# ENGINEERING ANALYSIS II

# LECTURE 1

# Basic Probability and Statistics

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# Basic Probability and Statistics

- **Statistics** is the area of science that deals with collection, organization, analysis, and interpretation of data. It also deals with methods and techniques that can be used to draw conclusions about the characteristics of a large number of data points commonly called a **population**.

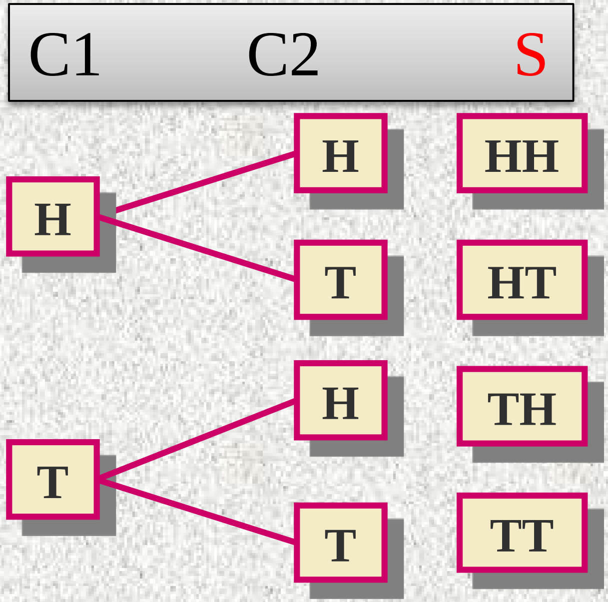
**Probability:** which measures the likelihood that an event will occur, is an important part of statistics. It is the basis of inferential statistics , where decisions are made under conditions of uncertainty

# Basic Concepts

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Sample Space (*S*)** is the set of all possible outcomes of an experiment The outcomes are called **sample points** in ***S***
- An **event** is the subset of the sample space.

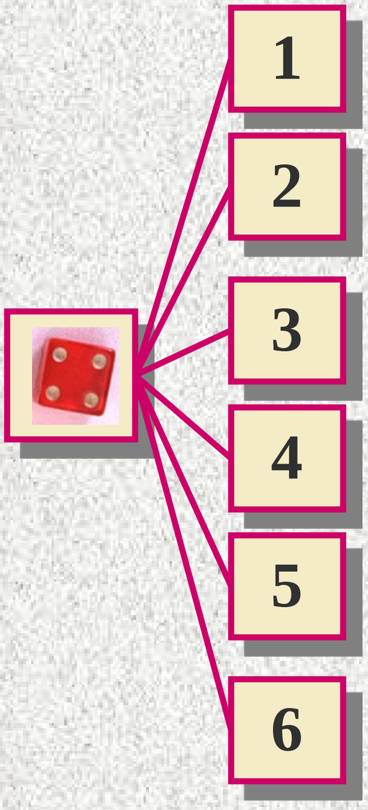
# Examples

| EXPERIMENT | SAMPLE SPACE(S) |
|---|---|
| Toss one coin | H, T |
| Roll a die | 1, 2, 3, 4, 5, 6 |
| Answer a true-false question | True, False |
| Toss two coins | HH, HT, TH, TT |



# Simple events

- **The die toss:**




$E_1$

$E_2$

$E_3$

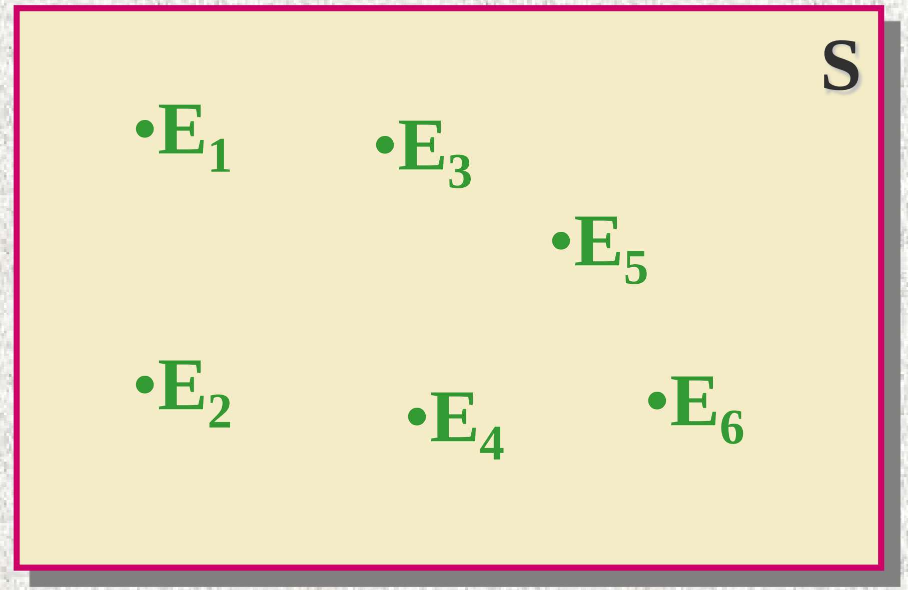
$E_4$

$E_5$

$E_6$

## Sample space:

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



# Basic Concepts

- Two events are **mutually exclusive** if, when one event occurs, the other cannot, and vice versa.

- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2
  - C: observe a 6
  - D: observe a 3

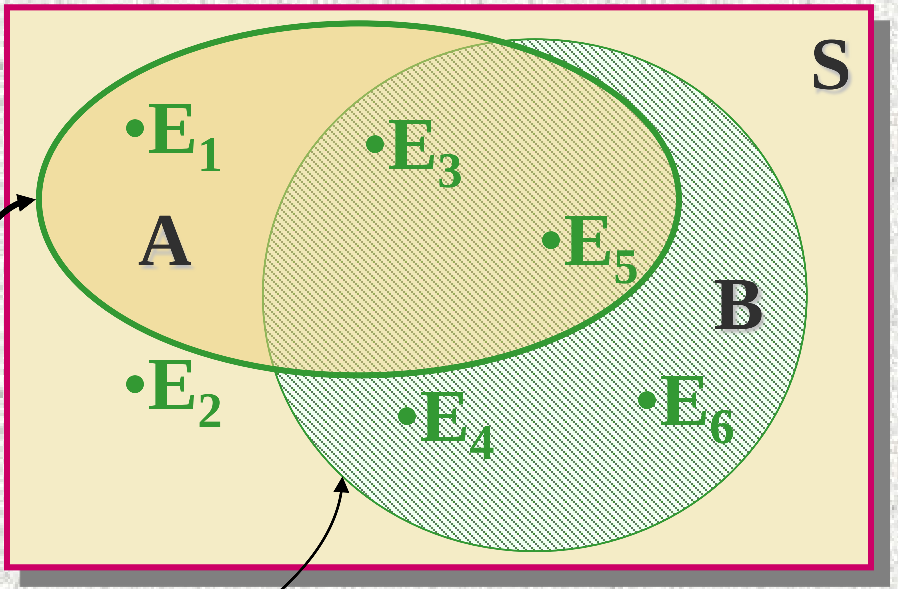
Not Mutually Exclusive

Mutually Exclusive

B and C?
B and D?

# Basic Concepts

- An **event** is a collection of one or more **simple events.**




- **The die toss:**
  - A: an odd number
  - B: a number > 2

**A ={$E_1$, $E_3$, $E_5$}**

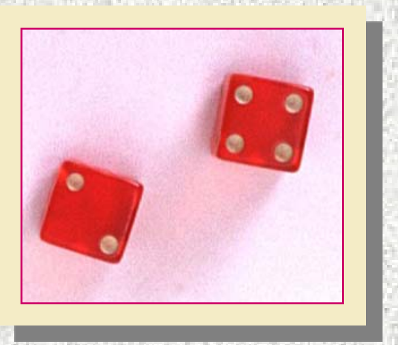
**B ={$E_3$, $E_4$, $E_5$, $E_6$}**

# Classical Probability

- The probability of an event A is equal to the sum of the probabilities of the simple events contained in A
- If the simple events in an experiment are **equally likely**, you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

# Empirical Probability



- The probability of an event A measures "how often" A will occur. We write **P(A).**
- Suppose that an experiment is performed *n* times. The relative frequency for an event A is

$$P(A) = \frac{\text{times event A occurs}}{\text{repetitions of the experiment}}$$

$$P(A) = \frac{f}{n}$$

# The Probability of an Event

- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0. If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.

- The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

# Finding Probabilities

- Probabilities can be found using
  - Estimates from empirical studies
  - Common sense estimates based on equally likely events.
- **Examples 1:**
  - Toss a fair coin. P(Head) = 1/2
  - Suppose that 10% of the U.S. population has red hair. Then for a person selected at random,

P(Red hair) = .10

# Example 2

Toss a fair coin twice. What is the probability of observing at least one head?

| 1st Coin | 2nd Coin | $E_i$ | $P(E_i)$ |
| --- | --- | --- | --- |
| H | H | HH | 1/4 |
| | T | HT | 1/4 |
| T | H | TH | 1/4 |
| | T | TT | 1/4 |

P(at least 1 head)

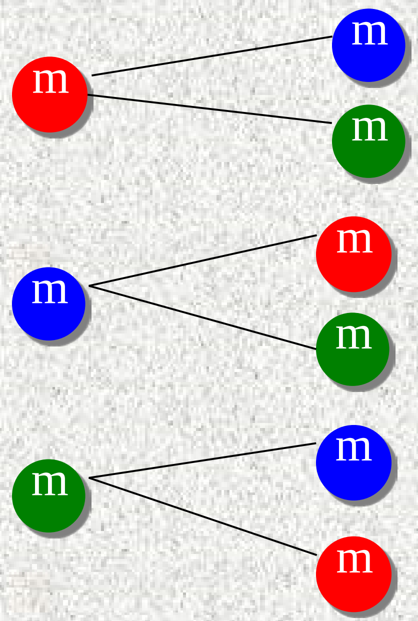
$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

# Example 3

A bowl contains three balls, one red, one blue and one green. A child selects two balls at random. What is the probability that at least one is red?

1st ball    2nd ball




| $E_i$ | $P(E_i)$ |
|---|---|
| **RB** | 1/6 |
| **RG** | 1/6 |
| **BR** | 1/6 |
| **BG** | 1/6 |
| **GB** | 1/6 |
| **GR** | 1/6 |

P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

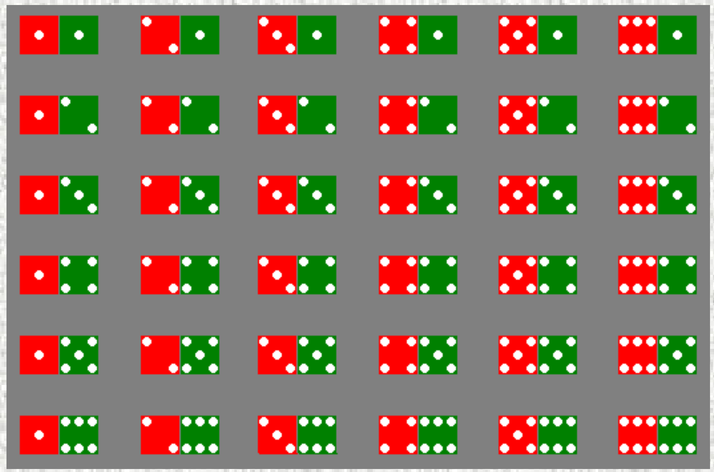
# Example 4

**Two dice are rolled .**

1-What is the probability that sum of dice greater than 3?

2- What is the probability that sum of dice equal 11?

3- What is the probability that sum of dice at least 9?



1-P(sum >3)=1- P(sum $\leq$3)= $1-\frac{3}{36}=\frac{33}{36}=\frac{11}{12}$

2-P(sum =11)=$\frac{2}{36}=\frac{1}{18}$

3-P(sum at least 9)=$\frac{10}{36}=\frac{5}{18}$

# ENGINEERING ANALYSIS II

## LECTURE 2

# Event Relations

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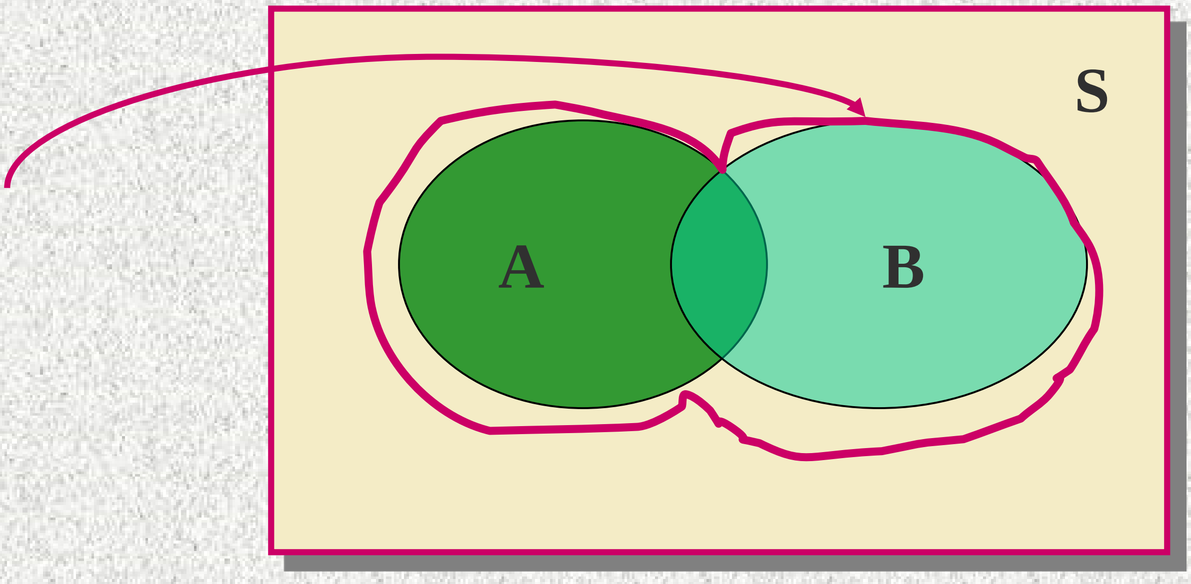
# Event Relations

A **Venn diagram** is useful for displaying the relationships among event in a sample space.

The beauty of using events, rather than simple events, is that we can **combine** events to make other events using logical operations: and, or and not.

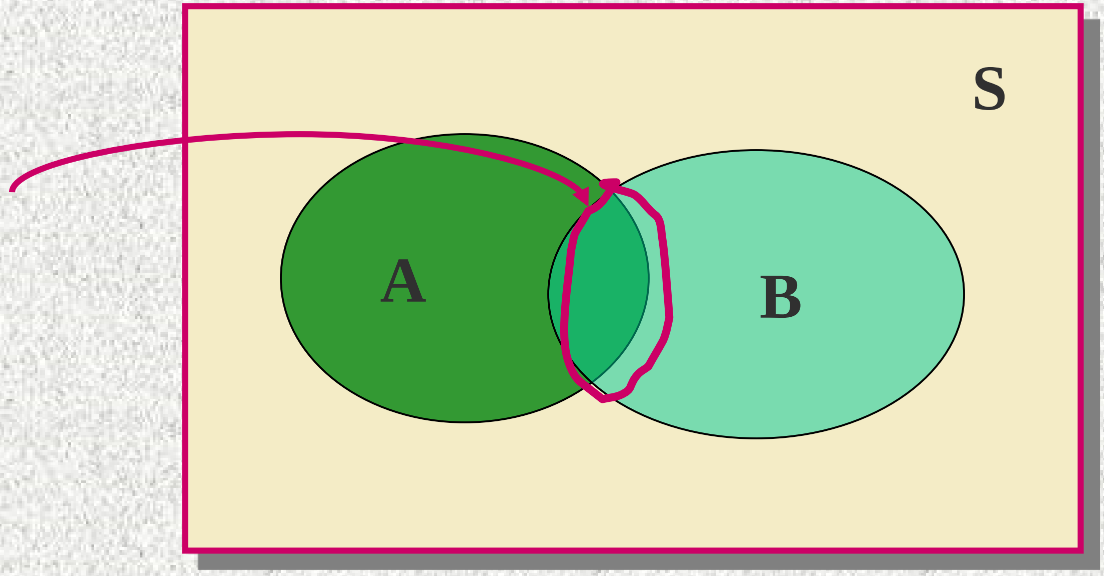
## 1- Union

The **union** of two events, **A** and **B**, is the event that either **A or B or both** occur when the experiment is performed. We write **A ∪ B**



# 2-Intersection

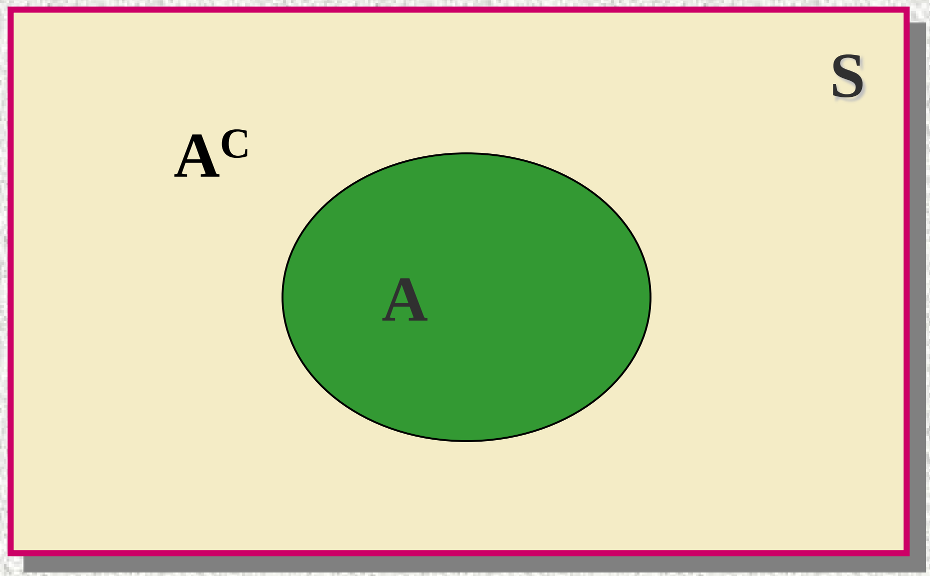
The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B**.




- If two events A and B are **mutually exclusive**, then P(**A ∩ B) = 0**.

# 3-Complement

The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event **A**. We write $A^C$.



# Example

Select a student from the classroom and record his/her **hair color** and **gender.**

– **A:** student has brown hair

– **B:** student is female

– **C:** student is male

Mutually exclusive; $B = C^C$

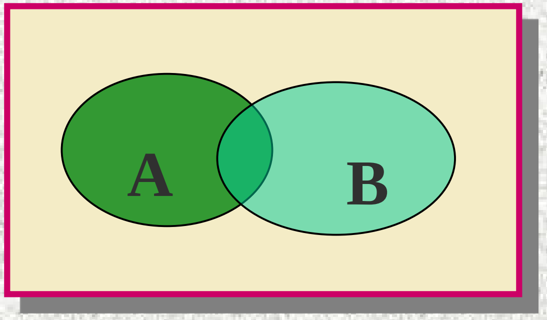
What is the relationship between events **B** and **C**?

- **$A^C$:** Student does not have brown hair
- **$B \cap C$:** Student is both male and female = $\varnothing$
- **$B \cup C$:** Student is either male and female = all students = S

# Calculating Probabilities for Unions and Complements

- There are special rules that will allow you to calculate probabilities for composite events.
- **The Additive Rule for Unions:**
- For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



# Example: Additive Rule

**Example 1**: Suppose that there were 120 students in the classroom, and that they could be classified as follows:

**A:** brown hair

**P(A) = 50/120**

**B:** female

**P(B) = 60/120**

| | **Brown** | **Not Brown** |
|---|---|---|
| **Male** | 20 | 40 |
| **Female** | 30 | 30 |

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 50/120 + 60/120 - 30/120$

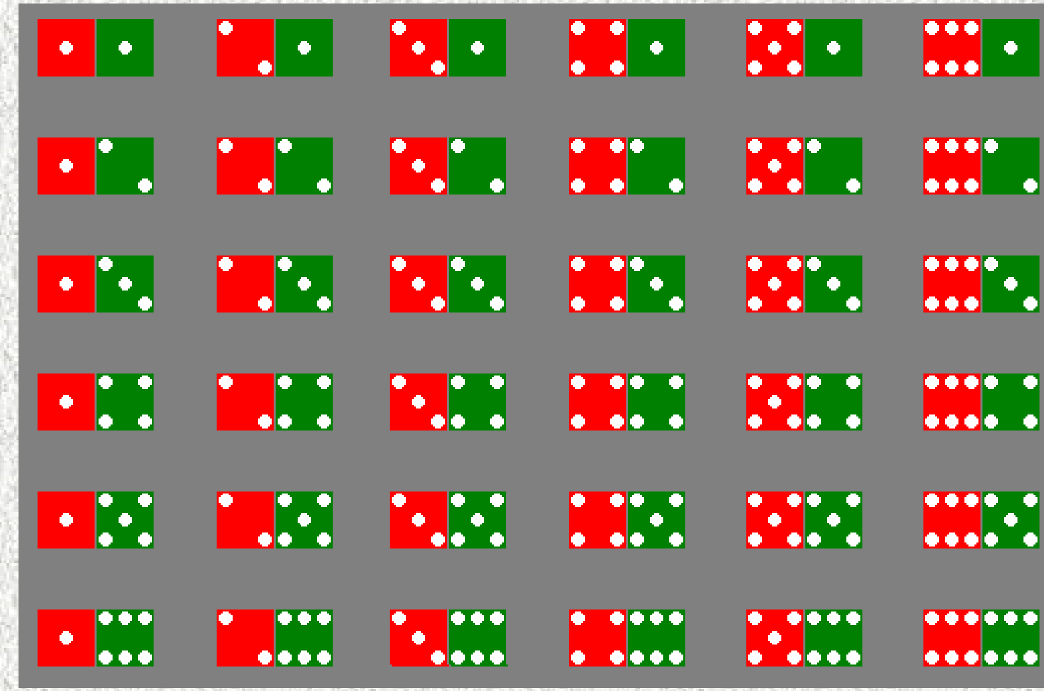
$= 80/120 = 2/3$

Check: $P(A \cup B)$

$= (20 + 30 + 30)/120$

# Example 2 :

Two dice Rolled. Find P(A∪B) when red die show 1 and green die show 1



A: red die show 1

B: green die show 1

Find P(A∪B) ?

P(A∪B) = P(A) + P(B) – P(A∩B)
= 6/36 + 6/36 – 1/36
= 11/36

# A Special Case

When two events A and B are **mutually exclusive** ,
$P(A \cap B) = 0$ and $\mathbf{P(A \cup B) = P(A) + P(B)}$.

**Example 3:** Suppose that there were 120 students in the classroom, and that they could be classified as follows:

**A:** male with brown hair
$P(A) = 20/120$
**B:** female with brown hair
$P(B) = 30/120$

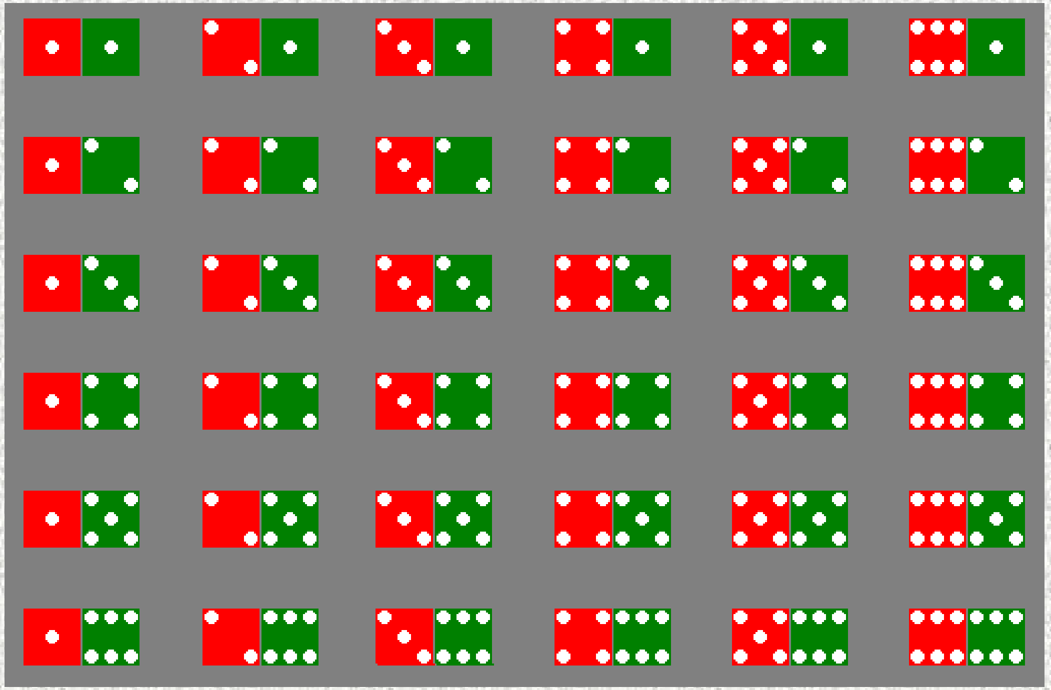
| | Brown | Not Brown |
|---|---|---|
| Male | 20 | 40 |
| Female | 30 | 30 |

A and B are mutually exclusive, so that →

$P(A \cup B) = P(A) + P(B)$
$= 20/120 + 30/120$
$= 50/120$

# Example 4:

Two Dice Rolled. Find P(A∪B) when event A dice add to 3 and event B dice add to 6



A: dice add to 3

B: dice add to 6

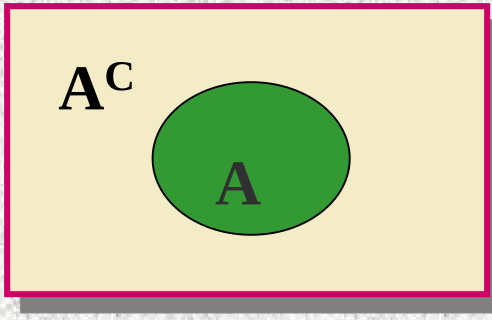
A and B are mutually exclusive, so that

$P(A \cup B) = P(A) + P(B)$

$= 2/36 + 5/36$

$= 7/36$

# Calculating Probabilities for Complements




- We know that for any event **A:**
  - $\mathbf{P(A \cap A^C) = 0}$
- Since either **A** or $\mathbf{A^C}$ must occur,

  $\mathbf{P(A \cup A^C) = 1}$
- so that $\mathbf{P(A \cup A^C) = P(A) + P(A^C) = 1}$

$$\mathbf{P(A^C) = 1 - P(A)}$$

# Example 5

Select a student at random from the classroom. Define:

| | Brown | Not Brown |
|---|---|---|
| Male | 20 | 40 |
| Female | 30 | 30 |

**A:** male

P(A) = 60/120

**B:** female

P(B) = ?

A and B are complementary, so that

P(B) = 1- P(A)
= 1- 60/120 = 60/120

# Conditional Probabilities

The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

"given"

# Defining Independence

- We can redefine independence in terms of conditional probabilities:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\mathbf{P(A \cap B) = P(A|B)\ P(B)}$$

- If the events **A** and **B** are independent, then the probability that both **A** and **B** occur is

$$\mathbf{P(A \cap B) = P(A)\ P(B)}$$

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are **dependent**.

# Example 1

Toss a fair coin twice. Define

- A: head on second toss
- B: head on first toss

| c2 c1 | S | |
|---|---|---|
| H H | HH | 1/4 |
| H T | HT | 1/4 |
| T H | TH | 1/4 |
| T T | TT | 1/4 |

P(A|B) = ½

P(A|not B) = ½

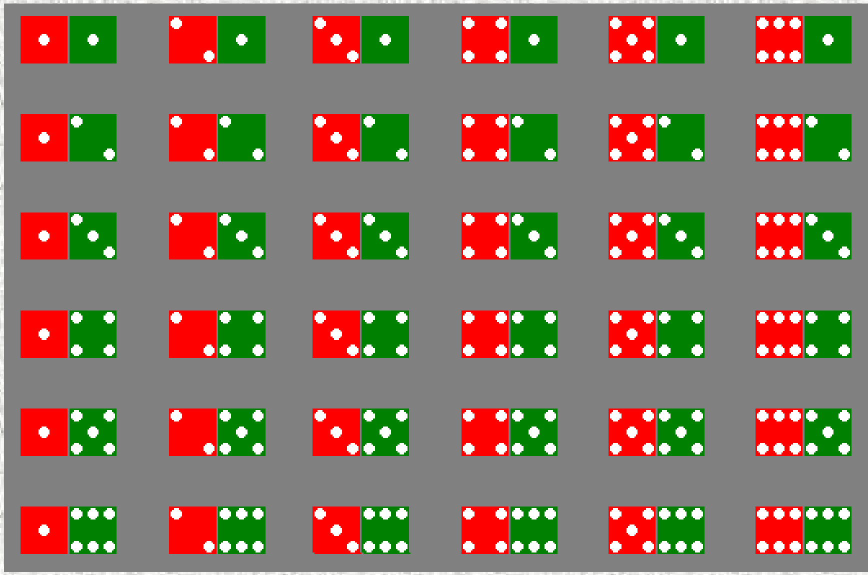
P(A) does not change, whether B happens or not…

A and B are independent!

# Example 2: Two Dice

Toss a pair of fair dice. Define

- A: red die show 1
- B: green die show 1



P(A|B) = P(A and B)/P(B)

=1/36 / 1/6=1/6=P(A)

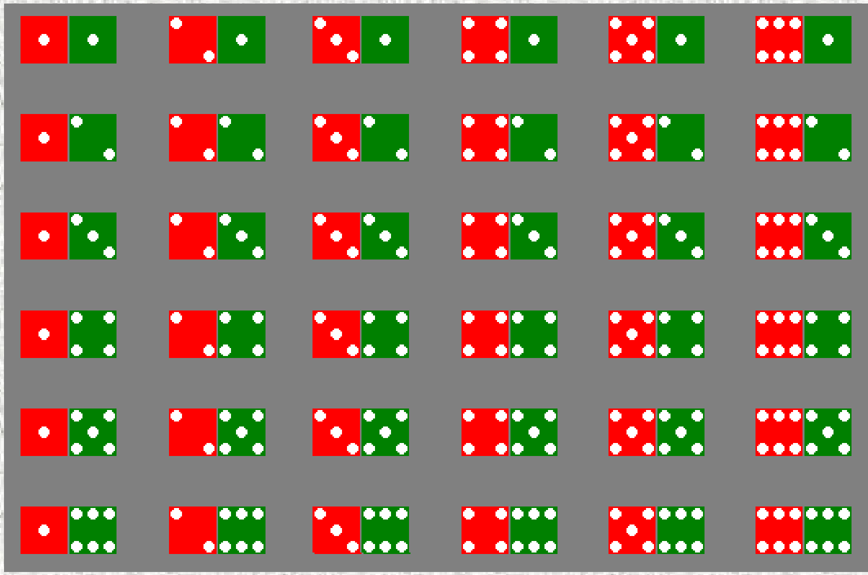
P(A) does not change, whether B happens or not…

A and B are independent!

# Example 3: Two Dice

Toss a pair of fair dice. Define

- A: add to 3
- B: add to 6



P(A|B) = P(A and B)/P(B)

=0/36/5/36=0

P(A) does change when B happens → A and B are dependent! In fact, when B happens, A can't

# Example 4

In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk?

Define H: high risk N: not high risk

P(exactly one high risk) = P(HNN) + P(NHN) + P(NNH)

= P(H)P(N)P(N) + P(N)P(H)P(N) + P(N)P(N)P(H)

= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1)= $3(.1)(.9)^2$ = .243

# Example 5

Suppose we have additional information in the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high risk female?

Define H: high risk F: female

From the example, P(F) = .49 and P(H|F) = .08.
Use the Multiplicative Rule:

$$P(\text{high risk female}) = P(H \cap F)$$

$$= P(F)P(H|F) = .49(.08) = .0392$$

# ENGINEERING ANALYSIS II

## LECTURE 3

# Counting Rules

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# Counting Rules

When outcomes are **equally likely** to occur (like when tossing a coin or rolling a die), you can use counting rules to find out how many outcomes are possible and then use that number to find probabilities.

Sample space of throwing 3 dice has 216 entries

Sample space of throwing 4 dice has 1296 entries, …

# 1-Multiplication Rule

- If an experiment is performed in two stages, with ***m*** ways to accomplish the first stage and ***n*** ways to accomplish the second stage, then there are ***mn*** ways to accomplish the experiment.
- This rule is easily extended to ***k*** stages, with the number of ways equal to $n_1 n_2 n_3 \ldots n_k$

**Example 1:** Toss three coins.

The total number of simple events is:

$2 \times 2 \times 2 = 8$

| c3 | c2 | c1 | S |
|---|---|---|---|
| H | H | H | HHH |
| H | H | T | HHT |
| H | T | H | HTH |
| H | T | T | HTT |
| T | H | H | THH |
| T | H | T | THT |
| T | T | H | TTH |
| T | T | T | TTT |

**Example2:** Toss three dice. The total number of simple events is:

$$6 \times 6 \times 6 = 216$$

**Example 3:** How many 4 or 5 digit telephone numbers are possible, assuming the first is not zero?

ans : $9 \times 10 \times 10 \times 10 + 9 \times 10 \times 10 \times 10 \times 10 = 99\,000$

**Example 4:**

(a)How many different car number plates are Possible with 3 letters followed by 3 digit ? Ans: 26*26*26*10*10*10 $= 26^3 \times 10^3$

(b) How many of these number plates begin with ABC

Solution: $1 \times 1 \times 1 \times 10 \times 10 \times 10 = 10^3$

(c) If a plate is chosen at random, what is the probability that it begins with ABC?)

$$P(ABC) = \frac{10^3}{26^3 \times 10^3} = \frac{1}{26^3}$$

# 2-Permutation Rule

Is an ordered arrangement of objects

- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)(n-2)...(2)(1)$ and $0! \equiv 1$.

**Example1 :** How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

**The number of permutations of n objects using all of them is n!**

**Example 1:** In how many ways can 5 people line up in a queue?

ans 5! = 120

**Example2:** A lock consists of five parts and can be assembled in any order. A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

The order of the choice is important!

$$P_5^5 = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120$$

# Permutations with repeated elements

- If a bag contains some objects in which m1 are of type 1, m2 are of type 2, ....... mk are of type k. The number of permutation is:

$$\frac{(m_1 + m_2 + \ldots + m_k)!}{m_1! m_2! \ldots m_k!}$$

**Example :** How many ways can you permute the letters: **B A N A N A** ?

Of the **6** letters, there are **3 A's**, **2 N's**, and **1 B**.
The **2 N's** could be rearranged in **2! = 2** different ways.
The **3 A's** could be rearranged in **3! = 6** different ways.
So we need to divide 6! by both 6 and 2.

The number of ways to rearrange the letters in **BANANA** is $\frac{6!}{3!\,2!\,1!} = 60$

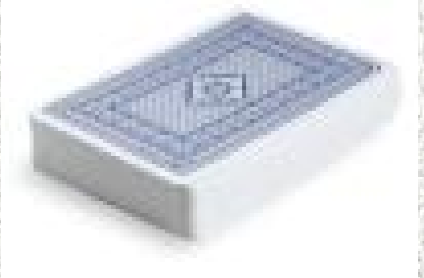
# 3-Combination Rule

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is $C_r^n = \frac{n!}{r!(n-r)!}$

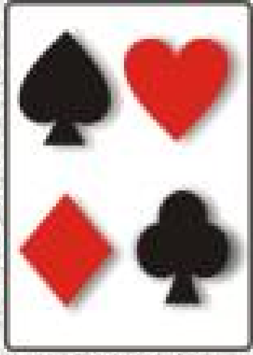
**Example 1:** Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$



# Example 2

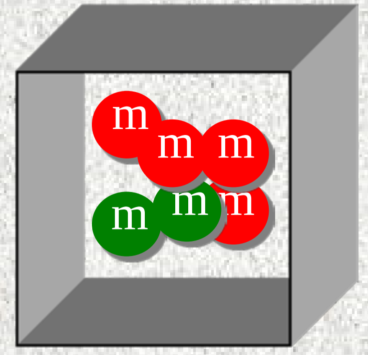


A deck of cards consists of 52 cards, 13 "kinds" each of four suits (spades, hearts, diamonds, and clubs). The 13 kinds are Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen (Q), King (K). In many poker games, each player is dealt five cards from a well shuffled deck.

There are $C_5^{52} = \dfrac{52!}{5!(52-5)!} = \dfrac{52(51)(50)(49)48}{5(4)(3)(2)1} = 2{,}598{,}960$

possible hands

# Probability Calculations Using Combinations / Permutations




**Example 1**

- A box contains six balls, four red and two green. A child selects two balls at random. What is the probability that exactly one is red?

The order of the choice is not important!

$$C_2^6 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 M & Ms.

$$C_1^2 = \frac{2!}{1!1!} = 2$$

ways to choose 1 green M & M.

$$C_1^4 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red M & M.

$4 \times 2 = 8$ ways to choose 1 red and 1 green M&M.

P(exactly one red) = 8/15

**Example 2:** 4 chocolates are chosen at random from a box containing 6 with hard centers , and 8 with soft centers.

(a) Calculate the probability that 3 of the chocolates have soft centers.

(b) Calculate the probability that at least 3 of the chocolates have soft centers.

(a) Total number of ways of selecting 4 chocolates = ${}^{14}C_4$ = 1 001

Number of ways of selecting 3 soft centers (and 1 hard) = ${}^{8}C_3 \times {}^{6}C_1$

= 336

$$P(\text{3 soft}) = \frac{336}{1001} = 0.3357$$

(b) Numbers of ways of selecting at least 3 soft = ${}^{8}C_3 \times {}^{6}C_1 + {}^{8}C_4 \times {}^{6}C_0$

$= 336 + 70$

$= 406$

P( at least 3 soft) = $\frac{406}{1001}$ or 0.4056

**Example 3:** A 4 digit security number is made using the digits 0, 1,……..9. If a number is made up at random, what is the probability that it contains the same digit repeated 3 times in a row.

(a) Total number of security codes = $\mathbf{10^4}$ = 10000.

(b) Total number of ways of getting **3** of the same in a row

$10 \times (1 \times 1 \times 1 \times 9 + 9 \times 1 \times 1 \times 1) = 180$

(c) $P(\text{3 in a row}) = \frac{180}{10000} = 0{\cdot}018$

## Example 4

From a group of 3 Indians, 4 Pakistanis, and 5 Americans, a sub-committee of four people is selected by lots. Find the probability that the sub-committee will consist of

i) 2 Indians and 2 Pakistanis.

ii) 1 Indians, 1 Pakistanis and 2 Americans.

iii) 4 Americans.

Sol: Total no. of people = 3 + 4 + 5 = 12

$\therefore$ 4 people can be chosen from 12 people = $12C_4$ ways

$$= \frac{12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4} = 495 \text{ ways}$$

i) 2 Indians can be chosen from 3 Indians = $3C_2$ ways

2 Pakistanis can be chosen from 4 Pakistanis = $4C_2$ ways

$\therefore$ No. of favourable cases = $3C_2 \times 4C_2$

$$\therefore \text{Prob.} = \frac{3C_2 \times 4C_2}{495} = \frac{2}{55}$$

ii) 1 Indian can be chosen from 3 Indians = $3C_1$ ways

1 Pakistani can be chosen from 4 Pakistanis = $4C_1$ ways

2 Americans can be chosen from 5 Americans = $5C_2$ ways

Favourable events = $3C_1 \times 4C_1 \times 5C_2$

$\therefore$ Prob. $= \frac{3C_1 \times 4C_2 \times 5C_2}{495} = \frac{8}{33}$

iii) 4 Americans can be chosen from 5 Americans = $5C_4$ ways

$\therefore$ Prob. $= \frac{5C_4}{495} = \frac{1}{99}$

**Example 5**

A bag contains 7 white, 6 red & 5 black balls. Two balls are drawn at random. Find the prob. that they both will be white.

Total no. of balls = 7 + 6 + 5=18

From there 18 balls, 2 balls can be drawn in $18C_2$ ways

i.e) $\frac{18 \times 17}{1 \times 2} = 153$

2 white balls can be drawn from 7 white balls = $7C_2$ ways

$= 21$

$\therefore$ Favourable cases = 21

P(drawing 2 white balls) = $\frac{21}{153} = \frac{7}{51}$

**Example 6**

A bag contains 10 white, 6 red, 4 black & 7 blue balls. 5 balls are drawn at random. What is the prob. that 2 of them are red and one is black?

Sol: Total no. of balls = 10 + 6 + 4 + 7 =27

5 balls can be drawn from these 27 balls = $27C_5$ ways

$$= \frac{27 \times 26 \times 25 \times 24 \times 23}{1 \times 2 \times 3 \times 4 \times 5}$$

= 80730 ways

Total no. of exhaustive events = 80730

2 red balls can be drawn from 6 red balls = $6C_2$ ways

$$= \frac{6 \times 5}{1 \times 2} = 15 \text{ ways}$$

1 black balls can be drawn from 4 black balls = $4C_1$ ways

= 4

$\therefore$ No. of favourable cases = 15 × 4 = 60

$$\text{Probability} = \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

$$= \frac{60}{80730} = \frac{6}{8073}$$

**Example 7.** In how many ways can 5 boys and 4 girls be arranged on a bench if

a) there are no restrictions? **9! or $_9P_9$**

b) boys and girls alternate?

**BGBGBGBGB = 5 * 4 * 4 * 3 * 3 *2 * 2 * 1 *1= 5! * 4!**
**or $_5P_5 * {}_4P_4$**

c)boys and girls are in separate groups?

**Boys & Girls or Girls & Boys**

$$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$$

$$\text{or } {}^5P_5 \times {}^4P_4 \times 2$$

d) Anne and Jim wish to stay together?

**(AJ) _ _ _ _ _ _ _**

$$= 2 \times 8! \text{ or } 2 \times {}^8P_8$$

**Example 8.** There are 6 boys who enter a boat with 8 seats, 4 on each side. In how many ways can

a) they sit anywhere? **Solution :** $_{8}P_{6}$

b) two boys A and B sit on the port side and another boy W sit on the starboard side?

**Solution :** A & B = $_{4}P_{2}$

W = $_{4}P_{1}$

Others = $_{5}P_{3}$

Total = $_{4}P_{2} * {}_{4}P_{1} * {}_{5}P_{3}$

# ENGINEERING ANALYSIS II

## LECTURE 4

# Discrete Random Variables

Prepared by: Mr. Abdullah I. Abdullah

# Random Variables and Probability Distributions

## 1 Concept of a Random Variable:

· In a statistical experiment, it is often very important to allocate numerical values to the outcomes.

**A random variable :**is a function that assigns areal number to each outcome in the sample space of a random experiment

**Example 1:**

· Experiment: testing two components. (D=defective, N=non-defective)

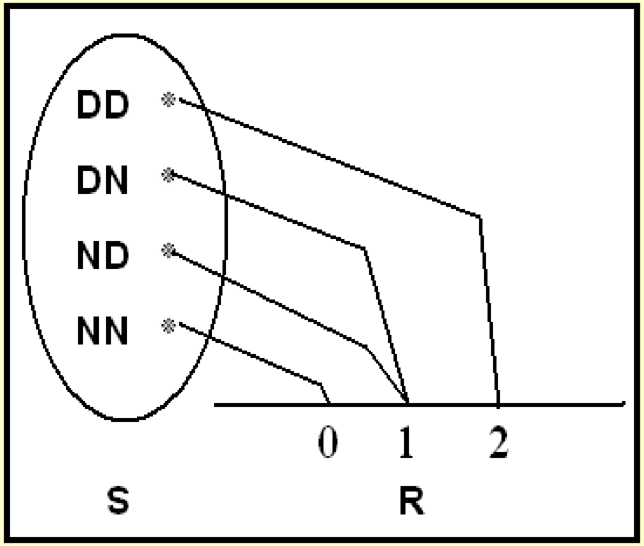
· Sample space: *S*={DD,DN,ND,NN}

· Let X = number of defective components when two components are tested.

· Assigned numerical values to the outcomes are:

| c2 | c1 | S |
|---|---|---|
| D | D | DD |
| D | N | DN |
| N | D | ND |
| N | N | NN |

| Sample point (Outcome) | Assigned Numerical Value (x) |
|---|---|
| DD | 2 |
| DN | 1 |
| ND | 1 |
| NN | 0 |




- ❑Notice that, the set of all possible values of the random variable X is {0, 1, 2}.

***Definition 1:***

A random variable X is a function that associates each element in the sample space with a real number (i.e., X : $S \rightarrow$ **R**.)

***Notation:*** " X " denotes the random variable .

" x " denotes a value of the random variable X.

## Types of Random Variables:

· A random variable X is called a **discrete** random variable if its set of possible values is countable, i.e.,
.$x \in \{x_1, x_2, \ldots, x_n\}$ or $x \in \{x_1, x_2, \ldots\}$

· A random variable X is called a **continuous** random variable if it can take values on a continuous scale, i.e.,
.$x \in \{x: a < x < b;\ a, b \in R\}$

· In most practical problems:

o A discrete random variable represents count data, such as the number of defectives in a sample of *k* items.

o A continuous random variable represents measured data, such as height.

# 2 Discrete Probability Distributions

- A discrete random variable X assumes each of its values with a certain probability.

**Example 1:**

- Experiment: tossing a non-balance coin 2 times independently.
- H= head , T=tail
- Sample space: $S$={HH, HT, TH, TT}
- Suppose P(H)=½ P(T) ⇔ P(H)=1/3 and P(T)=2/3
- Let X= number of heads

| c2 | c1 | S |
|---|---|---|
| H | H | HH |
| H | T | HT |
| T | H | TH |
| T | T | TT |

| Sample point (Outcome) | Probability | Value of X (x) |
|---|---|---|
| HH | P(HH)=P(H) P(H)=1/3×1/3 = 1/9 | 2 |
| HT | P(HT)=P(H) P(T)=1/3×2/3 = 2/9 | 1 |
| TH | P(TH)=P(T) P(H)=2/3×1/3 = 2/9 | 1 |
| TT | P(TT)=P(T) P(T)=2/3×2/3 = 4/9 | 0 |

- The possible values of X are: 0, 1, and 2.
- X is a discrete random variable.
- Define the following events:

| Event (X=x) | Probability = P(X=x) |
|---|---|
| (X=0)={TT} | P(X=0) = P(TT)=4/9 |
| (X=1)={HT,TH} | P(X=1) =P(HT)+P(TH)=2/9+2/9=4/9 |
| (X=2)={HH} | P(X=2) = P(HH)= 1/9 |

- The possible values of X with their probabilities are:

| X | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| P(X=x)=f(x) | 4/9 | 4/9 | 1/9 | 1.00 |

The function f(x)=P(X=x) is called the probability function (probability distribution) of the discrete random variable X.

***Definition 3:***

The function f(x) is a probability function of a discrete random variable X if, for each possible values x, we have:

1) $f(x) \geq 0$
2) $\sum_{all\ x} f(x) = 1$
3) f(x)= P(X=x)

***Note:***

$$P(X \in A) = \sum_{\text{all } x \in A} f(x) = \sum_{\text{all } x \in A} P(X = x)$$

***Example 2:***

For the previous example, we have:

| X | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| f(x)= P(X=x) | 4/9 | 4/9 | 1/9 | $\sum_{x=0}^{2} f(x) = 1$ |

$P(X<1) = P(X=0)=4/9$

$P(X\leq 1) = P(X=0) + P(X=1) = 4/9+4/9 = 8/9$

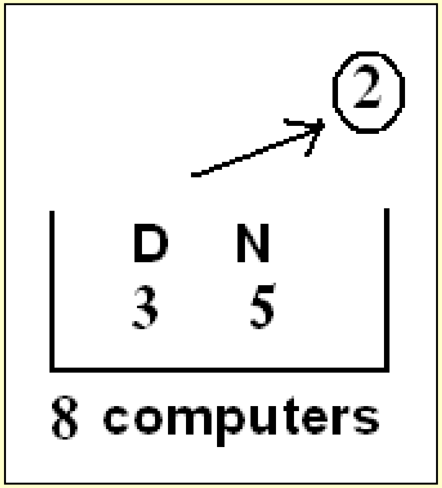
$P(X\geq 0.5) = P(X=1) + P(X=2) = 4/9+1/9 = 5/9$

$P(X>8) = P(\phi) = 0$

$P(X<10) = P(X=0) + P(X=1) + P(X=2) = P(S) = 1$

***Example 3:***

A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective and 5 are non-defective. If a school makes a random purchase of 2 of these computers, find the probability distribution of the number of defectives.




***Solution:***

We need to find the probability distribution of the random variable: X = the number of defective computers purchased.

Experiment: selecting 2 computers at random out of 8

$n(S) = \binom{8}{2}$ equally likely outcomes

The possible values of X are: x=0, 1, 2.

Consider the events:

$$(X=0)=\{0D \text{ and } 2N\} \Rightarrow n(X=0)=\binom{3}{0}\times\binom{5}{2}$$

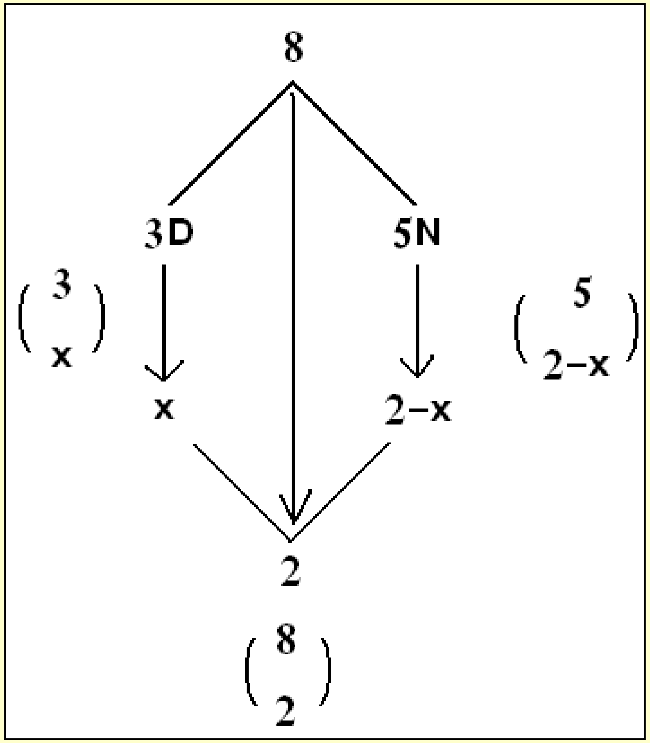
$$(X=1)=\{1D \text{ and } 1N\} \Rightarrow n(X=1)=\binom{3}{1}\times\binom{5}{1}$$

$$(X=2)=\{2D \text{ and } 0N\} \Rightarrow n(X=2)=\binom{3}{2}\times\binom{5}{0}$$

$$f(0)=P(X=0)=\frac{n(X=0)}{n(S)}=\frac{\binom{3}{0}\times\binom{5}{2}}{\binom{8}{2}}=\frac{10}{28}$$

$$f(1) = P(X=1) = \frac{n(X=1)}{n(S)} = \frac{\binom{3}{1} \times \binom{5}{1}}{\binom{8}{2}} = \frac{15}{28}$$

$$f(2) = P(X=2) = \frac{n(X=2)}{n(S)} = \frac{\binom{3}{2} \times \binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}$$




In general, for x=0,1, 2, we have:

$$f(x) = P(X=x) = \frac{n(X=x)}{n(S)} = \frac{\binom{3}{x} \times \binom{5}{2-x}}{\binom{8}{2}}$$

The probability distribution of X is:

| x | 0 | 1 | 2 | Total |
|---|---|---|---|---|
| f(x)= P(X=x) | $\frac{10}{28}$ | $\frac{15}{28}$ | $\frac{3}{28}$ | 1.00 |

$$f(x) = P(X = x) = \begin{cases} \dfrac{\binom{3}{x} \times \binom{5}{2-x}}{\binom{8}{2}}; & x = 0, 1, 2 \\ 0; \textit{ otherwise} \end{cases}$$

Hypergeometric Distribution

**Definition.4:**

The cumulative distribution function (CDF), F(x), of a discrete random variable X with the probability function f(x) is given by:

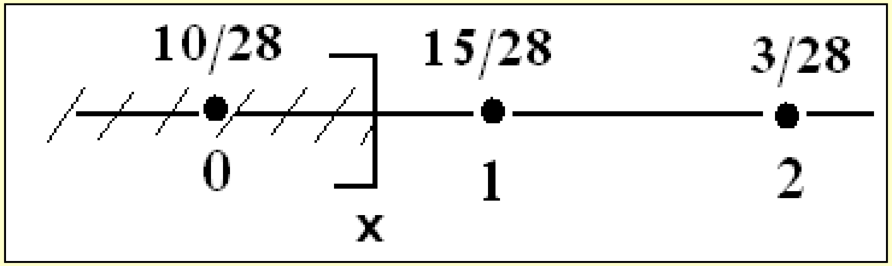
$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t) = \sum_{t \leq x} P(X = t); \quad \text{for } -\infty < x < \infty$$

## *Example 1:*

Find the CDF of the random variable X with the probability function:

| X | 0 | 1 | 2 |
|---|---|---|---|
| F(x) | $\frac{10}{28}$ | $\frac{15}{28}$ | $\frac{3}{28}$ |

## *Solution:*




F(x)=P(X≤x) for $-\infty<x<\infty$

For x<0: F(x)=0

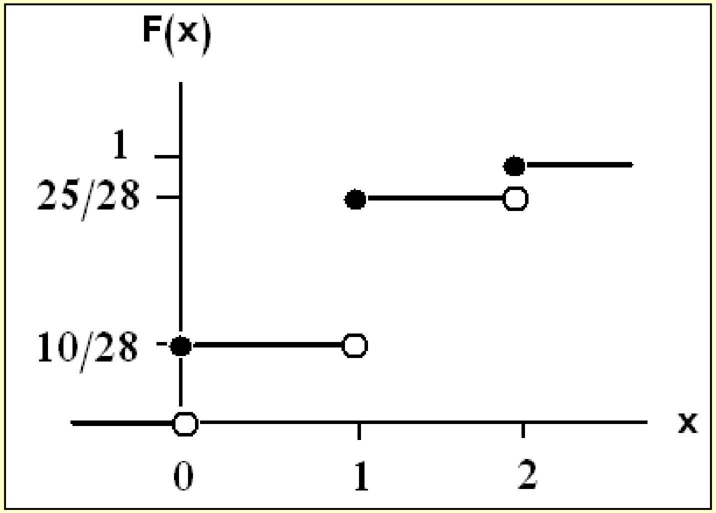
For 0≤x<1: F(x)=P(X=0)= $\frac{10}{28}$

For 1≤x<2: F(x)=P(X=0)+P(X=1)= $\frac{10}{28}+\frac{15}{28}=\frac{25}{28}$

For x≥2: F(x)=P(X=0)+P(X=1)+P(X=2)= $\frac{10}{28}+\frac{15}{28}+\frac{3}{28}=1$

The CDF of the random variable X is:

$$F(x) = P(X \le x) = \begin{cases} 0 \;;\; x < 0 \\ \frac{10}{28} \;;\; 0 \le x < 1 \\ \frac{25}{28} \;;\; 1 \le x < 2 \\ 1 \;;\; x \ge 2 \end{cases}$$




*Note:*

F(–0.5) = P(X≤–0.5)=0

F(1.5)=P(X≤1.5)=F(1) = $\frac{25}{28}$

F(3.8) =P(X≤3.8)=F(2)= 1

***Result:***

P(a **<** X ≤ b) = P(X ≤ b) – **P(X** ≤ a) = F(b) – **F(a)**

**P(a ≤ X ≤ b) = P(a <** X ≤ b) + P(X=a) = F(b) – **F(a) + f(a)**

**P(a < X < b) = P(a <** X ≤ b) – **P(X=b) = F(b) – F(a) – f(b)**

***Result:***

Suppose that the probability function of X is:

| x | $x_1$ | $x_2$ | $x_3$ | … | $x_n$ |
|---|---|---|---|---|---|
| F(x) | $f(x_1)$ | $f(x_2)$ | $f(x_3)$ | … | $f(x_n)$ |

Where $x_1 < x_2 < \ldots < x_n$. Then:

$F(x_i) = f(x_1) + f(x_2) + \ldots + f(x_i)$ ; $i=1, 2, \ldots, n$

$F(x_i) = F(x_{i-1}) + f(x_i)$ ; $i=2, \ldots, n$

$f(x_i) = F(x_i) - F(x_{i-1})$

***Example 2:***

In the previous example,

$P(0.5 < X \leq 1.5) = F(1.5) - F(0.5) = \frac{25}{28} - \frac{10}{28} = \frac{15}{28}$

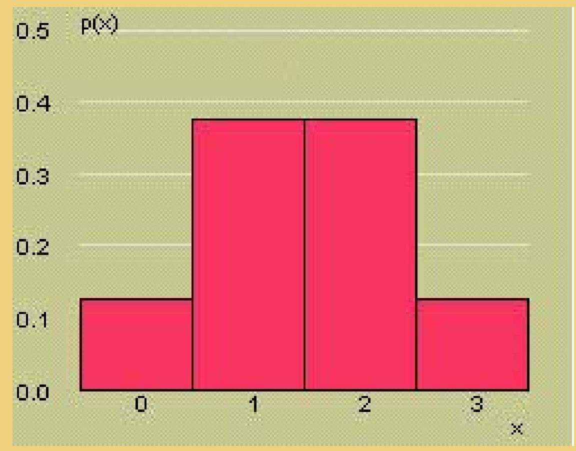
$P(1 < X \leq 2) = F(2) - F(1) = 1 - \frac{25}{28} = \frac{3}{28}$

**Example 3:** Toss a fair coin three times and define **$x$ = number of heads.**

| Outcome | Probability | $x$ |
|---|---|---|
| HHH | 1/8 | 3 |
| HHT | 1/8 | 2 |
| HTH | 1/8 | 2 |
| THH | 1/8 | 2 |
| HTT | 1/8 | 1 |
| THT | 1/8 | 1 |
| TTH | 1/8 | 1 |
| TTT | 1/8 | 0 |

$P(x = 0) = 1/8$
$P(x = 1) = 3/8$
$P(x = 2) = 3/8$
$P(x = 3) = 1/8$

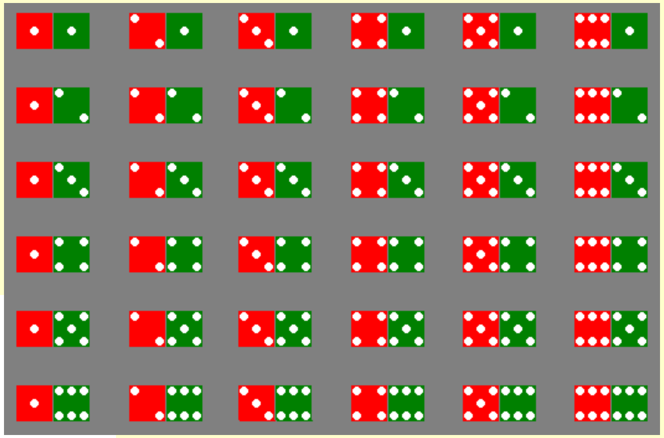
| $x$ | $p(x)$ |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |

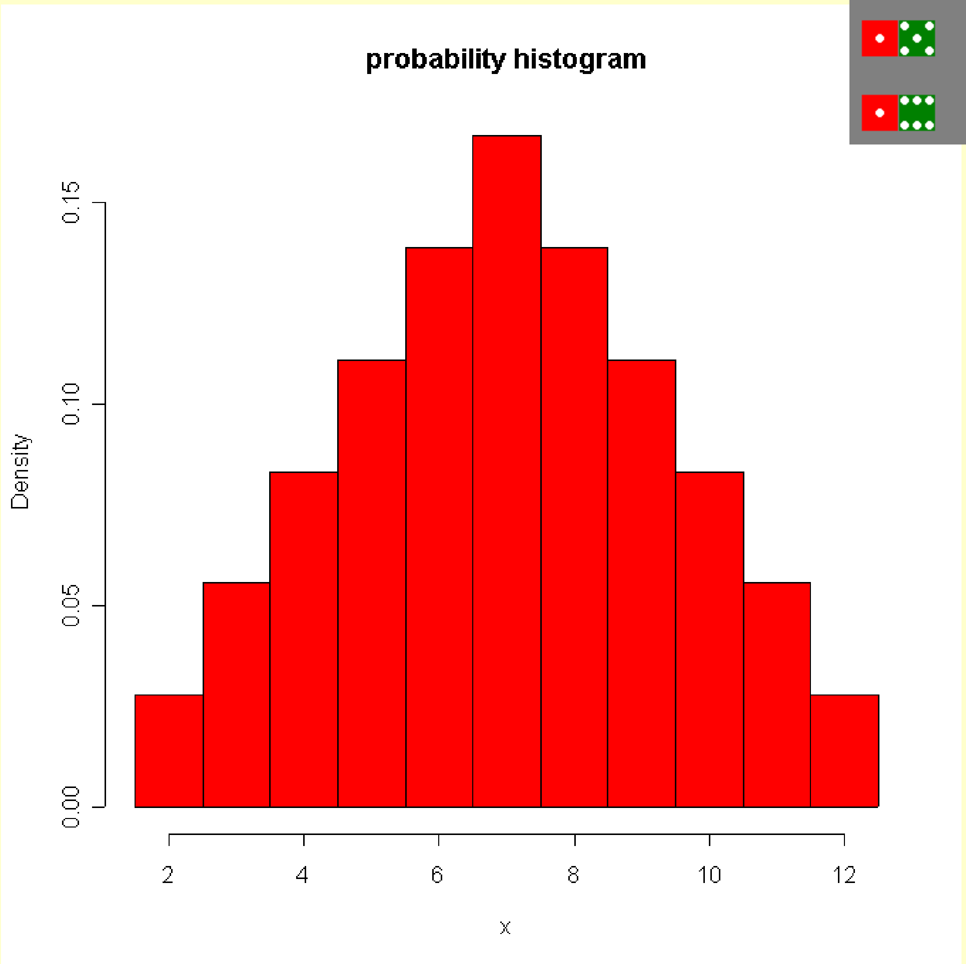



Probability Histogram for $x$

| c3 | c2 | c1 | S |
|---|---|---|---|
| H | H | H | HHH |
| H | H | T | HHT |
| H | T | H | HTH |
| H | T | T | HTT |
| T | H | H | THH |
| T | H | T | THT |
| T | T | H | TTH |
| T | T | T | TTT |

**Example 4** Toss two dice and define ***x* = sum of two dice.**






| *x* | *p(x)* |
|---|---|
| 2 | 1/36 |
| 3 | 2/36 |
| 4 | 3/36 |
| 5 | 4/36 |
| 6 | 5/36 |
| 7 | 6/36 |
| 8 | 5/36 |
| 9 | 4/36 |
| 10 | 3/36 |
| 11 | 2/36 |
| 12 | 1/36 |

# 3 -The Mean and Standard Deviation

The **mean** or **expected value** of a random variable *x* is the average value that we should expect for *x* over many trials of the experiment.

Often, we are also interested in how much the values of a random variable differ from trial to trial. To measure this, we can define the variance and standard deviation for a random variable

Let *x* be a discrete random variable with probability distribution *p(x)*.Then the mean, variance and standard deviation of *x* are given as

$$\text{Mean}: \mu = \sum xp(x)$$

$$\text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\text{Standard deviation}: \sigma = \sqrt{\sigma^2}$$

**Example 5:** Toss a fair coin 3 times and record *x* the number of heads.

| $x$ | $p(x)$ | $xp(x)$ | $(x-\mu)^2 p(x)$ |
| --- | --- | --- | --- |
| 0 | 1/8 | 0 | $(-1.5)^2(1/8)$ |
| 1 | 3/8 | 3/8 | $(-0.5)^2(3/8)$ |
| 2 | 3/8 | 6/8 | $(0.5)^2(3/8)$ |
| 3 | 1/8 | 3/8 | $(1.5)^2(1/8)$ |

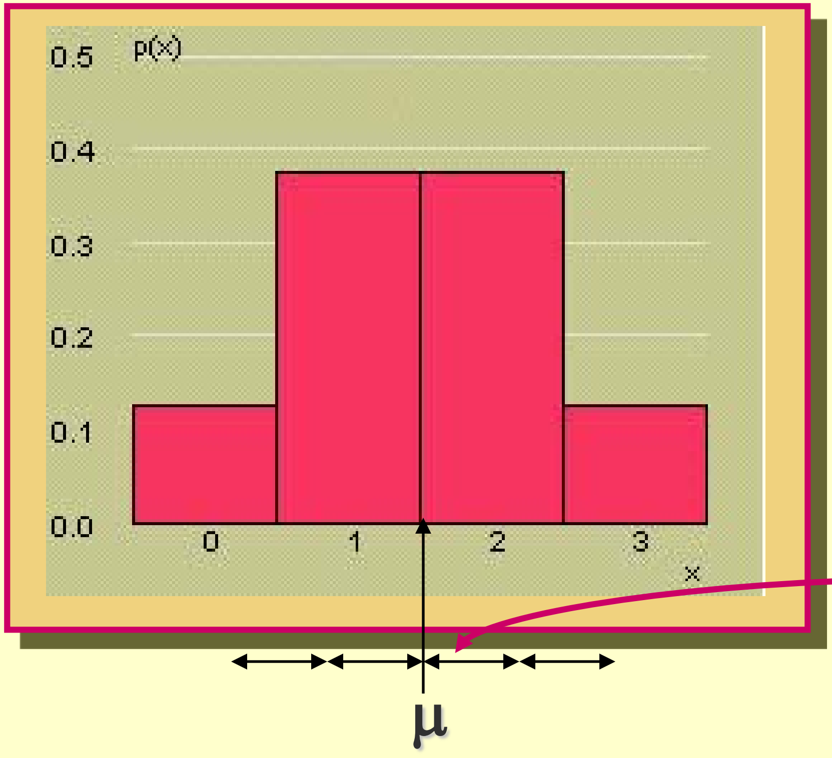
$$\mu = \sum xp(x) = \frac{12}{8} = 1.5$$

$$\sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\sigma^2 = .28125 + .09375 + .09375 + .28125 = .75$$

$$\sigma = \sqrt{.75} = .688$$

**Example 6:** The probability distribution for *x* the number of heads in tossing 3 fair coins.




- Shape? Symmetric; mound-shaped
- Outliers? None
- Center? $\mu = 1.5$
- Spread? $\sigma = .688$

# ENGINEERING ANALYSIS II

## LECTURE 5

## Binomial Distribution

Prepared by: Mr. Abdullah I. Abdullah

- The **Binomial Distribution** ***applies ONLY*** to cases where there are ***only 2*** possible outcomes: heads or tails, success or failure, defective or good item, etc.

## ***Requirements justifying use of the Binomial Distribution:***

1. The experiment must consist of ***n identical*** **trials.**
2. Each trial must result in ***only one*** **of** ***2 possible*** **outcomes.**
3. The outcomes of the trials **must be** ***statistically independent.***
4. All trials must have ***the same probability*** for a particular outcome.

# Binomial Distribution

**The Probability of x Successes out of n Attempts is:**

$$P(X = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

**p = Probability of a Success**

**q = Probability of a Failure , q = 1 – p**

**$(p + q)^n = 1$**

$$p(X = x) = b(x, n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

# Mean [$\mu$ , $E(x)$] and Standard Deviation $\sigma$ of the Binomial Distribution

$$\mu = \mathrm{E}(x) = n\,p$$

$$var(x) = \sigma^2 = n\,p\,q$$

$$s.d = \sigma = \sqrt{n\,p\,q}$$

**Example 1.** Suppose we have 10 balls in a bowl, 3 of the balls are red and 7 of them are blue. Define success $S$ as drawing a red ball. If we sample *with replacement*, Let's say $n$=20 , what is the probability that exactly 5 trial. P(X=5) ?

$P(S)$=0.3 for every trial.

$$p(X=x) = b(x, n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$p(X=5) = \binom{20}{5} 0.3^5 (1-0.3)^{20-5}$$

$$= 15504\,(0.3^5)(0.7^{15}) = 0.1789$$

The mean and variance are

$$\mu = \mathrm{E}(x) = n\,p = 20 * 0.3 = 6$$

$$var(x) = \sigma^2 = n\,p\,q = 20 * 0.3 * 0.7 = 4.2$$

**Example 2**. The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

(a) at least 10 survive?

(b) from 3 to 7 survive?

(c) exactly 5 survive?

$$p(X = x) = b(x, n, p) = \binom{n}{x} p^x (1 - p)^{n-x}$$

**Solution.** Probability of success= $p = 0.4$, and the probability of failure= $q = 0.6$.

$n = 15$ and $X$ : no. of surviving patients

(a) $P(X \geq 10) = 1 - P(X \leq 9) = 1 - B(9;15,0.4) = 1\text{-}0.9662 = 0.0338$

(b) $P(3 < X < 7) = P(4 \leq X \leq 6) = b(4;15,0.4) + b(5;15,0.4) + b(6;15,0.4) = 0.509$

(c) $P(X = 5) = b(5;15,0.4) = \binom{15}{5}(0.4)^5 (0.6)^{15-5} = 0.186$

**Example 3.** Assuming that 6 in 10 automobile accidents are due mainly to speed violation,

(a) find the probability that among 8 automobile accidents 6 will be due mainly to a speed violation.

(b) Find the mean and variance of the number of automobile accidents for 8 automobile accidents.

$$p(X = x) = b(x, n, p) = \binom{n}{x} p^x (1 - p)^{n-x}$$

**Solution.** Probability of success= $p = 6/10$, and the probability of failure

$q = 1 - 0.6 = 0.4$

$n = 8$ and $X$ : no. of automobile accidents

(a) $P(X = 6) = b(6; 8, 6/10) = \binom{8}{6}\left(\frac{6}{10}\right)^6 \left(1 - \frac{6}{10}\right)^{8-6} = 0.2090$

(b) The mean of the number of automobile accidents is

$$\mu = E(X) = np = 8 * \frac{6}{10} = 4.8 .$$

The variance of the no. of auto. accidents is

$$\sigma^2 = Var(X) = npq = 8 * \frac{6}{10} * \frac{4}{10} = 1.92 .$$

**Example 4.** A traffic control engineer reports that 75% of the vehicles passing through a checkpoint are from within the state. What is the probability that fewer than 2 of the next 9 vehicles are from out of the state?

$$p(X = x) = b(x, n, p) = \binom{n}{x} p^x (1 - p)^{n-x}$$

**Solution.** Probability of success= $p = 0.25$ , and the probability of failure= $q = 1 - 0.25 = 0.75$ .

$n = 9$ and $X$ : no. of vehicles passing through the checkpoint

$$P(X < 2) = P(X \le 1) = b(0;9,0.25) + b(1;9,0.25)$$

$$= \binom{9}{0}(0.25)^0(0.75)^9 + \binom{9}{1}(0.25)^1(0.75)^8 = 0.3$$

# ENGINEERING ANALYSIS II

## LECTURE 6

# Continuous Random Variables

Prepared by: Mr. Abdullah I. Abdullah

# Continuous Random Variable

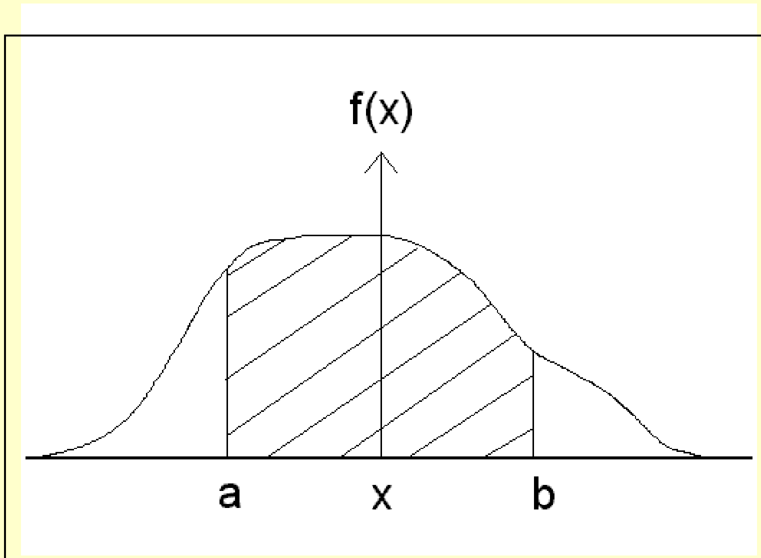
A variable with many possible values at all intervals

Examples

| Experiment | **Random Variable** | **Possible Values** |
|---|---|---|
| Weigh 100 People | Weight | 45.1, 78, ... |
| Measure Part Life | Hours | 900, 875.9, ... |
| Ask Food Spending | Spending | 54.12, 42, ... |
| Measure Time Between Arrivals | Inter-Arrival Time | 0, 1.3, 2.78, ... |

# Continuous Probability Distributions

For any continuous random variable, X, there exists a non-negative function f(x), called the probability density function (p.d.f) through which we can find probabilities of events expressed in term of X.




$f: R \rightarrow [0, \infty)$

$P(a < X < b) = \int_a^b f(x)\,dx$

= area under the curve of f(x) and over the interval (a,b)

$P(X \in A) = \int_A f(x)\,dx$

= area under the curve of f(x) and over the region A

***Definition 1:***

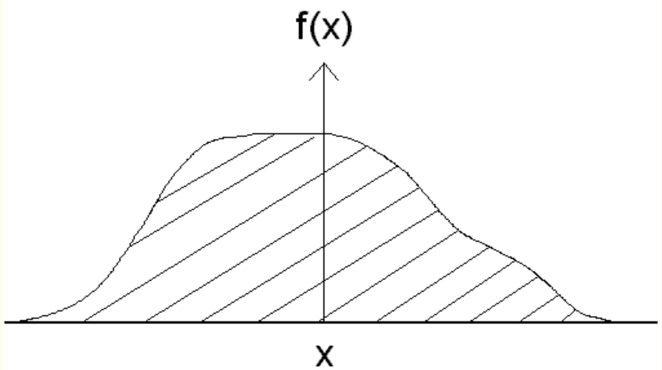
The function f(x) is a probability density function (pdf) for a continuous random variable X, defined on the set of real numbers, if:

1. $f(x) \geq 0 \quad \forall\, x \in R$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$
3. $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx \qquad \forall\, a, b \in R;\ a \leq b$

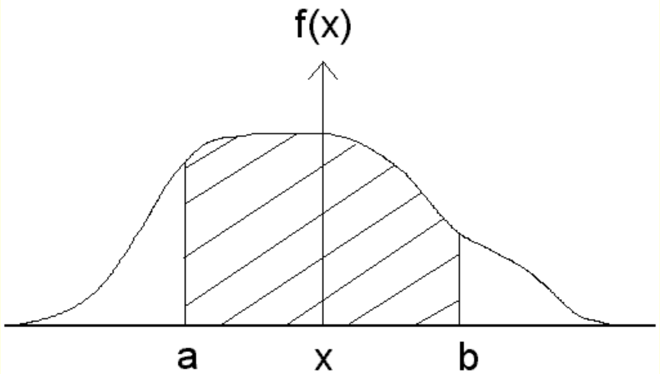
***Note:***

For a continuous random variable X, we have:

1. $f(x) \neq P(X=x)$ (in general)
2. $P(X=a) = 0$ for any $a \in R$
3. $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$
4. $P(X \in A) = \int_{A} f(x)\,dx$

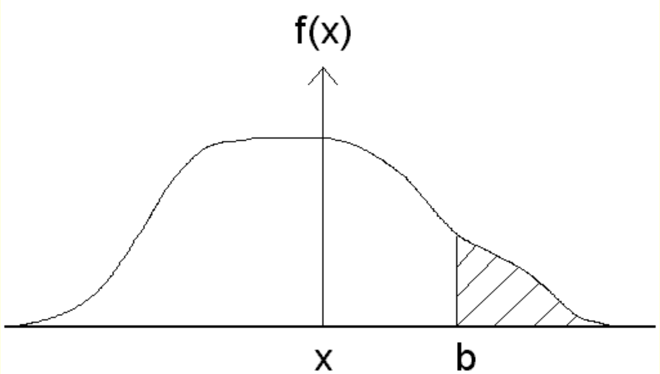



$$Total\ area = \int_{-\infty}^{\infty} f(x)\, dx = 1$$



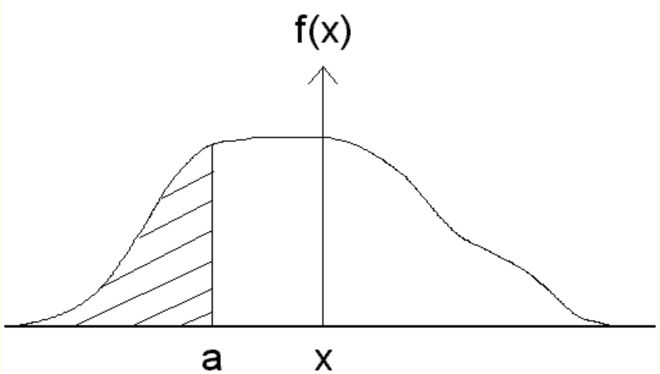

$$area = P(a \le X \le b)$$

$$= \int_a^b f(x) dx$$




$$area = P(X \ge b)$$

$$= \int_b^{\infty} f(x)\, dx$$




$$area = P(X \le a)$$

$$= \int_{-\infty}^{a} f(x) dx$$

## *Example 1:*

Suppose that the error in the reaction temperature, in $^{o}$C, for a controlled laboratory experiment is a continuous random variable X having the following probability density function:

$$f(x)=\begin{cases}\frac{1}{3}x^2 \; ; -1<x<2 \\ 0\; ;\; elsewhere\end{cases}$$

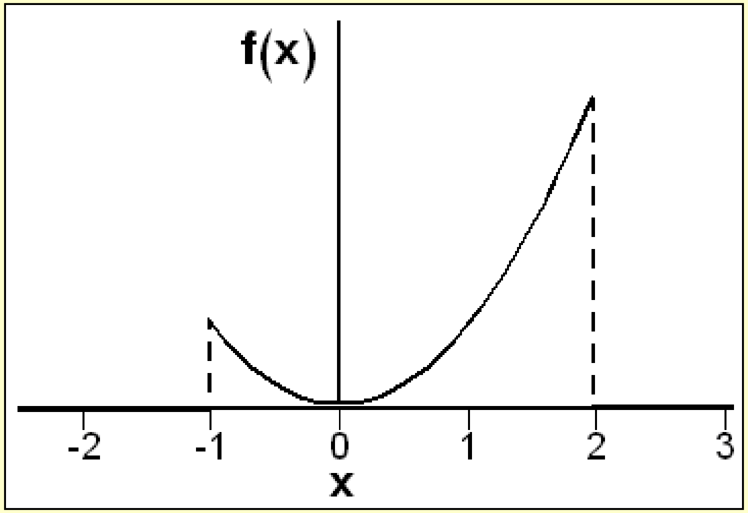
1. Verify that (a) $f(x) \geq 0$ and (b) $\int_{-\infty}^{\infty} f(x)\,dx = 1$
2. Find $P(0<X\leq 1)$

## *Solution:*

X = the error in the reaction temperature in $^{o}$C.

X is continuous r. v.

$$f(x)=\begin{cases}\frac{1}{3}x^2 \; ; -1<x<2 \\ 0\; ;\; elsewhere\end{cases}$$

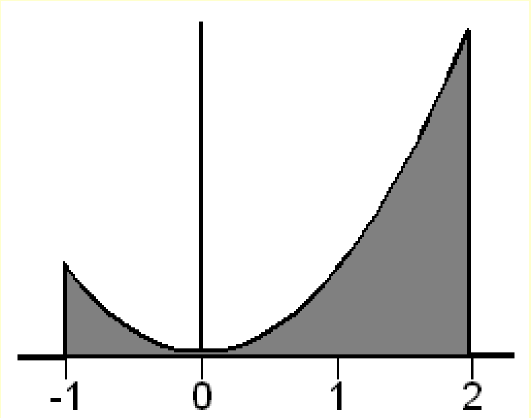


1. (a) $f(x) \geq 0$ because f(x) is a quadratic function.

(b) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{2} \frac{1}{3}x^2\,dx + \int_{2}^{\infty} 0\,dx$$

$$= \int_{-1}^{2} \frac{1}{3}x^2\,dx = \left[\frac{1}{9}x^3 \,\middle|\, \begin{matrix} x=2 \\ x=-1 \end{matrix}\right]$$

$$= \frac{1}{9}(8-(-1)) = 1$$

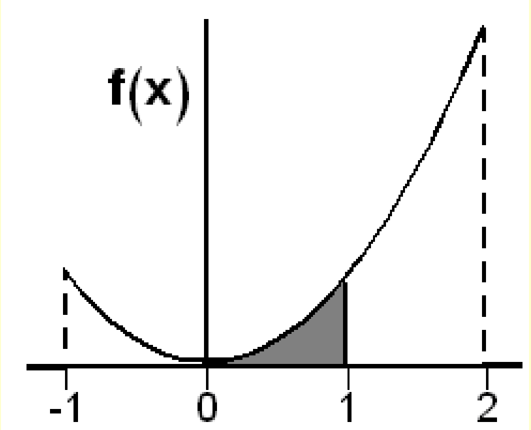



2. $P(0<X\leq 1) = \int_{0}^{1} f(x)\,dx = \int_{0}^{1} \frac{1}{3}x^2\,dx$

$$= \left[\frac{1}{9}x^3 \,\middle|\, \begin{matrix} x=1 \\ x=0 \end{matrix}\right]$$

$$= \frac{1}{9}(1-(0))$$

$$= \frac{1}{9}$$



***Definition 2:***

The cumulative distribution function (CDF), F(x), of a continuous random variable X with probability density function f(x) is given by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt; \qquad \text{for } -\infty < x < \infty$$

***Result:***

$P(a < X \leq b) = P(X \leq b) - \mathbf{P(X} \leq a) = F(b) - \mathbf{F(a)}$

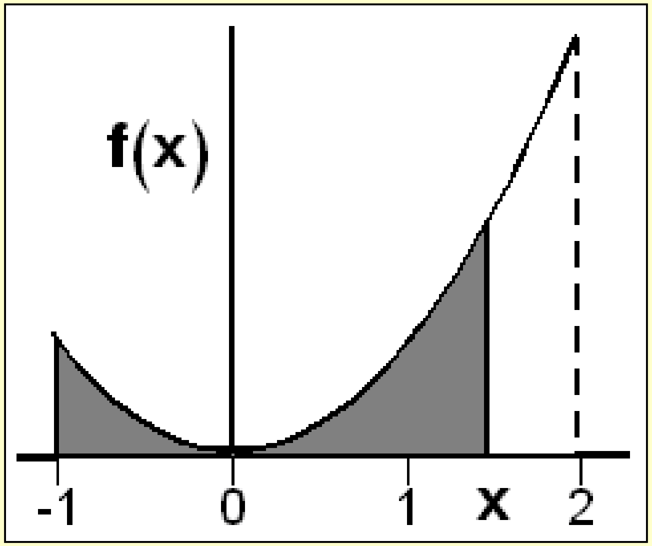
***Example 2:***

in Example 1,

1. Find the CDF
2. Using the CDF, find $P(0<X\leq 1)$.

**Solution:**

$$f(x)=\begin{cases}\frac{1}{3}x^2 \; ; -1<x<2\\ 0\; ;\; elsewhere\end{cases}$$




For x< –1:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

For –1≤x<2:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{x} \frac{1}{3}t^2\,dt$$

$$= \int_{-1}^{x} \frac{1}{3}t^2\,dt$$

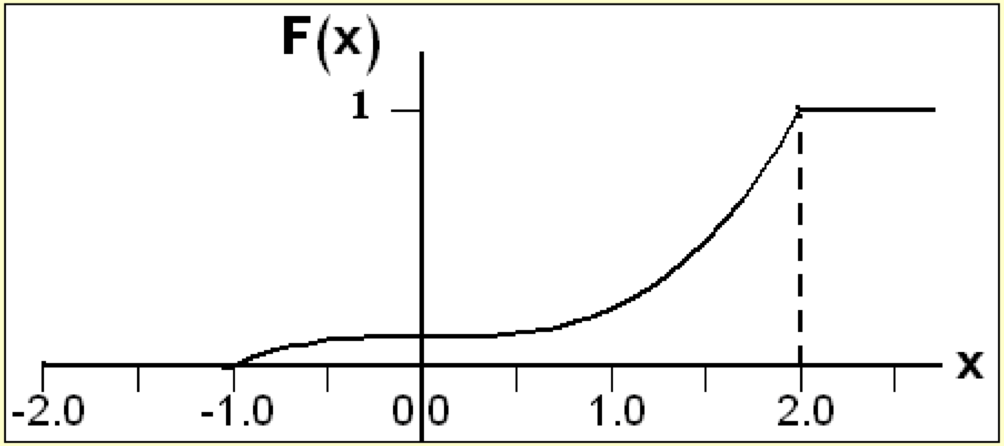
$$= \left[\frac{1}{9}t^3 \;\Big|_{t=-1}^{t=x}\right] = \frac{1}{9}(x^3-(-1)) = \frac{1}{9}(x^3+1)$$

For x≥2:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{2} \frac{1}{3} t^2\,dt + \int_{2}^{x} 0\,dt = \int_{-1}^{2} \frac{1}{3} t^2\,dt = 1$$

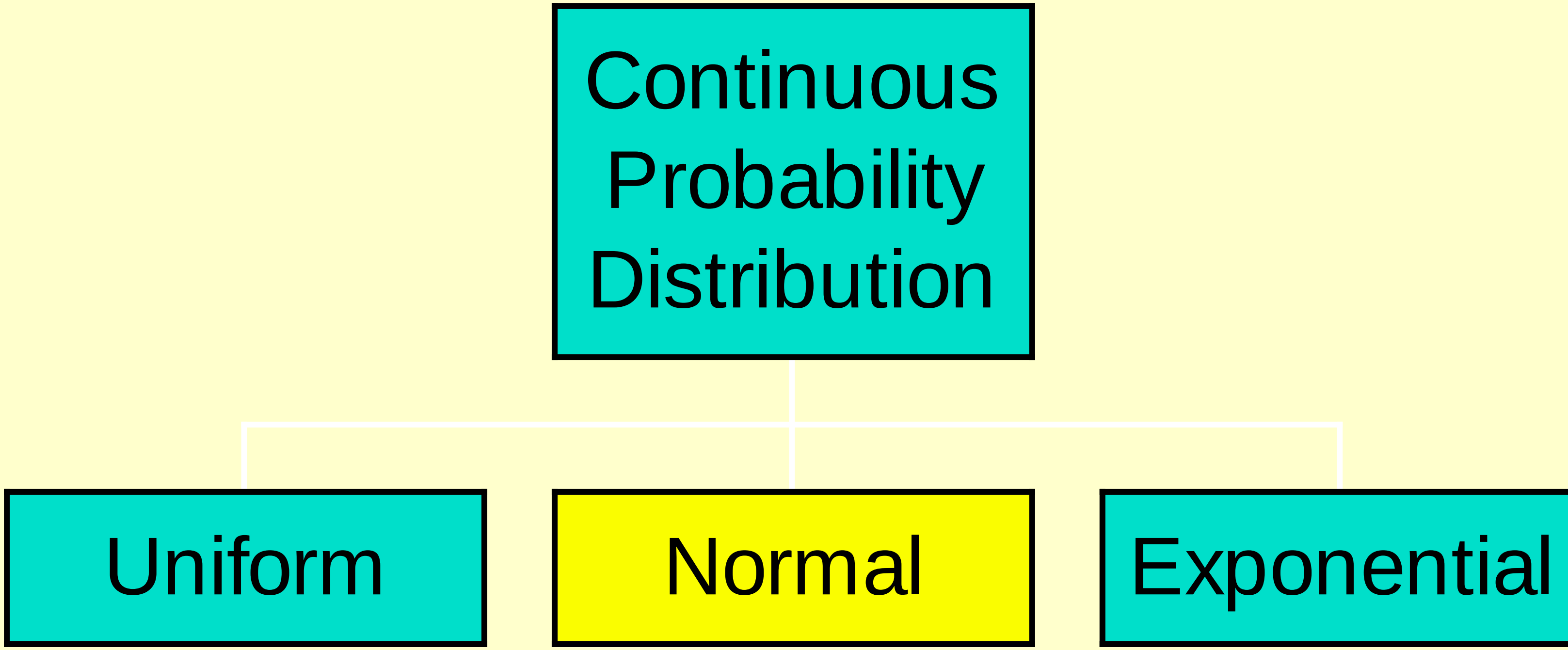
Therefore, the CDF is:

$$F(x) = P(X \le x) = \begin{cases} 0 \ ; x < -1 \\ \frac{1}{9}(x^3 + 1) \ ; -1 \le x < 2 \\ 1 \ ; x \ge 2 \end{cases}$$




2. Using the CDF,

P(0<X≤1) = F(1) – F(0) = $\frac{2}{9} - \frac{1}{9} = \frac{1}{9}$


Continuous Probability Distribution
Uniform
Normal
Exponential

# Normal Distribution

1. 'Bell-Shaped' & Symmetrical
2. Mean, Median, Mode Are Equal
3. 'Middle Spread' Is 1.33 $\sigma$
4. Random Variable Has Infinite Range


**Mean**
**Median**
**Mode**

# Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{1}{2}\right)\left(\frac{x-\mu}{\sigma}\right)^2}$$

- $x$ = Value of Random Variable ($-\infty < x < \infty$)
- $\sigma$ = Population Standard Deviation
- $\pi$ = 3.14159
- e = 2.71828
- $\mu$ = Mean of Random Variable x

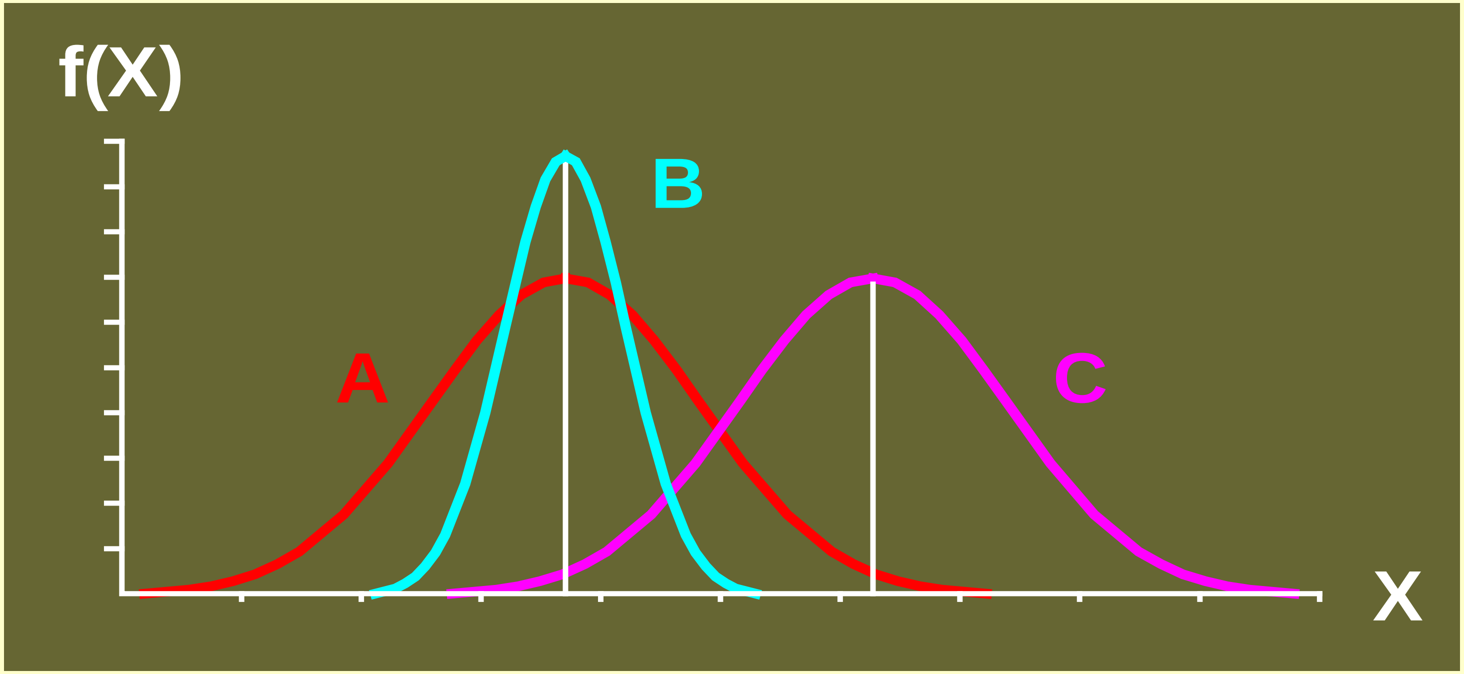
**Don't memorize this!**

•The random variable X has a normal distribution (N) with mean μ and standard deviation σ.:    X is N(μ,σ) :

•X is N(40,1)

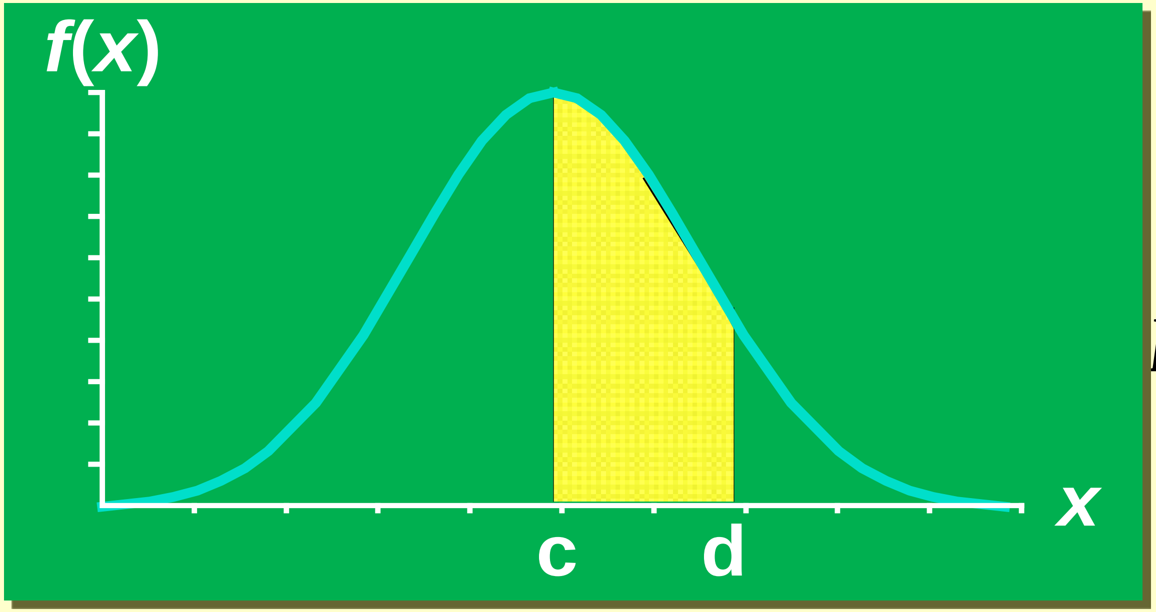
•X is N(10,5)

•X is N(50,3)



# Normal Distribution Probability

**Probability is area under curve!**




$$P(c \le x \le d) = \int_c^d f(x)\,dx$$

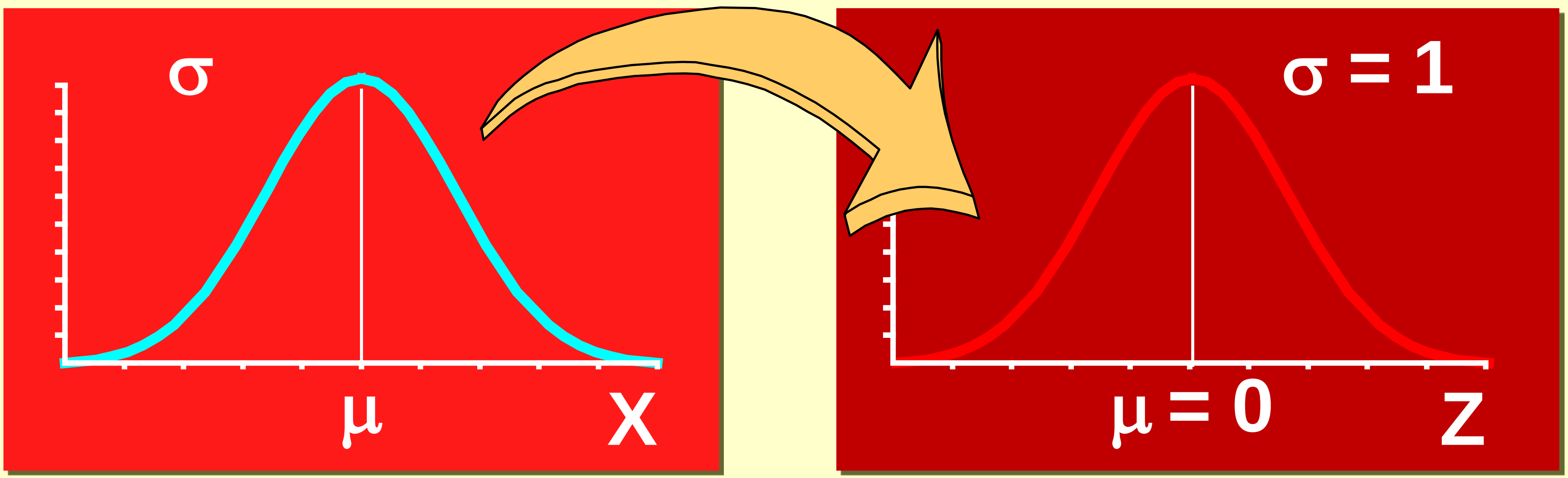
# Standardize the Normal Distribution

$$Z = \frac{X - \mu}{\sigma}$$

Z is N(0,1)

**Normal Distribution**

**Standardized Normal Distribution**



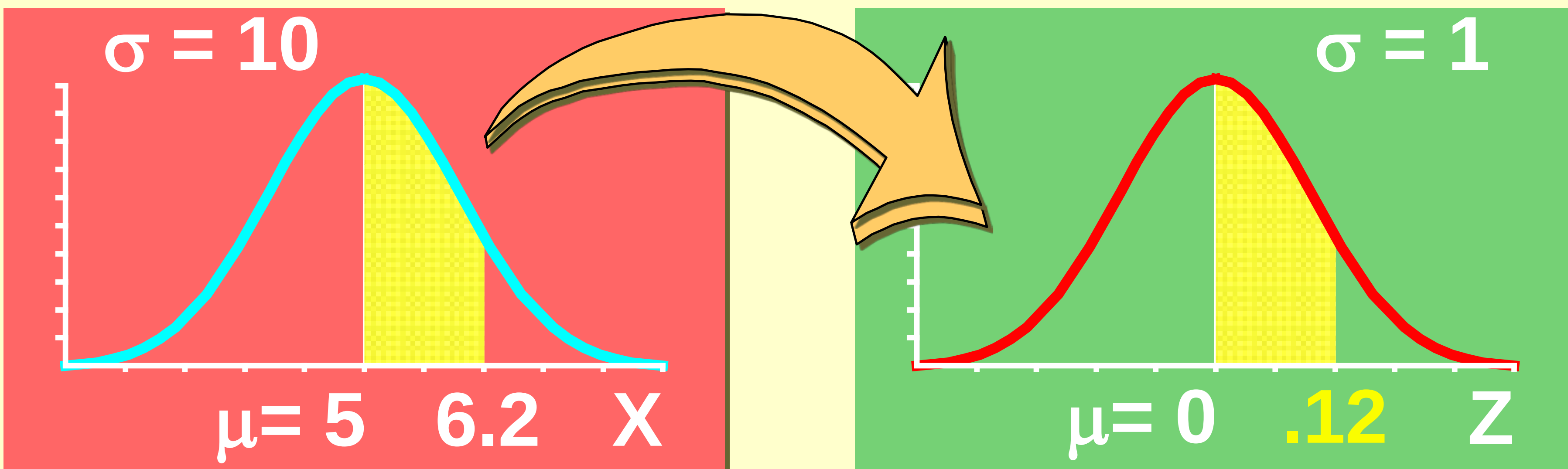

***One table!***

# Standardizing Example

$$Z = \frac{X - \mu}{\sigma} = \frac{6.2 - 5}{10} = .12$$

**Normal Distribution**

**Standardized Normal Distribution**

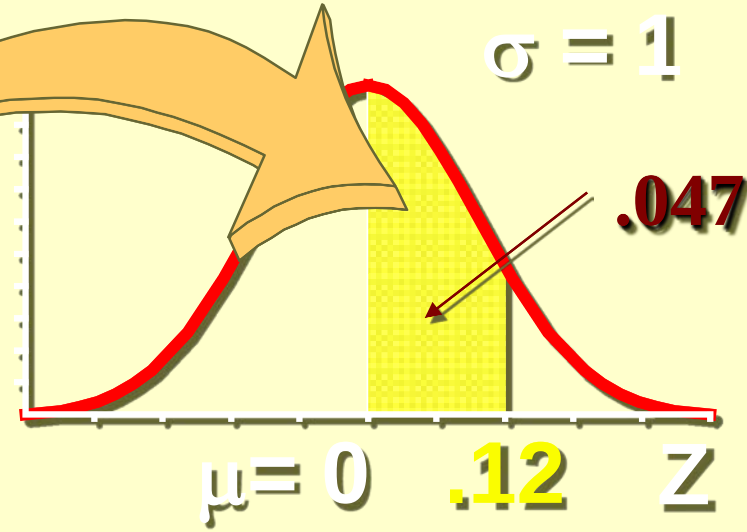


# Obtaining the Probability

**Standardized Normal Probability Table (Portion)**

| Z | .00 | .01 | .02 |
|---|---|---|---|
| 0.0 | .0000 | .0040 | .0080 |
| 0.1 | .0398 | .0438 | **.0478** |
| 0.2 | .0793 | .0832 | .0871 |
| 0.3 | .1179 | .1217 | .1255 |

**Probabilities**




**Shaded area exaggerated**

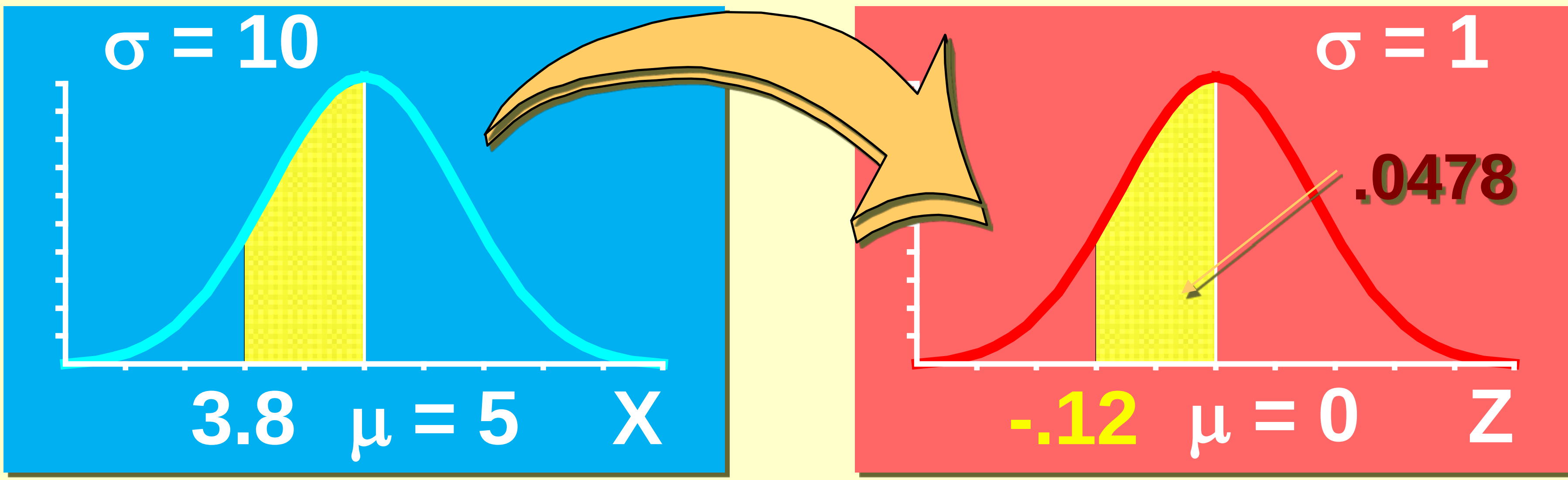
# Example

$$P(3.8 \le X \le 5)$$

$$Z = \frac{X - \mu}{\sigma} = \frac{3.8 - 5}{10} = -.12$$

**Normal Distribution**

**Standardized Normal Distribution**




**Shaded area exaggerated**

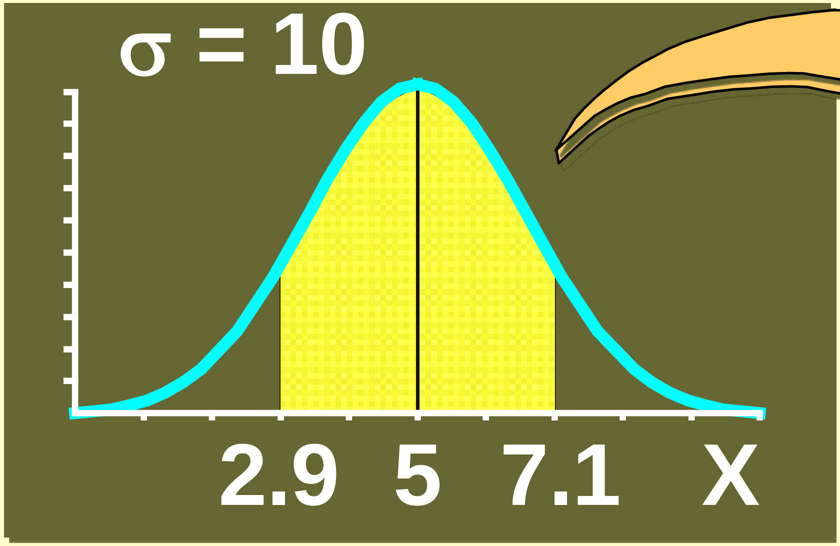
# Example

# $P(2.9 \le X \le 7.1)$

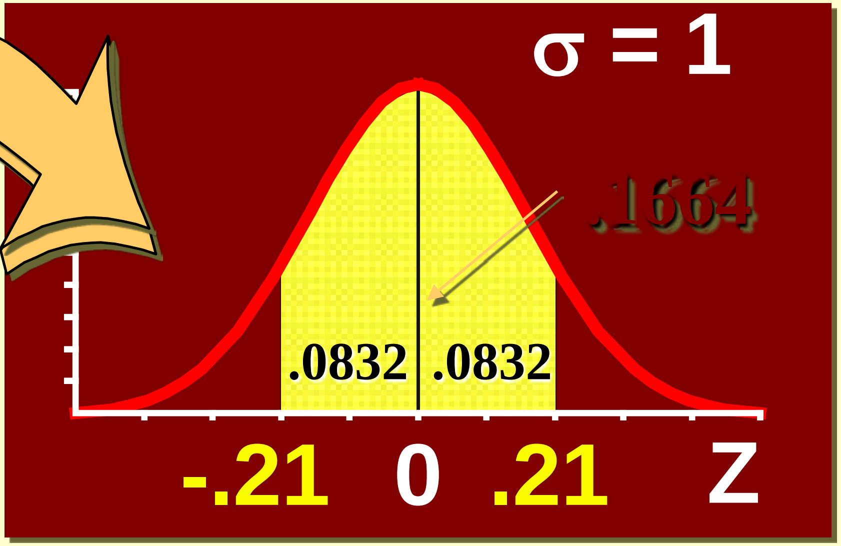
$$Z = \frac{X - \mu}{\sigma} = \frac{2.9 - 5}{10} = -.21$$

$$Z = \frac{X - \mu}{\sigma} = \frac{7.1 - 5}{10} = .21$$

**Normal Distribution**




**Standardized Normal Distribution**



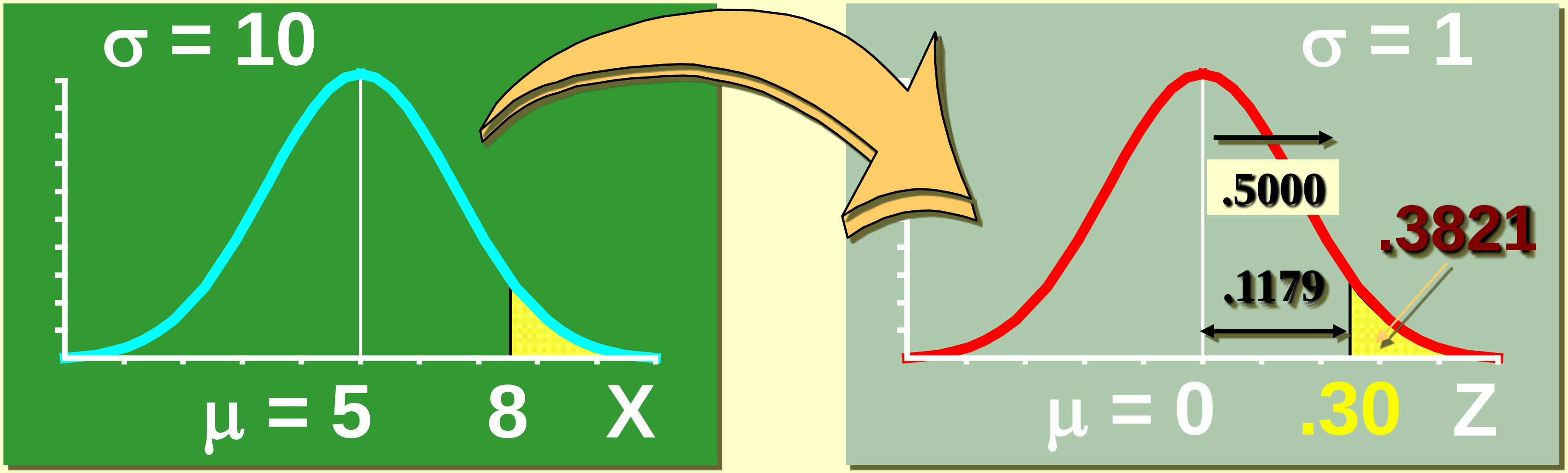

**Shaded area exaggerated**

# Example
# $P(X \geq 8)$

$$Z = \frac{X - \mu}{\sigma} = \frac{8 - 5}{10} = .30$$

**Normal Distribution**

**Standardized Normal Distribution**




**Shaded area exaggerated**

# Example

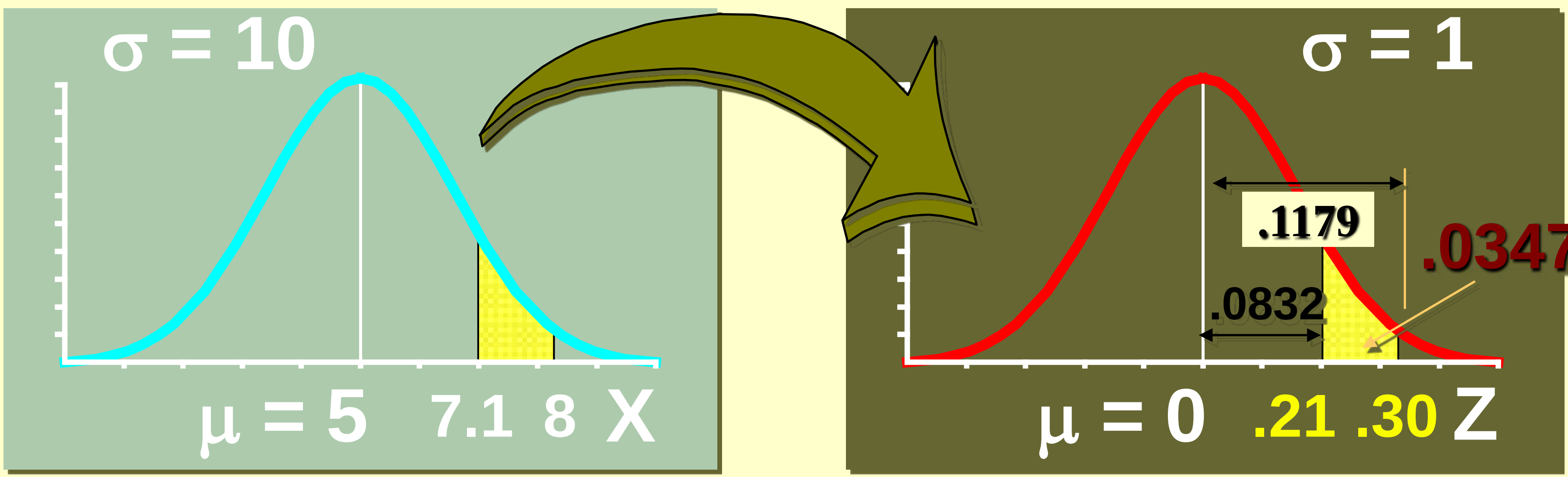
# $P(7.1 \le X \le 8)$

$$Z = \frac{X - \mu}{\sigma} = \frac{7.1 - 5}{10} = .21$$

$$Z = \frac{X - \mu}{\sigma} = \frac{8 - 5}{10} = .30$$

**Normal Distribution**

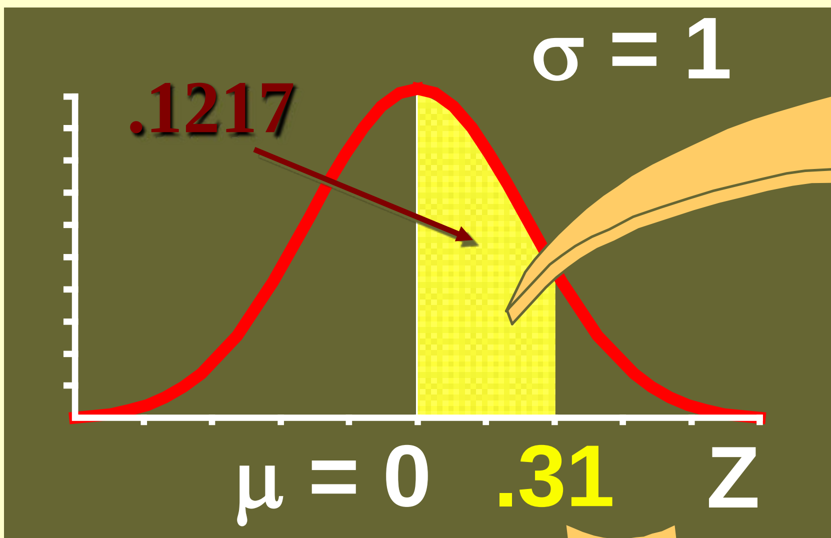
**Standardized Normal Distribution**




**Shaded area exaggerated**

# Finding Z Values for Known Probabilities

**What is Z given P(Z) = .1217?**



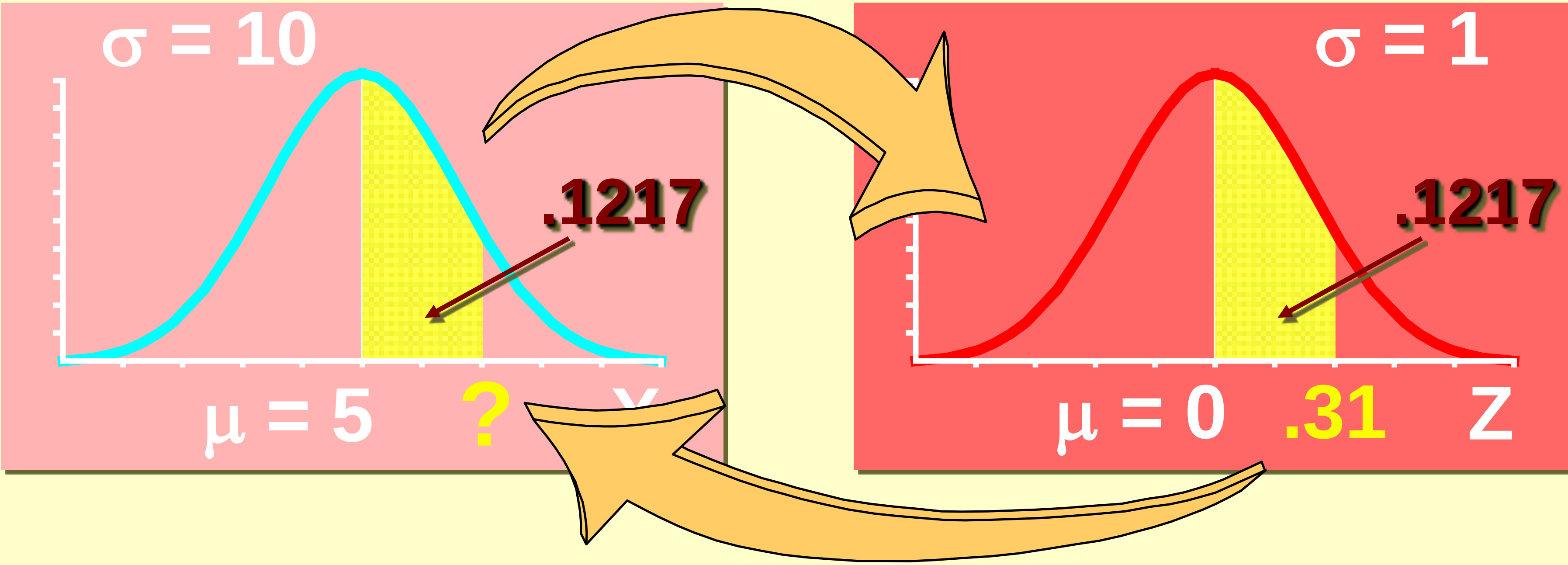

**Shaded area exaggerated**

**Standardized Normal Probability Table (Portion)**

| Z | .00 | .01 | 0.2 |
|---|---|---|---|
| [illegible] | .[illegible]000 | .0040 | .0080 |
| 0.[illegible] | .0398 | .0438 | .0478 |
| 0.2 | .0793 | .0832 | .0871 |
| 0.3 | .1179 | .1217 | .1255 |

# Finding *X* Values for Known Probabilities

**Normal Distribution** **Standardized Normal Distribution**




$$X = \mu + Z \cdot \sigma = 5 + (.31)(10) = 8.1$$

**Shaded areas exaggerated**

# Numerical Analysis

## LECTURE 7

## Roots of Single Equations

Prepared by: Mr. Abdullah I. Abdullah

# Why Numerical Analysis

Numerical methods are capable of handling large systems of equations, different degrees of nonlinearities which are common in engineering practice.

Numerical methods can handle any complicated physical geometries which are often impossible to solve analytically.

# 1-FIXED POINT ITERATION METHOD

**Fixed point :** A point, say, **s** is called a fixed point if it satisfies the equation $\mathbf{x = g(x)}$.

**Fixed point Iteration :** The transcendental equation $\mathbf{f(x) = 0}$ can be converted algebraically into the form $\mathbf{x = g(x)}$ and then using the iterative scheme with the recursive relation

$$x_{i+1} = g(x_i), \qquad i = 0, 1, 2, \ldots\ldots\ldots$$

with some initial guess $\mathbf{x_0}$ is called the fixed point iterative scheme.

## Algorithm - Fixed Point Iteration Scheme

Given an equation $f(x) = 0$

Convert $f(x) = 0$ into the form $x = g(x)$

Let the initial guess be $x_0$

Do

$$\mathbf{x_{i+1} = g(x_i)}$$

while (none of the convergence criterion C1 or C2 is met)

- C1. Fixing a priori the total number of iterations **N** .
- C2. By testing the condition $| \mathbf{x_{i+1} - g(x_i)} |$ (where **i** is the iteration number)less than some tolerance limit, say epsilon, fixed a priori.

## Numerical Example :

Find a root of $x^4 - x - 10 = 0$ ⇨ $x(x^3-1) = 10$ ⇨ $x_{i+1} = 10 / (x_i^3 - 1)$

1- Consider **g1(x) = 10 / (x³-1)** and the fixed point iterative scheme

$x_{i+1} = 10 / (x_i^3 - 1)$, **i = 0, 1, 2, . . .**let the initial guess $x_0$ be **2.0**

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 2 | 1.429 | 5.214 | 0.071 | -10.004 | -9.978E-3 | -10 | -9.99E-3 | -10 |

So the iterative process with **g1** gone into an infinite loop without converging.

2-Consider another function **$g2(x) = (x + 10)^{1/4}$** and the fixed point iterative scheme

$$\mathbf{x_{i+1} = (x_i + 10)^{1/4}, \quad i = 0, 1, 2, \ldots}$$

let the initial guess $x_0$ be **1.0, 2.0 and 4.0**

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $x_i$ | 1.0 | 1.82116 | 1.85424 | 1.85553 | 1.85558 | 1.85558 | |
| $x_i$ | 2.0 | 1.861 | 1.8558 | 1.85559 | 1.85558 | 1.85558 | |
| $x_i$ | 4.0 | 1.93434 | 1.85866 | 1.8557 | 1.85559 | 1.85558 | 1.85558 |

That is for **g2** the iterative process is converging to **1.85558 with** any initial guess.

3-Consider **$g3(x) = (x+10)^{1/2}/x$** and the fixed point iterative scheme

$$x^4 - x - 10 = 0$$

$$x_{i+1} = (x_i+10)^{1/2}/x_i, \quad i = 0, 1, 2, \ldots$$

$$x^2 = (x_i+10)^{1/2}$$

let the initial guess $x_0$ be **1.8**,

$$x_{i+1} = (x_i+10)^{1/2}/x_i$$

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | . . . | 98 |
|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 1.8 | 1.9084 | 1.80825 | 1.90035 | 1.81529 | 1,89355 | 1.82129 | . . . | 1.8555 |

That is for g3 with any initial guess the iterative process is converging but very slowly to

# Geometric interpretation of convergence with g1, g2 and g3

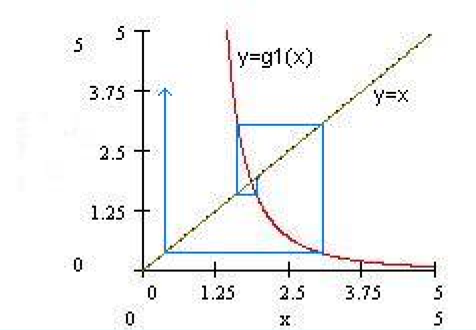



Fig g1

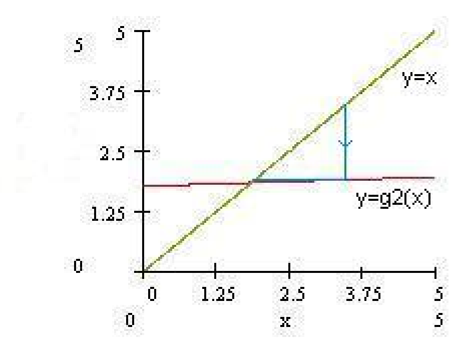



Fig g2

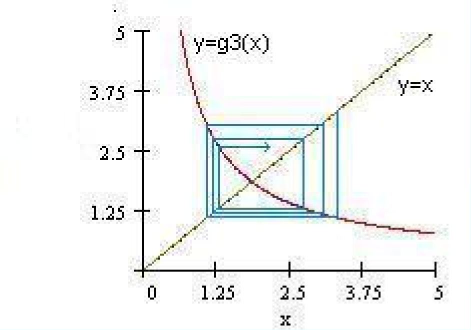



Fig g3

- Fig g1, the iterative process does not converge for any initial approximation.
- Fig g2, the iterative process converges very quickly to the root which is the intersection point of y = x and y = g2(x) as shown in the figure.
- Fig g3, the iterative process converges but very slowly.

# 2-Newton-Raphson Method

**The method is based on the first order Taylor expansion, i.e.:**

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

When we hit the root, $f(x_{i+1}) = 0,$ then:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

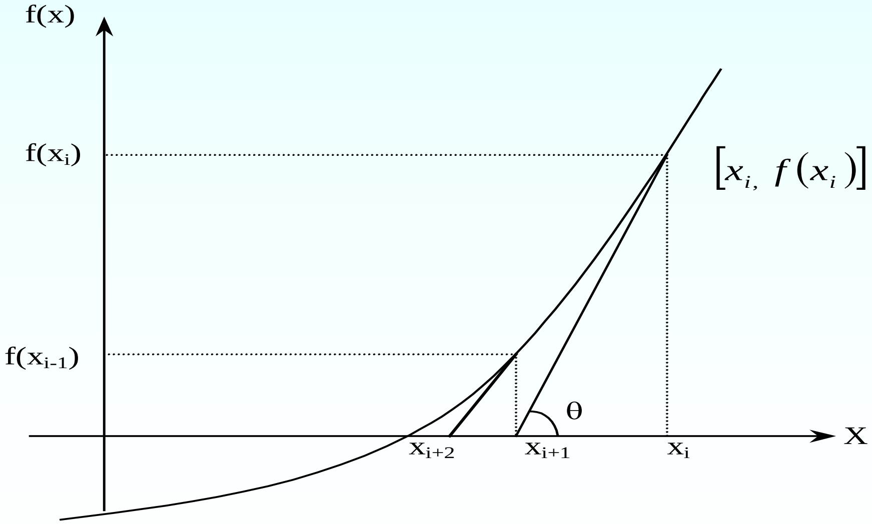
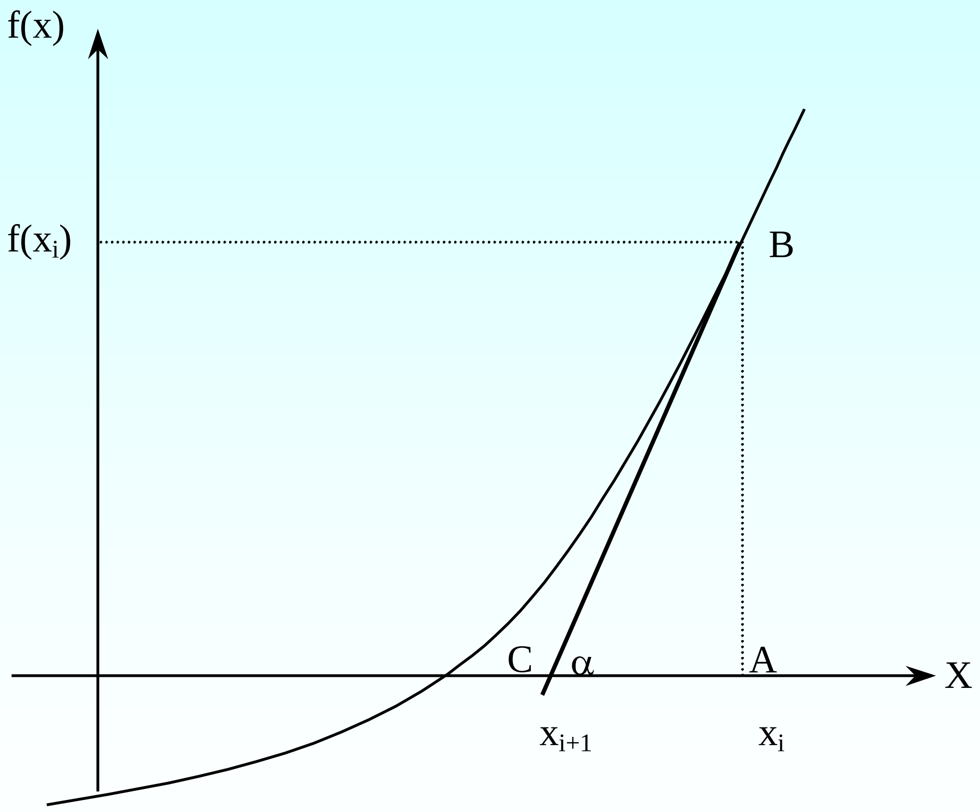



**Figure 1** Geometrical illustration of the Newton-Raphson method.

# Derivation




$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

**Figure 2** Derivation of the Newton-Raphson method.

# Procedure:

1-Assume an initial guess for the root = x0, and calculate the first estimate of the root x1 from : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

2-Repeat step 1 several times until convergence is achieved, i.e.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$ until $\epsilon_a < \epsilon_s$

**Example1:** Use Newton-Raphson method to estimate the root of

$f(x)=e^{-x}-x.$ Choose x0 = 0

| Iter | $x_i$ | $f(x_i)$ | $f'(x_i)=-e^{-x}-1$ | $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|
| 1 | 0. | 1. | -2. | 0.5 |
| 2 | 0.5 | 0.1065 | -1.6065 | 0.5663 |
| 3 | 0.5663 | 0.0013 | -1.5676 | 0.5671 |
| 4 | 0.5671 | 0.0000 | -1.5676 | 0.5671 |

Repeat with x0 = 5

| Iter | $x_i$ | $f(x_i)$ | $f'(x_i)=-e^{-x}-1$ | $x_{i+1}=x_n-\frac{f(x_i)}{f'(x_i)}$ |
|---|---|---|---|---|
| 1 | 5. | -4.9933 | -1.0067 | 0.04016 |
| 2 | 0.04016 | 0.92048 | -1.9606 | 0.5096 |
| 3 | 0.5096 | 0.0911 | -1.6007 | 0.5665 |
| 4 | 0.5665 | 0.0010 | -1.5675 | 0.5671 |

**Example2:** Use the Newton- Raphson method ,with 1.5 as starting point ,to find solution of f(x)= x-2sinx

f(x)=x-2sinx , $x_0$=1.5 , $f'(x)$ = 1-2cosx

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n}$$

$$= \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2\cos x_n}$$

| | |
|---|---|
| x1 | 2.0765582 |
| x2 | 1.9105066 |
| x3 | 1.8956220 |
| x4 | 1.89549427 |
| x5 | 1.89549426703 |
| x6 | 1.89549426703 |

# Numerical Analysis

## LECTURE 8

## Numerical Solution of Ordinary Differential Equations (ODE)

Prepared by: Mr. Abdullah I. Abdullah

# Euler Method

We will use Euler's method to solve an ODE under the form:

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

At x=0 , we are given the value of y=y0 Let us call x=0 as xo

Now since we know the slope of $y$ with respect to $x$, that is, $f(x,y)$, then at x=x0 , the slope is $f(x_0, y_0)$ Both x0 and y0 are known from the initial condition $y(x_0) = y_0$.

# Euler's Method

$$\frac{dy}{dx} = f(x,y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

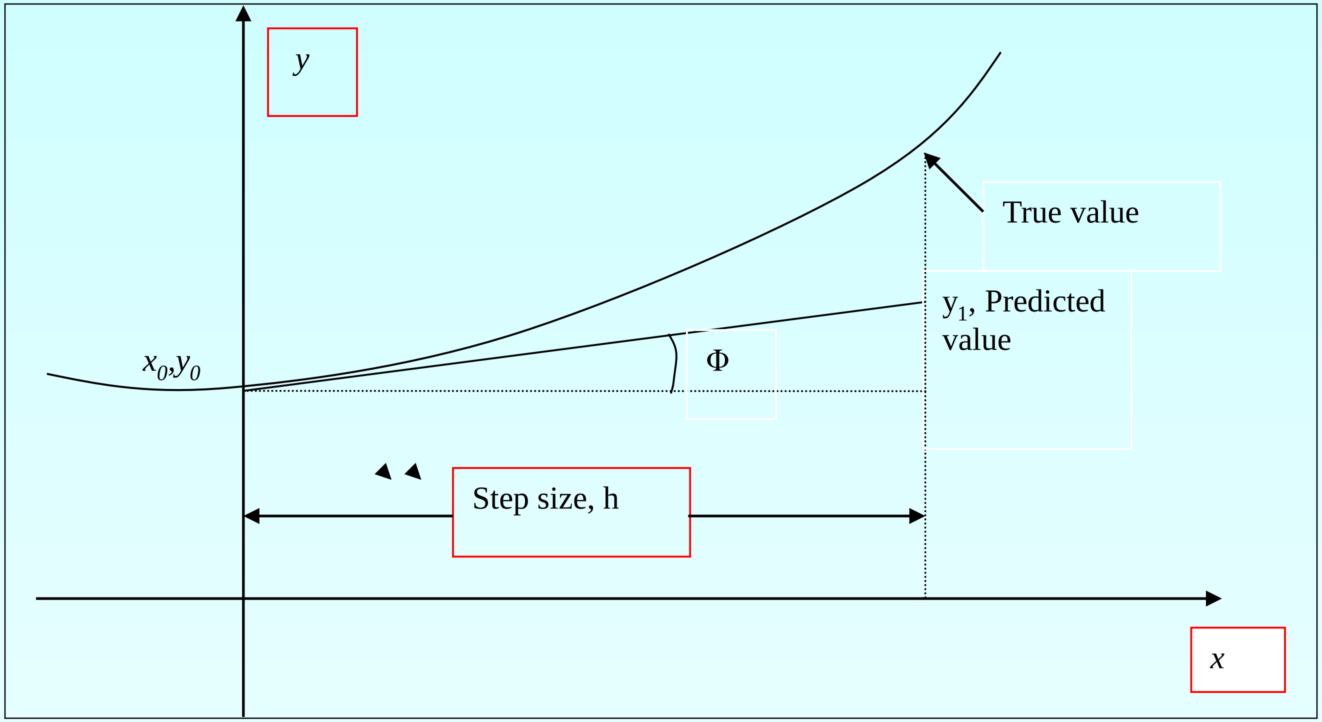



**Figure 1** Graphical interpretation of the first step of Euler's method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

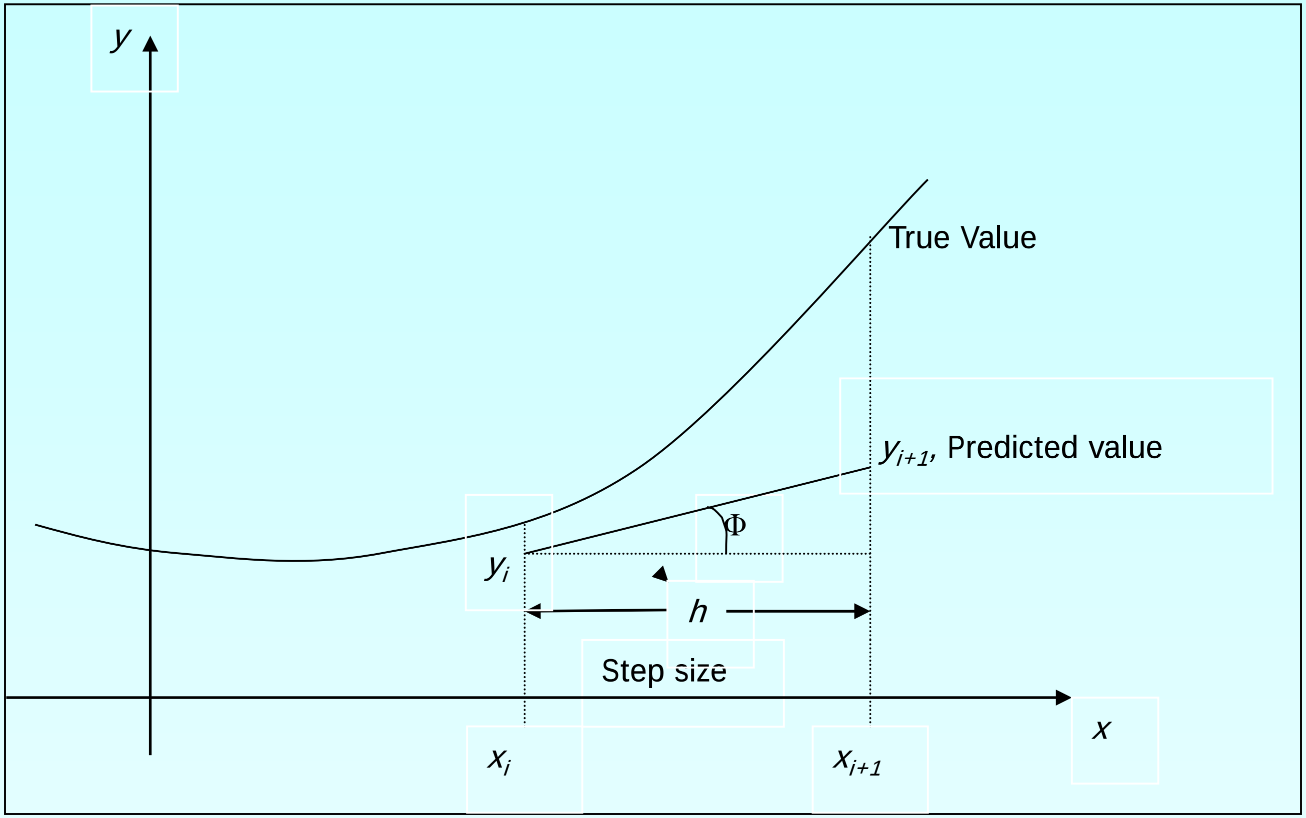



**Figure 2.** General graphical interpretation of Euler's method

**Example 1:** Solve the differential equation $\frac{dy}{dx} = 2x^2 + 2y \quad , y(0) = 1$

by Euler Method ,also find $y$ on $0 \le x \le 0.3$ Using h=0.1

$$y_{i+1} = y_i + hf(y_i, x_i)$$

$$y(0.1) = y(0) + (0.1)(2(0)^2 + 2(1))$$

$$= 1.2$$

$$y_{i+1} = y_i + h(2x_i^2 + 2y_i)$$

| $i$ | $x_i$ | $y(x_i)$ by *Euler* |
|---|---|---|
| *0* | *0* | *1* |
| *1* | *0.1* | *1.2* |
| *2* | *0.2* | *1.442* |
| *3* | *0.3* | *1.7384* |

**Example 2 :** Use Euler's method with h=0.14 to obtain a numerical solution of $y' = 1 + (y - x)^2$ subject to $y(1) = 1.59$ giving approximate values of y for $1 \le x \le 2.4$

$$y_{i+1} = y_i + h \quad f(x_i, y_i)$$

$$h = x_{i+1} - x_i$$

$$y_{n+1} = y_n + h[1 + (y_n - x_n)^2]$$

$$x_{i+1} = x_i + h$$

$1 \le x \le 2.4$ $\quad$ $y(1) = 1.59$

$$y_1 = y_0 + 0.14[1 + (y_0 - x_0)^2]$$

$$y_1 = 1.59 + 0.14[1 + (1.59 - 1)^2] = 1.7787$$

$$y_2 = y_1 + h[1 + (y_1 - x_1)^2]$$

$$y_2 = 1.7787 + 0.14[1 + (1.7787 - 1.14)^2] = 1.9758$$

$$y_{n+1} = y_n + h[1 + (y_n - x_n)^2]$$

$$y_3 = 1.9758 + 0.14[1 + (1.9758 - 1.24)^2] = 2.1836$$

$$y_4 = 2.1836 + 0.14[1 + (2.1836 - 1.42)^2] = 2.4052$$

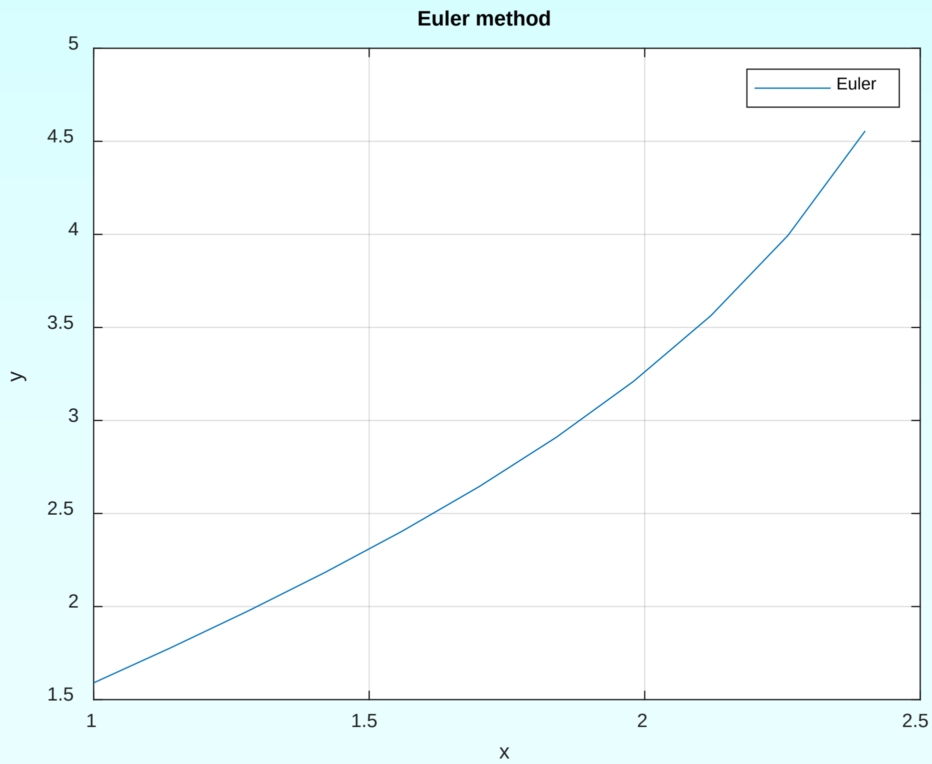
$$1 \leq x \leq 2.4$$

$$x_{i+1} = x_i + h$$

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 1 | 1.59 |
| 1 | 1.14 | 1.7787 |
| 2 | 1.28 | 1.9758 |
| 3 | 1.42 | 2.1836 |
| 4 | 1.56 | 2.4052 |
| 5 | 1.7 | 2.6453 |
| 6 | 1.84 | 2.9104 |
| 7 | 1.98 | 3.2108 |
| 8 | 2.12 | 3.5629 |
| 9 | 2.26 | 3.9944 |
| 10 | 2.4 | 4.5554 |

```
clear
clc
%%%%%%%    dy/dx=1+(y-x)^2  ,  1<x<2.4    ,y(1)=1.59 ,h=0.14
a=1;              % Initial time   a<x<b
b=2.4;            % Final time
h=0.14;           % Time step
N=((b-a)/h)+1;   % Number of steps
y0=1.59;     % Initial value y(a)
for i=1:N
   y(1)=y0;
   x(1)=a;
   y(i+1)=y(i)+(1+((y(i)-x(i))^2))*h;
   x(i+1)=x(i)+h;
end
```

```
V=[x',y']
plot(x,y)
title('Euler method')
ylabel('y')
xlabel('x')
grid
legend('Euler')
```




V =[x y ]

| x | y |
|---|---|
| 1.0000 | 1.5900 |
| 1.1400 | 1.7787 |
| 1.2800 | 1.9759 |
| 1.4200 | 2.1836 |
| 1.5600 | 2.4053 |
| 1.7000 | 2.6453 |
| 1.8400 | 2.9104 |
| 1.9800 | 3.2108 |
| 2.1200 | 3.5629 |
| 2.2600 | 3.9944 |
| 2.4000 | 4.5555 |

# Numerical Analysis

## LECTURE 9

## Numerical Integration

**Prepared by: Mr. Abdullah I. Abdullah**

# Trapezoidal Rule of Integration

## What is Integration

**Integration:**

The process of measuring the area under a function plotted on a graph.

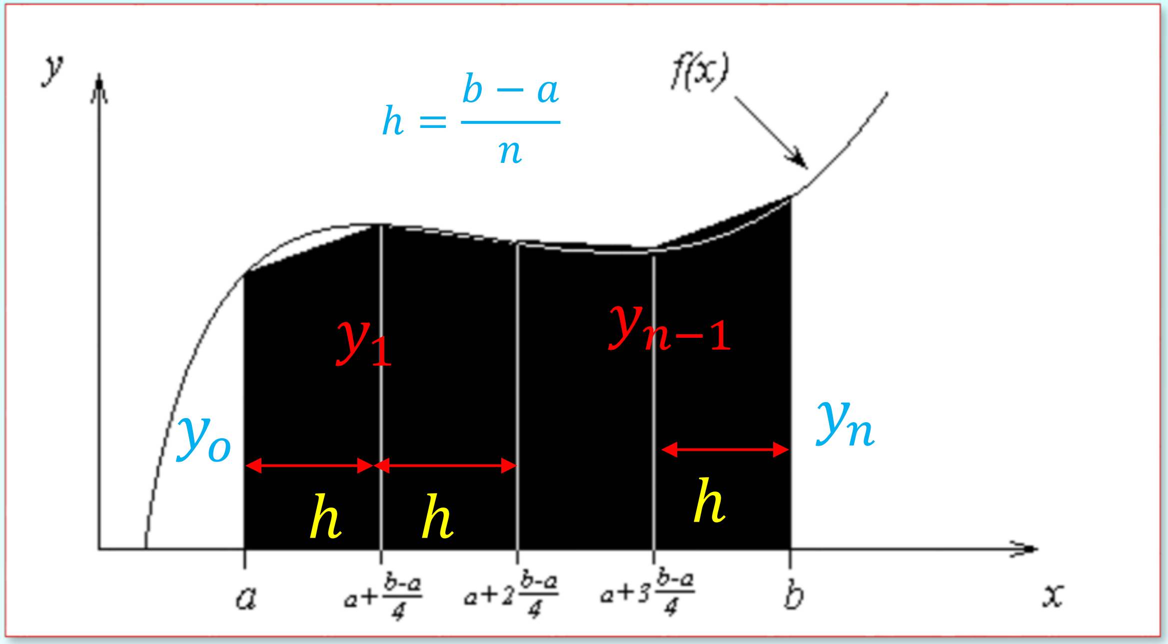
$$I = \int_a^b f(x)dx$$

Where:

*f(x)* is the integrand

a= lower limit of integration

b= upper limit of integration



# Basis of Trapezoidal Rule

$$\text{Total area} = \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + \ldots\ldots + \frac{1}{2}h(y_{n-1} + y_n),$$

$$= \frac{1}{2}h[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + \ldots\ldots + (y_{n-1} + y_n)]$$

$$= \frac{1}{2}h[y_0 + 2y_1 + 2y_2 + \ldots\ldots + 2y_{n-1} + y_n].$$

Remember the area under the curve is represented by $y = f(x)$

and so we have the formula known as the **trapezium rule:**

$$\int_a^b y\,dx \approx \frac{1}{2}h[y_0 + 2y_1 + 2y_2 + \ldots\ldots + 2y_{n-1} + y_n].$$

Remember that yi represents the height of the given function at the point , *xi* thus yi is called the **ordinate**. Hence the trapezium rule can also be written as

$$\text{Area} \approx \frac{\text{width of block}}{2}\left[(\text{first ordinate}) + (\text{last ordinate}) + 2(\text{other ordinates})\right].$$

**Example1:** Evaluate $\int_0^1 x^2 dx$ using the trapezium rule with 4 intervals.

$$h = \left(\frac{b-a}{n}\right) = \left(\frac{1-0}{4}\right) = \frac{1}{4} = 0.25$$

Calculate function value $f(x)$ (use width $h = 0.25$ for 4 intervals i.e. $n = 0,1,2,3,4$ so going from *a* to *b* (*0* to *1*) we have x=*0, 0.25, 0.5, 0.75* and *1*):

$$\text{Area} \approx \frac{\text{width of block}}{2}\left[(\text{first ordinate}) + (\text{last ordinate}) + 2(\text{other ordinates})\right].$$

| n | x | $y_n$ | $y=x^2$ | | |
|---|---|---|---|---|---|
| 0 | 0 | $y_0$ | 0 | | |
| 1 | 0.25 | $y_1$ | 0.0625 | | |
| 2 | 0.5 | $y_2$ | 0.25 | $2(y_1+y_2+y_3)$ | 1.75 |
| 3 | 0.75 | $y_3$ | 0.5625 | $y_0+2(y_1+y_2+y_3)+y_4$ | 2.75 |
| 4 | 1 | $y_4$ | 1 | $1/8[y_0+2(y_1+y_2+y_3)+y_4]$ | 0.344 |

$$\int_0^1 x^2 dx \approx \frac{0.25}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right] \approx \frac{0.25}{2}\left[0 + 1.75 + 1\right] \approx \frac{1}{8}\left[2.75\right] \approx 0.344$$

**Example2:** Evaluate $\int_0^1 e^{-x^2}\,dx$ using the trapezium rule with 4 intervals.

$$h = \left(\frac{b-a}{n}\right) = \left(\frac{1-0}{4}\right) = \frac{1}{4} = 0.25$$

| n | x | $y_n$ | $z=-x^2$ | $e^z$ | | |
|---|---|---|---|---|---|---|
| 0 | 0 | $y_0$ | 0 | 1 | | |
| 1 | 0.25 | $y_1$ | -0.063 | 0.9394 | | |
| 2 | 0.5 | $y_2$ | -0.25 | 0.7788 | $2(y_1+y_2+y_3)$ | 4.576 |
| 3 | 0.75 | $y_3$ | -0.563 | 0.5698 | $y_0+2(y_1+y_2+y_3)+y_4$ | 5.944 |
| 4 | 1 | $y_4$ | -1 | 0.3679 | $1/8[y_0+2(y_1+y_2+y_3)+y_4]$ | 0.743 |

$$\int_0^1 e^{-x^2}\,dx \approx \frac{0.25}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right] \approx \frac{0.25}{2}\left[1 + 4.576 + 0.368\right] \approx \frac{1}{8}\left[5.944\right] \approx 0.743$$

# Home work

Evaluate $\int_0^{2\pi} \sin\frac{1}{2}x\,dx$ using the trapezium rule with 4 intervals

Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx$ using the trapezium rule with 8 intervals