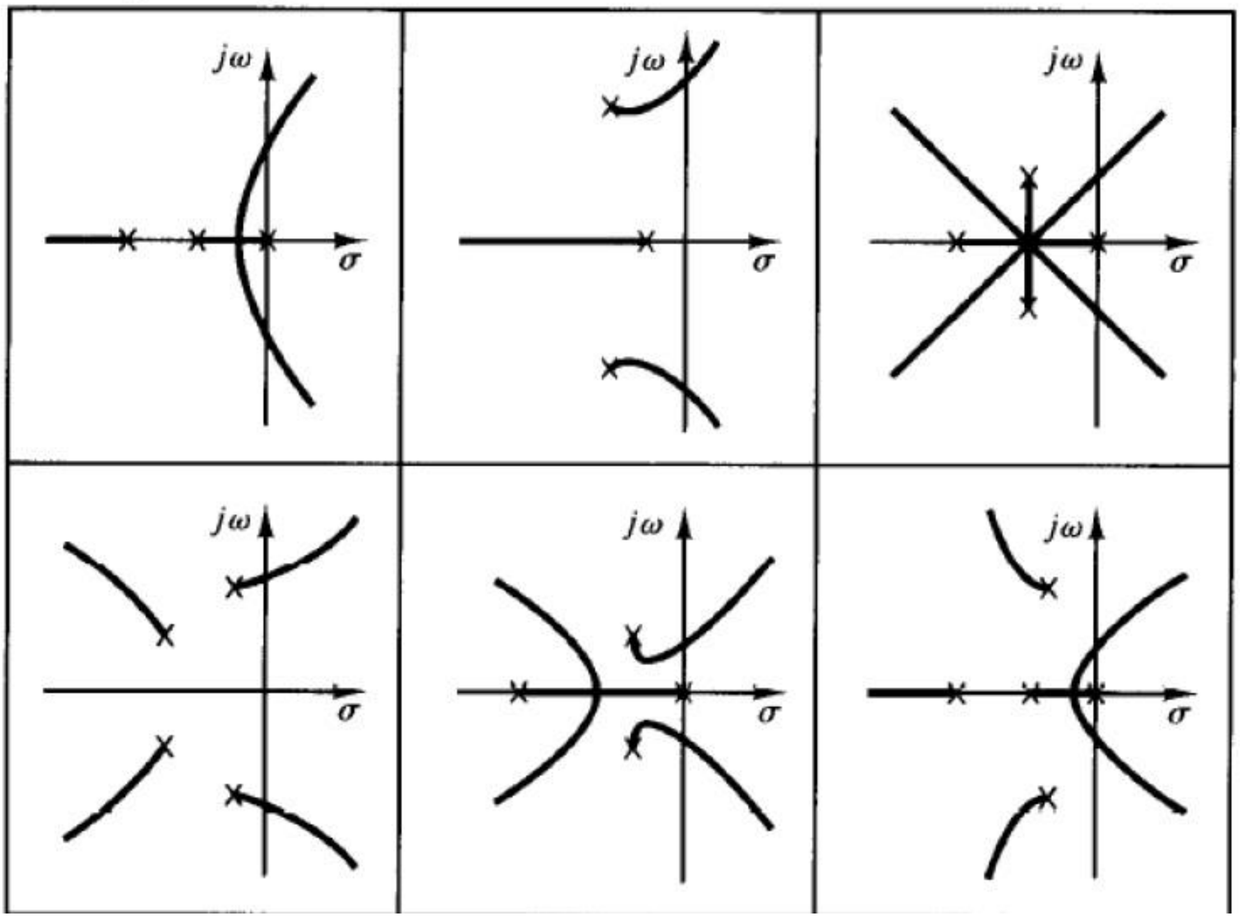
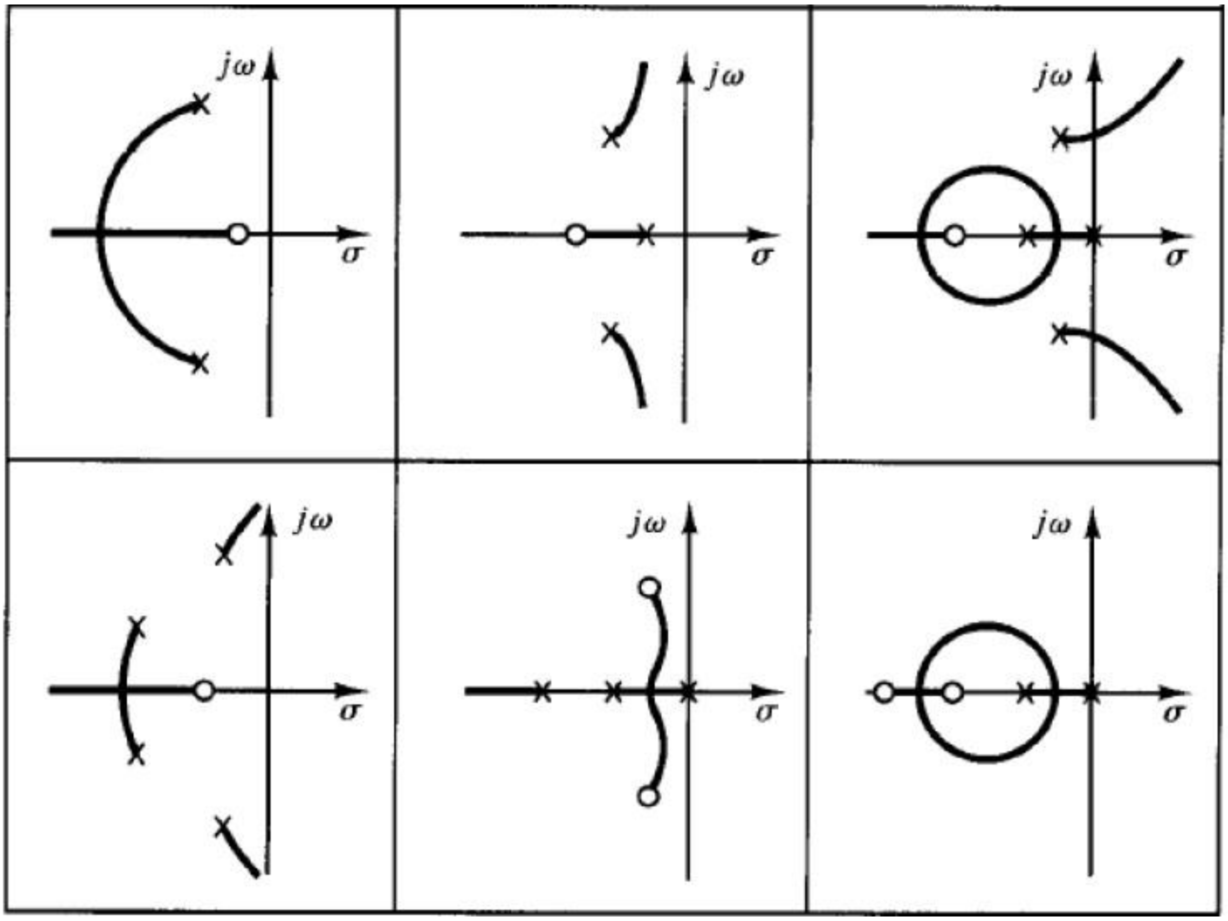


Table Open-Loop Pole–Zero Configurations and the Corresponding Root Loci





Ex.: Find all roots of $x^3 + 4x^2 - 3x - 18 = 0$

Solution: $a_n = 18 : \mp 1, \mp 2, \mp 3, \mp 6, \mp 9, \mp 18$

$f(2) = 8 + 16 - 6 - 18 = 0$, 2 is a root of the eq.

There are two methods to factorize f(x): *long division* & *fast division*.

First method: Fast division

	1	4	-3	-18
2	↓	2	12	18
	1	6	9	0

$\Rightarrow x^2 + 6x + 9 = 0$

$\Rightarrow (x-2)(x^2 + 6x + 9) = 0$

$(x-2)(x+3)(x+3) = 0$

$x_1 = 2, \quad x_2 = -3, \quad x_3 = -3$

The roots are 2, -3, -3

Second method: long division

$$x^3 + 4x^2 - 3x - 18 = 0$$

$$\Rightarrow (x-2)(x^2 + 6x + 9) = 0$$

$$(x-2)(x+3)(x+3) = 0$$

$$\begin{array}{r l}
 & x^2 + 6x + 9 \\
(x-2) & \overline{)\, x^3 + 4x^2 - 3x - 18} \\
 & \underline{\mp x^3 \pm 2x^2} \\
 & 6x^2 - 3x \\
 & \underline{\mp 6x^2 \pm 12x} \\
 & 9x - 18 \\
 & \underline{\mp 9x \pm 18} \\
 & \mathbf{0}
\end{array}$$

FREQUENCY RESPONSE

Review : Trigonometry

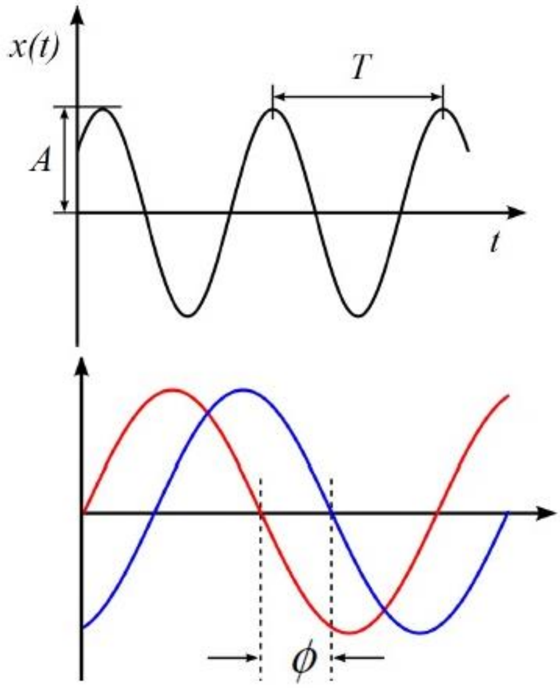
- General definition:

$$x(t) = A\sin\left(\omega t + \phi\right)$$

- Where:
  - A=amplitude
  - ω=angular frequency

$$\omega = 2\pi f$$

  - ϕ =phase shift
  - Period, T=1/f.



Review : Logarithms

- Let us refresh the laws of logarithms:

$$y = x^n \Leftrightarrow \log_x y = n$$

$$\log_{10} x = \log x$$

$$\log_e x = \ln x$$

$$\log\left(xy\right) = \log x + \log y$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log x^n = n\log x$$

$$\log 10 = 1 \; ; \; \log 1 = 0$$

$$\log z = \log\left|z\right| + j\arg(z);\ \text{where } z = a + jb$$

Log-log and semilog scale

- Normally, we can choose a particular scale for our graph. The scale may be a 1-to-1 or 1-to-n scale. In another word, the scale may increase in ones or tens - it is up to you.

- But consider the following function:

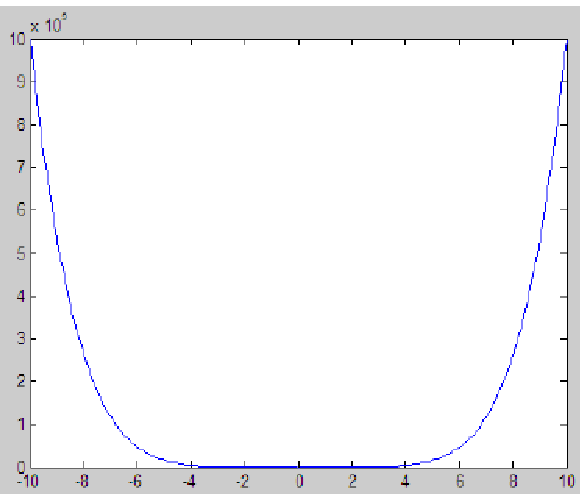
$$y = x^6$$

- Let us create a table of values:

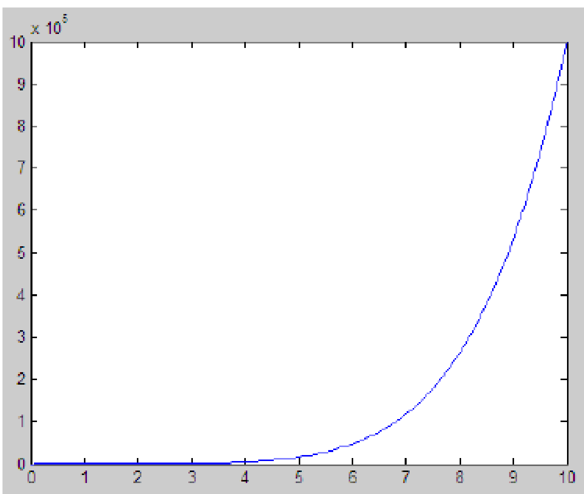
x	0	1	2	3	4	5	6	7	8	9	10
y	0	1	64	729	4,096	15,526	46,656	117,649	262,144	531,441	1,000,000

Log-log and semilog scale

- The plot for this graph is given by:



$-10 \leq x \leq 10$



$0 \leq x \leq 10$

Log-log and semilog scale

- The graph may appear simpler, however, consider the variations in x and y.
- As x varies from 1 to 10, y varies from 1 to 1,000,000.
- Thus, several of these points would not be discernible on a graph so valuable information may be lost.
- This problem may be overcome by using a semilog scale which accommodates the large variation in y.
- The log y is plotted against x, rather than y against x.
- For the example before, we have: $\log y = \log x^6 = 6\log x$

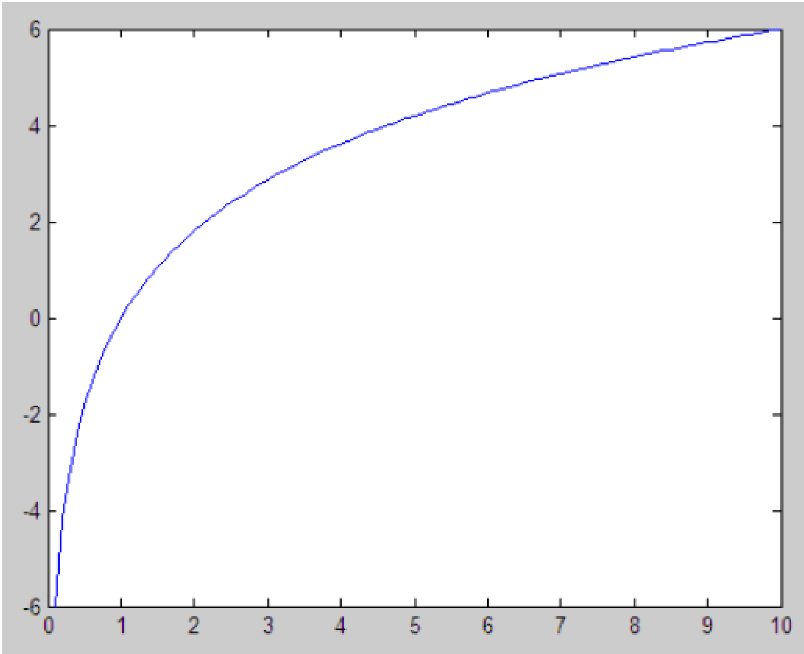
Log-log and semilog scale

- So, in the previous example, as x varies from 1 to 10, "log y" varies from 0 to 6.
- Let us tabulate the results:

x	0	1	2	3	4	5	6	7	8	9	10
log y	-	0	1.806	2.863	3.612	4.193	4.67	5.07	5.42	5.72	6

- The plot of this graph is given in the next slide.

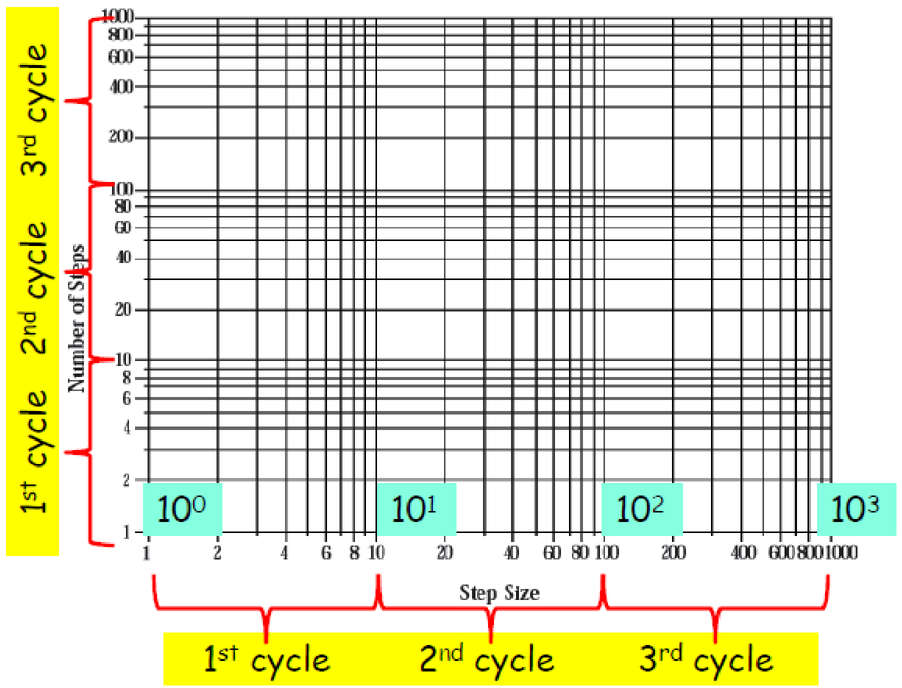
Log-log and semilog scale



Log-log and semilog scale

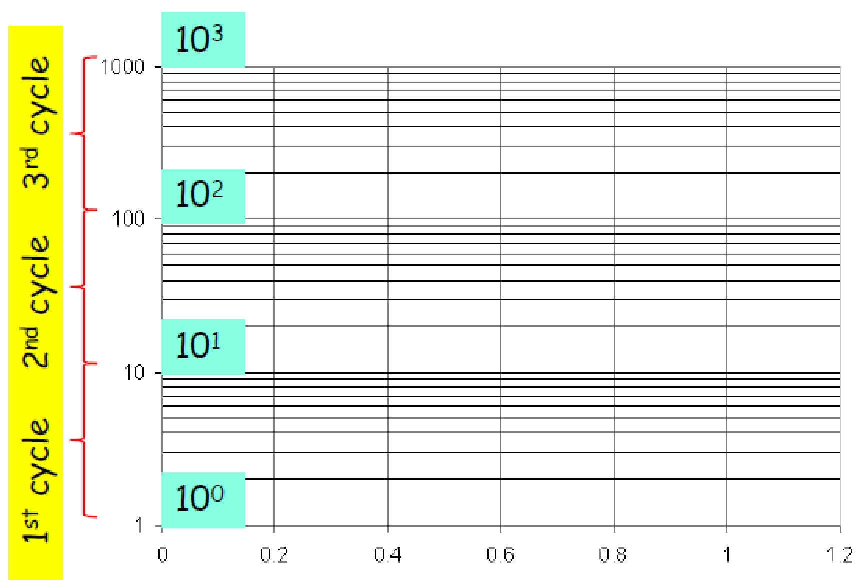
- Thus, in effect, the use of the log scale has compressed a large variation into one which is much smaller and easier to observe.
- Presentation of data on a logarithmic scale can be helpful when the data covers a large range of values – the logarithm reduces this to a more manageable range.
- A plot in which both scales are logarithmic is known as the log-log plot where log y is plotted against log x.
- We need to use log-log and semilog paper in producing these plots.
- Because each tickmark is a power of 10, they are referred to as a decade.

Log-log Graph Paper



Semilog Graph Paper

- The vertical axis has a log scale but a horizontal axis has a normal (linear and equally-spaced) scale, as shown in the following figure.

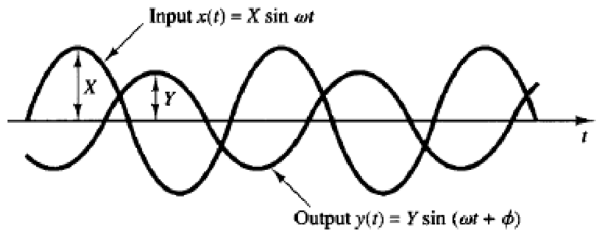


Frequency Response

- Engineers are often interested in how a circuit will respond to a sinusoidal input (i.e. current, voltage).

- Suppose that we have a linear system represented by the following diagram:

- If a sinusoidal input is given, then the output signal is also sinusoidal signal of the same frequency, but with different amplitude and phase.



Frequency Response

- Frequency response of a control system is defined as the steady-state response of the system when the sinusoidal input is applied at the input terminals.

- The sinusoidal input signal when applied to a linear system results in an output signal, which is sinusoidal in steady-state and differs from the input waveform only in amplitude and phase angle.

- Frequency response method determines experimentally the properties of complicated control systems without any difficulty as the sinusoidal test signals for various ranges of frequencies and amplitudes are easily available.

Frequency Response

- Frequency response function describing the sinusoidal steady-state behaviour of the system can be obtained by replacing $s=j\omega$ in the transfer function $G(s)$ of the system.

- The function $G(j\omega)$ representing the sinusoidal steady-state behaviour of the system is a function of complex variable having magnitude and phase angle.

- The magnitude and phase angle of function $G(j\omega)$ for various frequencies are represented by various graphical plots in different coordinates which give better insight for the analysis and design of control systems.

Frequency Response

- We take note that any curve giving information regarding the amplitude (gain) or phase shift of the frequency function is known as the frequency response of the system.

- Typical examples of frequency response graph are:
  - Polar plot (a.k.a. Nyquist plot)
  - Bode plot

In F-R we vary the frequency of the sinusoidal input signal over a certain range and study the resulting steady-state response

The frequency ranges needed for the test are:

0.001 - 10Hz for large time constant

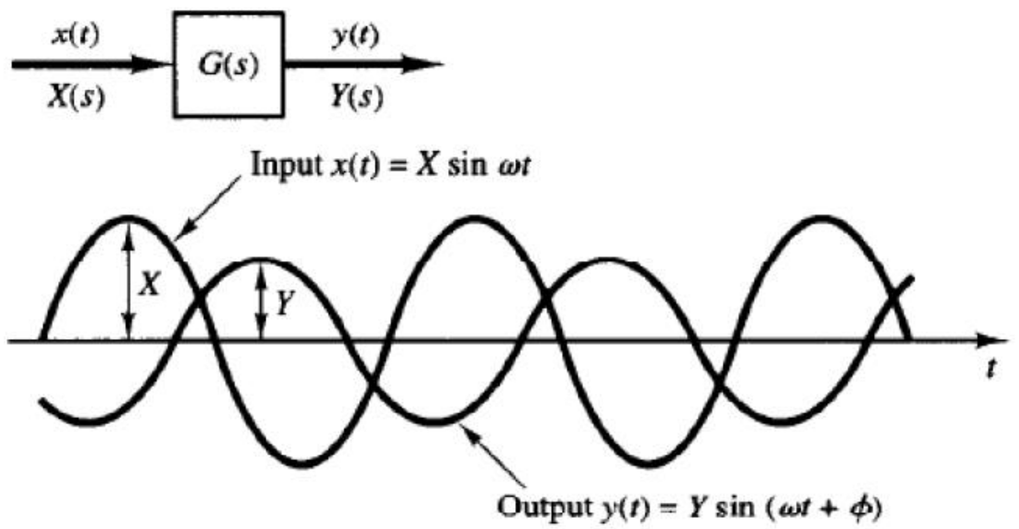
0.1 – 1000Hz for small time constant

The performance of most control system is judged based on the time responses due to certain test signals. This is contrast to the analysis and design of communication systems for which the F-R is of more importance , since the most of the signals to be processed are sinusoidal

The time response is usually more difficult to determine analytically , especially for higher – order system, the F.R tests are simple and can be made accurately

The sinusoidal transfer function G(jω) of any linear control system is obtained by substituting **iω for S** in the T.F .

Consider the linear time- invariant system shown below

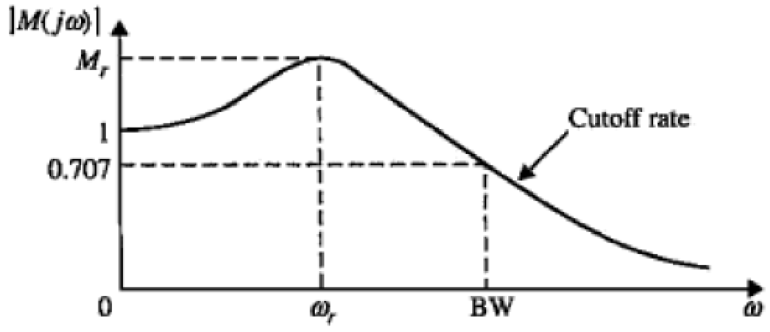


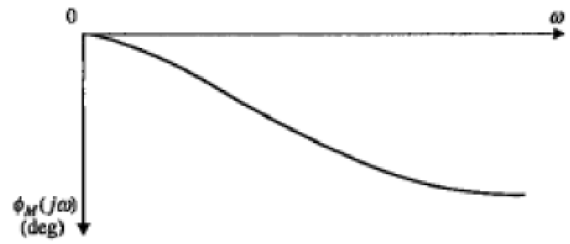
$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right| = \text{amplitude ratio of the output sinusoid to the input sinusoid}$$

$$\angle G(j\omega) = \angle \frac{Y(j\omega)}{X(j\omega)} = \text{phase shift of the output sinusoid with respect to the input sinusoid}$$

Frequency response of closed- loop systems:

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad M(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)H(j\omega)}$$





Typical gain and phase characteristics of a control system

Frequency – response specifications:

1-Resonant peak Mr:

The resonant peak M_r is the maximum value of $|M(j\omega)|$.

In general, the magnitude of M_r gives indication on the relative stability of a stable closed-loop system. Normally, a large M_r corresponds to a large maximum overshoot of the step response. For most control systems, it is generally accepted in practice that the desirable value of M_r should be between 1.1 and 1.5.

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \zeta \le 0.707$$

2-Resonant frequency ωr:

The resonant frequency ω_r is the frequency at which the peak resonance M_r occurs.

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

- For the prototype second-order system, $M_r = 1$ and $\omega_r = 0$ when $\zeta \ge 0.707$.

3-Bandwidth BW:

The range of frequencies (of the input) over which the system will respond satisfactorily. **OR** The **BW** indicates how well the system will track an input sinusoid

The specification of the bandwidth may be determined by the following factors:

1. The ability to reproduce the input signal. A large bandwidth corresponds to a small rise time, or fast response. Roughly speaking, we can say that the bandwidth is proportional to the speed of response.
2. The necessary filtering characteristics for high-frequency noise.

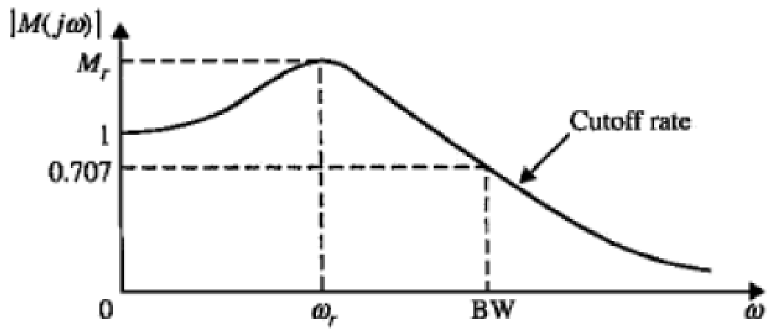
For the system to follow arbitrary inputs accurately, it is necessary that the system have a large bandwidth. From the viewpoint of noise, however, the bandwidth should not be too large. Thus, there are conflicting requirements on the bandwidth, and a compromise is usually necessary for good design. Note that a system with large bandwidth requires high-performance components. So the cost of components usually increases with the bandwidth.

$$BW = \omega_n\left[\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}\right]^{1/2}$$

4-Cutoff Rate:

Cutoff rate. The cutoff rate is the slope of the log-magnitude curve near the cutoff frequency. The cutoff rate indicates the ability of a system to distinguish the signal from noise.

It is noted that a closed-loop frequency response curve with a steep cutoff characteristic may have a large resonant peak magnitude, which implies that the system has relatively small stability margin.

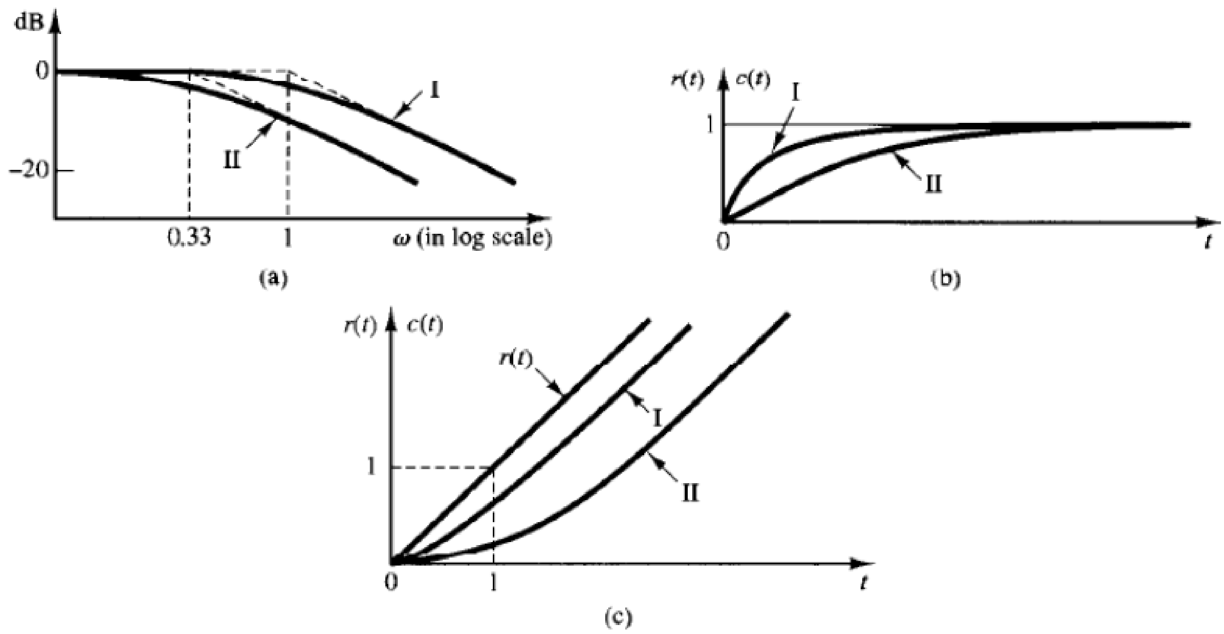


Example:

Consider the following two systems:

$$\text{System I: } \frac{C(s)}{R(s)} = \frac{1}{s+1}, \qquad \text{System II: } \frac{C(s)}{R(s)} = \frac{1}{3s+1}$$

Compare the bandwidths of these two systems. Show that the system with the larger bandwidth has a faster speed of response and can follow the input much better than the one with a smaller bandwidth.



Figs ,a-closed-loop F.R ,b-unit-step response ,c-unit-ramp response

5- Gain margin:

Gain margin is the amount of gain in decibels (dB) that can be added to the loop before the closed-loop system becomes unstable.

Gain margin, a measure of relative stability, is defined as the magnitude of the reciprocal of the open-loop transfer function, evaluated at the frequency ω_π at which the phase angle (see chapter 6) is $-180°$. That is,

$$\text{gain margin} \equiv \frac{1}{|GH(\omega_\pi)|}$$

where $\arg GH(\omega_\pi) = -180° = -\pi$ radians and ω_π is called the **phase crossover** frequency.

6-Phase margin:

Phase margin ϕ_{PM}, a measure of relative stability, is defined as 180° plus the phase angle ϕ_1 of the open-loop transfer function at unity gain. That is,

$$\phi_{PM} \equiv [180 + \arg GH(\omega_1)] \text{ degrees}$$

where $|GH(\omega_1)| = 1$ and ω_1 is called the **gain crossover** frequency.

Gain margin:

It is the maximum value of gain a control system can have to operate in stable state. Small variation over this value leads to unstable system. the gain margin occurs at phase cross over frequency.

Phase crossover frequency:

The phase crossover frequency is the frequency at which the phase angle first reaches $-180°$.

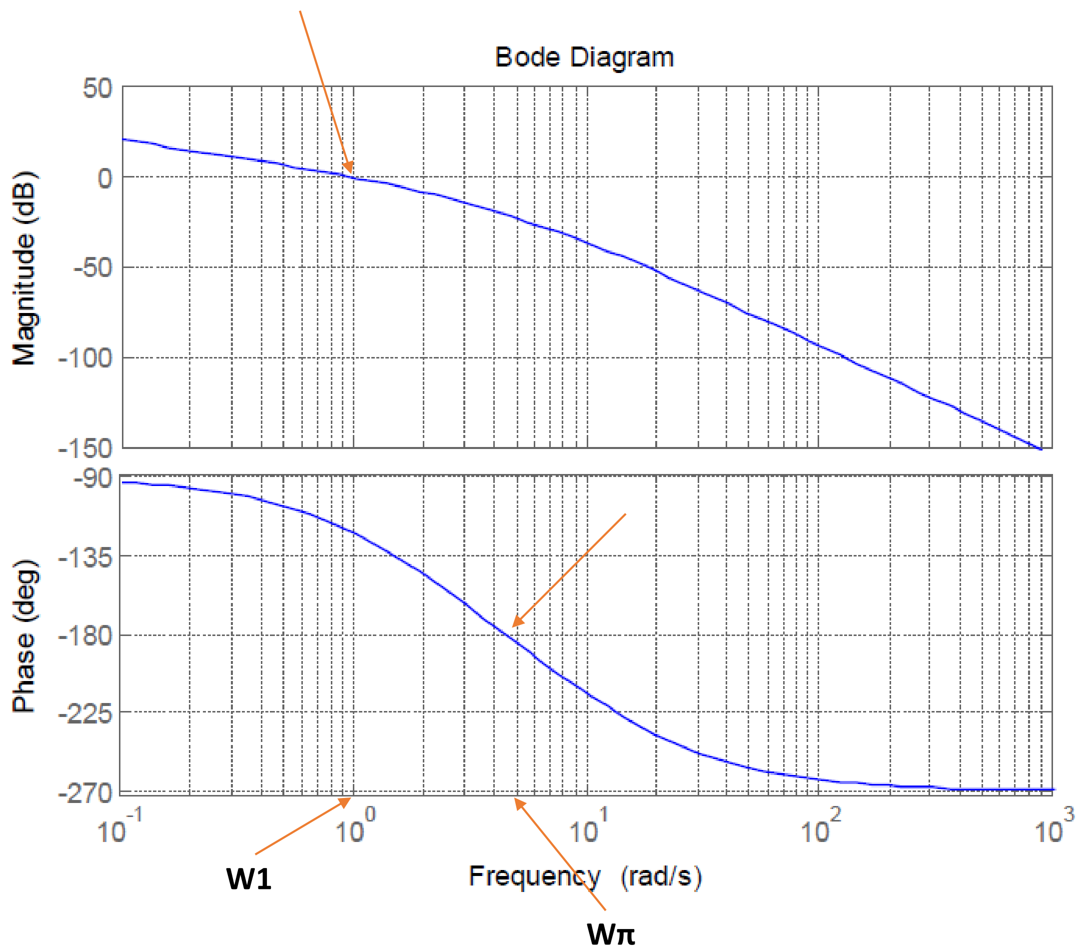
Phase margin:

It is the amount of phase lag added to gain cross over frequency to bring system from stable to unstable.

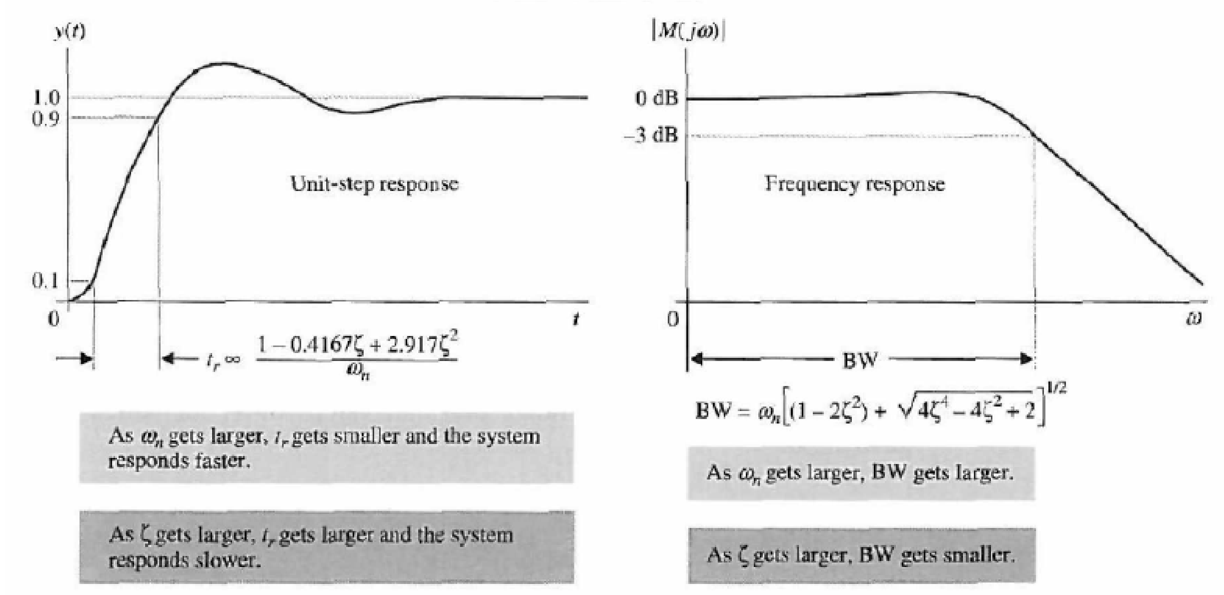
phase margin = 180 + phase angle at which gain cross over frequency occurs

Gain crossover frequency:

It is the frequency at which magnitude of open loop transfer function is unity or else the frequency at which the gain of the system is zero



The correlation between pole locations , unit step response and the magnitude of the frequency response for the prototype –second order system are summarized in figure below.



Example: The specification on a second-order unity feedback control system with closed-loop T.F

$$M(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

are that the maximum overshoot must not exceed 10% and the rise time must be less than 0.1 sec. Find the corresponding limiting values of M_r and BW analytically.

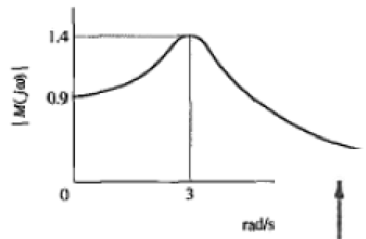
Maximum overshoot $= 0.1$ Thus, $\zeta = 0.59$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.05 \qquad t_r = \frac{1-0.416\zeta+2.917\zeta^2}{\omega_n} = 0.1 \text{ sec}$$

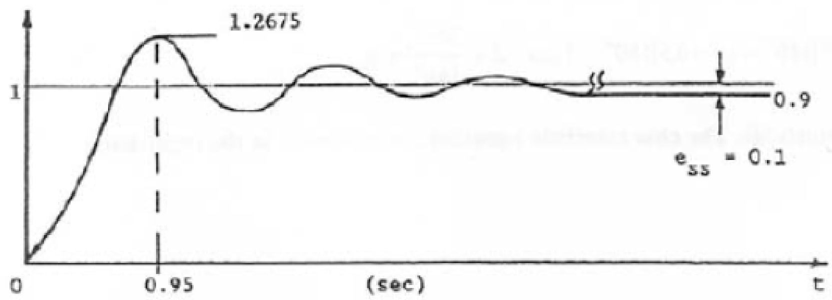
Thus, minimum $\omega_n = 17.7$ rad / sec Maximum $M_r = 1.05$

Minimum BW = $\omega_n\left(\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}\right)^{1/2} = 20.56$ rad/sec

Example: The closed-loop frequency response M(ωr)versus frequency of a second –order system (proto-type) is shown in figure below. Sketch the corresponding unit- step response of the system.



solution:



$M_r = 1.4 = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ Thus, $\zeta = 0.387$ Maximum overshoot $= e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.2675\ (26.75\%)$

$\omega_r = 3 \text{ rad/sec} = \omega_n\sqrt{1-2\zeta^2} = 0.8367\omega_n$ rad/sec $\quad \omega_n = \dfrac{3}{0.8367} = 3.586$ r ad/sec

$t_{max} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{3.586\sqrt{1-(0.387)^2}} = 0.95$ sec $\quad$ At $\omega = 0$, $|M| = 0.9$.

There are three commonly used representation of sinusoidal transfer function:

1-Bode diagram or logarithmic plot

2-Nyquist plot or polar plot

3-Nichols plot

Bode Plot

- The Bode plot is generally:
  - Plots of frequency response. Gain and phase are displayed in separate plots.
  - Logarithmic plots.
  - The horizontal axis is frequency - plotted on a log scale. It can be either f or ω.
  - The vertical axis is gain, expressed in decibels - a logarithmic measure of gain.
  - Sometimes, the vertical axis is simply a gain on a logarithmic scale.

Bode Plot

- And in general, a Bode plot consists of two components (subplot):
  - The ratio of the amplitudes of the output signal and the input signal is plotted against frequency.
  - The phase shift between the input and output signal is plotted against the frequency.

- The Bode plot allows us to experimentally determine the transfer function without the tedious process of detailed modeling.

Bode Plot

- Consider the following system:

- If the input signal is given as:

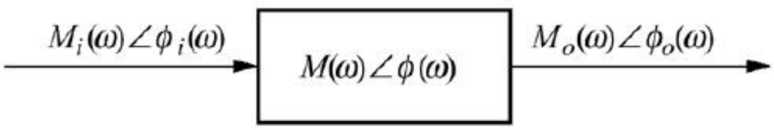
$$x(t) = \underbrace{X}_{\substack{\text{input}\\ \text{amplitude}}} \sin\left(\underbrace{\omega}_{\text{frequency}} t \right)$$

- Then, the output signal is:

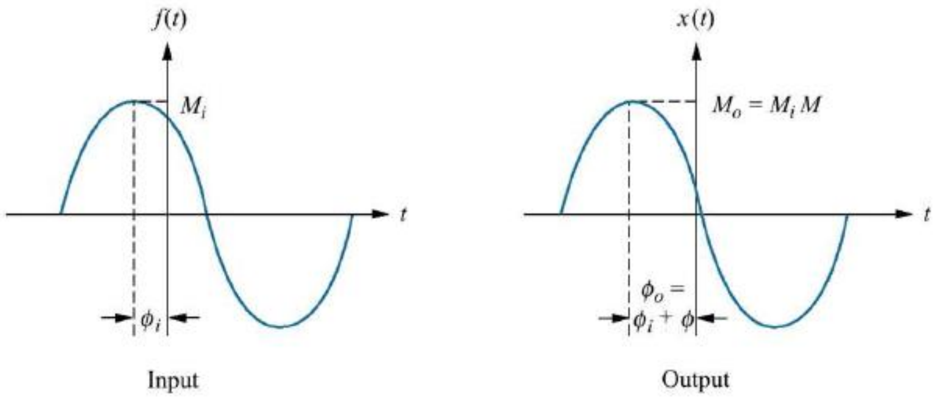
$$y(t) = \underbrace{Y}_{\substack{\text{output}\\ \text{amplitude}}} \sin\left(\omega t + \underbrace{\phi}_{\substack{\text{phase}\\ \text{shift}}} \right)$$

Bode Plot

- Let us view this pictorially: given the following system:



- The output then is:



Bode Plot

- Note that the transfer function is given as:

$$G(s) = \frac{Y(s)}{X(s)}$$

- Therefore, the magnitude expression is:

$$|G(s)| = \left|\frac{Y(s)}{X(s)}\right|$$

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right| = \sqrt{[\operatorname{Re} G(j\omega)]^2 + [\operatorname{Im} G(j\omega)]^2}$$

- Note that we substitute s=jω because we are dealing with sinusoidal signal.

- The magnitude is plot in decibels (dB). $|G(s)| = 20\log_{10}\left[G(j\omega)\right]$

A bit on decibel…

- A decibel (dB) is a ratio between two numbers on a logarithmic scale.

- A decibel is not itself a number, and cannot be treated as such in normal calculations.

- Widely used when dealing with sinusoidal function and waves.

- It confers a number of advantages, such as:
  - the ability to conveniently represent very large or small numbers, a logarithmic scaling that roughly corresponds to the human perception of sound and light; and
  - the ability to carry out multiplication of ratios by simple addition and subtraction.

Bode Plot

- The phase shift is given as:

$$\angle G(s) = \angle \frac{Y(s)}{X(s)}$$

$$\angle G(j\omega) = \angle \frac{Y(j\omega)}{X(j\omega)}$$

$$\angle G(j\omega) = \phi = \tan^{-1}\left[\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right]$$

- Note that a positive angle is known as "phase-lead" and a negative angle is known as "phase-lag"

Bode form and bode gain:

Example: Determine the bode form and bode gain for the open-loop T.F:

$$GH(s) = \frac{K(s+2)}{s^2(s+4)(s+6)}$$

Solution: convert s-domain to frequency domain s= jω

$$GH(j\omega) = \frac{K(j\omega+2)}{(j\omega)^2(j\omega+4)(j\omega+6)}$$

The bode form is :

$$GH(j\omega) = \frac{K/12(j\omega/2+1)}{(j\omega)^2(j\omega/4+1)(j\omega/6+1)}$$ **, Bode gain = k /12**

Example

- Sketch the Bode plot for the following transfer function:

$$G(s) = \frac{2}{s}$$

- Note : a transfer function of the form $G(s)=1/s$ is said to be an integrator.

Solution to Example

- We take note that the transfer function is in the normal form. Therefore: $$G(s) = \frac{2}{s}$$

- Putting s=jω yield: $$G(j\omega) = \frac{2}{j\omega}$$

- And the magnitude is found by:

$$|G(j\omega)| = \frac{2}{\sqrt{0^2 + \omega^2}} = \frac{2}{\omega}$$

Solution to Example

- The magnitude in decibel is given by:

$$|G(j\omega)| = 20\log\left(\frac{2}{\omega}\right) = 20\log 2 - 20\log\omega = 6.02 - 20\log\omega$$

- We note that the gain (K) is K=6.02.

- To determine the vertical intercept, we simply let ω=1, and hence the second term is dropped out. Therefore, the graph's vertical intercept at 20log(2).

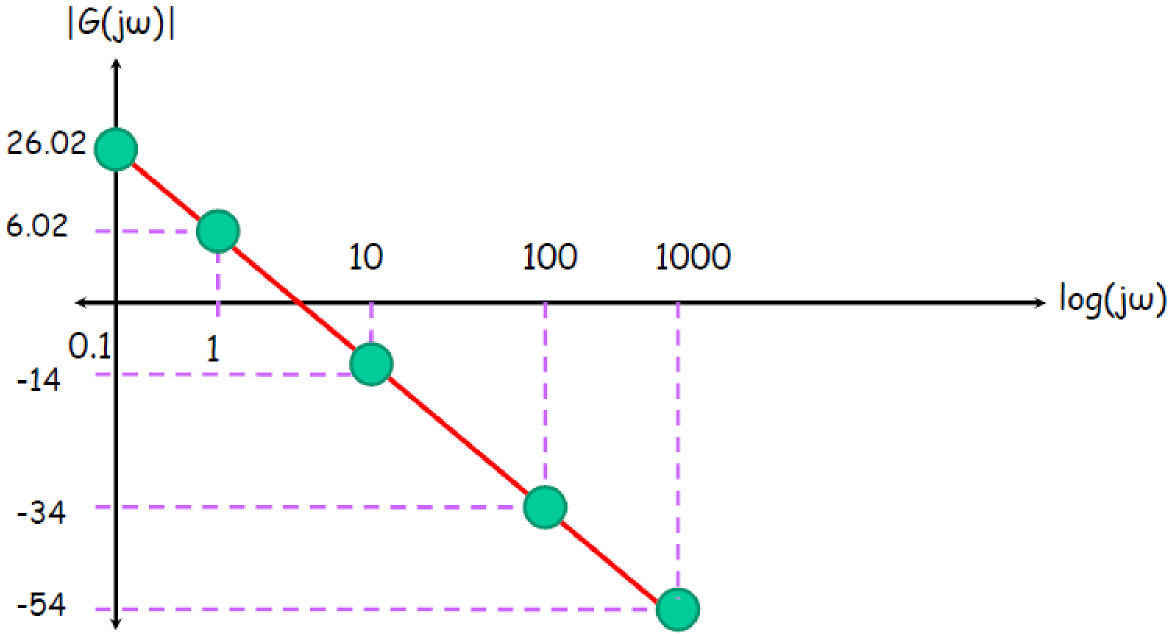
Solution to Example 1

- Let us generate the table of values for this function:

ω	G(jω)
0.1	26.0206
1	6.0206
10	-13.9794
100	-33.9794
1000	-53.9794
10000	-73.9794

Solution to Example

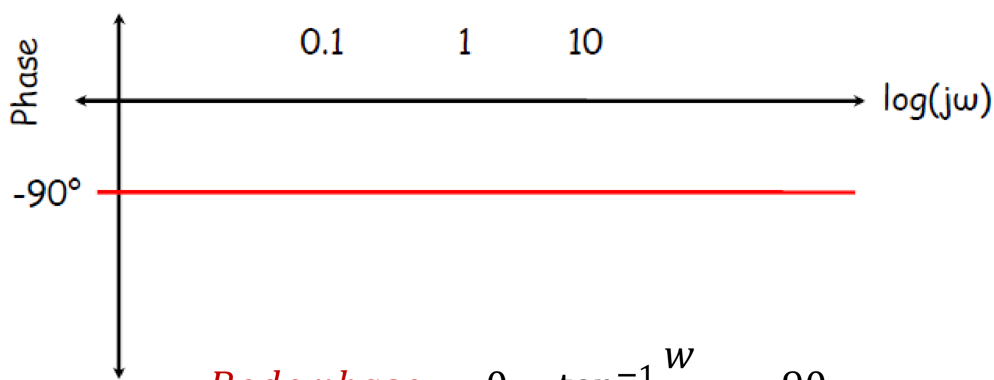
- Hence, our Bode plot of the magnitude is:



Solution to Example

- The constant is positive, so it has not much of an effect in terms of the phase.

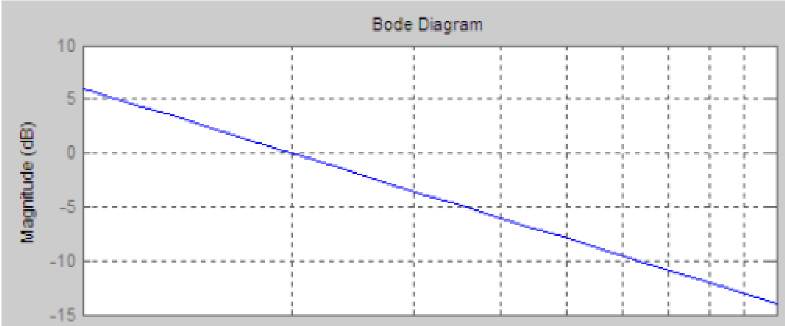
- We have a pole at origin, then, there will be a phase shift of -90°. And since there are no poles located at the origin, the our Bode phase plot is just a simple straight line.



$$\textit{Bode phase} = 0 - \tan^{-1}\frac{w}{0} = -90$$

Solution to Example

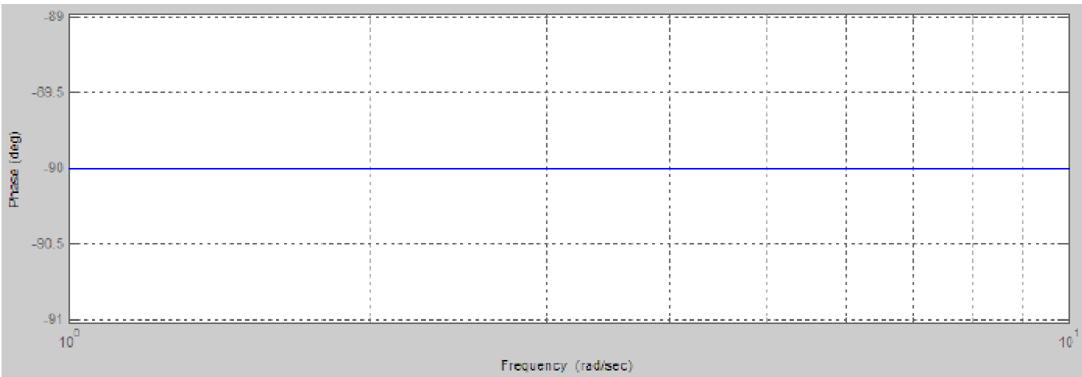
- We can also verify our plot with MATLAB.




```
clc;
num=[2];
den=[1 0];
sys=tf(num,den);
sgrid on;
bode(sys)
```

Solution to Example

- MATLAB verification:




```
clc;
num=[2];
den=[1 0];
sys=tf(num,den);
sgrid on;
bode(sys)
```

Example

- Sketch the magnitude Bode plot for the following system:

$$G(s) = s$$

- Note : this transfer function is said to a differentiator.

Solution to Example

- Since the function is in the standard Bode form, therefore, we have:

$$G(s) = s$$
$$G(j\omega) = j\omega$$

- Determining the magnitude gives:

$$|G(j\omega)| = \sqrt{0^2 + \omega^2} = \omega$$

- Putting in decibel yield:

$$|G(j\omega)| = 20\log\omega$$

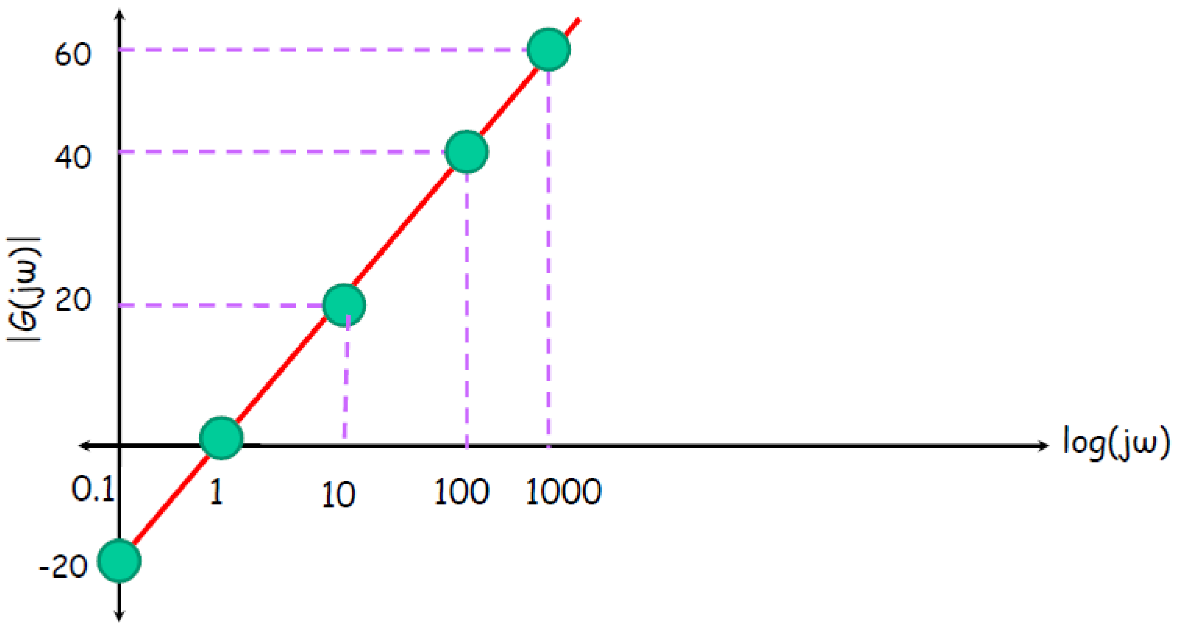
Solution to Example

- Let us generate the table of values for this function:

ω	$G(j\omega)$
0.1	-20
1	0
10	20
100	40
1000	60
10000	80

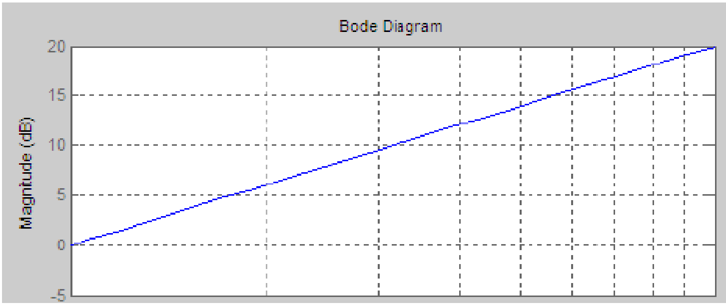
Solution to Example

- Next, we plot this function:



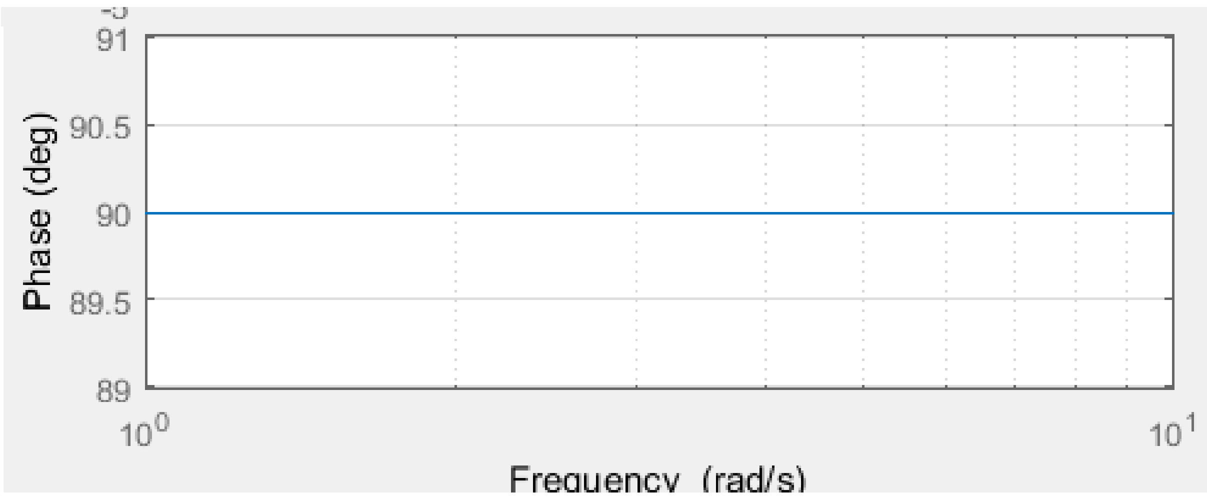
Solution to Example

- MATLAB verification:




```
clc;
num=[1 0];
den=[1];
sys=tf(num,den);
sgrid on;
bode(sys)
```

$$\textit{Bode phase} = \tan^{-1}\frac{w}{0} = 90$$



How to Sketch a Bode Plot

- Just like a root locus, there are few rules that you need to follow:
  1. Determine the transfer function of the system.

$$G(s)=\frac{Y(s)}{X(s)}=\frac{K(s+z_1)}{s(s+p_1)}$$

  2. Rewrite the transfer function into standard Bode form:

$$G(s)=\frac{K\,z_1\left(\frac{s}{z_1}+1\right)}{s\,p_1\left(\frac{s}{p_1}+1\right)}$$

Factorize!

Pause for a moment.

- Exercise : Rewrite the following transfer function in standard Bode form.

$$G(s)=\frac{30(s+10)}{(s+3)(s+50)}$$

- Answer:

$$G(s)=\frac{2\left(\frac{s}{10}+1\right)}{\left(\frac{s}{3}+1\right)\left(\frac{s}{50}+1\right)}$$

How to sketch a Bode plot?

- Continuing on:
  3. Replace the term "s" with s=jω. Hence:

$$G(s)=\frac{K\,z_1\left(\frac{j\omega}{z_1}+1\right)}{j\omega\,p_1\left(\frac{j\omega}{p_1}+1\right)}$$

  4. Next, find the magnitude for the above transfer function in decibels (dB).

$$|G(s)|=20\log\left[\frac{K\,z_1\left(\frac{j\omega}{z_1}+1\right)}{j\omega\,p_1\left(\frac{j\omega}{p_1}+1\right)}\right]$$

How to sketch a Bode plot?

- Continuing on by expanding the previous expression:

$$|G(s)|=\underbrace{20\log\left[K\,z_1\left(\frac{j\omega}{z_1}+1\right)\right]}_{\text{expand}}-\underbrace{20\log\left[j\omega\,p_1\left(\frac{j\omega}{p_1}+1\right)\right]}_{\text{expand}}$$

$$|G(s)|=20\left[\log K+\log z_1+\log\left(\frac{j\omega}{z_1}+1\right)\right]-20\left[\log(j\omega)+\log p_1+\log\left(\frac{j\omega}{p_1}+1\right)\right]$$

- From the above equation, we should observe the significance of each of the above expression:
  - Constant term, K.
  - Poles and zeros at origin, jω.
  - Poles and zeros not at the origin : $\log\left(\frac{j\omega}{p_1}+1\right)$ or $\log\left(\frac{j\omega}{z_1}+1\right)$

Example

- Sketch the Bode magnitude plot for the following transfer function:

$$G(s) = \frac{1}{2s+100}$$

- Note, we will use BOTH methods to plot the graph. You know,

Solution to Example : Method (1)

- Step 1 : Repose the above equation in standard Bode form.

$$G(s) = \frac{1}{2s+100} = \frac{1}{2(s+50)} = \frac{1}{2} \cdot \frac{1}{50\left(\frac{s}{50}+1\right)} = \frac{1/100}{\frac{s}{50}+1} = \frac{0.01}{0.02s+1}$$

- If you notice, the standard Bode form for this particular transfer function is of the following form:

$$G(s) = \frac{K}{\frac{s}{p_1}+1}$$

- And we note that $K=0.01$ and $p_1=50$.

Solution to Example : Method (1)

- We note since we have a pole, then the graph will have a ramp going downwards at $p_1=50$.

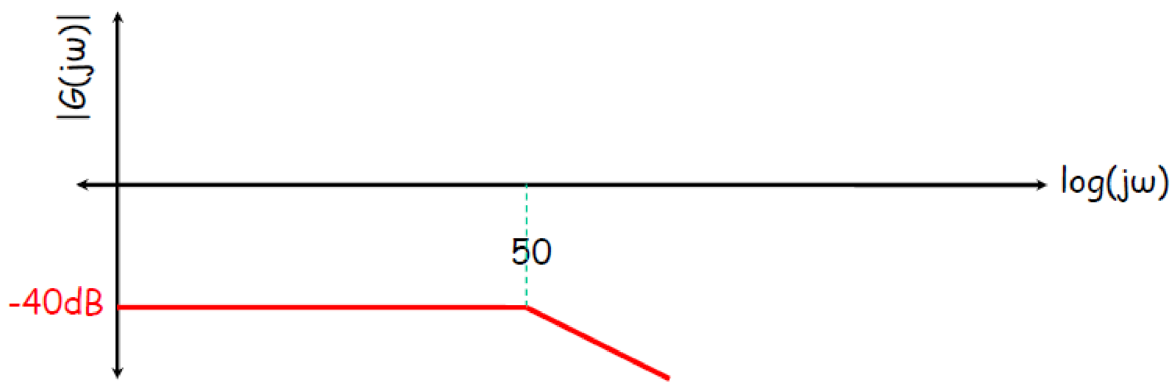
- The magnitude of this transfer function is, after substituting s=jω:

$$|G(j\omega)| = 20\log\left(\frac{0.01}{\sqrt{1^2+(0.02\omega)^2}}\right) = 20\left[\log 0.01 - \frac{1}{2}\log\left(1+0.0004\omega^2\right)\right]$$

$$|G(j\omega)| = 20\log 0.01 - 10\log\left(1+0.0004\omega^2\right)$$

$$|G(j\omega)| = \underbrace{-40}_{\text{constant (K)}} \underbrace{-10\log\left(1+0.0004\omega^2\right)}_{\text{poles not at origin}}$$

Solution to Example : Method (1)



- Now, this is Bode plot of the magnitude G(jω).

Solution to Example : Method (2)

- Step 1 : Determine the magnitude of the equation.

$$G(s)=\frac{1}{2s+100}$$

$$G(j\omega)=\frac{1}{2j\omega+100}$$

$$\left|G(j\omega)\right|=\frac{1}{\sqrt{(2\omega)^2+100^2}}=\frac{1}{\sqrt{4\omega^2+100^2}}$$

$$\left|G(j\omega)\right|=20\log\left[\frac{1}{\sqrt{4\omega^2+100^2}}\right]=20\log 1-10\log\left(4\omega^2+100^2\right)$$

$$\left|G(j\omega)\right|=-10\log\left(4\omega^2+100^2\right)$$

$$Bode\ phase\ =0-\tan^{-1}\frac{w}{50}$$

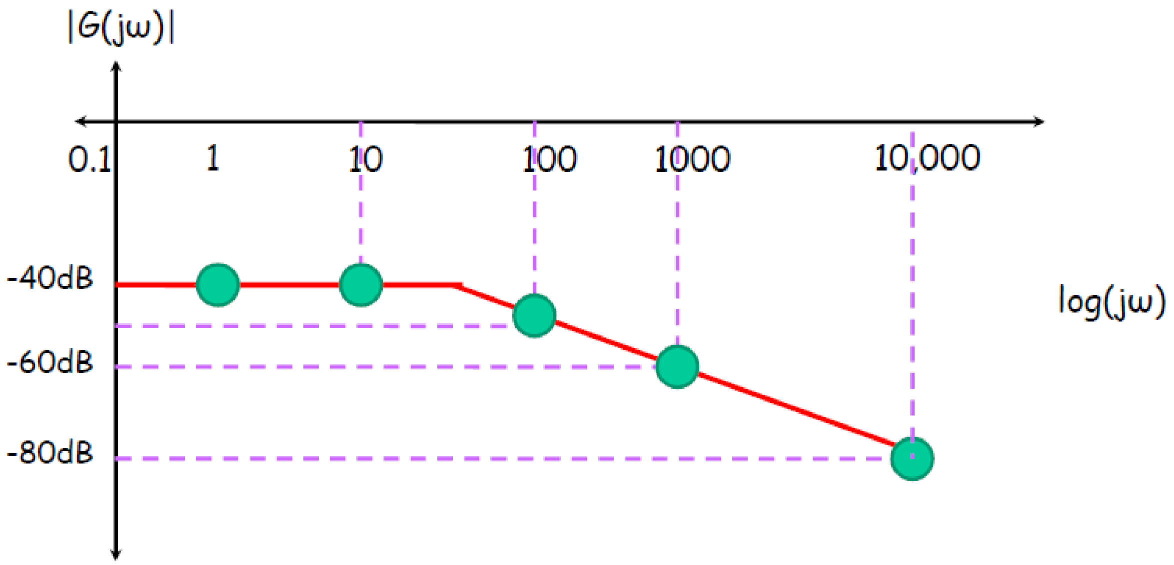
Solution to Example : Method (2)

- Step 2 : Let us tabulate the results.

ω	G(jω)	Phase
0.1	-40	-0.114
1	-40.0004	-1.145
10	-40.0432	-11.309
100	-43.0103	-63.434
1000	-60.0432	-87.137
10000	-80.0004	-89.713

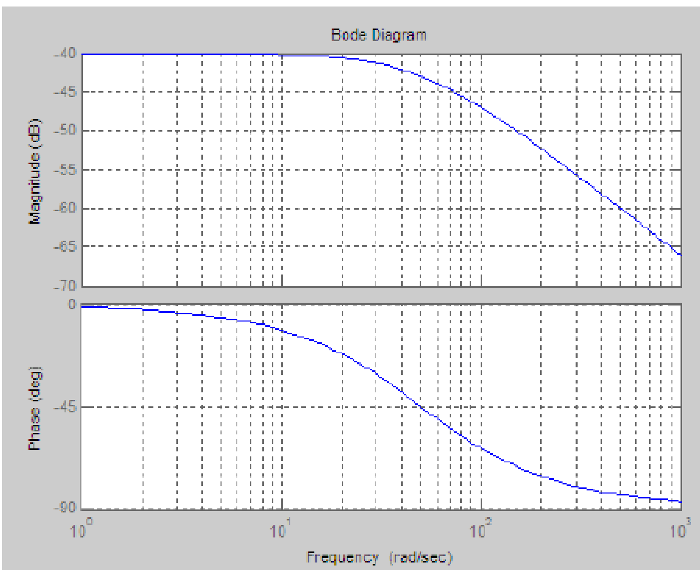
Solution to Example : Method (2)

- Let us now sketch the Bode plot:



Solution to Example

- We can verify our solution with MATLAB.




```
num=[1];
den=[2 100];
sys=tf(num,den);
sgrid on;
bode(sys)
```

Example

- Sketch the Bode plot of the magnitude $G(s)$ for the following system:

$$G(s) = \frac{5}{s(s+1)}$$

$$G(j\omega) = \frac{5}{j\omega(j\omega+1)}$$

$$\text{Bode phase} = 0 - \tan^{-1}\frac{w}{0} - \tan^{-1}\frac{w}{1}$$
$$= 0 - 90 - \tan^{-1} w$$

Solution to Example

- calculate the magnitude in decibel after substituting $s=j\omega$.

$$|G(j\omega)| = 20\log G(j\omega) = 20\log\left[\frac{5}{j\omega(j\omega+1)}\right]$$

$$|G(j\omega)| = 20\left[\log 5 - \left(\log j\omega + \log(j\omega+1)\right)\right]$$

$$|G(j\omega)| = 14 - \underbrace{20\log j\omega}_{\substack{\text{pole at origin} \\ \text{i.e. at } \omega=1}} - \underbrace{20\log(j\omega+1)}_{\substack{\text{pole not at origin} \\ \text{i.e. but at } p_1=1}}$$

- The vertical intercept is found by approximating $\omega=0.1$.

$$|G(j\omega)| = 14 - 20\log j\omega - 20\log(j\omega+1) = 33.17 \approx 33$$

Solution to Example

- However, making approximation such as ω=0.1 is not accurately correct. The correct method is:

$$G(j\omega)=\frac{5}{j\omega(j\omega+1)}=\frac{5}{j\omega-\omega^2}=\frac{5}{-\omega^2+j\omega}$$

Substitute s=jω

$$|G(j\omega)|=\frac{5}{\sqrt{(-\omega^2)^2+\omega^2}}=\frac{5}{\sqrt{\omega^4+\omega^2}}$$

Calculate the magnitude.

$$|G(j\omega)|=20\log\left[\frac{5}{\sqrt{\omega^4+\omega^2}}\right]$$

Express the magnitude in decibels.

$$|G(j\omega)|=20\log 5-10\log(\omega^4+\omega^2)$$

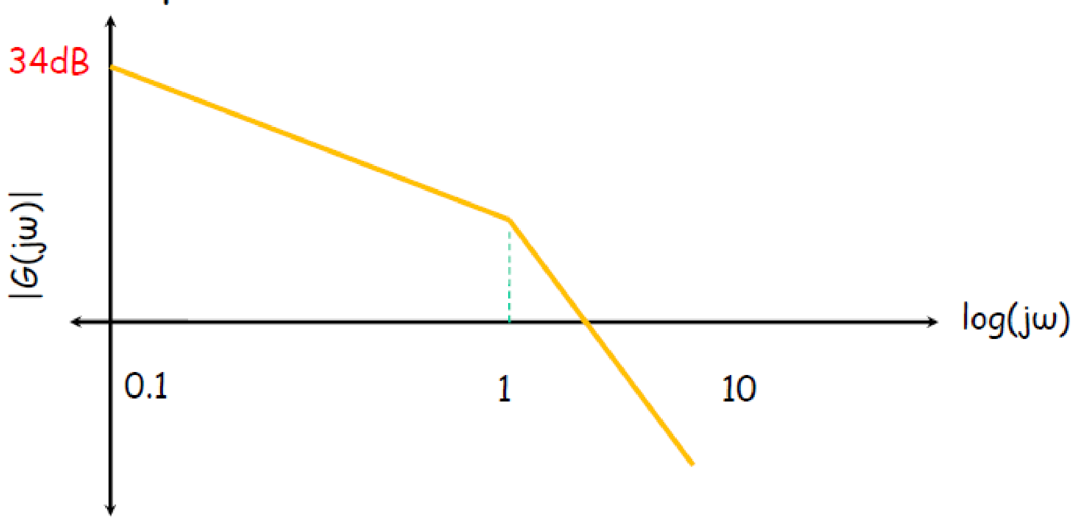
$$|G(j\omega)|=14-10\log(\omega^4+\omega^2)$$

$$|G(j\omega)|=14-10\log(0.1^4+0.1^2)\simeq 34$$

Substitute s=0.1.

Solution to Example

- Superposing all the lines gives the following the final plot:



Solution to Example

- First, write the transfer function in standard Bode form. We see that the transfer function is already in standard Bode form:

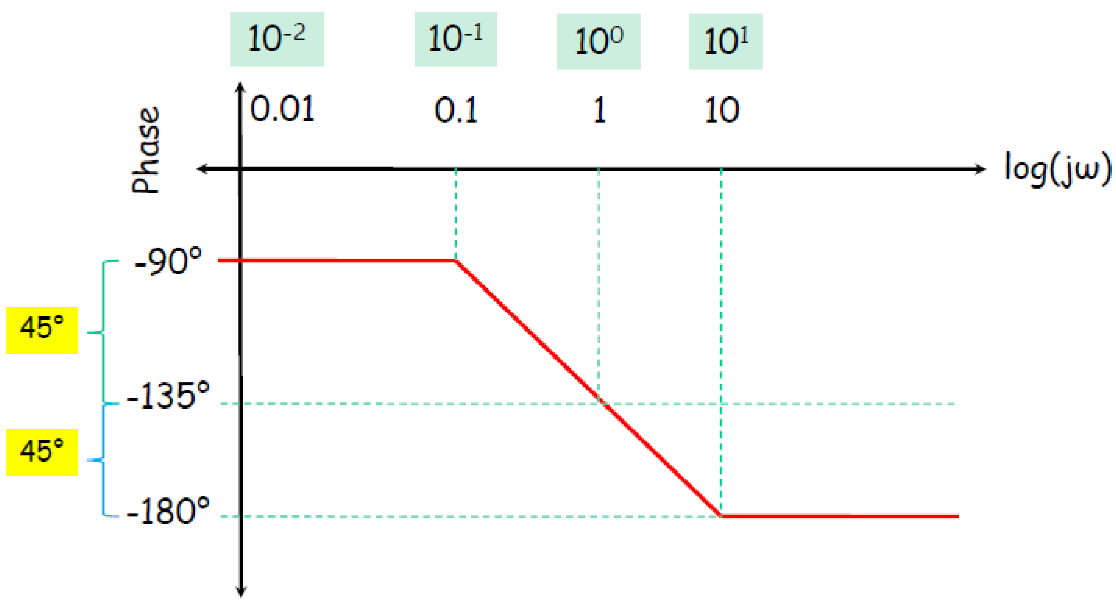
$$G(s) = \frac{5}{s(s+1)}$$

- The gain, K is 5. It has no effect on the Bode phase plot. But we have two poles - one pole is at the origin (ω=1) and the other pole not located at the origin is also at (ω=1).

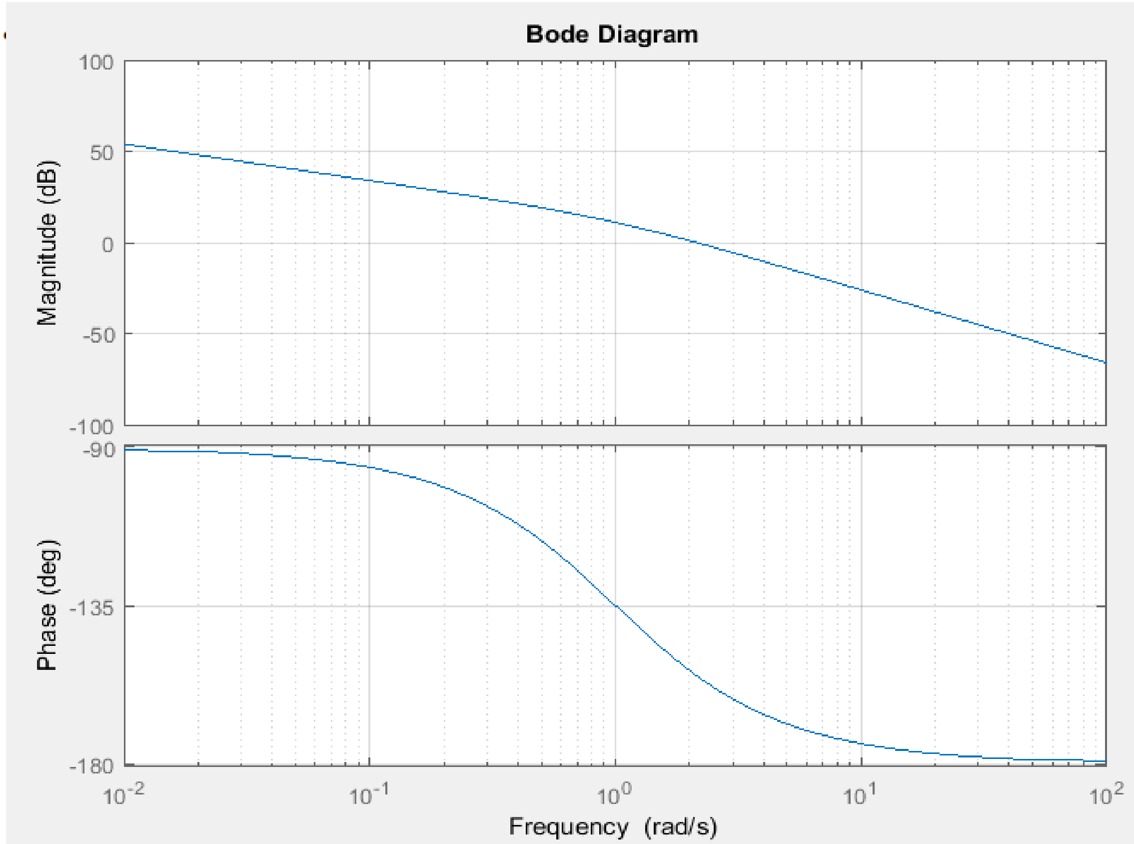
- It simply means that the pole at the origin will shift the graph -90° and for the other pole, at ω=1, there will be a slope going down at 45° per decade. The starting point for the slope to go down is at $p_1/10=1/10=0.1$ and the ending point for the slope is at p_1*10.

Solution to Example

- Let us now sketch the Bode phase plot for individual term:



Solution to Example



Example: Determine accurate values of bode magnitude and phase angle for the following open-loop transfer function.

$$GH(s) = \frac{20}{s(s+2)(s+10)}$$, **corner frequencies are 2 , 10**

$$GH(j\omega) = \frac{1}{j\omega(0.5j\omega+1)(0.1j\omega+1)}$$, **bode gain =1**

Magnitude equation:

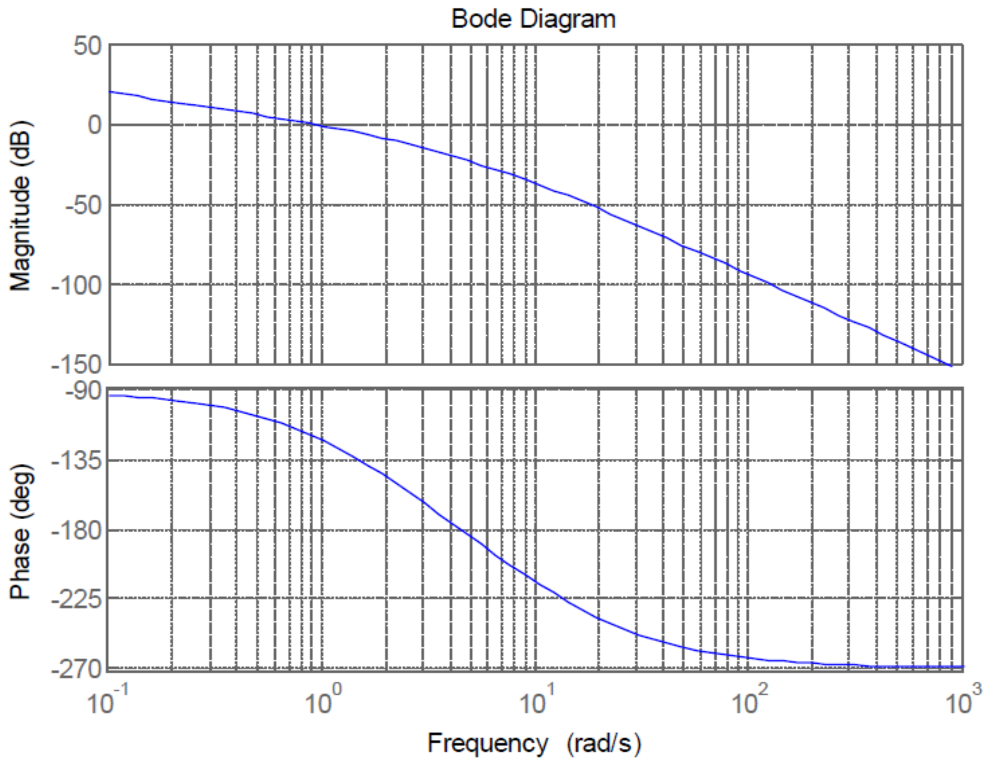
$|GH(j\omega)|_{db} = 20\text{Log}1 - 20\text{Log}\omega - 20\text{Log}\sqrt{1+(0.5\omega)^2} - 20\text{Log}\sqrt{1+(0.1\omega)^2}$

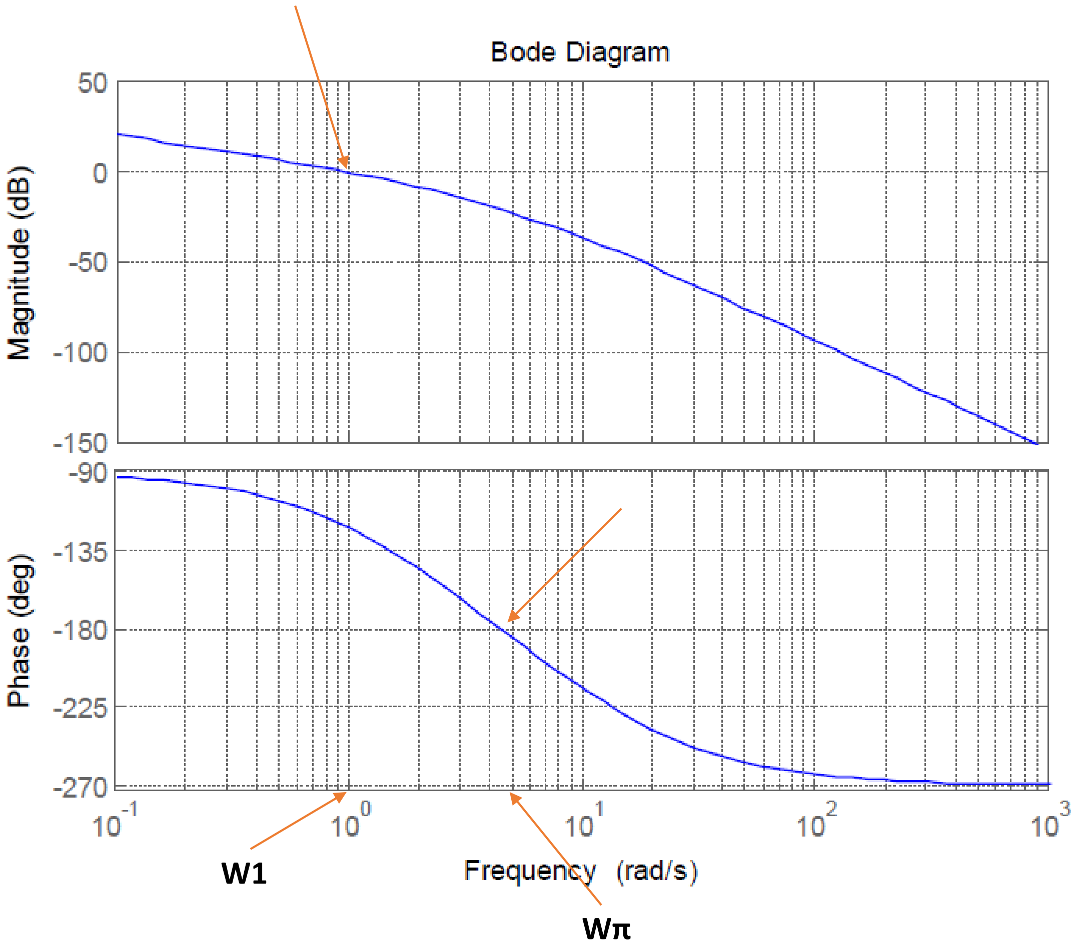
Phase equation:

Phase $= 0 - 90 - \tan^{-1} 0.5\omega - \tan^{-1} 0.1\omega$

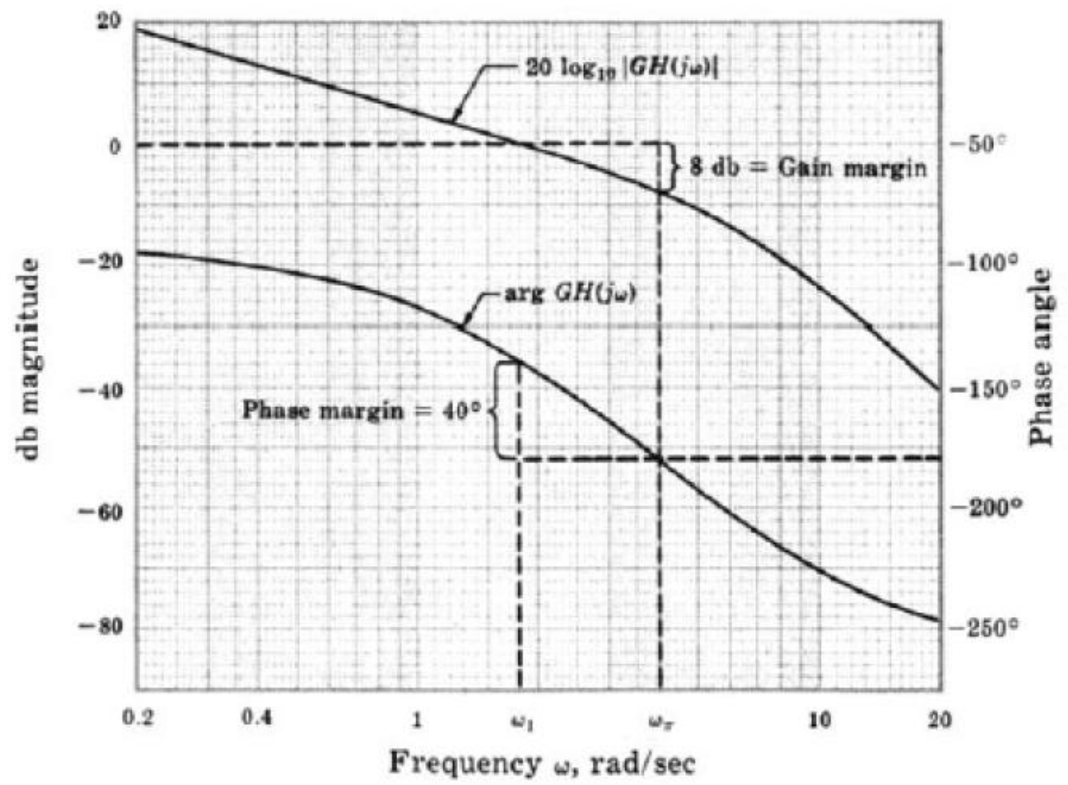
Frequency ω(rad/sec)	Magnitude (db)	Phase (degree)
0.1	20	-93.5
10	-37	-213.5

Magnitude and phase plot can be approximated by straight-line segments(a symptotic or corner plot) which allow the simple sketching without detailed computation





Example: The system whose bode plots are shown in fig below has a gain margin G.M = 8 db and a phase margin P.M = 40 deg



Bode Plots of Systems with Pure Time Delays

The stability analysis of a closed-loop system with a pure time delay in the loop can be conducted easily with the Bode plot.

Example:

Consider that the loop transfer function of a closed-loop system is

$$L(s) = \frac{Ke^{-T_d s}}{s(s+1)(s+2)}$$

fig below shows the Bode plot of $L(j\omega)$ with $K = 1$ and $T_d = 0$. The following results are obtained:

Gain-crossover frequency = 0.446 rad/sec

Phase margin = 53.4°

Phase-crossover frequency = 1.416 rad/sec

Gain margin = 15.57 dB

Thus, the system with the present parameters is stable.

The effect of the pure time delay is to add a phase of $-T_d\omega$ radians to the phase curve while not affecting the magnitude curve. The adverse effect of the time delay on stability is apparent, because the negative phase shift caused by the time delay increases rapidly with the increase in ω. To find the critical value of the time delay for stability, we set

$$T_d\omega_g = 53.4^\circ \frac{\pi}{180^\circ} = 0.932 \quad \text{radians}$$

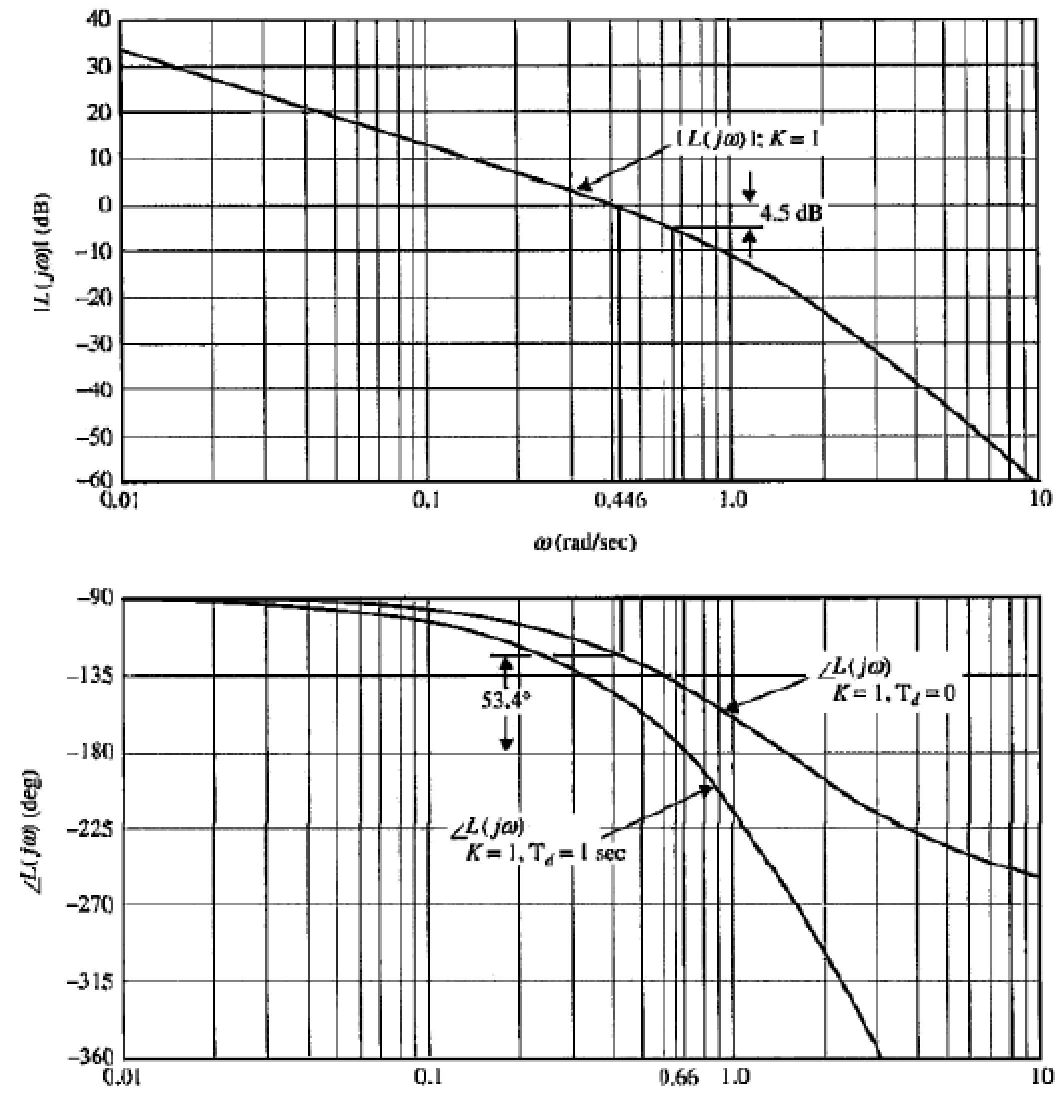
Solving for T_d from the last equation, we get the critical value of T_d to be 2.09 seconds.

Continuing with the example, we set T_d arbitrarily at 1 second and find the critical value of K for stability. Fig. 8-43 shows the Bode plot of $L(j\omega)$ with this new time delay. With K still equal to 1, the magnitude curve is unchanged. The phase curve droops with the increase in ω, and the following results are obtained:

Phase-crossover frequency = 0.66 rad/sec

Gain margin = 4.5 dB

Thus, using the definition of gain margin of Eq. (8-91), the critical value of K for stability is $10^{4.5/20} = 1.68$.



CONTROL SYSTEM DESIGN BY FREQUENCY RESPONSE (DESIGN BY K)

EXAMPLE:

$$GH(s)=\frac{K}{s(s+2)(s+20)}$$

Find: 1- Value of k for system to be critical stable (G.M=0 db)

2- Value of K for G.M =10 db

3- Value of k for P.M =50 DEG

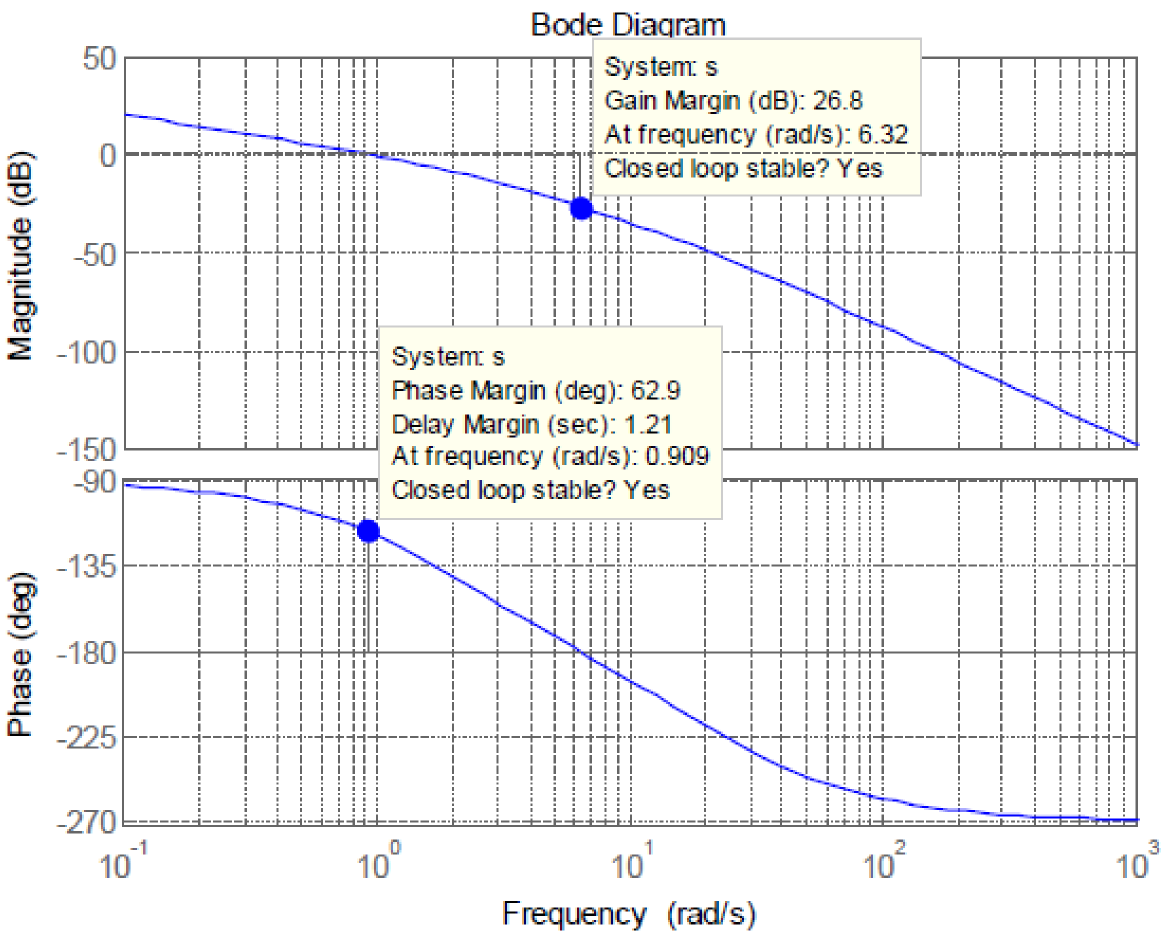
4- Value of K for ω_1 =4 rad / sec

Solution: The bode form is:

$$GH(s)=\frac{K/40}{j\omega(0.5j\omega+1)(0.05j\omega+1)}$$

Bode gain KB = K/40 = 0.025 K, Let KB =1 , K=40

We construct bode plot using Matlab program



1-For G.M=0 db, KB=26.8 – 0=26.8db ,

20logKB=26.8, KB=21.8, KB=0.025K,

K=875

2- For G.M=10 db, KB=26.8 – 10=16.8db ,

20logKB=16.8, KB=6.91, KB=0.025K,

K=276.7

3- $\omega_1 = 1.43$ rad/sec at phase =-180+50= - 130 ,

Magnitude = - 4.9 db at $\omega 1 = 1.43$ 20log KB=4.9 db,

KB=1.75, KB=0.025K,

K=70

4-Magnitude = - 19.2 db at $\omega_1 = 4$rad/sec,

20logKB=19.2, KB=9.1, KB=0.025K,

K=364.8

كيف نرسم مخطط بود للمثال الموجود في صفحة 6 في محاضرة Compensator باستخدام الماتلاب

$$G1(s) = \frac{40}{S(S+2)}$$

```
>> n =40
>>d=[1 2 0]
>>s=tf(n,d)
>>bode(s)
```

n معاملات البسط

d معاملات المقام

Design of Control system (compensator)

Compensation

- A closed-loop system is usually an unstable system. Hence, because it is unstable, there must be some kind of compensators than can compensate the stability of a closed loop system.

- Compensators are used to alter the output response of a system in order to accommodate to the set of desired criteria. This is achieved by introducing additional poles and/or zeros to the system transfer function.

- Sometimes, the word regulator or correction is used instead of the word compensator.

A device inserted into the system for the purpose of satisfying the specifications is called a *compensator*. The compensator compensates for deficit performance of the original system.

- We consider the following diagram where we have a sample root locus of a transfer function.

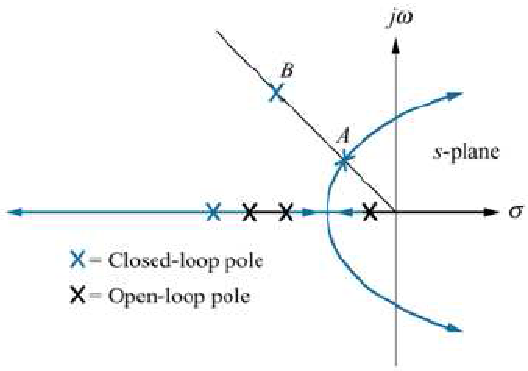


Figure (a)

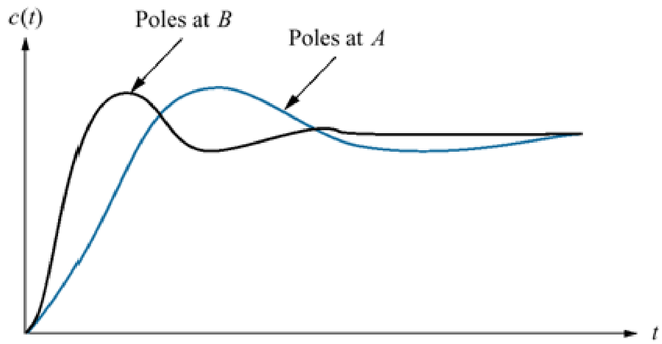


Figure (b)

- In the previous diagram, we assume that the desired output that we want is at point B. However, we note that currently, our output is at A. And we further note that at both points A and B, we desire a system with a "nice" percent overshoot and settling time.

- When we change the location of the root from point A to point B, the system response is speed-up, and interestingly, the percent overshoot is not affected.

- When we want to change from point A to point B, we can't simply adjust the gain of the system; rather, we need to introduce additional poles and/or zeros.

- The introduction of additional zeros and/or poles will speed-up the response of the system.

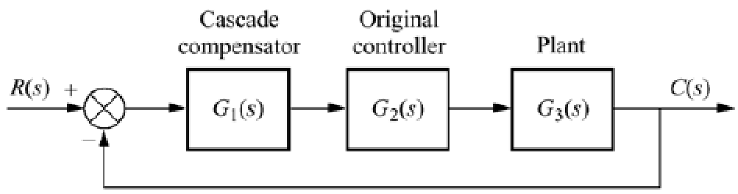
- When we introduce additional poles/zeros, we are actually improving the transient response of the system, as well as reducing the steady-state errors.

- Additional poles will eventually improve the steady-state characteristics, while additional zeros will improve the transient response.

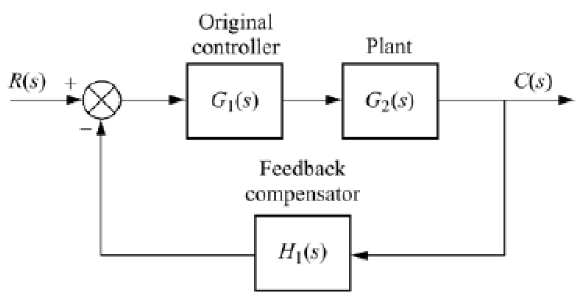
- Recall that poles are also called integrators in s-domain while zeros are called differentiators.

Compensation Configuration

- Two configuration of compensation are commonly used:
  - (a) cascade compensation.
  - (b) feedback compensation.



(a)



(b)

Compensator Design

- There are two (2) types of compensators:

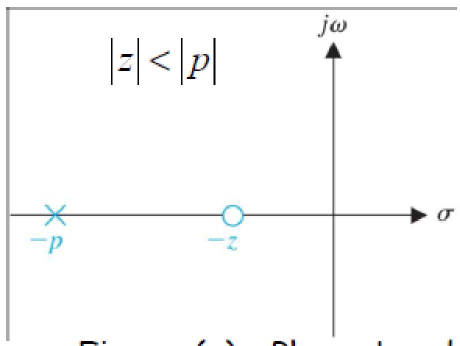


Figure (a) : Phase Lead

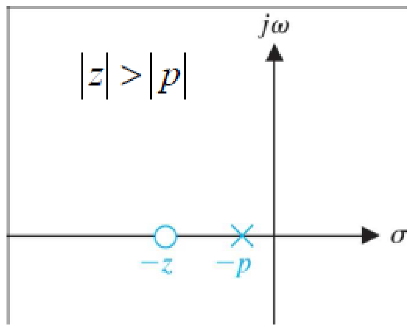


Figure (b) : Phase Lag

- We shall look at two ways of designing a compensator. The compensators will generally:
  - (1) improve the steady state error, by means of adding additional **Lag compensation.**
  - (b) improve the transient response by means of adding additional **Lead compensation.**

Performance Requirement

1. Time domain criteria（step response）

Overshoot, settling time, rising time, steady-state error

2. Frequency domain criteria

Open-loop frequency domain criteria:

crossover frequency, phase margin, gain margin

Closed-loop frequency domain criteria :

maximum value M_r, resonant frequency, bandwidth

Methods for Compensator Design

1. Frequency Response Based Method

Main idea: By inserting the compensator, the Bode diagram of the original system is altered to achieve performance requirements.

Original open-loop Bode diagram＋Bode diagram of compensator＋alteration of gain ＝open-loop Bode diagram with compensation

2. **Root Locus Based Method**

Main idea: Inserting the compensator introduces new open-loop zeros and poles to change the closed-loop root locus to satisfy the requirement.

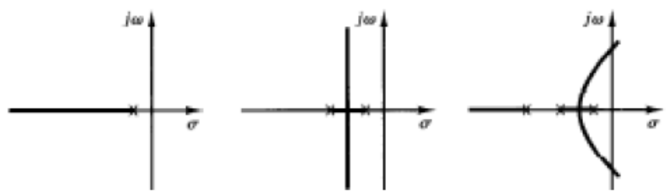
Design specifications:

Design in the frequency domain is simple and straightforward.

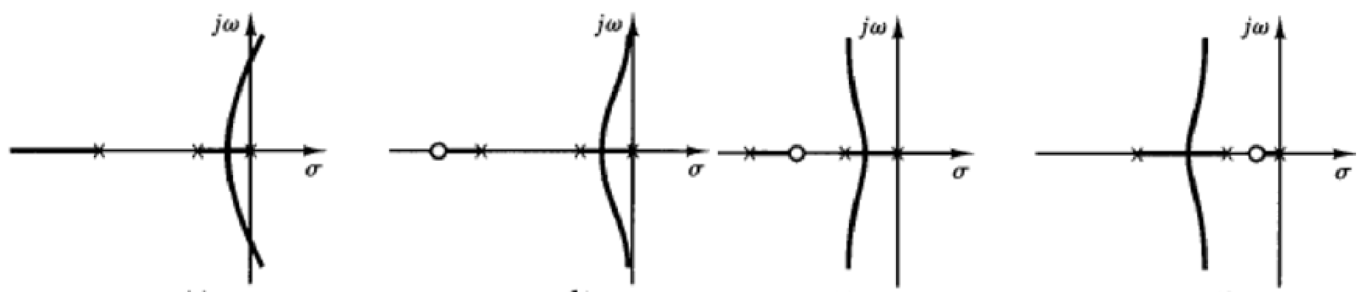
The transient – response performance is specified in terms P.M , G.M, BW(they give a rough estimate of the speed of transient response), M_r(they give a rough estimate of system damping) ,ω_r ,ω_1 and ω_π

The steady –state response is specified in term of steady –state errors (error constants) they give the steady-state accuracy

Effects of the addition of poles. The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response.



Effects of the addition of zeros. The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response.



(a) (b) (c) (d)

Compensator Realization

There are three types of compensator:

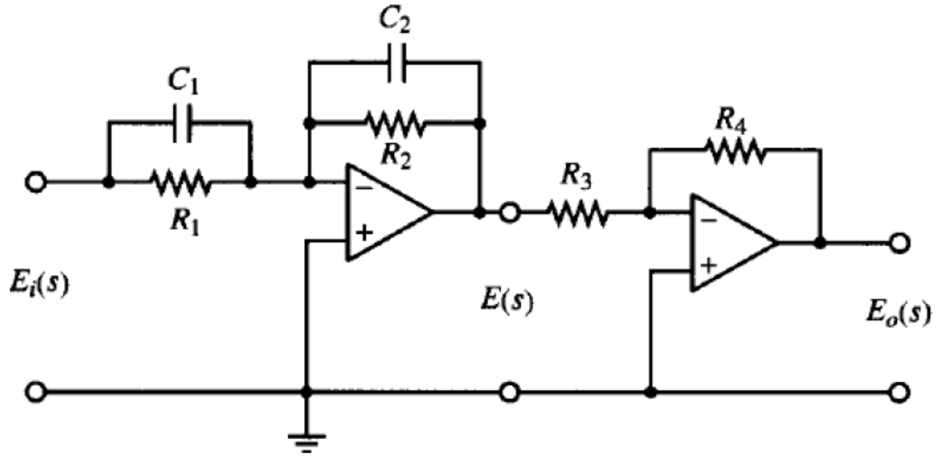
1-Lead compensator

2-Lag compensator

3-Lag-Lead compensator

There are many ways to realize analog compensator such as, electronic or electric network and mechanical system.

Compensator using operational amplifier are used in practice as shown in fig below



Electronic circuit that is a lead network if $R_1C_1 > R_2C_2$ and a lag network if $R_1C_1 < R_2C_2$

The transfer function of the circuit is

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4}{R_1R_3}\frac{R_1C_1s+1}{R_2C_2s+1} = \frac{R_4C_1}{R_3C_2}\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}$$

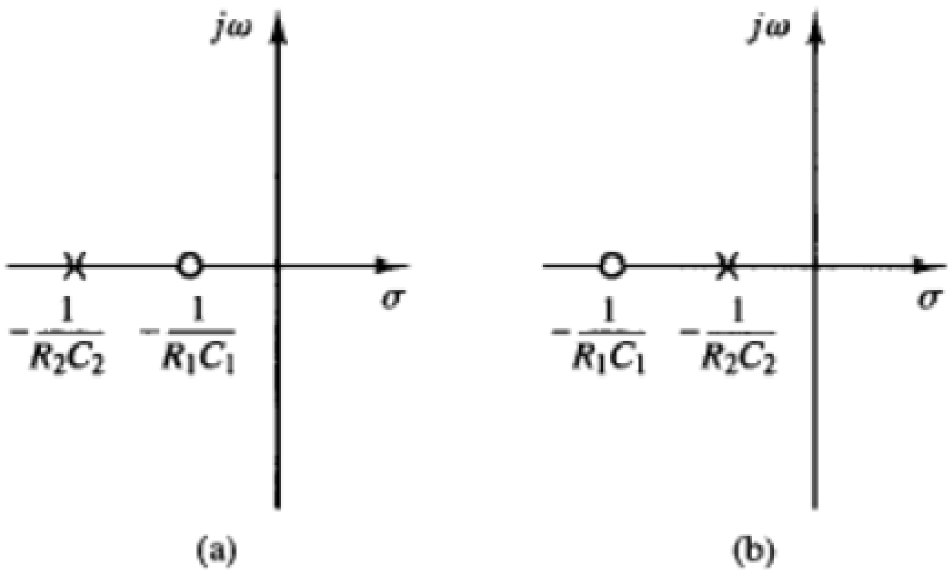
$$= K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

$$T = R_1C_1, \qquad \alpha T = R_2C_2, \qquad K_c = \frac{R_4C_1}{R_3C_2}$$

$$K_c\alpha = \frac{R_4C_1}{R_3C_2}\frac{R_2C_2}{R_1C_1} = \frac{R_2R_4}{R_1R_3}, \qquad \alpha = \frac{R_2C_2}{R_1C_1}$$

if $C_1 = C_2$ and $R_3 = R_4$, $K_C = 1$, $K_C\alpha$ = bode gain

$\alpha < 1$ for lead network and $\alpha > 1$ for lag network



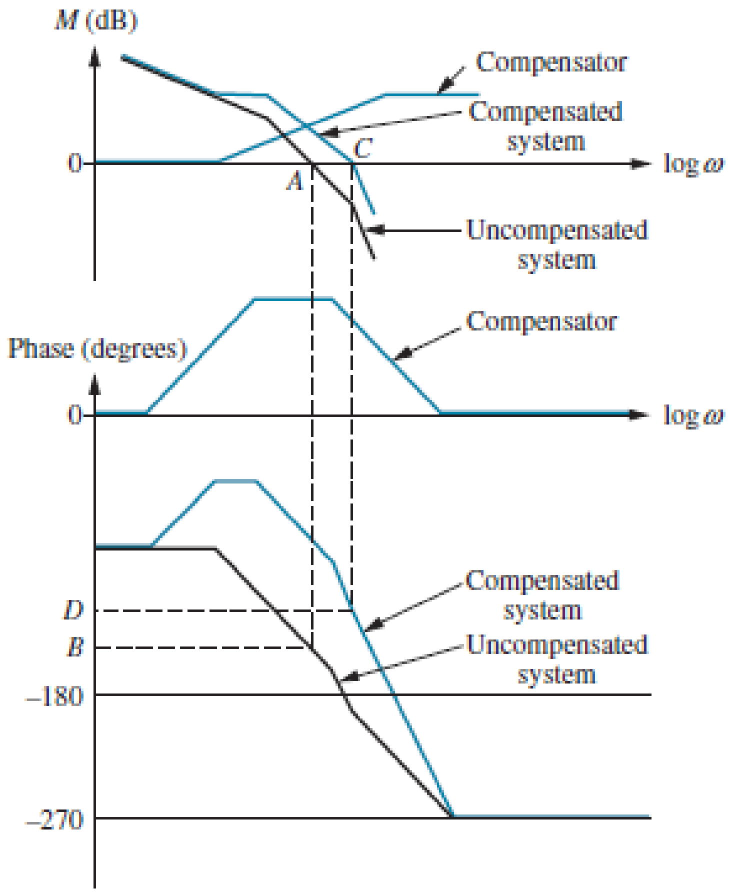
Pole–zero configurations: (a) lead network; (b) lag network.

Lead compensator

The lead compensator increases the bandwidth by increasing the gain crossover frequency. At the same time, the phase diagram is raised at higher frequencies. The result is a larger phase margin and a higher phase-margin (gain crossover) frequency. In the time domain, lower percent overshoots (larger phase margins) with smaller peak times (higher phase margin frequencies) are the results. The concepts are shown in figure.

The uncompensated system has a small phase margin (B) and a low phase margin frequency(gain crossover) (A). Using a phase lead compensator, the phase angle plot (compensated system) is raised for higher frequencies. At the same time, the gain crossover frequency in the magnitude plot is increased from A rad/s to C rad/s. These effects yield a larger phase margin (D), a higher phase-margin frequency (C), and a larger bandwidth.

One advantage of the frequency response technique over the root locus is that we can implement a steady-state error requirement and then design a transient response. This specification of transient response with the constraint of a steady state error is easier to implement with the frequency response technique than with the root locus.



1. Moving the zero $-1/T$ toward the origin should improve rise time and settling time. If the zero is moved too close to the origin, the maximum overshoot may again increase, because $-1/T$ also appears as a zero of the closed-loop transfer function.
2. Moving the pole at $-1/\alpha T$ farther away from the zero and the origin should reduce the maximum overshoot, but if the value of T is too small, rise time and settling time will again increase.

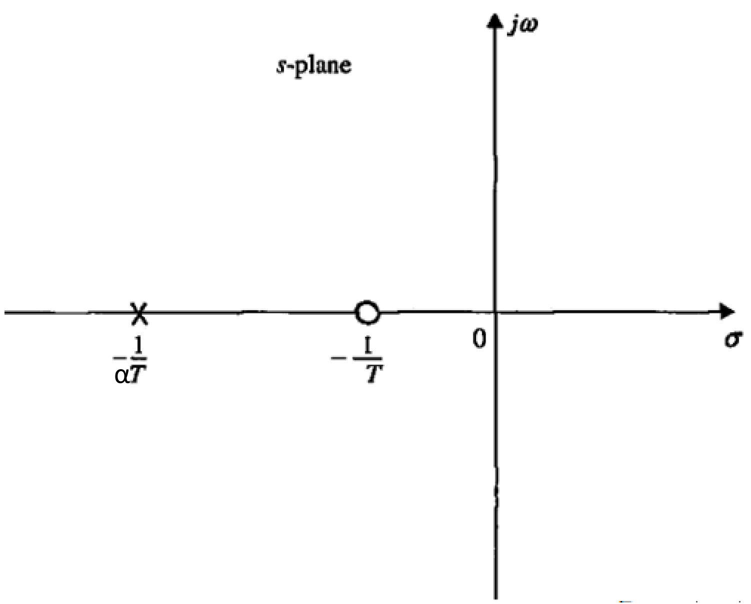


Figure Pole–zero configuration of the phase-lead controller.

The primary function of the lead compensator is to reshape the frequency-response curve to provide sufficient phase-lead angle to offset the excessive phase lag associated with the components of the fixed system.

Characteristics of lead compensators. Consider a lead compensator having the following transfer function:

$$K_c\alpha \frac{Ts + 1}{\alpha Ts + 1} = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \qquad (0 < \alpha < 1)$$

It has a zero at $s = -1/T$ and a pole at $s = -1/(\alpha T)$. Since $0 < \alpha < 1$, we see that the zero is always located to the right of the pole in the complex plane. Note that for a small value of α the pole is located far to the left. The minimum value of α is limited by the physical construction of the lead compensator. The minimum value of α is usually taken to be about 0.05. (This means that the maximum phase lead that may be produced by a lead compensator is about 65°.)

Figure shows the Bode diagram of a lead compensator when $K_c = 1$ and $\alpha = 0.1$. The corner frequencies for the lead compensator are $\omega = 1/T$ and $\omega = 1/(\alpha T) = 10/T$. The maximum value of the phase **Øm**, and the frequency at which it occurs, ***Wm,*** are derived as follows. Because ***Wm*** is the geometric mean of the two corner frequencies, we write

$$\log_{10}\omega_m = \frac{1}{2}\left(\log_{10}\frac{1}{aT} + \log_{10}\frac{1}{T}\right)$$

Thus,

$$\omega_m = \frac{1}{\sqrt{a}T}$$

To determine the maximum phase ϕ_m, the phase of $G_c(j\omega)$ is written

$$\angle G_c(j\omega) = \phi(j\omega) = \tan^{-1}\omega aT - \tan^{-1}\omega T$$

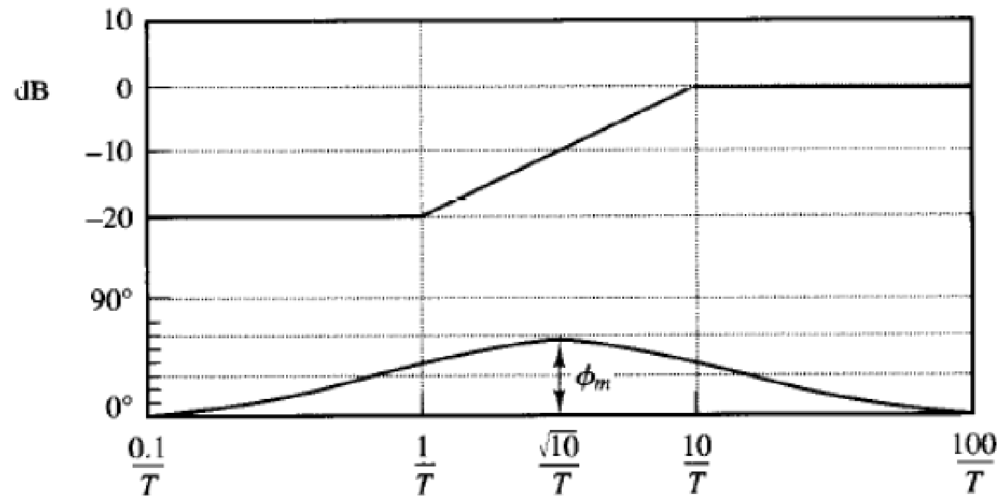
the phase angle at $\omega = \omega_m$ is ϕ_m, where

$$\sin\phi_m = \frac{1 - a}{1 + a}$$

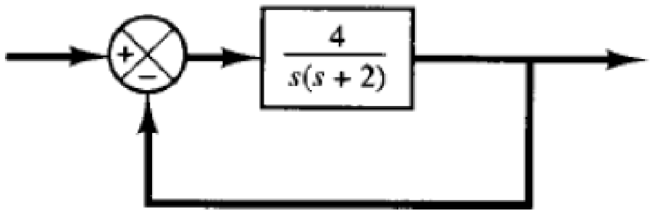
The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}T)$

$$\left|\frac{1 + j\omega T}{1 + j\omega\alpha T}\right|_{\omega = 1/(\sqrt{\alpha}T)} = \left|\frac{1 + j\dfrac{1}{\sqrt{\alpha}}}{1 + j\alpha\dfrac{1}{\sqrt{\alpha}}}\right| = \frac{1}{\sqrt{\alpha}}$$

Bode diagram of a lead compensator $\alpha(j\omega T + 1)/(j\omega\alpha T + 1)$, where $\alpha = 0.1$.



Example : consider the system shown in figure below. It is desired to design a lead compensator for the system so that the static-velocity error constant **Kv= 20** ,the phase margin is at least **50 degree**, and the gain margin is at least **10db**



Solution:

We shall use a lead compensator of the form

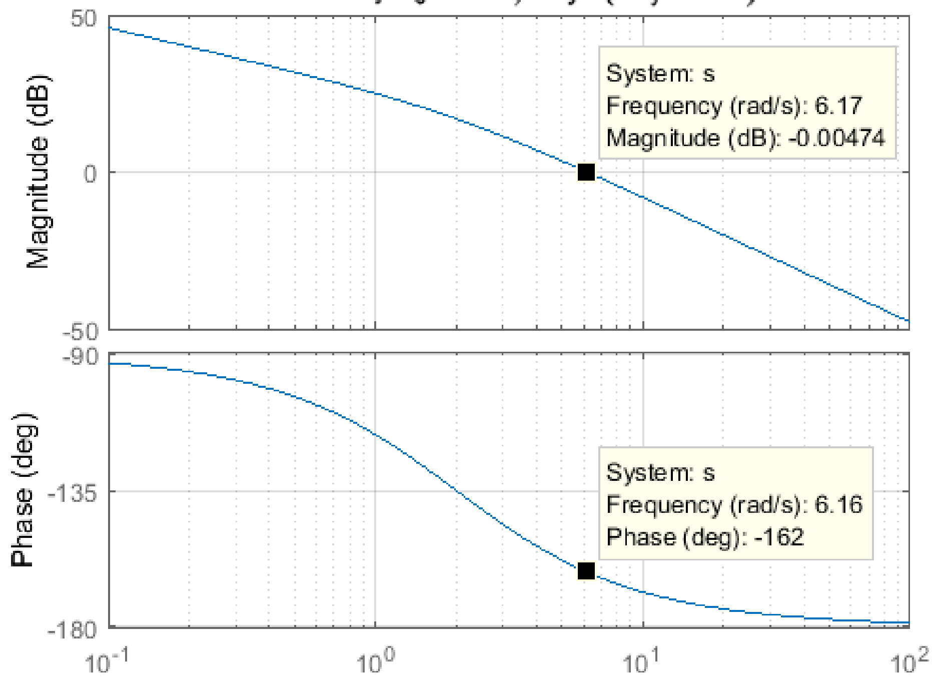
$$G_c(s) = K_c\alpha\frac{Ts + 1}{\alpha Ts + 1} = K_c\frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\alpha T}}$$

Define : $$G_1(s) = KG(s) = \frac{4K}{s(s + 2)}$$ where $K = K_c\alpha$.

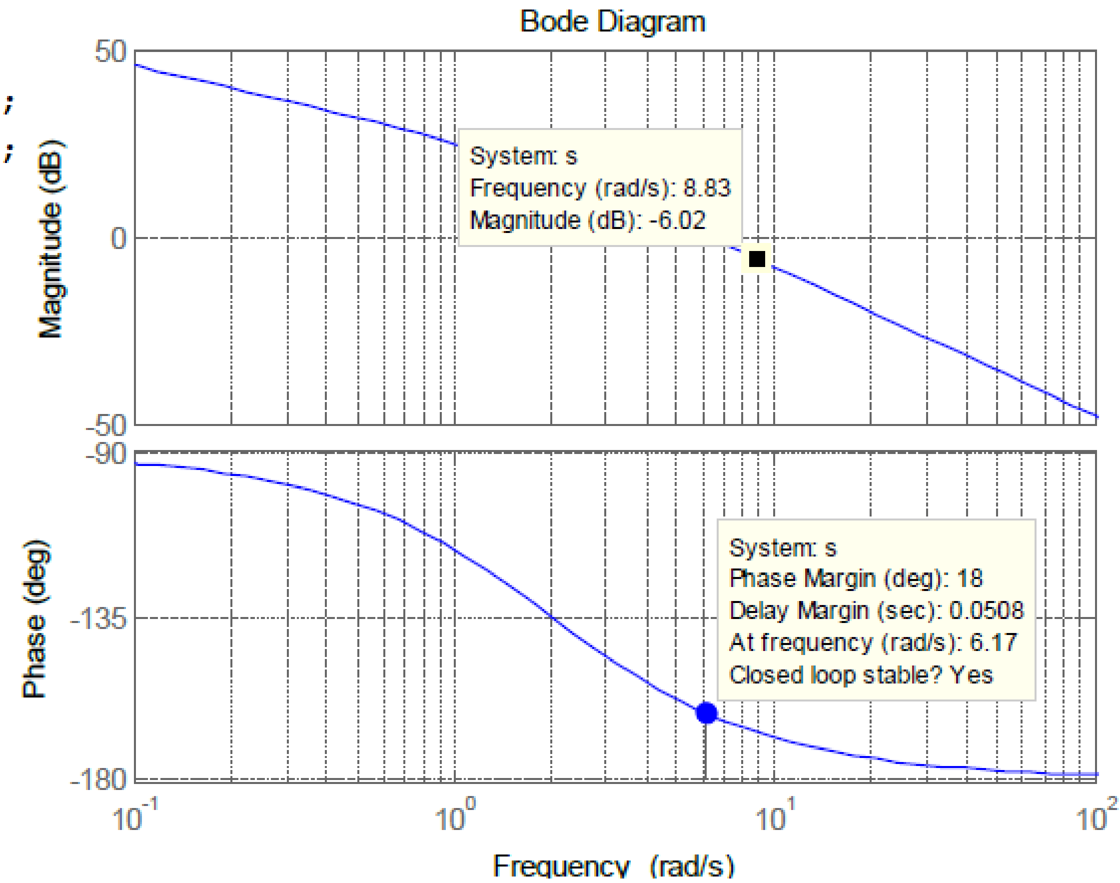
$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} s\frac{Ts+1}{\alpha Ts+1}G_1(s) = \lim_{s\to 0}\frac{s4K}{s(s+2)} = 2K = 20 \qquad K = 10$$

We shall next plot the Bode diagram of

$$G_1(j\omega) = \frac{40}{j\omega(j\omega+2)} = \frac{20}{j\omega(0.5j\omega+1)}$$




```
m=[40];
n=[1 2 0];
s=tf(m,n);
sgrid on;
bode(s)
```



From this plot phase margin = 18 , gain margin $=\infty$ db and ω_1 = 6.17rad/sec

$$\phi_m = 50 - 18 + 5 = 37$$

$$\sin \phi_m = \frac{1-\alpha}{1+\alpha} \qquad \alpha = 0.25$$

This means that 5° has been added to compensate for the shift in the gain crossover frequency

The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}T)$

$1/\sqrt{\alpha}$ = 6db

$G_1(j\omega)$ = - 6db correspond to ω_c = 8.8 rad / sec

$1/T = \sqrt{\alpha}\ \omega_c = 4.4$, $1/\alpha T = \omega_c/\sqrt{\alpha} = 17.6$

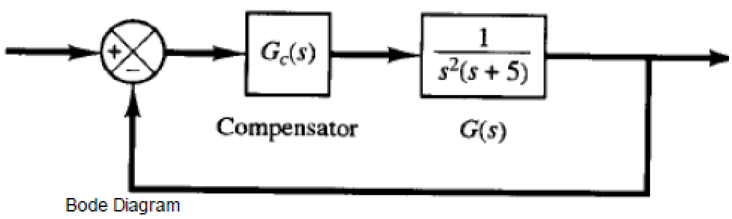
$K_c = K/\alpha = 10/0.25 = 40$

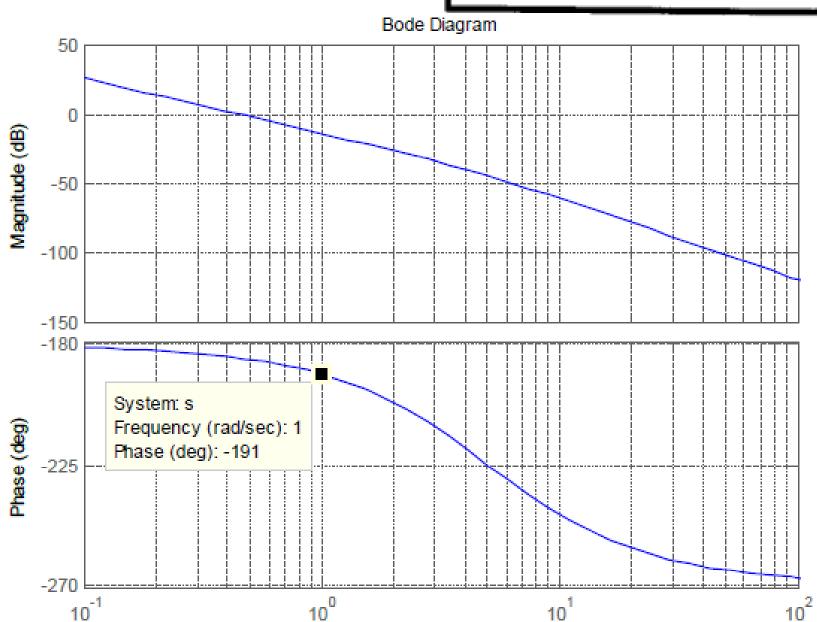
Thus , the transfer function of the compensator becomes

G_c=40(S+4.4) / (S+17.6)

Example: consider the system shown in figure below. Design a lead compensator such that the closed –loop system will have the P.M=50 G.M =10db and BW =1 rad/sec

Solution: BW is close to ω_1





From bode plot P.M =-11 degree at $\omega_1 = 1$ rad/sec

$$\sin \phi_m = \sin 61 = \frac{1-\alpha}{1+\alpha} = 0.8772$$

ϕ_m=50-(-11)=61

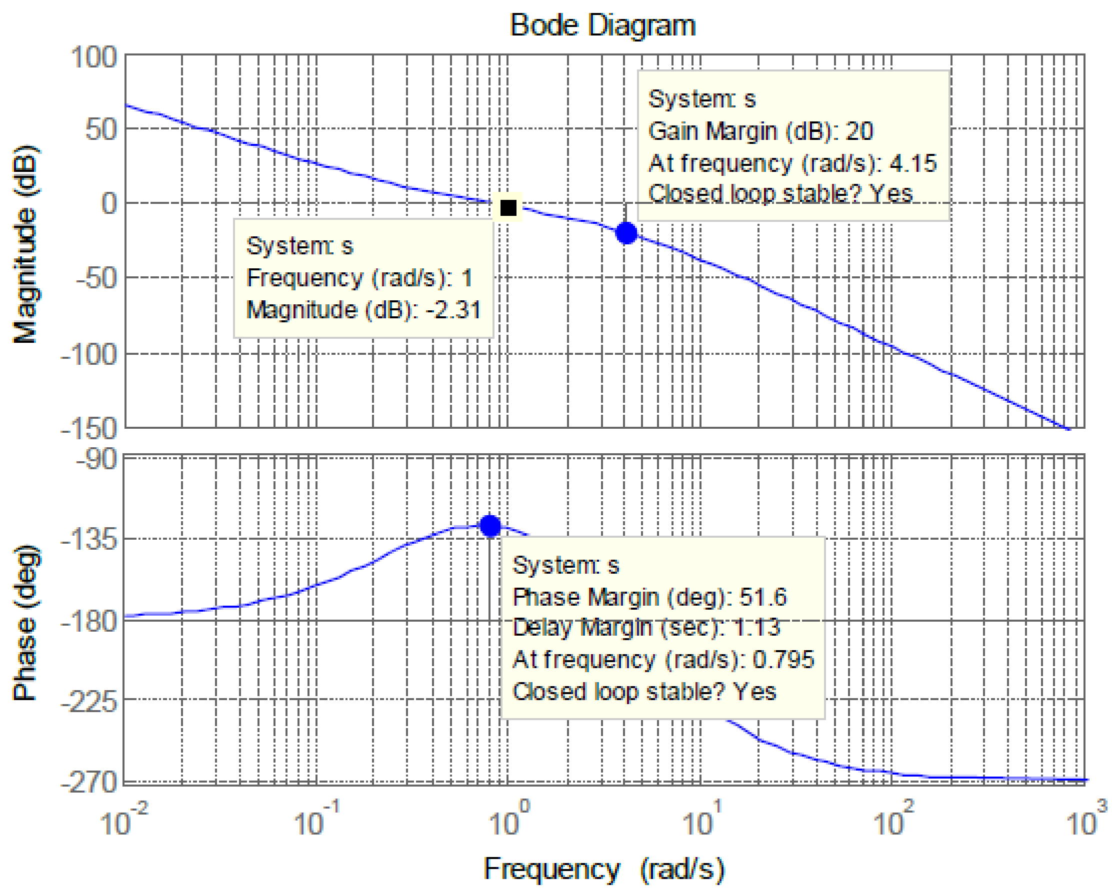
$$\omega_1 = \frac{1}{\sqrt{\alpha T}} = \frac{1}{\sqrt{0.06541 T}} = \frac{3.910}{T} = 1$$

α=0.06541

$$\frac{1}{T} = \frac{1}{3.910} = 0.2558 \qquad \frac{1}{\alpha T} = \frac{0.2558}{0.06541} = 3.910$$

$$G_c(j\omega)G(j\omega) = 0.06541 K_c \frac{3.910 j\omega + 1}{0.2558 j\omega + 1} \frac{0.2}{(j\omega)^2(0.2 j\omega + 1)}$$

K_c =19.9 from bode of Gc G(jω)



Notes:

1-lead compensator raises the overall phase angle curve in the low to mid-frequency and lowers the overall magnitude curve in low frequency depend on lead ratio (α) this lead to increase the gain –crossover frequency ($\omega1$)

2-if the orignal system has very low damping or is unstable, lead compensator may not be effective in improving the stability

3-several lead networks may be cascaded if a large phase –lead is required

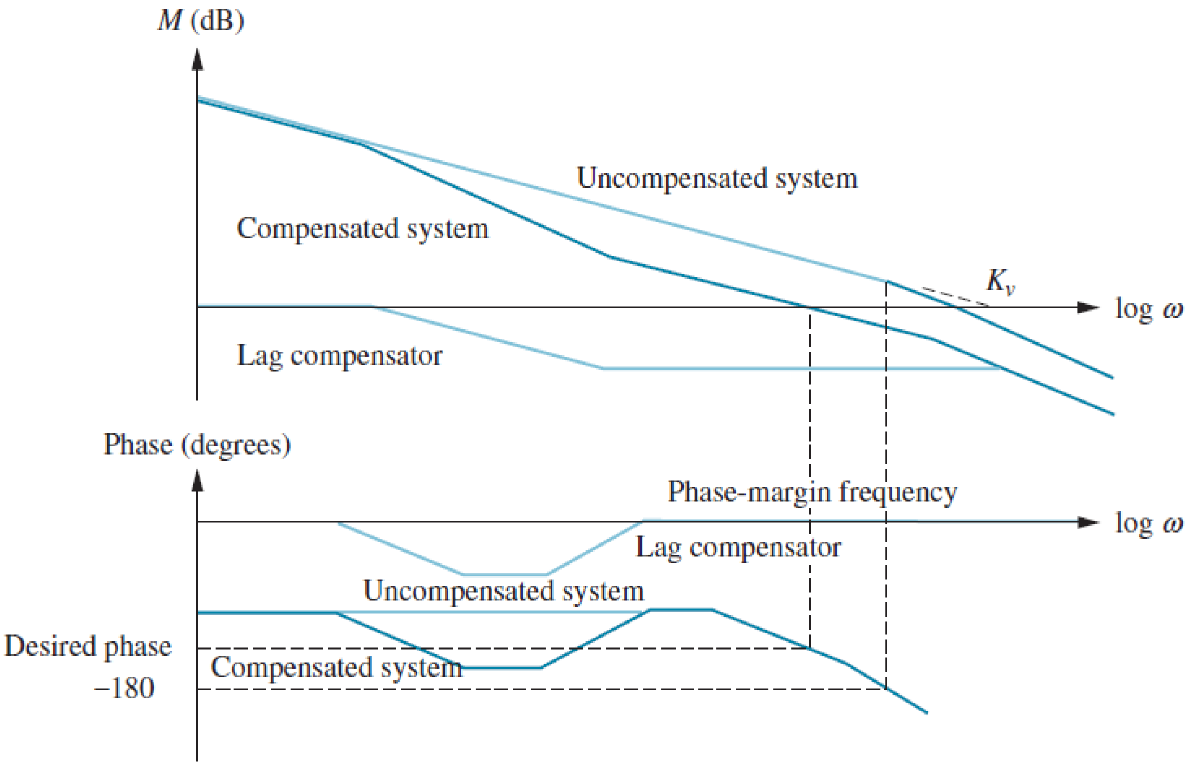
4-lead is basically a high-pass fliter(H.f passed, but L.f attenuated)

5-accentuate high-frequency noise effect

Lag compensator

The function of the lag compensator as seen on Bode diagrams is to (1) improve the static error constant by increasing only the low-frequency gain without any resulting instability, and (2) increase the phase margin of the system to yield the desired transient response. These concepts are illustrated in Figure below.

The uncompensated system is unstable since the gain at 180° is greater than 0 dB. The lag compensator, while not changing the low-frequency gain, does reduce the high-frequency gain. Thus, the low-frequency gain of the system can be made high to yield a large Kv without creating instability. This stabilizing effect of the lag network comes about because the gain at 180° of phase is reduced below 0 dB. Through judicious design, the magnitude curve can be reshaped, as shown in Figure below, to go through 0 dB at the desired phase margin. Thus, both Kv and the desired transient response can be obtained. We now enumerate a design procedure.

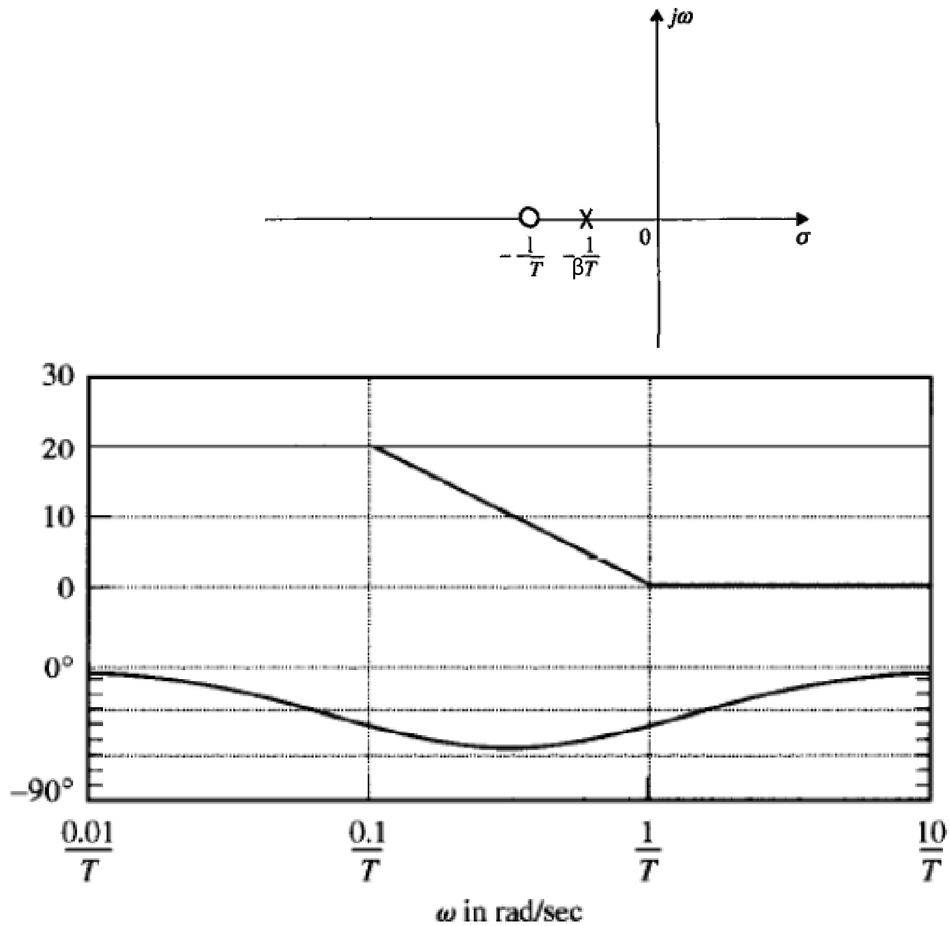


The primary function of a lag compensator is to provide attenuation in the high-frequency range to give a system sufficient phase margin.

Characteristics of lag compensators. Consider a lag compensator having the following transfer function:

$$G_c(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad (\beta > 1)$$

In the complex plane, a lag compensator has a zero at $s = -1/T$ and a pole at $s = -1/(\beta T)$. The pole is located to the right of the zero.



Bode diagram of a lag compensator $\beta(j\omega T+1)/(j\omega\beta T+1)$, with $\beta = 10$.

Example: consider the open-loop transfer function

$$G(s) = \frac{1}{s(s+1)(0.5s+1)}$$

It is desired to compensate the system so that the static velocity error constant K_v is 5 , the phase margin is at least 40°, and the gain margin is at least 10 dB.

Solution:

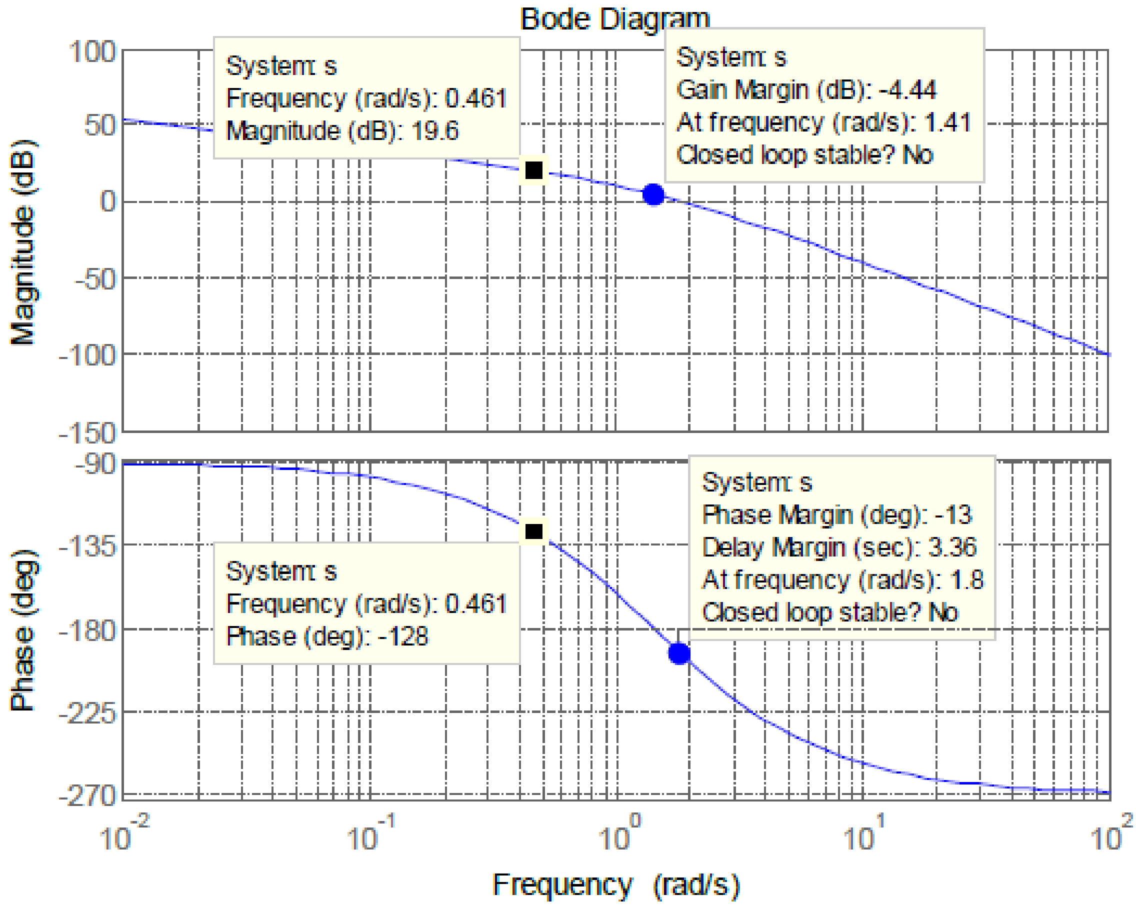
$$K_c\beta = K \qquad G_1(s) = KG(s) = \frac{K}{s(s+1)(0.5s+1)}$$

$$K_v = \lim_{s\to 0} sG_c(s)G(s) = \lim_{s\to 0} s\frac{Ts+1}{\beta Ts+1}G_1(s) = \lim_{s\to 0} sG_1(s)$$

$$= \lim_{s\to 0}\frac{sK}{s(s+1)(0.5s+1)} = K = 5$$

With $K = 5$, the compensated system satisfies the steady-state performance requirement. We shall next plot the Bode diagram of

$$G_1(j\omega) = \frac{5}{j\omega(j\omega+1)(0.5j\omega+1)}$$



The system unstable

The new gain crossover frequency ω_1 at the angle -180 +40 +12= -128 is 0.46 rad/sec

5° to 12° **must be added to compensate for the modification of the phase curve**

To prevent detrimental effects of phase lag due to the lag compensator

The pole and zero must be located lower than ω_1

$$\frac{1}{T} = 0.1 \times \text{lower corner frequency} = 0.1 \times 1 = 0.1$$

Or $\frac{1}{T}$ **may be chosen one decade below ω_1(if T do not become too large)**

The attenuation necessary to bring the magnitude curve down to 0db at this ω_1 is:

$20 \log 1/\beta = -19.6$, $\beta \cong 10$

$$\frac{1}{\beta T} = 0.01 \text{ rad/sec}$$

$$G_c(s) = K_c(10)\frac{10s+1}{100s+1} = K_c\frac{s+\frac{1}{10}}{s+\frac{1}{100}} \qquad K_c = \frac{K}{\beta} = \frac{5}{10} = 0.5$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s) = \frac{5(10s+1)}{s(100s+1)(s+1)(0.5s+1)}$$

Notes:

1-Lag compensator are essentially low –pass filter

2-lag compensator permits a high gain at low frequencies which improve the steady-state performance and reduce the gain in the higher frequencies to improve the phase margin(several lag networks in cascaded to provide higher attenuation)

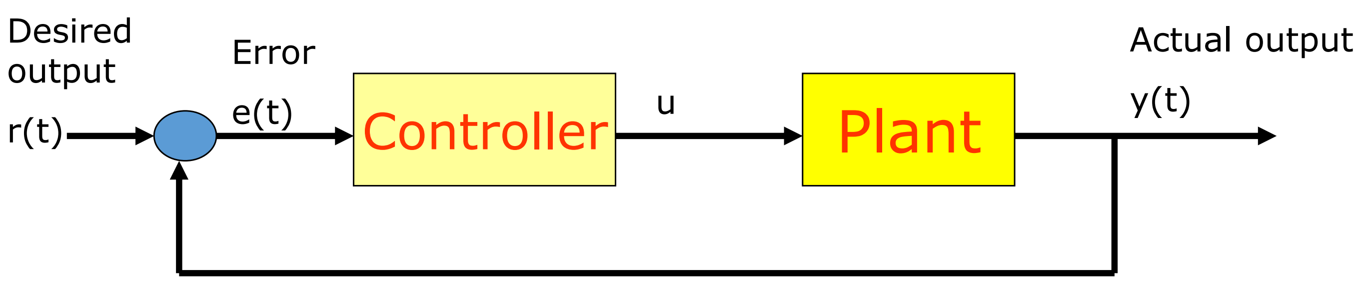
3-lag compensator decrease the BW because ω_1 reduced

4-reduced the high-frequency noise effect.

PID Controller

- PID control is the most widely used controller in the field of industrial automation.

- PID is an abbreviation of:
  - Proportional (P),
  - Integral (I), and
  - Derivative (D).

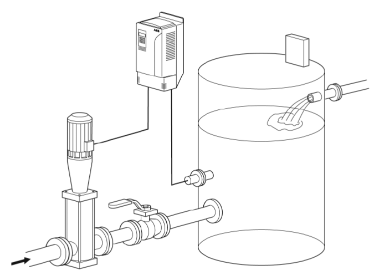


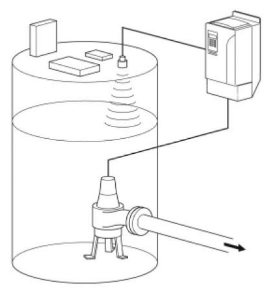
PID controllers:

- More than 50% of industrial controllers
- Easy to implement
- Applicable to most control systems
- Adjustable on site
- Satisfactory performance.

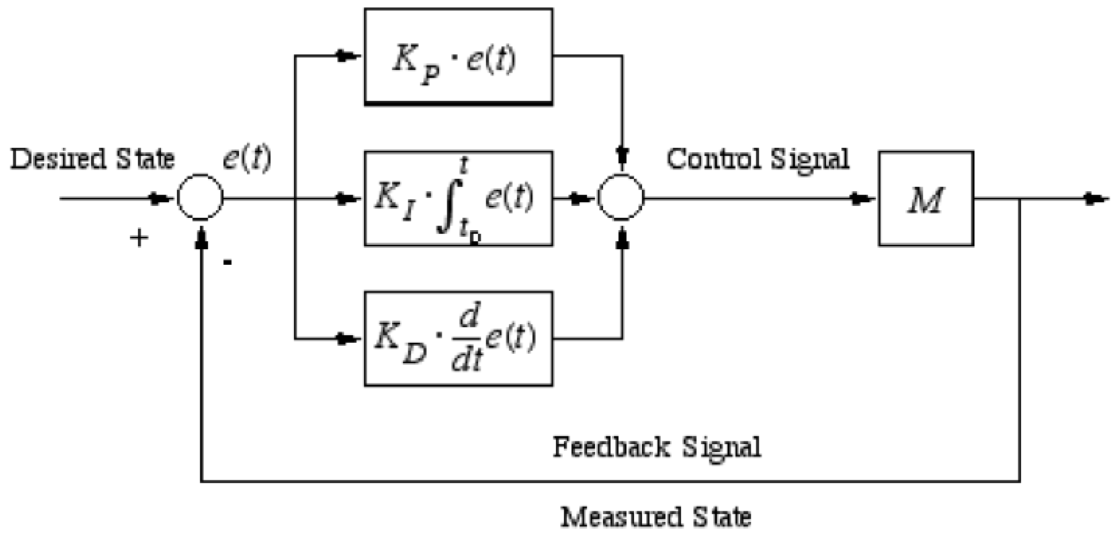
Widely used in automation and process industries.

- Motor Speed Control
- Pressure & Temperature Control
- Flow Control
- (Water) Level Control





- The block diagram of a PID control scheme is shown as follows:



The Proportional (P) + Integral (I) + Derivative (D) or (PID) controller used in feedback control system.

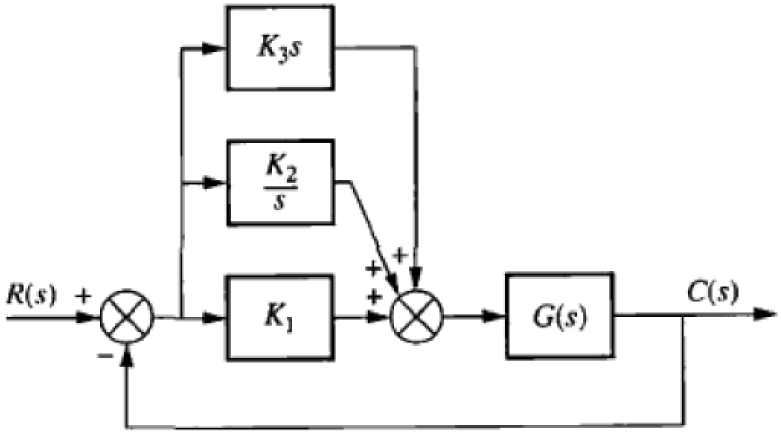
- The defining equation for a PID control scheme can be expressed in time domain as follows:

$$G_c(t) = K_p\, e(t) + K_i \int e(t)\, dt + K_d \frac{de(t)}{dt}$$

- Taking a forward Laplace transform yield:

$$G_c(s) = K_p\, E(s) + \frac{K_i}{s} E(s) + K_d\, s\, E(s)$$

where K_p, K_i and K_d are called the proportional gain, integral gain and derivative gain respectively. The variable e(t) or E(s) is known as the error signal.



PID controller

$$G_c(s) = K_1 + \frac{K_2}{s} + K_3 s \longrightarrow G_c(S) = \frac{K_3 S^2 + K_1 S + K_2}{S}$$

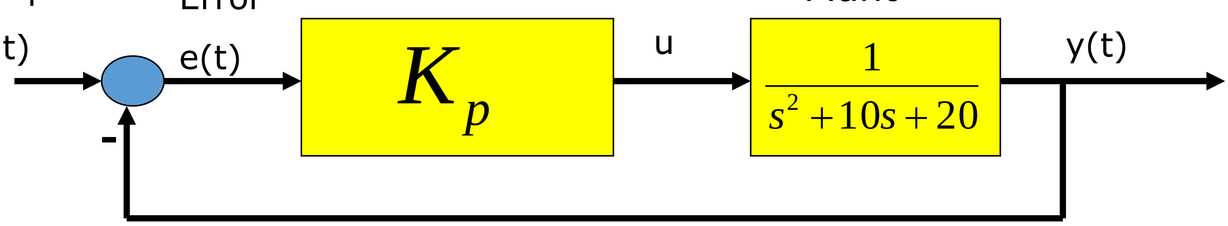
PID transfer function

K_1= K_P =Proportional gain, K_2 =K_i= integral gain K_3= K_d= derivative gain

The industrial controller may be classified according to their control as:

1-P controller 2- PI-controller 3-PD-controller 4- PID-controller

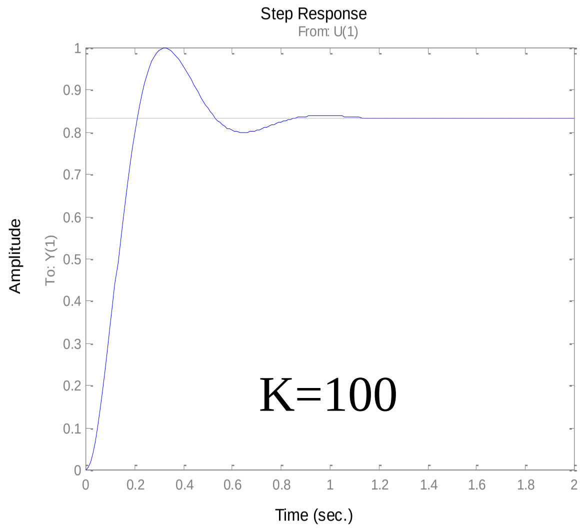
P controller (proportional)

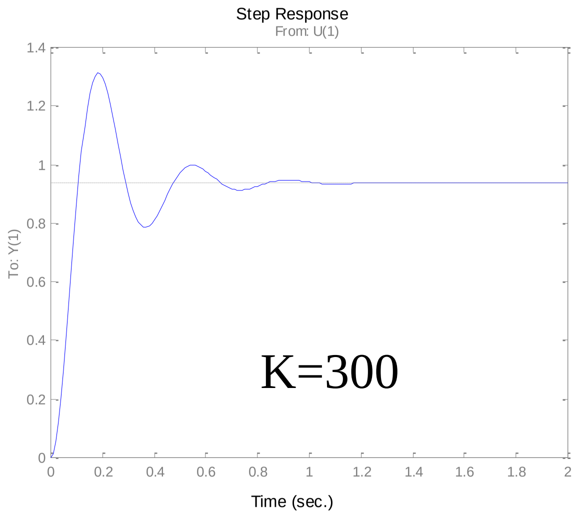


- The p-controller is a pure gain of value K_P .
- With P controller we can
  - Reduce steady state error
- But
  - Overshoot may increase

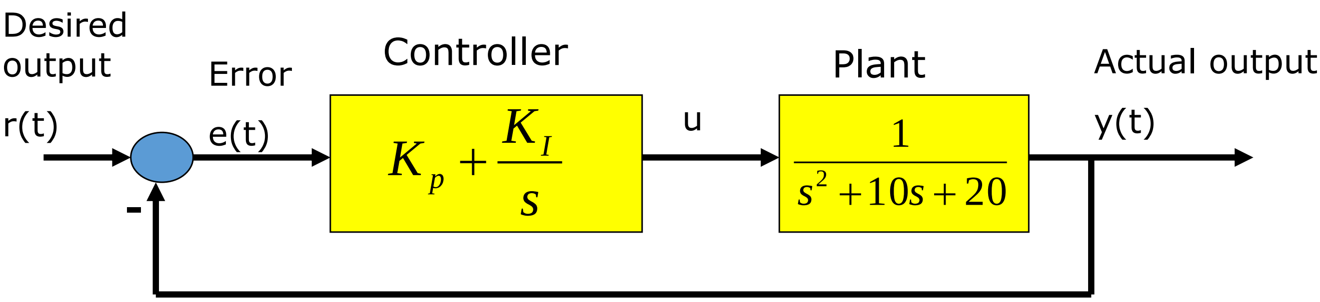
Proportional Control

By only employing proportional control, a steady state error occurs.





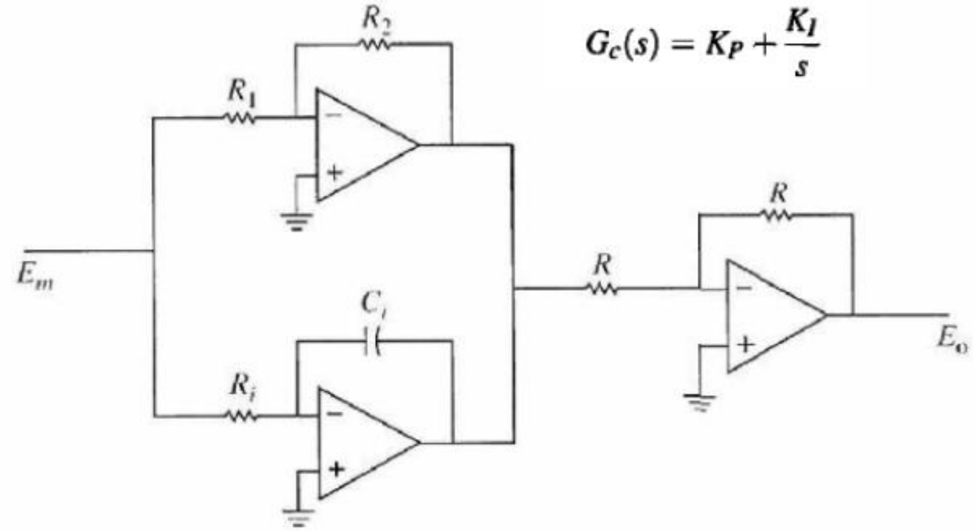
PI controller



- **Integral action of the PI controller**
- Improving the steady state response (eliminate e_{ss})
- Decrease rise time
- Increase overshoot
- Increase settling time
- Decreasing BW
- Filtering out-high-frequency response
- Increase the type and the order of system by one 1
- Adding a zero at $S= -K_i / K_P$ and pole at $S=0$
- To avoid an appreciable change in the root locus we place the zero near the origin

The basic electronic circuit used to realize PI-controller

$$G_c(s) = K_P + \frac{K_I}{s}$$

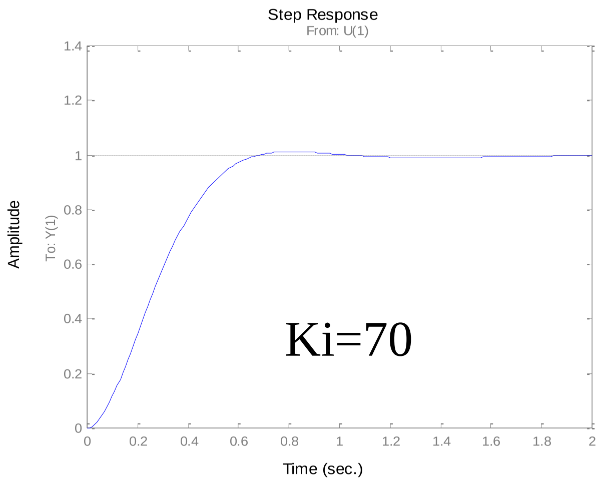


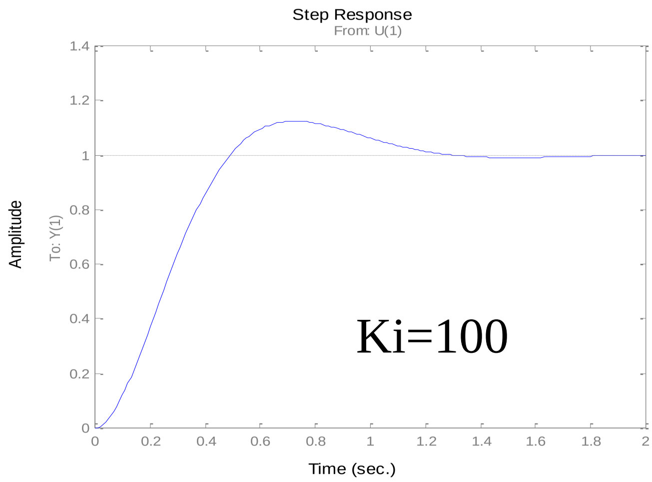
$$G_c(s) = \frac{E_o(s)}{E_{in}(s)} = \frac{R_2}{R_1} + \frac{1}{R_i C_i s} \qquad K_P = \frac{R_2}{R_1} \quad K_I = \frac{1}{R_i C_i}$$

$$G_c(s) = K_1 + \frac{K_2}{s} = \frac{K_1\left(s + \frac{K_2}{K_1}\right)}{s}$$

Proportional and Integral Control

The response becomes more oscillatory and needs longer to settle, the error disappears.





Example: Consider the open-loop T.F $G(S) = 0.25 / S + 0.1$ with unity feedback. Design PI-controller so that closed –loop time constant τ=2.5 sec and no e_{ss} with a constant input.

Zero of PI controller = 10 % of nearest real pole to origin point

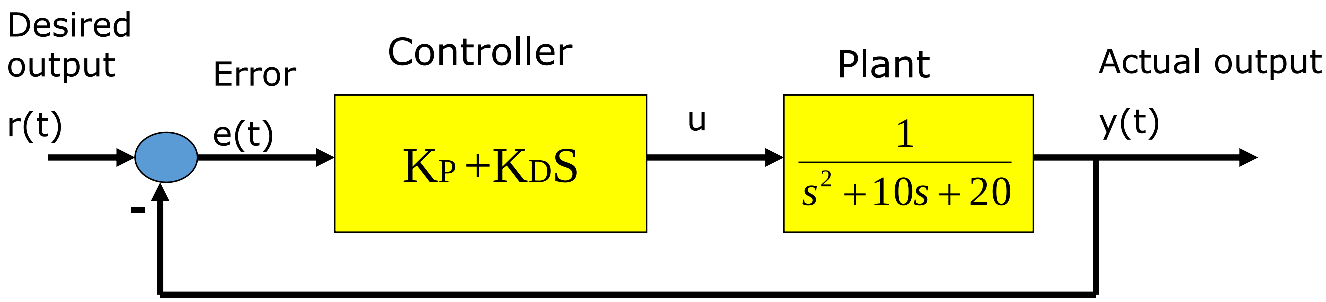
Solution: $K_i / K_P = 10\% (0.1) = 0.01$ $S = 1/T = 1/2.5 = 0.4$

From magnitude condition $|G_c(S)\, G(S)|_{S=-0.4} = 1$

$$|0.25\, K_P (S + K_i / K_P) / S (S + 0.1)|_{S=-0.4} = 1$$

$K_P = 1.231$ $K_i = 0.01231$

PD controller

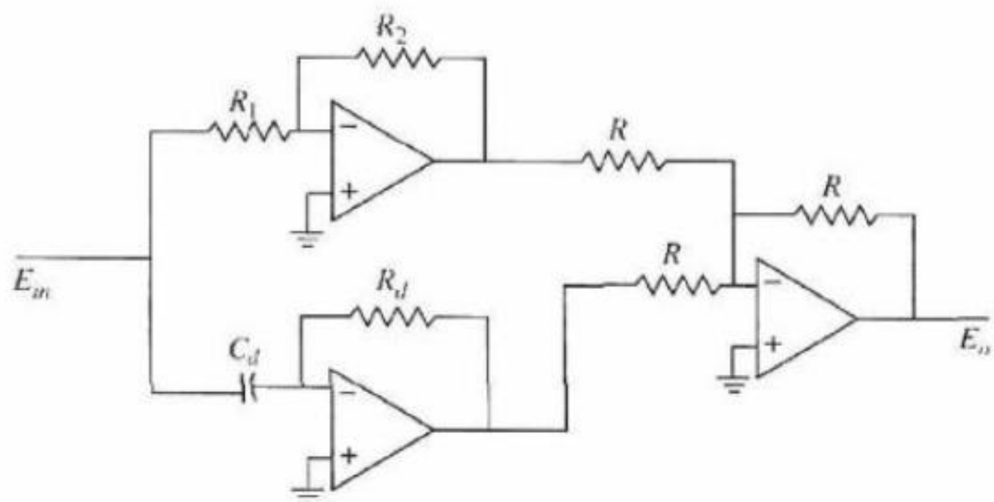


- **Derivative action of the PD controller**
- Decrease overshoot
- Decrease settling time
- Improving damping and reducing overshoot
- Reducing settling time (increasing bandwidth)
- Adding a simple zero at $S = -k_p / k_D$

The basic electronic circuit used to realize PD-controller

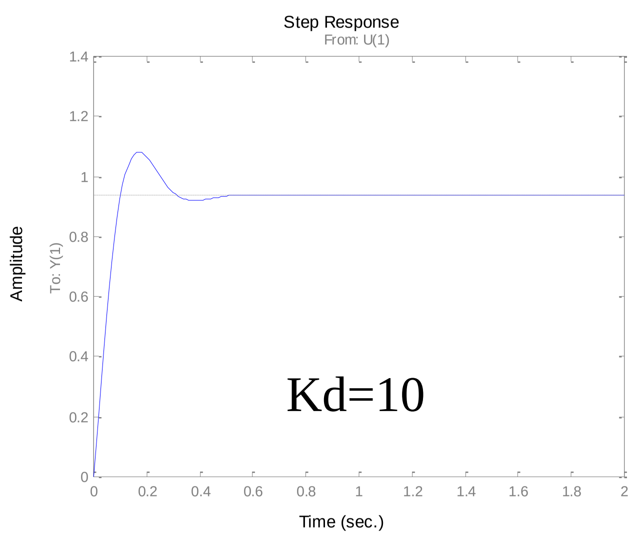
$$u(t) = K_P e(t) + K_D \frac{de(t)}{dt} \qquad \frac{E_o(s)}{E_{in}(s)} = \frac{R_2}{R_1} + R_d C_d s$$

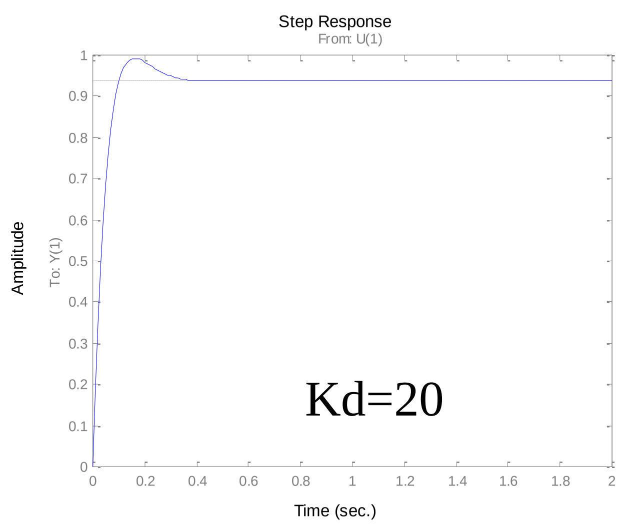
$$K_P = R_2/R_1 \quad K_D = R_d C_d \qquad G_c(s) = K_P + K_D s$$



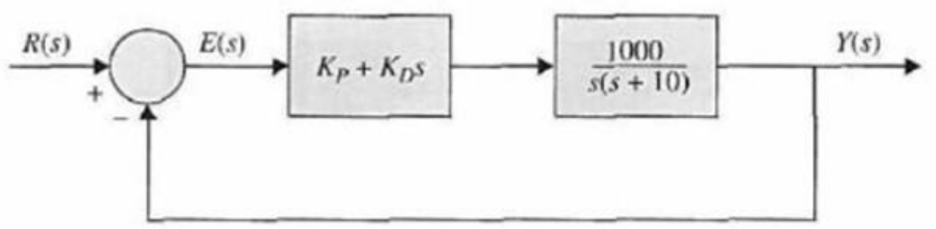
Proportional and Derivative Control

The overshoot will be reduce , and improve the transient response





Example: a control system with a PD controller is shown



Find the values of K_P and K_D so that the ramp-error constant K_v is 1000 and the damping ratio is 0.5.

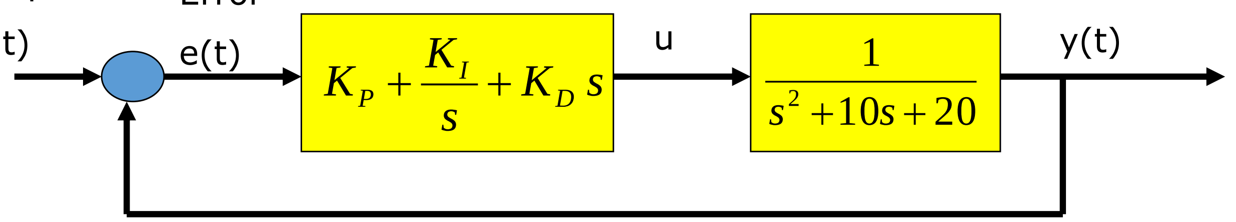
Solution:

$$K_v = \lim_{s\to 0} s \frac{1000(K_p + K_D s)}{s(s+10)} = \frac{1000K_p}{10} = 100K_p = 1000 \qquad \text{Thus} \qquad K_p = 10$$

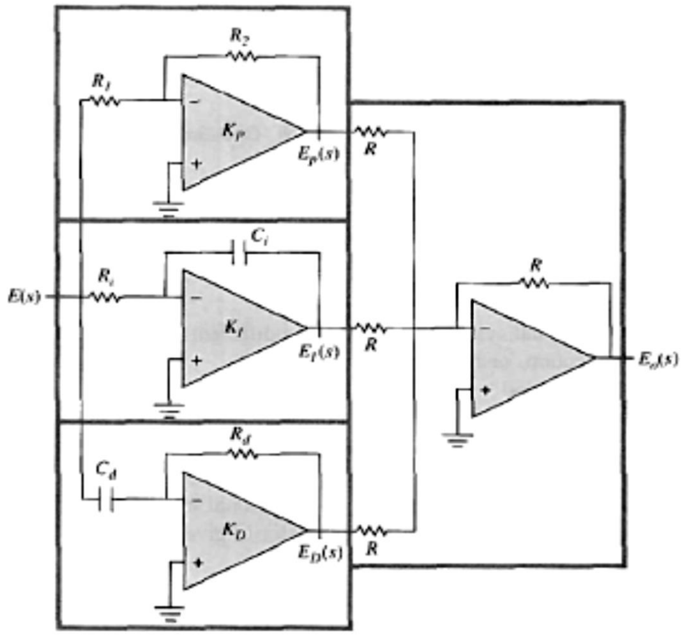
Characteristic Equation: $s^2 + (10 + 1000K_D)s + 1000K_p = 0$ $\qquad s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$

$$\omega_n = \sqrt{1000K_p} = \sqrt{10000} = 100 \text{ rad/sec} \qquad 2\zeta\omega_n = 10 + 1000K_D = 2\times 0.5\times 100 = 100 \qquad K_D = \frac{90}{1000} = 0.09$$

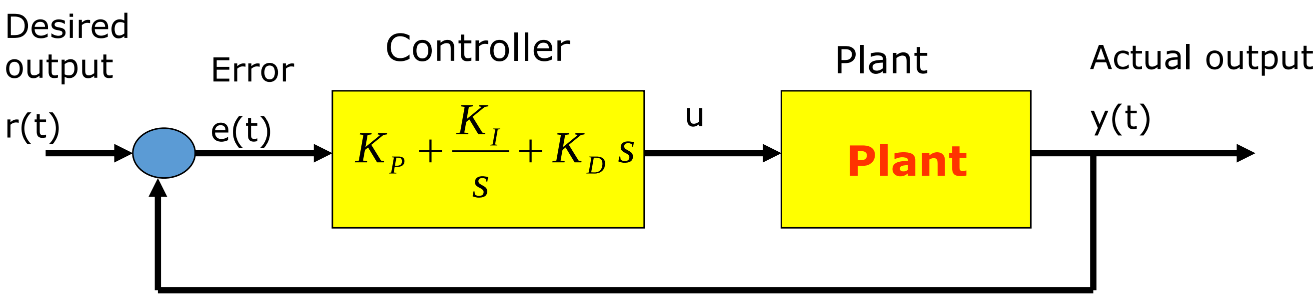
PID controller



- PID-controller Improving both :
- Transient Response and the Steady-State Error.



$$G(s) = K_p + \frac{K_I}{s} + K_D s$$



- **How do we select the PID parameters?**

Closed loop response	Rise time	overshoot	Settling time	Steady state error
Kp	decrease	increase	Small change	decrease
K_I	decrease	increase	increase	eliminate
K_D	Small change	decrease	decrease	Small change

Tips for Designing a PID Controller

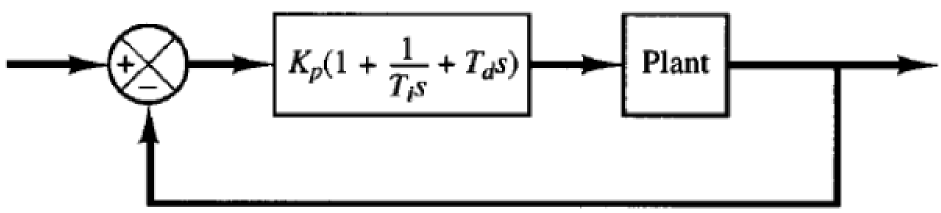
1. Obtain an open-loop response and determine what needs to be improved
2. Add a proportional control to improve the rise time
3. Add a derivative control to improve the overshoot
4. Add an integral control to eliminate the steady-state error
5. Adjust each of Kp, Ki, and Kd until you obtain a desired overall response.

We need to select the PID parameters so that we have

- Fast rise time
- Small overshoot
- No steady state error

Tuning Rules for PID Controllers

PID control of plants. Figure shows a PID control of a plant. If a mathematical model of the plant can be derived, then it is possible to apply various design techniques for determining parameters of the controller that will meet the transient and steady-state specifications of the closed-loop system. However, if the plant is so complicated that its mathematical model cannot be easily obtained, then analytical approach to the design of a PID controller is not possible. Then we must resort to experimental approaches to the tuning of PID controllers.

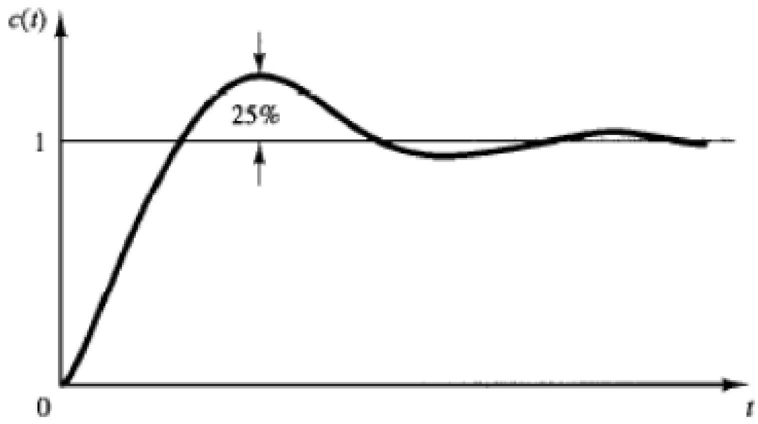


The process of selecting the controller parameters to meet given performance specifications is known as controller tuning. Ziegler and Nichols suggested rules for tuning

$T_i = K_P / K_i$ integral time , $Td = K_d / K_P$ derivative time

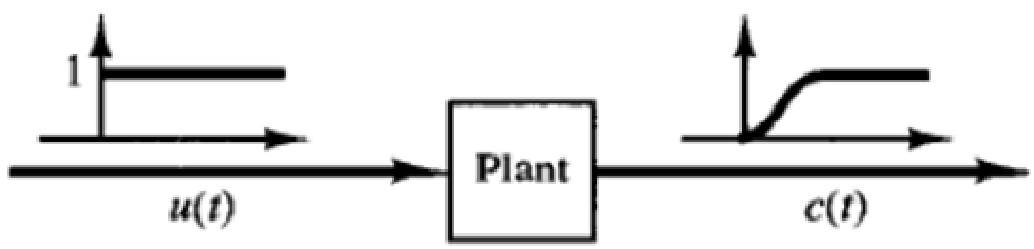
Ziegler–Nichols rules for tuning PID controllers. Ziegler and Nichols proposed rules for determining values of the proportional gain K_p, integral time T_i, and derivative time T_d based on the transient response characteristics of a given plant. Such determination of the parameters of PID controllers or tuning of PID controllers can be made by engineers on site by experiments on the plant. (Numerous tuning rules for PID controllers have been proposed since the Ziegler–Nichols proposal. They are available in the literature. Here, however, we introduce only the Ziegler–Nichols tuning rules.)

There are two methods called Ziegler–Nichols tuning rules. In both methods, they aimed at obtaining 25% maximum overshoot in step response



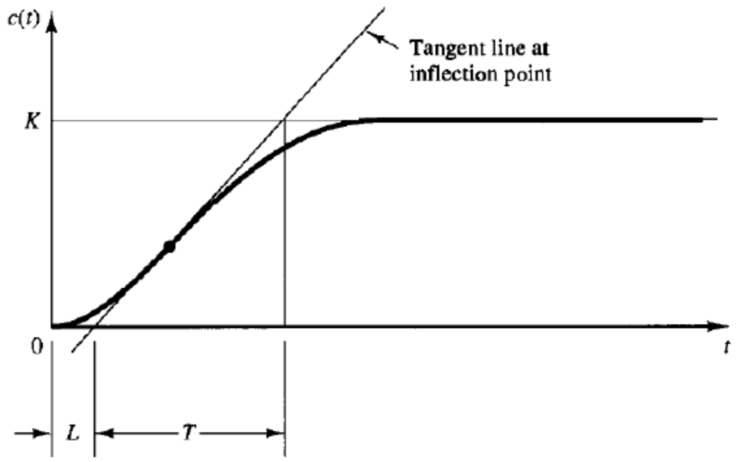
Open Loop Method

First method. In the first method, we obtain experimentally the response of the plant to a unit-step input, as shown in Figure



(If the response does not ex- hibit an S-shaped curve, this method does not apply.) Such step-response generated experimentally or from a dynamic simulation of the plant.

The S-shaped curve may be characterized by two constants, delay time L and time constant T. The delay time and time constant are determined by drawing a tangent line at the inflection point of the S-shaped curve and determining the intersections of the tangent line with the time axis and line $c(t) = K$, as shown in Figure



function $C(s)/U(s)$ may then be approximated by a first-order system with a transport lag as follows:

$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts + 1}$$

Ziegler–Nichols Tuning Rule Based on Step Response of Plant

Type of Controller	K_p	T_i	T_d
P	$\frac{T}{L}$	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{LK}$	$2L$	$0.5L$

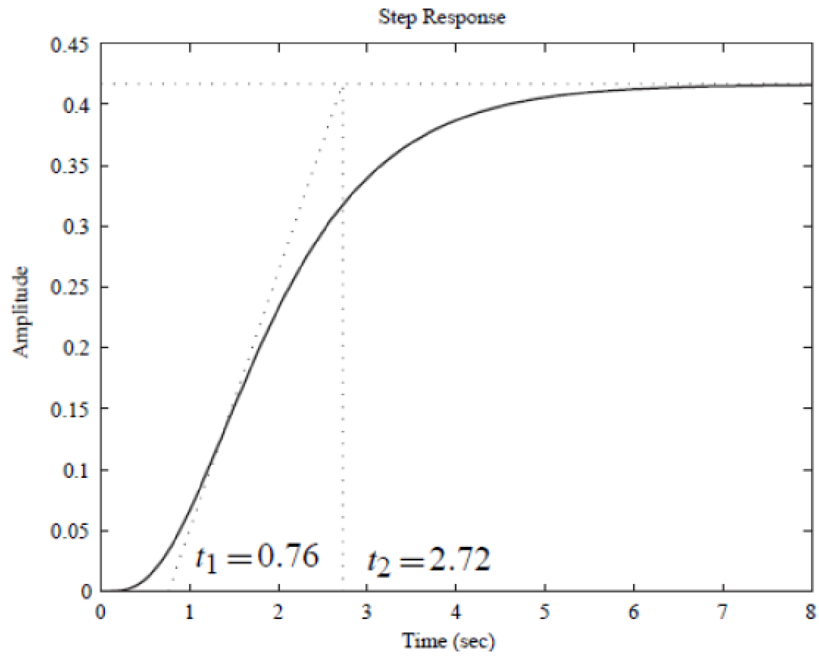
$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = 1.2\frac{T}{L}\left(1 + \frac{1}{2Ls} + 0.5Ls\right) = 0.6T\frac{\left(s + \frac{1}{L}\right)^2}{s}$$

Thus, the **PID** controller has a pole at the origin and double zeros at $s = -1/L$.

Example: Consider a fourth-order plant

$$G(s) = \frac{10}{(s+1)(s+2)(s+3)(s+4)}.$$

The open-loop step response is shown in Figure



$L = 0.76$, and $T = 2.72 - 0.76 = 1.96$ with a steady-state value of 0.4167.

K= 0.4167

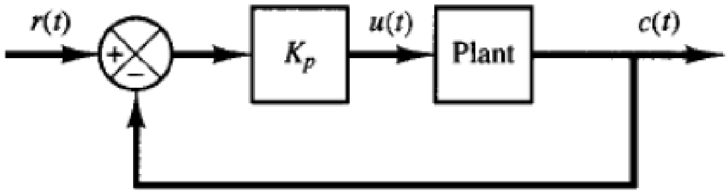
$$G_{\text{PID}}(s) = 7.4274\left(1 + \frac{1}{1.52s} + 0.38s\right).$$

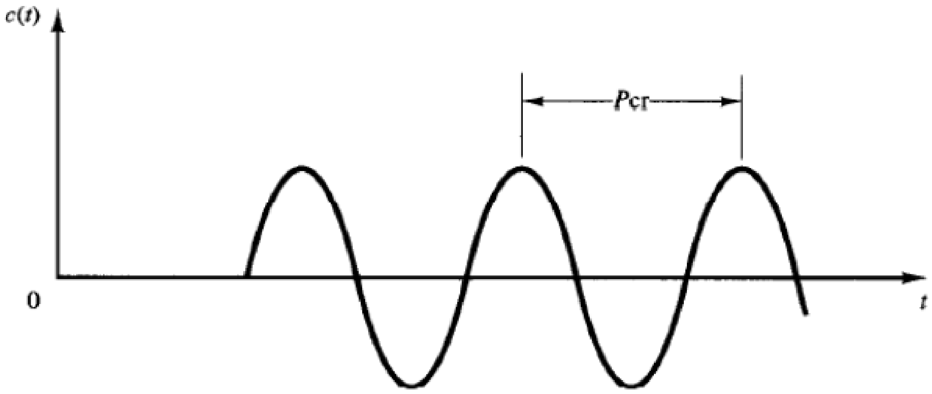
Kp= 1.2 T / KL , Ki=0.6 T/ KL^2, Kd= 0.6 T/K ,R=K/T

KP=1.2/RL ,Ki=0.6/RL^2 , Kd=0.6/R

Close Loop Method

Second method. In the second method, we first set $T_i = \infty$ and $T_d = 0$. Using the proportional control action only increase K_p from 0 to a critical value K_{cr} where the output first exhibits sustained oscillations. (If the output does not exhibit sustained oscillations for whatever value K_p may take, then this method does not apply.) Thus, the critical gain K_{cr} and the corresponding period P_{cr} are experimentally determined Ziegler and Nichols suggested that we set the values of the parameters K_p, T_i, and T_d according to the formula shown in Table





Ziegler–Nichols Tuning Rule Based on Critical Gain K_{cr} and Critical Period P_{cr}

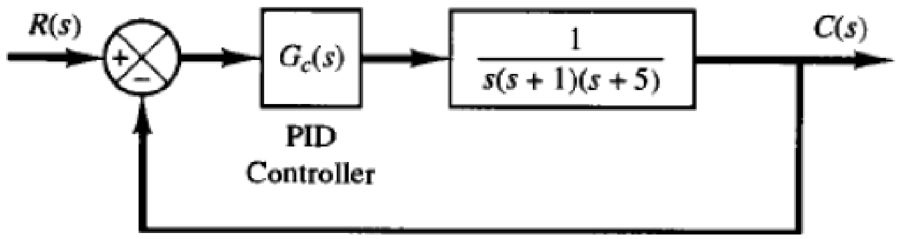
Type of Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = 0.6K_{cr}\left(1 + \frac{1}{0.5P_{cr}s} + 0.125\,P_{cr}s\right)$$

$$= 0.075K_{cr}P_{cr}\frac{\left(s + \frac{4}{P_{cr}}\right)^2}{s}$$

PID controller has a pole at the origin and double zeros at $s = -4/P_{cr}$.

Example: Consider the control system shown in Figure



$$\frac{C(s)}{R(s)} = \frac{K_p}{s(s+1)(s+5)+K_p}$$

$$s^3 + 6s^2 + 5s + K_p = 0$$

s^3	1	5
s^2	6	K_p
s^1	$\frac{30 - K_p}{6}$	
s^0	K_p	

$$K_{cr} = 30$$

$$\omega^2 = 5 \text{ or } \omega = \sqrt{5}.$$

$$P_{cr} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5}} = 2.8099$$

$$K_p = 0.6K_{cr} = 18$$

$$T_i = 0.5P_{cr} = 1.405$$

$$T_d = 0.125P_{cr} = 0.35124$$

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

$$= 18\left(1 + \frac{1}{1.405s} + 0.35124s\right)$$

$$= \frac{6.3223(s + 1.4235)^2}{s}$$