ENGINEERING ANALYSIS I

LECTURE 1

Ordinary Differential Equations

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Chapter 1

Ordinary Differential Equations (ODEs)

A differential equation is an equation that involves one or more derivatives, or differentials. المعادلة التفاضلية هي معادلة تتضمن مشتقة واحدة أو أكثر للدالة المجهولة

Differential equations are classified by:

1. ***Type*:** Ordinary or partial. نوع
2. ***Order*:** The order of differential equation is the highest order derivative that occurs in the equation. المرتبة
3. ***Degree*:** The exponent of the highest power of the highest order derivative. الدرجة

A differential equation is an **ordinary D.Eqs**. if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variable, the D.Eqs. is a **partial D.Eqs**.

$$2\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=2y$$ is a partial D.eqs.

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$$

Ex1:

$$\frac{dy}{dx} = 5x + 3$$ 1st order-1st degree

Ex2:

$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx}\right)^5$$ 3rd order-2nd degree

Ex3:

$$4\frac{d^3y}{dx^3} + \sin x\frac{d^2y}{dx^2} + 5xy = 0$$ 3rd order-1st degree

Exercise: Find the order and degree of these differential equations.

1. $\frac{dy}{dx} + \cos x = 0$ ans:1st order-1st degree
2. $3dx + 4y^2dy = 0$ ans:1st order-1st degree
3. $\frac{d^2y}{dx^2} + y = y^2$
4. $(y'')^2 + 2y' = x^2$
5. $y''' + 2(y'')^2 = xy$

Solution

The solution of the differential equation in the unknown function y and the independent variable x is a function y(x) that satisfies the differential equation.

Ordinary Differential Equations:

Ordinary Differential Equations are equations involve derivatives.

A. First Order D.Eqs.

1- Variable Separable. قابل للفصل
2- Homogeneous. متجانسة
3- Linear. خطية
4- Exact. تامة

1- Variable Separable:

A first order D.Eq. can be solved by integration if it is possible to collect all y terms with dy and all x terms with dx, that is, if it is possible to write the D.Eq. in the form

$f(x)\ dx + g(y)\ dy = 0$ then the general solution is:

$\int f(x)dx + \int g(y)dy = c$ where **c** is an arbitrary constant.

Ex.1:

Solve $\dfrac{dy}{dx} = e^{x+y}$

Sol.:

$$\frac{dy}{dx} = e^{x} \cdot e^{y}$$

$$\frac{\mathrm{dy}}{\mathrm{e^{y}}} = \mathrm{e^{x}dx}$$

$$\int e^{-y}dy = \int e^{x} \cdot dx$$

$$-\int e^{-y} \cdot (-dy) = \int e^{x}dx \quad \Rightarrow \quad -e^{-y} = e^{x} + c$$

Ex.2:

Solve $(1+x)\frac{dy}{dx} = x(y^2+1)$

Sol.:

$$\int \frac{dy}{(y^2+1)} = \int \frac{x}{x+1} dx$$

$$\tan^{-1} y = \int dx - \int \frac{1}{x+1} dx$$

$$\tan^{-1} y = x - \ln|x+1| + c$$

Ex.3: Solve $\frac{dy}{dx} = (y-x)^2 \quad \cdots(1)$

Sol.: *put* $y - x = \mathrm{u}$, $\frac{\mathrm{dy}}{\mathrm{dx}} - 1 = \frac{du}{dx} \Rightarrow \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{du}{dx} + 1 \quad \ldots.. \ (2)$

$$\frac{du}{dx} + 1 = u^2 \quad \Rightarrow \quad \int \frac{\mathrm{du}}{\mathrm{u}^2 - 1} = \int dx$$

$$\therefore \int \left[\frac{1/2}{u-1} + \frac{-1/2}{u+1} \right] du = \int dx$$

$$\frac{1}{2}[\ln(u-1) - \ln(u+1)] = x + c$$

$$\frac{1}{2}\ln\frac{u-1}{u+1} = x + c$$

$$\frac{u-1}{u+1} = e^{2x+c}$$

Exercise: Separate the variables and solve**.**

1. $x(2y-3)dx+(x^2+1)dy=0$ ans: $(x^2+1)(2y-3)=c$

2. $dy=e^{x-y}\,dx$ ans: $e^y=e^x+c$

3. $\sin x\frac{dy}{dx}+\cosh 2y=0$ ans: $\sinh 2y-2\cos x=c$

4. $xe^y dy+\frac{x^2+1}{y}dx=0$ ans: $e^y(y-1)+\frac{x^2}{2}+\ln|x|=c$

5. $\sqrt{2xy}\frac{dy}{dx}=1$ ans: $\frac{\sqrt{2}}{3}y^{\frac{3}{2}}=x^{\frac{1}{2}}+c$

2- Homogeneous:

Some times a D.Eq. which variables can't be separated can be transformed by a change of variables into an equation which variables can be separated. This is the case with any equation that can be put into form:

$$\frac{dy}{dx} = f(\frac{y}{x}) \ldots (1)$$

Such an equation is called homogenous.

Put $\frac{y}{x} = u \quad \Rightarrow \quad y = ux, \ \frac{dy}{dx} = u + x \cdot \frac{du}{dx}$ and (1) becomes

$$x \cdot \frac{du}{dx} + u = f(u)$$

Ex.1:

$$\text{Solve} \quad \frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

Sol.:

$$\frac{dy}{dx} = \frac{1 + \frac{y^2}{x^2}}{\frac{y}{x}} \quad \Rightarrow \text{homo.} \quad \text{Put} \quad \frac{y}{x} = u \quad \Rightarrow \frac{dy}{dx} = x \cdot \frac{du}{dx} + u$$

$$x \cdot \frac{du}{dx} + u = \frac{1 + u^2}{u} \quad \Rightarrow \quad x \cdot \frac{du}{dx} = \frac{1 + u^2 - u^2}{u}$$

$$x \cdot \frac{du}{dx} = \frac{1}{u} \quad , \quad \int u \cdot du = \int \frac{dx}{x}$$

$$\frac{u^2}{2} = \ln x + c \quad \Rightarrow \quad \frac{y^2}{2x^2} = \ln x + c$$

Ex.2: Solve the homogenous D.Eq $xdy + 2ydx = 0$

Sol.:

$$xdy = 2ydx \Rightarrow \frac{dy}{dx} = \frac{2y}{x} \quad \text{put} \quad \frac{y}{x} = u \qquad \Rightarrow \frac{dy}{dx} = x \cdot \frac{du}{dx} + u$$

$$x \cdot \frac{du}{dx} + u = 2u \qquad \ln|x| - \ln|u| = c \qquad \Rightarrow \frac{x}{u} = c \Rightarrow \frac{x^2}{y} = c$$

Exercise: Show that the following differential equations are homogenous and solve.

1. $(x^2+y^2)dx+xy\,dy=0$ ans: $x^2(x^2+2y^2)=c$
2. $x^2dy+(y^2-xy)dx=0$ ans: $y = \dfrac{x}{\ln x - c}$
3. $(xe^{\frac{y}{x}} + y)dx - xdy = 0$ ans: $\ln|x| + e^{\frac{-y}{x}} = c$

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LECTURE 2

Ordinary Differential Equations

First Order D.Eqs.

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3 - Linear

The equation of the form $\frac{dy}{dx} + p \cdot y = Q$ where P and Q are functions of only x or constant is called linear in y and $\frac{dy}{dx}$.

Find integrating factor $(I.f.) = e^{\int P dx}$, then the general solution is $y \cdot (I.f.) = \int (I.f.)\ Q \, .dx$

Ex.1: Solve $\frac{dy}{dx} - \frac{y}{x} = x \cdot e^x$

$$P(x) = -\frac{1}{x}, \quad Q(x) = x.e^x$$

$$(I.f.) = e^{\int \frac{-1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

Solution is

$$y \cdot \frac{1}{x} = \int \frac{1}{x} \cdot x e^x \cdot dx$$

$$\frac{y}{x} = e^x + c$$

Ex.2: Solve $\frac{dy}{dx} + x.y = x$

$P=x, \quad Q=x$

$(I.f.) = e^{\int x dx} = e^{\frac{x^2}{2}}$ ⇒ $y \cdot (I.f.) = \int (I.f.)\ Q \, .dx$

$y \cdot e^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}} \cdot x \cdot dx$

Let $u = \frac{x^2}{2}$

$\frac{du}{dx} = \frac{1}{2} 2x \rightarrow du = xdx$

$ye^u = \int e^u du$

$ye^u = e^u + c$

$y = 1 + ce^{-u} = 1 + ce^{-x^2/2}$

Exercise:

1. $\frac{dy}{dx} + 2y = e^{-x}$ ans: $y=e^{-x}+ce^{-2x}$
2. $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$ ans: $x^3y=c-\cos x$
3. $xdy + ydx = ydy$ ans: $x = \frac{y}{2} + \frac{c}{y}$

4- Exact

The equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

General Solution is

$$c = \int Mdx + \int (terms\ \ in\ \ N\ \text{do not contains}\ \ x)\text{dy}$$

Ex.1: Show that the following D.Eq. are exact D.Eq.

a) $(3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 + 2x \quad , \quad \frac{\partial N}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The D.Eq. is exact.

b) $[x\cos(x+y) + \sin(x+y)]dx + (x\cos(x+y)dy = 0$

$$\frac{\partial M}{\partial y} = -x\sin(x+y) + \cos(x+y)$$

$$\frac{\partial N}{\partial x} = -x\sin(x+y) + \cos(x+y)$$

the D.Eq. is exact.

Ex.2: Is the D.Eq. $\frac{dy}{dx} = -\frac{(x^2 + y^2)}{2xy}$ exact or not?

Sol.

$$2xydy = -(x^2 + y^2)dx$$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \therefore \text{the D.Eq. is exact}$$

Ex.3: Solve the exact D.Eqs. in Ex.1 (a) above $(3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy = 0$

Sol.

$$c = \int (3x^2y + 2xy)dx + \int 2y\mathrm{dy}$$

$$= 3\mathrm{y} \cdot \frac{\mathrm{x}^3}{3} + 2y \cdot \frac{\mathrm{x}^2}{2} + 2 \cdot \frac{\mathrm{y}^2}{3}$$

the solution is $\mathrm{x}^3\mathrm{y} + \mathrm{x}^2\mathrm{y} + \mathrm{y}^2 = c$

Ex.4: Solve $(x+y)dx+(x+y^2)dy=0$

Sol.

$$\frac{\partial M}{\partial y}=1 \quad , \quad \frac{\partial N}{\partial x}=1$$

$\therefore$ the D.Eq. is exact

$$c=\int Mdx+\int (terms\ \ in\ \ N\ \text{ do not contains } x)\text{dy}$$

$$=(x+y)dx+\int y^2\text{dy}$$

$$=\frac{x^2}{2}+xy+\frac{y^3}{3}$$

the solution is $\frac{x^2}{2}+xy+\frac{y^3}{3}=c$

Exercise:

1. $(2+ye^{xy})dx+(xe^{xy}-2y)dy=0$ ans: $c=2x+e^{xy}-y^2$
2. $(\tan x+\tan y)dy+(y\sec^2x+\sec x\tan x)dx=0$ ans: $c=y\tan x-\ln\cos y+\sec x$
3. $(2xy+y^2)dx+(x^2+2xy-y)dy=0$ ans: $x^2y+y^2x-y^2/2=c$

Problems:

Solve the following differential equations:

1- $y\ln y\,dx+(1+x^2)dy=0$

2- $e^{x+2y}dy-e^{y-2x}dx=0$

3- $(2x+y)dx+(x-2y)dy=0$

4- $x\,dy=(y+x\cos^2(\frac{y}{x}))dx$

5- $x(\ln y-\ln x)dy=y(1+\ln y-\ln x)dx$

6- $x\,dy+(2y-x^2-1)dx=0$

7- $\cos y\,dx+(x\sin y-\cos^2 y)dy=0$

8- $(1+y^2)dx+(2xy+y^2+1)dy=0$

9- $(e^x+\ln y)dx+(\frac{x+y}{y})dy=0$

10- $x(1+e^y)dx+\frac{1}{2}(x^2+y^2)e^y dy=0$

ENGINEERING ANALYSIS I

LECTURE 3

Ordinary Differential Equations

Second Order D.Eqs.

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Second Order Differential Equations:

The second order linear differential equations with constant coefficient has the general form is:

$$ay'' + by' + cy = F(x) \qquad \ldots(1),$$ where a, b and c are constants.

If $F(x) = 0$ then (1) is called homogenous.

If $F(x) \neq 0$ then (1) is called non homogenous.

Ex:

1) $y''-x^2y'+\sin x\, y=0$ is linear, 2nd order, homo.
2) $y''-(y')^2+ y=\sin x$ is non linear, 2nd order, non homo.
3) $y''+2yy'=\ln x$

a) Homogeneous.

b) Nonhomogeneous.

1. Undeterminant coefficients.

2. Variation of parameters.

a) The Second order linear homogenous D.Eq. with constant coefficient:

The general form is $ay'' + by' + cy = 0 \quad \ldots(2)$ where a, b and c are constants.

The general solution

Put y'=Dy and y''=D^2y in eq. (2) (D is an operator)

$\Rightarrow a\,D^2y+bDy+cy=0$

$\Rightarrow (aD^2 + bD + c)y = 0$ (using D-operator)

now substitute D by r and leave y then $ar^2 + br + c = 0$

is called **characteristic equation** of the differential equation and the solution of this equation (the roots r) give the solution of the differential equation where

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

There are two values of r :

1- real (equal and not equal).
2- complex.

Case 1: If $b^2 - 4ac \succ 0$ then r_1 and r_2 are distinct ($r_1 \neq r_2$) and real roots, and the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

Case 2: If $b^2 - 4ac = 0$ then $r_1 = r_2 = \mathrm{r}$, and the general solution is:

$$y = (c_1 + c_2 x)e^{rx}$$

Case 3: If $b^2 - 4ac \prec 0$ then the roots are two complex conjugate roots $r = \alpha \pm i\beta$, $i = \sqrt{-1}$, and the general solution is:

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Ex.1: Solve $y'' - 2y' - 3y = 0$

Solution:

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = 0 \quad , \quad y = 1 \ , \ y' = r \quad , \quad y'' = r^2$$

$$(r+1)(r-3) = 0$$

$$r + 1 = 0 \quad \Rightarrow \quad r = -1$$

$$r - 3 = 0 \quad \Rightarrow \quad r = 3$$

the general solution is

$$y = c_1 e^{-x} + c_2 e^{3x}$$

Ex.2: Solve $y'' - 6y' + 9y = 0$

Solution:

$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 3$$

$$\therefore \quad y = (c_1 + c_2 x)e^{3x}$$

Ex.3: Solve $y'' + y' + y = 0$

Solution:

$$y'' + y' + y = 0$$

$$r^2 + r + 1 = 0 \qquad \mathrm{a} = 1, \mathrm{b} = 1, \mathrm{c} = 1$$

$$r = \frac{-b \pm \sqrt{1 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} \qquad = \frac{-1 \pm \sqrt{3}i}{2}$$

$$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i \qquad \alpha = \frac{-1}{2} \quad , \quad \beta = \frac{\sqrt{3}}{2}$$

$$\therefore \quad y = e^{\frac{-1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

Exercise: solve

1. $4y''-12y'+5y=0$ ans: $y=c_1e^{(1/2)x}+ c_2e^{(5/2)x}$
2. $3y''-14y'-5y=0$ ans: $y=c_1e^{5x}+ c_2e^{(-1/3)x}$
3. $4y''+y=0$ ans: $y=c_1\cos(x/2)+ c_2\sin(x/2)$

b) The Second order linear non homogenous D.Eq. with constant coefficient:

The general form is: $ay'' + by' + cy = F(x) \quad \dots(3)$ where a, b and c are constants.

The general solution

If y_h is the solution of the homo. D.Eq. $ay'' + by' + cy = 0$, then the general solution of eq. (3) is:

$$y = y_h + y_p$$

y_h (complementary function)

y_p (porticular integral)

i) y_h is y homo.

ii) y_p (use the table)

Methods of finding y_p :

There are two methods:

1) Undetermined coefficients:

In this method y_p depends on the roots r1, and r2 of characteristic equation

and on the form of $F(x)$ in eq. (3) as follows:

$F(x)$	Choice of y_p	M.R.
kx^n nth degree polynomial	$k_n x^n + k_{n-1} x^{n-1} + k_{n-2} x^{n-2} + \cdots + k_0$	0
ke^{px}	ce^{px}	p
$k \sin \beta x$ *or* $k \cos \beta x$	$c_1 \cos \beta x + c_2 \sin \beta x$	$\mp i\beta$

Note: For repeated term (root), multiply by X .

Ex.1: Use the table to write y_p

1) $F(x) = 3x^2$, $k = 3$, $n = 2$

$$y_p = k_2 x^2 + k_1 x + k_0$$

2) $F(x) = \frac{-1}{2} e^{-3x}$, $k = \frac{-1}{2}$ $\Rightarrow$ c

$$y_p = ce^{-3x}$$

3) $F(x) = 2\cos 3x$, $k = 2$, $\beta = 3$

$$y_p = c_1 \cos 3x + c_2 \sin 3x$$

4) $F(x) = 3x^2 - 3x + 5 - 2e^{3x}$, $k = -3$, $c = -2$

$$y_p = k_2 x^2 + k_1 x + k_0 + ce^{3x}$$

5) $F(x) = 2\cos x - \frac{1}{2}\sin x$

$$y_p = c_1 \cos x + c_2 \sin x$$

6) $F(x) = \sin x - \cos 2x$

$$y_p = c_1 \cos x + c_2 \sin x + A\cos 2x + B\sin 2x$$

Ex.2: Solve $y'' - y' - 2y = 4x^2 \quad .... \quad (1)$

Solution:

$y'' - y' - 2y = 0$

the char. Eq. $r^2 - r - 2 = 0$

$$(r+1)(r-2) = 0$$

$$r_1 = -1, \; r_2 = 2$$

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

$f(x) = 4x^2$ is polynomial of second degree then

$$y_p = k_2 x^2 + k_1 x + k_0 \quad ... \; (2)$$

$$\Rightarrow \quad y_p' = 2k_2 x + k_1 \quad , \quad y_p'' = 2k_2$$

Substitution gives

$$2k_2 - (2k_2 x + k_1) - 2(k_2 x^2 + k_1 x + k_0) = 4x^2$$

$$coeff.of\ x^2 : -2k_2 = 4 \quad \Rightarrow \quad k_2 = -2$$

$$coeff.of\ x : -2k_2 - 2k_1 = 0 \quad \Rightarrow \quad k_1 = 2$$

$$const : 2k_2 - k_1 - 2k_0 = 0 \quad \Rightarrow \quad k_0 = -3$$

$$y_p = -2x^2 + 2x - 3$$

$$y_g = y_h + y_p = (c_1 e^{-x} + c_2 e^{2x}) - 2x^2 + 2x - 3$$

Ex.3: $y'' - y' - 2y = e^{3x}$

Solution:

$$y'' - y' - 2y = e^{3x} \quad \ldots (1)$$

$$y'' - y' - 2y = 0$$

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0 \Rightarrow r_1 = 2,\ r_2 = -1$$

$y_h = (c_1 e^{2x} + c_2 e^{-x})$, Put

$$y_p = ce^{3x} \quad \ldots (2)$$

$$y_p' = 3ce^{3x} \quad , \quad y_p'' = 9ce^{3x}$$

Substitute In (1)

$$9ce^{3x} - 3ce^{3x} - 2ce^{3x} = e^{3x}$$

$$9c - 3c - 2c = 1 \Rightarrow 4c = 1 \quad \Rightarrow \quad c = \frac{1}{4}$$

In (2) $\Rightarrow$ $y_p = \frac{1}{4} e^{3x}$

$$y_g = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4} e^{3x}$$

Modification rule <u>قاعدة التعديل</u>

1) اذا كان $F(x) = kx^n$ وكان احد جذري المعادلة القياسية $= 0$ ← يضرب y_p السابق في x.

2)

a - اذا كان $F(x) = ke^{px}$ وكان احد جذري المعادلة القياسية $= p$ ← يضرب y_p السابق في x.

b - اذا كان $F(x) = ke^{px}$ وكان جذري المعادلة القياسية $= p$ ← يضرب y_p السابق في x^2.

3) اذا كان $F(x) = \begin{cases} k\cos\beta x \\ k\sin\beta x \end{cases}$ وكان $\alpha = 0$, $r = \mp i\beta$ ← يضرب y_p السابق في x.

Ex.4: Solve $y''+y=\sin x$

Solution:

$y''+y=0$

$r^2+1=0,\ r^2=-1 \Rightarrow r=\pm i,\ \alpha=0,\ \beta=1$

$y_h=c_1\cos x+c_2\sin x$

$y_p=x(c_3\cos x+c_4\sin x),\ y'_p=x(-c_3\sin x+c_4\cos x)+(c_3\cos x+c_4\sin x)$

$y''_p=x(-c_3\cos x-c_4\sin x)+(-c_3\sin x+c_4\cos x)+(-c_3\sin x+c_4\cos x)$

Substitution gives

$-2c_3\sin x+2c_4\cos x=\sin x$

$-2c_3=1\Rightarrow c_3=-1/2,\ 2c_4=0\Rightarrow c_4=0$

$$y_g=c_1\cos x+c_2\sin x-\frac{x}{2}\cos x$$

ENGINEERING ANALYSIS I

LECTURE 4

Ordinary Differential Equations

Second Order D.Eqs.

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2- Variation of Parameters.

Consider the differential equation $ay'' + by' + cy = F(x)$

Let the general homogenous solution of the D.E. equation is $y_h = c_1u_1 + c_2u_2$

and the particular solution has the form $y_p = u_1v_1 + u_2v_2$ where v_1 and v_2 are unknown

functions of x which must be determined .

1-Solve the following linear equations for v'_1 and v'_2

$$v'_1u_1 + v'_2u_2 = 0$$

$$v'_1u'_1 + v'_2u'_2 = F(x)$$

(Wronskian equation)

which can be solved with respect to v'_1 and v'_2 by **Grammar rule** as follows

$$D = \begin{vmatrix} u_1 & u_2 \\ u'_1 & u'_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} 0 & u_2 \\ F(x) & u'_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} u_1 & 0 \\ u'_1 & F(x) \end{vmatrix}$$

and $v'_1 = \dfrac{D_1}{D}, \quad v'_2 = \dfrac{D_2}{D}$

2-Integrad v'_1 and v'_2 with respect to x we can find $\mathbf{v_1}$ and $\mathbf{v_2}$.

Ex.1:

Solve $y'' - y' - 2y = e^{3x}$ .... (1)

$y_h = c_1 e^{-x} + c_2 e^{2x}$, hence

$u_1 = e^{-x} \Rightarrow u_1' = -e^{-x}$

$u_2 = e^{2x} \Rightarrow u_2' = 2e^{2x}$

$y_p = v_1 u_1 + v_2 u_2$

$v_1' u_1 + v_2' u_2 = 0$

$v_1' u_1' + v_2' u_2' = F(x)$

$v_1'(e^{-x}) + v_2'(e^{2x}) = 0$

$v_1'(-e^{-x}) + v_2'(2e^{2x}) = e^{3x}$

Solving this system by Cramer rule gives

$$D = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^{x}, \quad D_1 = \begin{vmatrix} 0 & e^{2x} \\ e^{3x} & 2e^{2x} \end{vmatrix} = -e^{5x},$$

$$D_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{3x} \end{vmatrix} = e^{2x}$$

$$v'_1 = \frac{-e^{5x}}{3e^{x}} = \frac{-1}{3} e^{4x} \Rightarrow v_1 = \int \frac{-1}{3} e^{4x} = -\frac{1}{12} e^{4x},$$

$$v'_2 = \frac{e^{2x}}{3e^{x}} = \frac{1}{3} e^{x} \Rightarrow v_2 = \int \frac{1}{3} e^{x} = \frac{1}{3} e^{x}$$

$$\therefore y_p = -\frac{1}{2} e^{4x} e^{-x} + \frac{1}{3} e^{x} e^{2x} = \frac{1}{4} e^{3x}$$

the general solution is

$$y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4} e^{3x}$$

Ex.2: solve $y''+y=\sec x$

Solution:

$y''+y=0$

$r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r=\pm i \qquad \alpha=0,\ \beta=1$

$y_h=c_1\cos x + c_2\sin x, \quad u_1=\cos x,\ u_2=\sin x,\ f(x)=\sec x$

$y_p= v_1u_1+v_2u_2$

$= v_1\cos x + v_2\sin x$ then

$$v_1'(\cos x)+v_2'(\sin x)=0$$

$$v_1'(-\sin x)+v_2'(\cos x)=\sec x$$

$$D=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=\cos^2 x+\sin^2 x=1,$$

$$D_1=\begin{vmatrix}0 & \sin x\\ \sec x & \cos x\end{vmatrix}=-\sin x\ \sec x=-\sin x\frac{1}{\cos x}=-\tan x,$$

$$D_2=\begin{vmatrix}\cos x & 0\\ -\sin x & \sec x\end{vmatrix}=\cos x\sec x=1$$

$$v_1'=\frac{-\tan x}{1}=-\tan x$$

$$v_1=\int\frac{-\sin x}{\cos x}dx=\ln|\cos x|$$

$$v'_2=1\Rightarrow v_2=\int dx=x$$

$$y_p=\ln|\cos x|\cos x+x\sin x$$

$$y_g=c_1\cos x+c_2\sin x+\ln|\cos x|\cos x+x\sin x$$

Exercise:

Solve the following deferential equations using variation of parameters:

1. $y''-2y'+y = e^{x} \ln x$

2. $y''-2y'+y = \dfrac{e^{x}}{x^{5}}$

3. $y''+4y=\sin^{2}2x$

4. $y'' + y = \tan x$.

5. $4y'' - 4y' - 8y = 8e^{-t}$

6. $y'' - 2y' + y = \dfrac{e^{t}}{1+t^{2}} + 3e^{t}$

7. $4y'' + y = \cos x$

ENGINEERING ANALYSIS I

LECTURE 5

Laplace Transform

Prepared by: Abdurahman B. AYOUB

Laplace Transforms

• Important analytical method for solving *linear* ordinary differential equations.

- Application to nonlinear ODEs? Must linearize first.

• Laplace transforms play a key role in important process control concepts and techniques.

Examples:

- Transfer functions
- Frequency response
- Control system design
- Stability analysis

Definition

The Laplace Transform is an integral transformation of a function f(t) from the time domain into the complex frequency domain, giving F(s)

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

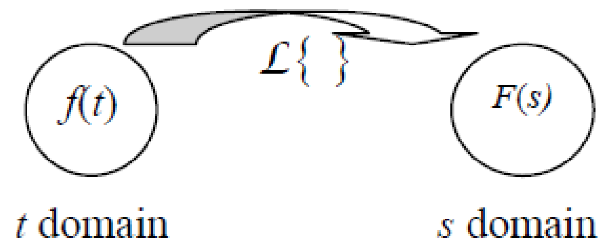
Where $\mathcal{L}$ interpreted as an operator and **s** is a complex variable: $\boldsymbol{s = a + bj}$

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}\,ds$$

$$\boldsymbol{f(t) \Leftrightarrow F(s)}$$

$$\mathcal{L}\{f(t)\} = F(s), \quad \mathcal{L}\{g(t)\} = G(s), \quad \text{and} \quad \mathcal{L}\{y(t)\} = Y(s).$$



The Laplace transform operator

Laplace Transforms of Simple Functions

1. Constant Function

Let $f(t) = a$ (a constant). Then from the definition of the Laplace transform.

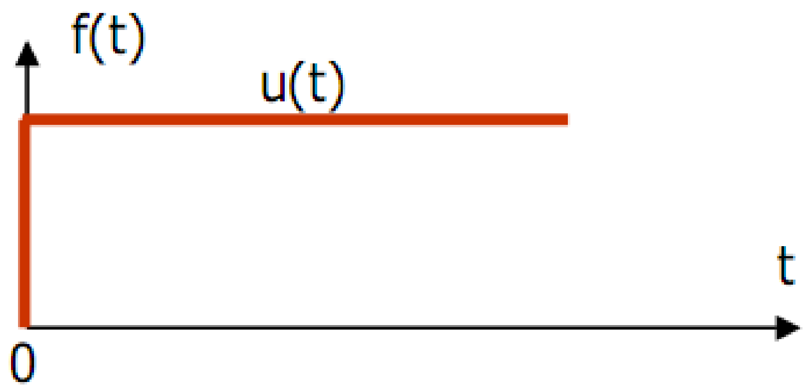
$$\mathcal{L}(a) = \int_0^\infty a e^{-st} dt = -\frac{a}{s} e^{-st}\Big|_0^\infty = 0 - \left(-\frac{a}{s}\right) = \boxed{\frac{a}{s}}$$

Ex. Find Laplace transform of $f(t) = 1$

$$F(s) = \frac{1}{s}$$

2.Step Function

Laplace Transform of the unit step function u(t)



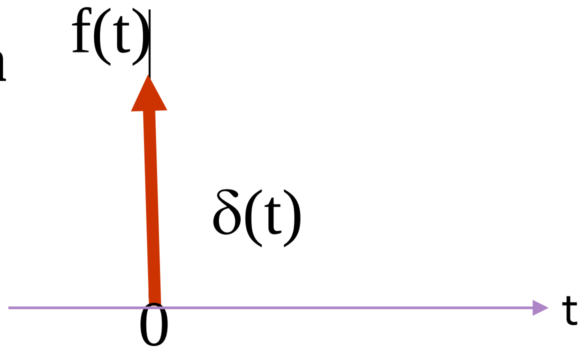
$$L[u(t)] = \int_0^\infty 1e^{-st}dt = \frac{-1}{s}e^{-st}\Big|_0^\infty = \frac{-1}{s}(e^{-s\infty} - e^{-s0}) = \frac{1}{s}$$

The Laplace Transform of a unit step function is:

$$L[u(t)] \Leftrightarrow \frac{1}{s}$$

3.Unit impulse function

$\delta(t) = 1 \qquad t = 0$



The Laplace transform of a unit impulse function:

In particular, if we let f(t) = δ(t) and take the Laplace

$$L[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt = e^{-0s} = 1$$

The Laplace Transform of a unit impulse function is:

$$L[\delta(t)] \Leftrightarrow 1$$

4 The Exponential Function e^{-at} :

$$L[e^{-at}u(t)] = \int_0^\infty e^{-at}e^{-st}\,dt = \int_0^\infty e^{-(s+a)t}\,dt$$

$$L[e^{-at}u(t)] = \left.\frac{-e^{-st}}{(s+a)}\right|_0^\infty = \frac{1}{s+a}\left(-e^{-s\infty} + e^{-s0}\right)$$

The Transform of Exponential Function e^{-at} :

$$e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

Ex. Find

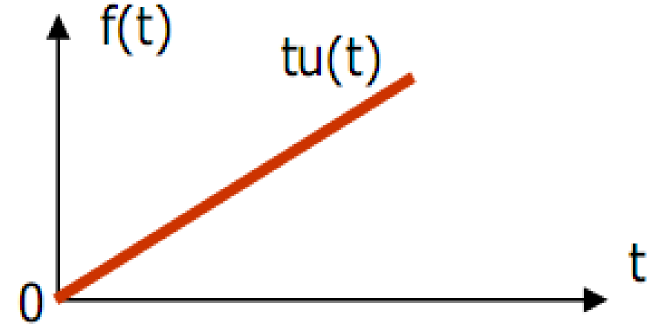
1. $\mathcal{L}(e^{-t}) = \dfrac{1}{s+1}$

2. $\mathcal{L}(e^{t}) = \dfrac{1}{s-1}$

3. $\mathcal{L}(e^{4t}) = \dfrac{1}{s-4}$

5 The Ramp Function tu(t)

$$L[tu(t)] = \int_0^\infty t e^{-st} dt$$



Remember

u = t
dv = e^{-st}dt

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

Type equation here.

$$L[tu(t)] = t\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)dt = \frac{1}{s}\int_0^\infty e^{-st} dt$$

The transform of a Ramp Function tu(t)

$$tu(t) \Leftrightarrow \frac{1}{s^2}$$

In general

$f(t) = t^n \ (n \geq 1)$

$F(s) = \dfrac{n!}{s^{n+1}}$

Ex.1 Find $\mathcal{L}(t) = \frac{1}{s^2}$ **Ex.2 Find** $\mathcal{L}(t^3) = \frac{6}{s^4}$

6 **The cosine function cos(ωt):**

$$L[\cos(\omega t)] = \int_0^\infty \frac{(e^{j\omega t} + e^{-j\omega t})}{2} e^{-st} dt = \frac{1}{2}\left[\int_0^\infty e^{-(s-j\omega)t} dt + \int_0^\infty e^{-(s+j\omega)t} dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right]$$

$$= \frac{s}{s^2+\omega^2}$$

The transform of a cosine function cos(ωt):

$$\cos(\omega t)u(t) \Leftrightarrow \frac{s}{s^2+\omega^2}$$

Ex. Find

$$\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$$

7 **The Sine function sin(ωt):**

$$L[\sin(\omega t)] = \int_0^\infty \frac{(e^{j\omega t} - e^{-j\omega t})}{2j} e^{-st} dt = \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t} dt - \int_0^\infty e^{-(s+j\omega)t} dt\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$= \frac{\omega}{s^2+\omega^2}$$

A transform pair

$$\sin(\omega t)u(t) \Leftrightarrow \frac{\omega}{s^2+\omega^2}$$

Ex. Find $\mathcal{L}(\sin 3t) = \frac{3}{s^2+9}$

Using Matlab with Laplace transform:

Example

Use Matlab to find the transform of te^{-4t}

The following is written in italic to indicate Matlab code

```
syms t,s
laplace(t*exp(-4*t),t,s)
ans =
      1/(s+4)^2
```

Standard Table of Laplace Transform

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
a	$\frac{a}{s}$
$t^n,\ n = 1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

ENGINEERING ANALYSIS I

LECTURE 6

Properties of Laplace Transform

Prepared by: Abdurahman B. AYOUB

Properties of Laplace Transform

1. Linearity

$$L\,[af(t)+bg(t)]=\int_0^{\infty}[af(t)+bg(t)]e^{-st}\,dt$$

$$=a\int_0^{\infty}f(t)e^{-st}\,dt+b\int_0^{\infty}g(t)e^{-st}\,dt=aF(s)+bG(s)$$

Ex.1 Find the Laplace Transforms of f(t)=$3\,e^{2t}-2\sin 3t$

$$L\,[f(t)]=[\frac{3}{s-2}-\frac{6}{s^2+9}]$$

Ex. 2

Find the Laplace transform of $\cos^2 t.$

Solution

$$L\,[\cos^2 t]=L\,[\frac{1+\cos 2t}{2}]$$

$$=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+2^2}\right)=\frac{s^2+2}{s(s^2+4)}$$

2.Shifting

The First Shifting Theorem: (frequency shifting property)

a) If $\mathcal{L}(f(t)) = F(s)$.

Then $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$

ملاحظة لحل المثال

$$L(t^n) = \frac{n!}{s^{n+1}} \quad where\ n = 0,1,2,3,---$$

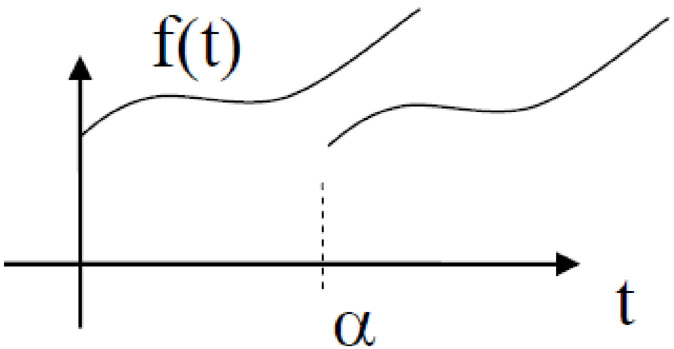
$$L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}, s > a$$

Ex.1 Find $L(e^{2t}t^3) = \left[\frac{6}{(s-2)^4}\right]$

b) The second shifting theorem:(Delay)

If $\mathcal{L}(f(t)) = F(s)$.

Then $\mathcal{L}\left\{f(t-a)u(t-a)\right\} = e^{-as}F(s)$



Ex.2

What is the Laplace transform of the function $f(t) = \begin{cases} 0, & t < 4 \\ 2t^3, & t \geq 4 \end{cases}$. $\quad f(t) = 2t^3u(t-4)$

$$L[f(t)] = L\{2[(t-4)^3 + 12(t-4)^2 + 48(t-4) + 64]\,u(t-4)\}$$

$$= 2e^{-4s}\left(\frac{3!}{s^4} + 12 \times \frac{2!}{s^3} + 48 \times \frac{1}{s^2} + \frac{64}{s}\right) = 4e^{-4s}\left(\frac{3}{s^4} + \frac{12}{s^3} + \frac{24}{s^2} + \frac{32}{s}\right)$$

Ex.3 Find

1. $\mathcal{L}[e^{3t}\cos 2t] = \dfrac{s-3}{(s-3)^2+4}$

2. $\mathcal{L}\{e^{2t}(t^3+5t-2)\}$
$= \dfrac{3!}{(s-2)^4} + 5\dfrac{1}{(s-2)^2} - 2\dfrac{1}{s-2}.$

3. $\mathcal{L}\{(t^2+4)e^{2t} - e^{-t}\cos t\}$
$= \dfrac{2}{(s-2)^3} + \dfrac{4}{s-2} - \dfrac{s+1}{(s+1)^2+1},$

3. Scaling

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L}$ (f(at)) = $\dfrac{1}{a}F(\dfrac{s}{a})$

Ex.
Find the Laplace transform of cos2*t*.

$\mathrm{L}\ [\cos t] = \dfrac{s}{s^2+1}$ $\qquad$ $\mathrm{L}\ [\cos 2t] = \dfrac{1}{2}\dfrac{\frac{s}{2}}{(\frac{s}{2})^2+1} = \dfrac{s}{s^2+4}$

4. Derivative
a)Time - differentiation property

(i) Transform of the First Derivative

If $\mathcal{L}$ (f(t)) = F(s).

Then $L\,[f'(t)] = sF(s) - f(0)$

$$L\,[f'(t)] = \int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}dt$$

where $u = e^{-st}$ $v' = f'(t)$

$u' = -se^{-st}$ $v = \int f'(t) = f(t)$

$$L\,[f'(t)] = sF(s) - f(0)$$

$$L\,[f'(t)] = sF(s) - f(0)$$

(ii)Transform of the Second Derivative

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

(iii) Higher order derivative:

$$L\,[f^{(n)}(t)] = s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

b)frequency - differentiation property

If $\mathcal{L}(f(t)) = F(s)$.

Then $\mathcal{L}(t\,f(t)) = -\frac{d}{ds}F(s)$

In general $\mathcal{L}(t^n f(t)) = -\frac{d^n}{ds^n}F(s)$

Ex.1

Find the Laplace transform of $t\,e^t$.

$$\mathrm{L}\ (e^t) = \frac{1}{s-1} \Rightarrow \mathrm{L}\ (te^t) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

Ex.2

If $f(t) = e^t$ Find the Laplace transform of $f'(t)$

Solution: 1st Method

$f(t) = e^t$, so $f'(t) = e^t$ and $\mathcal{L}(f) = \mathcal{L}(f') = \frac{1}{s-1}$

Second Method

Using the formula,

$$L\,[f'(t)] = sF(s) - f(0)$$

$$\mathcal{L}(f') = s(\frac{1}{s-1}) - 1$$

$$= \frac{1}{s-1}$$ which is the same

Ex.3 If f(t)=t (unit ramp), find the Laplace transform of $f'(t)$

Solution

$$\mathcal{L}(f') = s\left(\frac{1}{s^2}\right) - 0 = 1/s$$

5. Integration

a)Time - integration property

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L} \int_{o}^{t} f(\tau)d\tau = \frac{F(s)}{s}$

$$\Rightarrow \mathrm{L}[\int_0^t \int_0^t \dots \int_0^t f(t)dtdt\cdots dt] = \frac{1}{s^n}F(s)$$

b)frequency - integration property

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L} \left\{\frac{f(t)}{t}\right\} = \int_{s}^{\infty} F(s)\,ds$

$$\Rightarrow \int_s^\infty \int_s^\infty \cdots \int_s^\infty F(s)dsds\cdots ds = \mathrm{L}\ [\frac{1}{t^n}f(t)]$$

Ex.1 **Find** $L\left[\frac{1-e^{-t}}{t}\right]$

$$L\left[1-e^{-t}\right]=\frac{1}{s}-\frac{1}{s+1}$$

$$L\left[\frac{1-e^{-t}}{t}\right]=\int_{s}^{\infty}\left(\frac{1}{s}-\frac{1}{s+1}\right)ds$$

$$=\ln s-\ln(s+1)\Big|_{s}^{\infty}=\ln\frac{s}{s+1}\Big|_{s}^{\infty}$$

$$=0-\ln\frac{s}{s+1}=\ln\frac{s+1}{s}$$

Ex.2 If f(t)=1 (unit step), find the Laplace transform of $\int f(t)$

Using the formula,

If $\mathcal{L}$ (f(t)) = F(s).

Then $\mathcal{L}\int_{o}^{t} f(\tau)d\tau=\frac{F(s)}{s}$

$$\Rightarrow \quad F(s)=\frac{\frac{1}{s}}{s}=\frac{1}{s^{2}}$$

6. Convolution theorem

If $\quad \mathcal{L}\{ f(t) \} = F(s)$

and $\quad \mathcal{L}\{ g(t) \} = G(s)$

then the convolution of $f(t)$ and $g(t)$ is denoted by $(f * g)(t)$, is defined by

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

and the Laplace transform of the convolution of two functions is the product of the separate Laplace transforms:

$$\mathcal{L}\{ (f * g)(t) \} = F(s) G(s)$$

Ex.

Find the Laplace transform of $\int_0^t e^{t-\tau} \sin 2\tau \, d\tau.$

$$\mathcal{L}\ [e^t] = \frac{1}{s-1}, \quad \mathcal{L}\ [\sin 2t] = \frac{2}{s^2+4}$$

$$\therefore \mathcal{L}\ [\int_0^t e^{t-\tau} \sin 2t \, d\tau] = \mathcal{L}\ [e^t * \sin 2t] = \mathcal{L}\ [e^t] \cdot \mathcal{L}\ [\sin 2t] = \frac{1}{s-1} \cdot \frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

7. **Theorem:** **Initial Value**

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s\to\infty} sF(s)$ exists, then

Initial Value Theorem

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t) = f(0)$$

This theorem tell us that we don't need to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

Example: **Initial Value**

Given;

$$F(s) = \frac{(s+1)}{(s+1)^2 + 4^2}$$

Find f(0)

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\frac{(s+1)}{(s+1)^2 + 4^2} \quad \lim_{s\to\infty}\left[\frac{s^2+s}{s^2+2s+1+16}\right]$$

$$= \lim_{s\to\infty} \frac{s^2\left(1+\frac{1}{s}\right)}{s^2\left(1+\frac{1}{s}+\frac{17}{s^2}\right)} = 1$$

8. Theorem: Final Value

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s \to 0} sF(s)$ exists, then

Final Value Theorem

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)$$

This theorem tell us that we don't need to take the inverse of F(s) in order to find out the final value of f(t) in the time domain.
This is particularly useful in circuits and systems.

Example: **Final Value**

Given:

$$F(s) = \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} \qquad \text{note} \quad F^{-1}(s) = te^{-2t}\cos 3t$$

Find $f(\infty)$

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s\frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} = 0$$

Properties of Laplace Transform

Property	Original Function	Transformed Function
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$
Shifting	$f(t-a)\,u(t-a)$	$e^{-as}F(s)$
	$e^{at}f(t)$	$F(s-a)$
Scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-s^{n-3}f''(0)-\cdots-f^{(n-1)}(0)$
	$(-t)^nf(t)$	$\frac{d^nF(s)}{ds^n}$
Integration	$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
	$\frac{1}{t}f(t)$	$\int_s^\infty F(s)ds$
Convolution	$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
Periodic Function	$f(t)=f(t+T)$	$\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$
Initial Value Theorem	$\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$	
	$\lim_{t\to 0}\frac{f(t)}{g(t)}=\lim_{s\to\infty}\frac{F(s)}{G(s)}$	
Final Value Theorem	$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$	
	$\lim_{t\to\infty}\frac{f(t)}{g(t)}=\lim_{s\to 0}\frac{F(s)}{G(s)}$	

ENGINEERING ANALYSIS I

LECTURE 7

Inverse Laplace Transform

Prepared by: Abdurahman B. AYOUB

Inverse Laplace Transform

Definition

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}\,ds$$

$$\boldsymbol{f(t) \Leftrightarrow F(s)}$$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$ if $F(s) = \mathcal{L}\{f(t)\}$

Technique: Find the way back.

Example 1.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\sin 2t.$$

Example 2.

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{6}{(s+5)^4}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{(s+5)^4}\right\} = \frac{1}{3}e^{-5t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$= \frac{1}{3}e^{-5t}t^3.$$

Example 3

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}=\cos 2t+\frac{1}{2}\sin 2t$$

Example 4

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s+2)}\right\}=\mathcal{L}^{-1}\left\{\frac{3/4}{s-2}+\frac{1/4}{s+2}\right\}=\frac{3}{4}e^{2t}+\frac{1}{4}e^{-2t}$$

Here we used partial fraction to Find out:

$$\frac{s+1}{(s-2)(s+2)}=\frac{A}{s-2}+\frac{B}{s+2},\quad A=3/4,\quad B=1/4.$$

Example 5

Determine the Inverse Laplace Transform of $F(s) = \dfrac{3s^2 + 2s + 4}{(s+1)(s^2+4)}$

Solution

Using partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4}$$

$$3s^2 + 2s + 4 \equiv A(s^2 + 4) + (Bs + C)(s + 1)$$

Substituting s = −1

5 = 5A implying that A = 1

Equating coefficients of s^2 on both sides, 3 = A + B so that B = 2.

Equating constant terms on both sides, 4 = 4A + C so that C = 0

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}$$

$$f(t) = e^{-t} + 2\cos 2t$$

Example 6

Determine the Inverse Laplace Transform of $F(s) = \frac{1}{(s+2)^5}$

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$

$$f(t) = \frac{1}{24} t^4 e^{-2t}$$

Example 7

Determine the Inverse Laplace Transform of $F(s) = \dfrac{3}{(s-7)^2+9}$.

Solution
Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$

$$f(t) = e^{7t}\sin 3t$$

Example 8

Determine the Inverse Laplace Transform of $F(s) = \dfrac{s}{s^2+4s+13}$.

Solution
The denominator will not factories conveniently, so we complete the square. This gives

$$F(s) = \frac{s}{(s+2)^2+9}.$$

To use the First Shifting Theorem, we must include s + 2 in the numerator.

$$F(s) = \frac{(s+2)-2}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+3^2} - \frac{2}{3} \cdot \frac{3}{(s+2)^2+3^2}$$

$$f(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t$$

$$= \frac{1}{3}e^{-2t}[3\cos 3t - 2\sin 3t]$$

Example 9

Determine the Inverse Laplace Transform of $F(s) = \dfrac{8(s+1)}{s(s^2+4s+8)}$

Solution

Using partial fractions,

$$\frac{8(s+1)}{s(s^2+4s+8)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+8}.$$

$$8(s+1) \equiv A(s^2+4s+8) + (Bs+C)s$$

Substituting s = 0 gives ⇨ 8 = 8A so that A = 1.

Equating coefficients of s^2 on both sides, 0 = A + B which gives B = −1.

Equating coefficients of s on both sides, 8 = 4A + C which gives C = 4.

$$F(s) = \frac{1}{s} + \frac{-s+4}{s^2+4s+8}.$$

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t$$

Example 10

Determine the Inverse Laplace Transform of $F(s) = \dfrac{s+10}{s^2-4s-12}.$

This time, the denominator will factories, into (s +2)(s − 6).

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

$$s + 10 \equiv \mathrm{A}(s - 6) + \mathrm{B}(s + 2).$$

Putting s = −2

$$8 = -8A \text{ giving } A = -1.$$

Putting s = 6

$$16 = 8B \text{ giving } B = 2.$$

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

$$f(t) = -e^{-2t} + 2e^{6t}$$

Note:

If we did not factories the denominator,

$$F(s) = \frac{s + 10}{s^2 - 4s - 12}.$$

$$F(s) = \frac{(s-2) + 12}{(s-2)^2 - 4^2} = \frac{s-2}{(s-2)^2 - 4^2} + 3.\frac{4}{(s-2)^2 + 4^2}.$$

$$f(t) = e^{2t}[\cosh 4t + 3\sinh 4t]$$

Example 11

Determine the Inverse Laplace Transform of $F(s) = \dfrac{1}{(s-1)(s+2)}$.

Using Convolution Theorem.

$$F(s) = \frac{1}{(s-1)}.\frac{1}{(s+2)},$$

$$f(t) = \int_0^t e^T.e^{-2(t-T)}\,\mathrm{d}T = \int_0^t e^{(3T-2t)}\,\mathrm{d}T$$

$$= \left[\frac{e^{3T-2t}}{3}\right]_0^t = \frac{e^t}{3} - \frac{e^{-2t}}{3}$$

Example 12 Find $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+1}\right]$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

the second shifting theorem implies that

$$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+1}\right] = \sin(t-3)u_3(t)$$

Example 13. Invert $\frac{s-1}{s^2(s+1)}$.

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \int_0^t e^{-z}\,dz = 1 - e^{-t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)}\right] = \int_0^t (1-e^{-z})\,dz$$

$$= \left[z + e^{-z}\right]_0^t = t + e^{-t} - 1$$

$$\mathcal{L}^{-1}\left[\frac{s-1}{s^2(s+1)}\right] = 2(1-e^{-t}) - t$$

Example 12

Use Matlab to find the inverse transform of

$$F(s)=\frac{s(s+6)}{(s+3)(s^2+6s+18)}$$

syms s t

ilaplace(s(s+6)/((s+3)*(s^2+6*s+18)))*

ans =

*-exp(-3*t)+2*exp(-3*t)*cos(3*t)*

ENGINEERING ANALYSIS I

LECTURE 8

Differential Equations

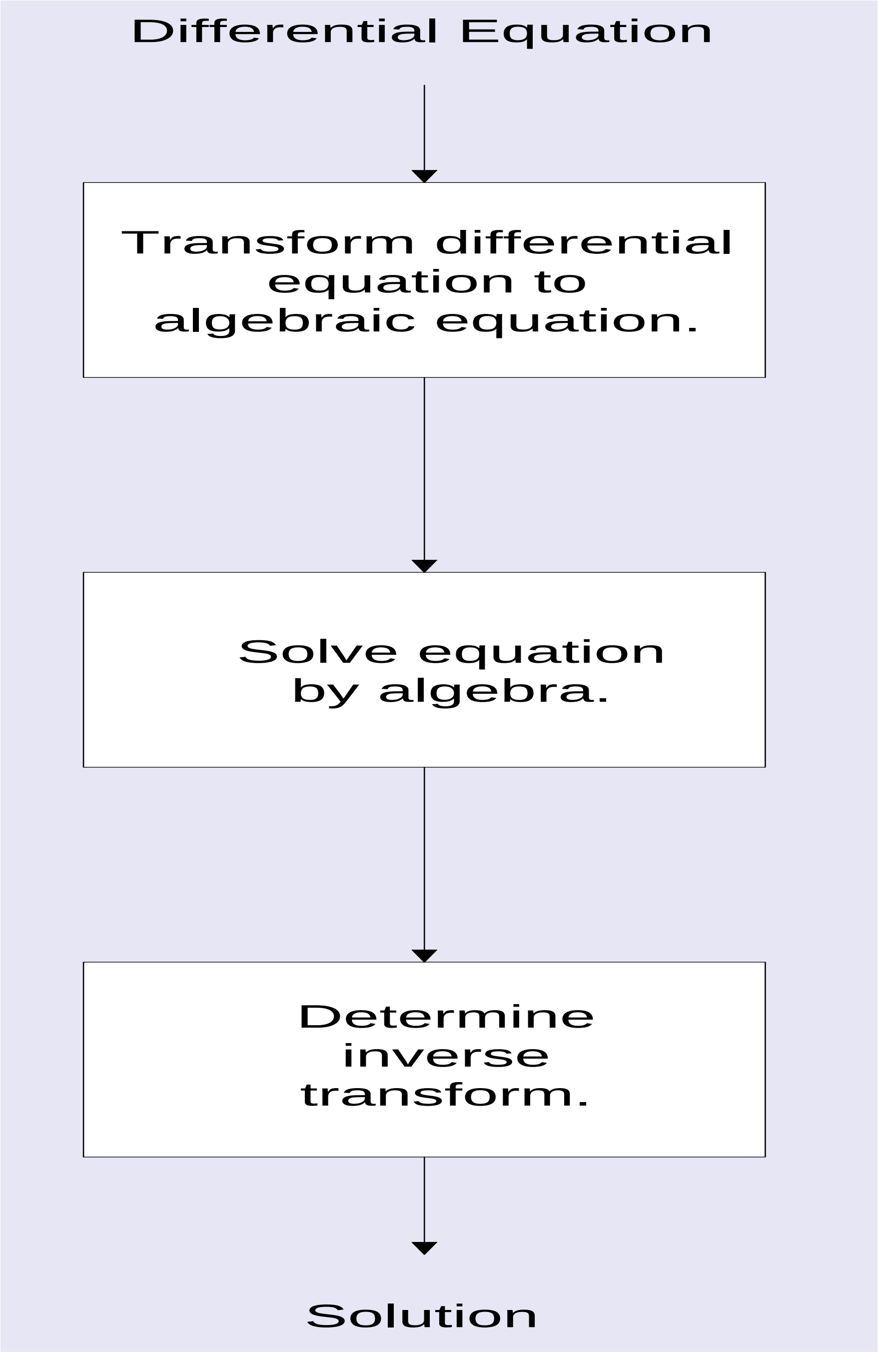
Laplace Transform Methods

Prepared by: Abdurahman B. AYOUB

Differential Equations
Laplace Transform Methods

The Laplace transform was developed by the French mathematician by the same name (1749-1827) and was widely adapted to engineering problems in the last century. Its utility lies in the ability to convert differential equations to algebraic forms that are more easily solved. The notation has become very common in certain areas as a form of engineering "language" for dealing with systems.

Figure 1.
Steps involved in
using the
Laplace transform.



Significant Operations for Solving Differential Equations

$$L[f'(t)] = sF(s) - f(0)$$

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$

$$L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

Example 1. Solve DE shown below.

$$\frac{dy}{dt}+2y=12 \qquad y(0)=10$$

$$L[f'(t)]=sF(s)-f(0)$$

$$L\left[\frac{dy}{dt}\right]+2L[y]=L[12]$$

$$sY(s)-10+2Y(s)=\frac{12}{s}$$

$$(s+2)Y(s)=10+\frac{12}{s}$$

$$Y(s)=\frac{10}{s+2}+\frac{12}{s(s+2)}$$

$$\frac{12}{s(s+2)} = \frac{A_1}{s} + \frac{A_2}{s+2}$$

$$A_1 = s\left[\frac{12}{s(s+2)}\right]_{s=0} = \left[\frac{12}{s+2}\right]_{s=0} = 6$$

$$A_2 = (s+2)\left[\frac{12}{s(s+2)}\right]_{s=-2} = \left[\frac{12}{s}\right]_{s=-2} = -6$$

$$Y(s) = \frac{10}{s+2} + \frac{6}{s} - \frac{6}{s+2} = \frac{6}{s} + \frac{4}{s+2}$$

$$y(t) = 6 + 4e^{-2t}$$

Example 2. Solve DE shown below.

$$\frac{dy}{dt} + 2y = 12\sin 4t \qquad y(0) = 10$$

$$L[f'(t)] = sF(s) - f(0)$$

$$sY(s) - 10 + 2Y(s) = \frac{12(4)}{s^2 + 16}$$

$$Y(s) = \frac{10}{s+2} + \frac{48}{(s+2)(s^2+16)}$$

$$\frac{48}{(s+2)(s^2+16)} = \frac{A}{s+2} + \frac{B_1 s + B_2}{s^2+16}$$

$$A=\left.\frac{48}{s^2+16}\right]_{s=-2}=\frac{48}{20}=2.4$$

$$\frac{48}{(s+2)(s^2+16)}=\frac{2.4}{s+2}+\frac{B_1 s+B_2}{s^2+16}$$

$$48=2.4(s^2+16)+(s+2)(B_1 s+B_2)$$

s^0: $48=2.4(16)+(2)(B_2)$ ⇒ $B_2=4.8$

s^2: $0=2.4+B_1$ ⇒ $B_1=-2.4$

$$Y(s) = \frac{10}{s+2} + \frac{2.4}{s+2} - \frac{2.4s}{s^2+16} + \frac{4.8}{s^2+16}$$

$$y(t) = 12.4e^{-2t} - 2.4\cos 4t + 1.2\sin 4t$$

Example 3. Solve DE shown below.

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 24 \qquad y(0) = 10 \text{ and } y'(0) = 0$$

$$L[f''(t)] = s^2 F(s) - sf(0) - f'(0) \quad , \quad L[f'(t)] = sF(s) - f(0)$$

$$s^2 Y(s) - 10s - 0 + 3\left[sY(s) - 10\right] + 2Y(s) = \frac{24}{s}$$

$$Y(s) = \frac{24}{s(s^2 + 3s + 2)} + \frac{10s + 30}{s^2 + 3s + 2}$$

$$= \frac{24}{s(s+1)(s+2)} + \frac{10s + 30}{(s+1)(s+2)}$$

$$\frac{24}{s(s+1)(s+2)} = \frac{12}{s} - \frac{24}{s+1} + \frac{12}{s+2}$$

$$\frac{10s+30}{(s+1)(s+2)} = \frac{20}{s+1} - \frac{10}{s+2}$$

$$F(s) = \frac{12}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

$$f(t) = 12 - 4e^{-t} + 2e^{-2t}$$

Example 4. Solve DE shown below.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 20 \qquad y(0) = 0 \text{ and } y'(0) = 10$$

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0) \quad , \quad L[f'(t)] = sF(s) - f(0)$$

$$s^2Y(s) - 0 - 10 + 2\left[sY(s) - 0\right] + 5Y(s) = \frac{20}{s}$$

$$Y(s) = \frac{20}{s(s^2+2s+5)} + \frac{10}{s^2+2s+5}$$

$$\frac{20}{s(s^2+2s+5)} = \frac{4}{s} + \frac{As+B}{(s^2+2s+5)}$$

$$20 = 4(s^2+2s+5) + s(As+B)$$

$$S: \quad 0 = 4(2) + (B)$$

$$B = -8$$

$$s^2: \quad 0 = 4 + A$$

$$A = -4$$

$$Y(s) = \frac{4}{s} + \frac{-4s-8}{s^2+2s+5} + \frac{10}{s^2+2s+5} = \frac{4}{s} + \frac{-4s+2}{s^2+2s+5}$$

$$s^2+2s+5 = s^2+2s+1+5-1 = (s+1)^2+(2)^2$$

$$Y(s) = \frac{4}{s} + \frac{-4(s+1)}{(s+1)^2+(2)^2} + \frac{3(2)}{(s+1)^2+(2)^2}$$

$$y(t) = 4 - 4e^{-t}\cos 2t + 3e^{-t}\sin 2t$$

Example 5. Solve the initial value problem (IVP) by Laplace transform

$$L[f''(t)] = s^2F(s) - sf(0) - f'(0)$$
$$L[f'(t)] = sF(s) - f(0)$$

$$y'' - y' - 2y = e^{2t}, \qquad y(0) = 0, y'(0) = 1$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = \mathcal{L}\{e^{2t}\}$$

$$s^2Y(s) - 1 - sY(s) - 2Y(s) = \frac{1}{s-2}$$

$$(s^2 - s - 2)Y(s) = \frac{1}{s-2} + 1 = \frac{s-1}{s-2}$$

$$Y(s) = \frac{s-1}{(s-2)(s^2-s-2)} = \frac{s-1}{(s-2)^2(s+1)}$$

$$\frac{s-1}{(s-2)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}.$$

$$s - 1 = A(s-2)^2 + B(s+1)(s-2) + C(s+1)$$

Set $s = -1$, we get $A = -\frac{2}{9}$.

Set $s = 2$, we get $C = \frac{1}{3}$

Set $s = 0$ we get $B = \frac{2}{9}$.

$$Y(s) = -\frac{2}{9}\frac{1}{s+1} + \frac{2}{9}\frac{1}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y\} = -\frac{2}{9}e^{-t} + \frac{2}{9}e^{2t} + \frac{1}{3}te^{2t}.$$

ENGINEERING ANALYSIS I

LECTURE 9

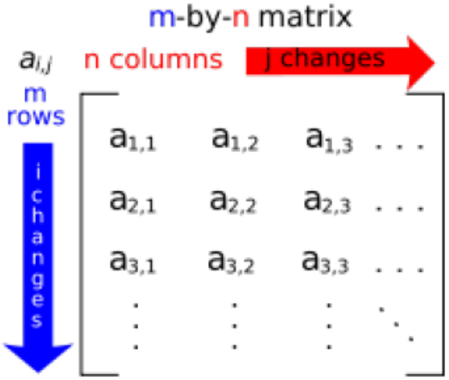
MATRIX THEORY

Prepared by: Abdurahman B. AYOUB

MATRIX THEORY

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets



A_{mxn}

$$\begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

2x4 matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix}$$

3x3 matrix

Matrix algebra has at least two advantages:

- Reduces complicated systems of equations to simple expressions

يختزل أنظمة المعادلات المعقدة إلى تعبيرات بسيطة

- Adaptable to systematic method of mathematical treatment and well suited to computers

قابل للتكيف مع أسلوب منهجي في المعالجة الرياضية ومناسب تمامًا لأجهزة الكمبيوتر

TYPES OF MATRICES

1. Column matrix or vector:

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \end{bmatrix}$$

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix} \qquad m \neq n \qquad \begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix}$$

4. Square matrix

The number of rows is equal to the number of columns

(a square matrix **A** has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} \neq 0$ for some or all $i = j$

6. Unit or Identity matrix - I

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A diagonal matrix with ones on the main diagonal

i.e. $a_{ij} = 0$ for all $i \neq j$

$a_{ij} = 1$ for some or all $i = j$

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad a_{ij} = 0$$

For all i,j

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

Lower triangular matrix

$$\begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Upper triangular matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

Basic Operations

1-EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \mathbf{A} = \mathbf{B}$$

2-ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Commutative Law:

A + **B** = **B** + **A**

Associative Law:

A + (**B** + **C**) = (**A** + **B**) + **C** = **A** + **B** + **C**

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A 2x3 **B** 2x3 **C** 2x3

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

C= A - B

3-SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then **kA = Ak**

Ex. If k=4 and

$$A=\begin{bmatrix}3 & -1\\ 2 & 1\\ 2 & -3\\ 4 & 1\end{bmatrix}$$

$$4\times\begin{bmatrix}3 & -1\\ 2 & 1\\ 2 & -3\\ 4 & 1\end{bmatrix}=\begin{bmatrix}3 & -1\\ 2 & 1\\ 2 & -3\\ 4 & 1\end{bmatrix}\times 4=\begin{bmatrix}12 & -4\\ 8 & 4\\ 8 & -12\\ 16 & 4\end{bmatrix}$$

Properties:

- k (**A** + **B**) = k**A** + k**B**
- (k + g)**A** = k**A** + g**A**
- k(**AB**) = (k**A**)**B** = **A**(k)**B**
- k(g**A**) = (kg)**A**

4-MULTIPLICATION OF MATRICES

The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\mathbf{A} \quad x \quad \mathbf{B} \quad = \quad \mathbf{C}$$

$$(2x3) \quad (3x2) \quad (2x2)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1\times 4)+(2\times 6)+(3\times 5) & (1\times 8)+(2\times 2)+(3\times 3) \\ (4\times 4)+(2\times 6)+(7\times 5) & (4\times 8)+(2\times 2)+(7\times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. **AI** = **IA** = **A**
2. **A**(**BC**) = (**AB**)**C** = **ABC** - (associative law)
3. **A**(**B**+**C**) = **AB** + **AC** - (first distributive law)
4. (**A**+**B**)**C** = **AC** + **BC** - (second distributive law)

5-TRANSPOSE OF A MATRIX

If : $A = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$ Then transpose of A, denoted A^T is:

$$A^T = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$(A \times B)^T = B^T \times A^T$$

$$A = (A^T)^T = A$$

$$(A + B)^T = A^T + B^T$$

$$(kA)^T = k(A)^T$$

6-SYMMETRIC MATRICES

A Square matrix is symmetric if it is equal to its transpose: $\mathbf{A} = \mathbf{A}^T$

i.e. $a_{i,j} = a_{j,i}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

7-Skew Symmetric Matrix

A skew symmetric matrix is a square matrix where $a_{i,j} = -a_{j,i}$ i.e. $(A = -A^T)$

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

8-INVERSE OF A MATRIX

The inverse of a square matrix, A, if it exists, is the unique matrix $\mathbf{A}^{-1}$ where: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$$A^{-1} = \frac{adjA}{|A|}$$

9-DETERMINANT OF A MATRIX

Each square matrix **A** has a unit scalar value called the determinant of **A**, denoted by det **A** or |**A**|

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \qquad |A| = (3)(2) - (1)(1) = 5$$

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

10-ADJOINT MATRICES

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

Example 1:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$\text{C} = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Example 2

$$C=\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} \quad adjA = C^T$$

If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ Find A^{-1}

$$A \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check $\quad \mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example 3

If $A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ Find A^{-1}

$$A^{-1} = \frac{adjA}{|A|}$$

The determinant of **A** is

|**A**| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2

Minor M_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The elements of the cofactor matrix are

$$c_{11} = +(-1), \quad c_{12} = -(-2), \quad c_{13} = +(3),$$
$$c_{21} = -(-1), \quad c_{22} = +(-4), \quad c_{23} = -(7),$$
$$c_{31} = +(-1), \quad c_{32} = -(-2), \quad c_{33} = +(5),$$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

Example 4

$$C=\begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix} \quad adjA = C^T$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}$$

Check inverse

A^{-1} A=I

$$-\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$|A| \neq 0$ ⇒ A matrix possessing an inverse is called Nonsingular or Invertible

ENGINEERING ANALYSIS I

LECTURE 10

Eigenvalues and Eigenvectors

Prepared by: Abdurahman B. AYOUB

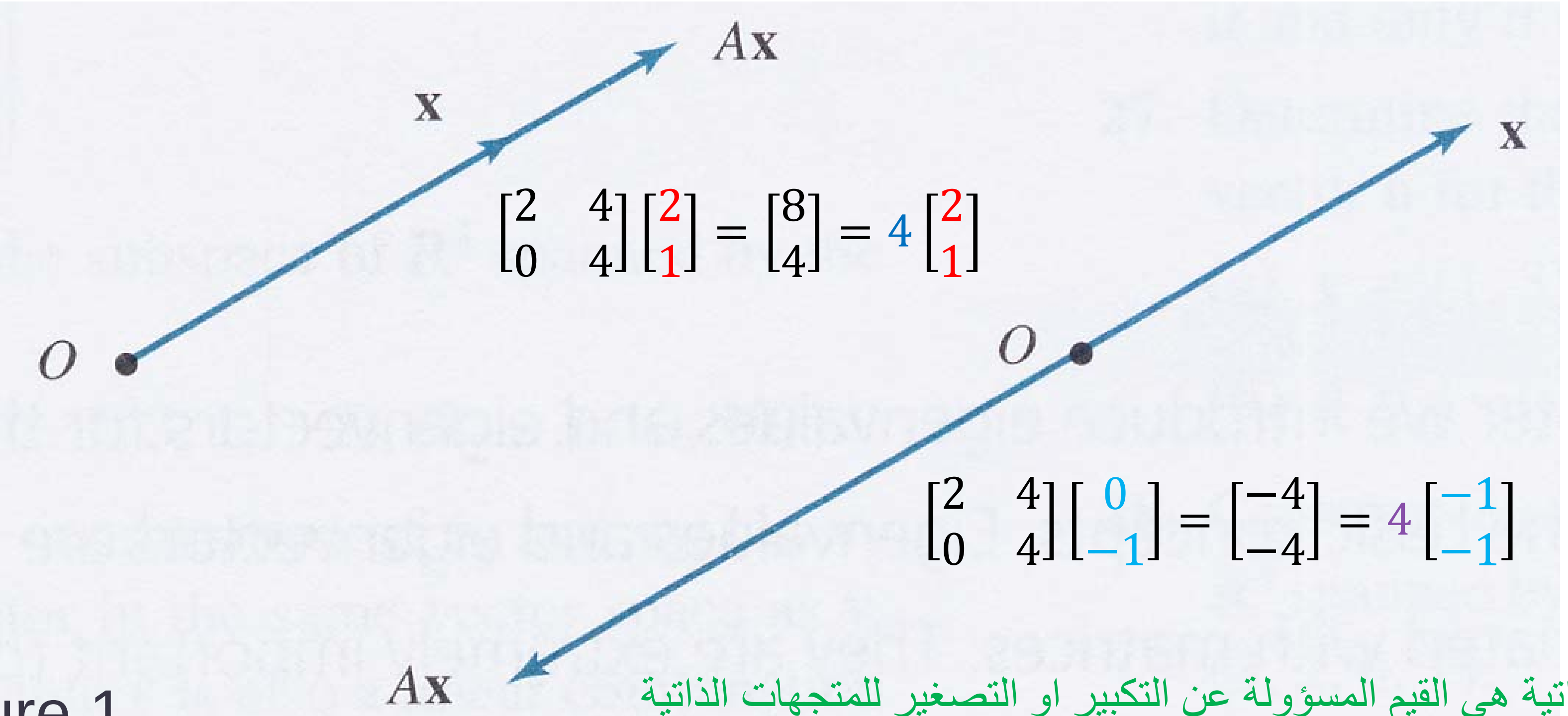
Eigenvalues and Eigenvectors

Definition المتجهات الذاتية والقيم الذاتية

Let A be an $n \times n$ matrix. A <u>*scalar*</u> λ is called an **eigenvalue** of A if there exists a nonzero vector $\mathbf{X}$ in $\mathbf{R}^n$ such that

$$A\mathbf{X} = \lambda\mathbf{X}.$$

The vector $\mathbf{X}$ is called an **eigenvector** corresponding to λ.



$$\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = 4\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Figure 1

القيم الذاتية هي القيم المسؤولة عن التكبير او التصغير للمتجهات الذاتية

Computation of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{X}$. Thus $A\mathbf{X} = \lambda\mathbf{X}$. This equation may be written

$$A\mathbf{X} - \lambda\mathbf{X} = \mathbf{0}$$

given

$$(A - \lambda I_n)\mathbf{X} = \mathbf{0} \quad \text{------(1)}$$

Solving the equation $|A - \lambda I_n| = 0$ -------(2) for λ leads to all the eigenvalues of A.

On expending the determinant $|A - \lambda I_n|$, we get a polynomial in λ. This polynomial is called the **characteristic polynomial** of A.

The equation $|A - \lambda I_n| = 0$ is called the **characteristic equation** of A.

Example 1:

Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$.

Let us first derive the characteristic polynomial of *A*. $|A - \lambda I_2| = 0$

$$A - \lambda I_2 = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{pmatrix}.$$

$$|A - \lambda I_2| = (-3-\lambda)(1-\lambda) - (2)(-2) = \lambda^2 + 2\lambda + 1$$

$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$ A **has a repeated Eigen value:** $\lambda 1 = \lambda 2 = -1$

Now we have to solve the system $[A - \lambda I_n]X = 0$ **Here** $\lambda 1 = -1$ **, so that,**

$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \Rightarrow \begin{matrix} -2x_1 + 2x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{matrix}, \Rightarrow x_1 = x_2.$$

Let $x_1 = \alpha$ $\therefore$ $x_2 = \alpha$

A vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ **is therefore an eigenvector associated with the Eigen value** $\lambda 1 = -1$

$$X = \alpha\begin{pmatrix}1\\1\end{pmatrix}$$ **where α is a real number. Normalized**

$$X = \frac{\alpha\begin{bmatrix}1\\1\end{bmatrix}}{\sqrt{\alpha^2 + \alpha^2}} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

Example 2:

IF $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ Find the Eigen values and the Eigen vector of the matrix A .

$|A - \lambda I_n| = 0$ -------(2)

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\{(1-\lambda)(2-\lambda)-1\} - (1-\lambda)$$

$$= \lambda^3 - 4\lambda^2 + 3\lambda$$

$$= \lambda(\lambda - 1)(\lambda - 3)$$

$(A-\lambda I_n)\mathbf{X} = \mathbf{0}$ ------(1)

When $\lambda = 0$ $\Rightarrow$ $[A-\lambda I_n]X = 0$ $\Rightarrow$ $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_2 = x_3, x_1 = x_2 = x_3$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k \neq 0 \Rightarrow \hat{X}_1 = \frac{k * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{(k)^2 + (k)^2 + (k)^2}} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}}$$

When $\lambda = 1$ $\Rightarrow$ $[A-\lambda I_n]X = 0$ $\Rightarrow$ $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_2 = 0, x_1 = x_2 - x_3 = -x_3$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, k \neq 0 \Rightarrow \hat{X}_2 = \frac{k * \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{(-k)^2 + 0^2 + k^2}} = \frac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{2}}$$

When $\lambda = 3 \Rightarrow [A - \lambda I_n]X = 0 \Rightarrow \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\left[\begin{array}{ccc|} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|} -1 & -1 & -1 \\ -2 & -1 & 0 \\ 0 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|} -1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|} -1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$x_2 = -2x_3, x_1 = x_2 + x_3 = -x_3$$

$$X_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, k \neq 0 \Rightarrow \hat{X}_3 = \frac{k * \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}{\sqrt{(-k)^2 + (-2k)^2 + k^2}} = \frac{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}}{\sqrt{6}}$$

Example 3: Find the eigenvalues and eigenvectors of the matrix

$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \Rightarrow A - \lambda I_2 = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix} \Rightarrow |A - \lambda I_n| = 0$

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = 2 \text{ or } -1$$

For $\underline{\lambda = 2}$ $\Rightarrow$ $(A - 2I_2)\mathbf{X} = \mathbf{0}$ $\Rightarrow$ $\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$ $\Rightarrow$ $\begin{matrix} -6x_1 - 6x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{matrix}$

$x_1 = -x_2$ $\Rightarrow$ Let x_2=1 $\Rightarrow$ x_1=-1

For $\lambda = -1$ $\Rightarrow$ $\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$ $\Rightarrow$ $\begin{matrix} -3x_1 - 6x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{matrix}$ $\Rightarrow$ $x_1 = -2x_2$

Let x_1=-2

x_2=1

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$X_1 = \frac{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}{\sqrt{(-1)^2 + 1^2}} = \frac{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}{\sqrt{2}} \qquad X_2 = \frac{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}{\sqrt{(-2)^2 + 1^2}} = \frac{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}{\sqrt{5}}$$

Example 4:

Find the eigenvalues of matrix A = $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix}$

First the eigenvalues: $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 1-\lambda & -3 \\ 3 & -3 & -3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)(-3-\lambda)-9] - [(-3-\lambda)+9] + 3[-3-3(1-\lambda)] = 0$$

$$= (1-\lambda)[(-3+2\lambda+\lambda^2-9] - [6-\lambda] + [-9-9+9\lambda]$$

$$= (-12+2\lambda+\lambda^2) + (12\lambda - 2\lambda^2 - \lambda^3) + (-24+10\lambda)$$

$$= -\lambda^3 - \lambda^2 + 24\lambda - 36 = \lambda^3 + \lambda^2 - 24\lambda + 36 = 0$$

$\lambda = 2$	1	1	-24	36
2		2	6	-36
	1	3	-18	0

$$(\lambda-2)(\lambda^2+3\lambda-18) = (\lambda-2)(\lambda-3)(\lambda+6)$$

$$\lambda_1 = 2 \quad , \lambda_2 = 3 \quad , \lambda_3 = -6$$

Property 1: Sum of eigenvalues

For any square matrix A:

sum of eigenvalues = sum of diagonal terms of A (called the trace of A)

Formally, for an n × n matrix A: $\text{trace }(A) = \sum_{i=1}^{n} \lambda_i$

Property 2: Product of eigenvalues

For any square matrix A:

product of eigenvalues = determinant of A , $\det(A) = \lambda_1 \lambda_2 \lambda_3 \ldots \ldots \ldots \lambda_n$

$$A = \begin{bmatrix} 7 & 4 & 6 \\ -3 & -1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = 2$

trace (A) = 7-1+2= 8

trace (A) = 5+1+2= 8

$\det(A) = 5 * 1 * 2 = 10$

$\det(A) = 10$

H.W.

1-Find the eigenvalues and eigenvectors of the matrix $B = \begin{bmatrix} 7 & 4 & 6 \\ -3 & -1 & -8 \\ 0 & 0 & 1 \end{bmatrix}$.

2-Given that 2 is an eigenvalue , Find the eigenvalues of matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Example 5. Given that 2 is an eigenvalue for $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find a basis of its Eigen space.

$$A - 2I = \begin{bmatrix} 4-2 & -1 & 6 \\ 2 & 1-2 & 6 \\ 2 & -1 & 8-2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$2x_1 - x_2 + 6x_3 = 0, \text{ or } x_2 = 2x_1 + 6x_3,$$

where we select x1 and x3 as free variables only to avoid fractions. Solution set in parametric form is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 + 6x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

A basis for the Eigen space:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}.$$

ENGINEERING ANALYSIS I

LECTURE 11

Cayley Hamilton Theorem & Diagonalization of Matrices

Prepared by: Abdurahman B. AYOUB

The Cayley Hamilton Theorem

The Cayley Hamilton theorem is one of the most powerful results in linear algebra. This theorem basically gives a relation between a square matrix and its characteristic polynomial. One important application of this theorem is to find inverse and higher powers of matrices.

Theorem: Every square matrix satisfies its characteristic equation.

Char. Equation: $f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$

By Caley-Hamilton Theorem $f(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = 0$

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ $|A - \lambda I_n| = 0$, Characteristic equation is $\lambda^2 - 4\lambda - 5 = 0$.

$A^2 - 4A - 5I = 0$

One can check that $A^2 = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$, $4A = \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix}$. So

$$A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

Example 1

Find inverse of the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ applying Cayley- Hamilton theorem.

$$\det(A - \lambda I) = -\lambda^3 + \lambda^2 + \lambda - 1 = 0.$$

By the Cayley-Hamilton theorem

$$-A^3 + A^2 + A - I = 0$$

Or $$-A^3 + A^2 + A = I$$

$$A^{-1} = -A^2 + A + I$$

or $$A^{-1} = -A^2 + A + I = -\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Computation of powers of A

Example 2 Use the Cayley-Hamilton theorem to find $\mathbf{M}^6$ if $\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

$$\mathbf{M} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix} \Rightarrow \det(M - \lambda I) = 0$$

Characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$

$\mathbf{M}^2 - 5\mathbf{M} + 6\mathbf{I} = 0$

$$\Rightarrow \quad \mathbf{M}^2 = 5\mathbf{M} - 6\mathbf{I}$$

$$\Rightarrow \quad \mathbf{M}^4 = (5\mathbf{M} - 6\mathbf{I})^2$$

$$= 25\mathbf{M}^2 - 60\mathbf{M} + 36\mathbf{I}$$

$$= 25(5\mathbf{M} - 6\mathbf{I}) - 60\mathbf{M} + 36\mathbf{I}$$

$$= 65\mathbf{M} - 114\mathbf{I}$$

$$\mathrm{M}^6 = \mathrm{M}^4 \times \mathrm{M}^2$$

$$= (65\mathbf{M} - 114\mathbf{I})(5\mathbf{M} - 6\mathbf{I})$$

$$= 325\mathbf{M}^2 - 960\mathbf{M} + 684\mathbf{I}$$

$$= 325(5\mathbf{M} - 6\mathbf{I}) - 960\mathbf{M} + 684\mathbf{I}$$

$$= 665\mathbf{M} - 1266\mathbf{I}$$

$$= 665\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} - 1266\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -601 & 665 \\ -1330 & 1394 \end{bmatrix}$$

Example 2:

Compute e^A for $A = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$ using the exponential series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots.$$ **Scalar polynomial**

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots\ldots.$$ **Matrix polynomial**

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & -4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (-2-\lambda)(2-\lambda) + 4 = 0$$

$$-4 + 2\lambda - 2\lambda + \lambda^2 + 4 = 0 \Rightarrow \lambda^2 = 0 \Rightarrow A^2 = 0$$

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^m = 0 \text{ if } m > 1$$

$$e^A = I + A = \begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$$

Diagonalization of Matrices

Definition

A *square matrix* A is said to be **diagonalizable** if there exists a matrix P such that $D = P^{-1}AP$ is a diagonal matrix.

Theorem

Let A be an $n \times n$ matrix.

(a) If A has n linearly independent eigenvectors, it is diagonalizable. *The matrix P whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $P^{-1}AP$ to give a diagonal matrix D. The diagonal elements of D will be the eigenvalues of A.*

(b) If A is diagonalizable, then it has n linearly independent eigenvectors

Example 1:

Show that the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable.

$|A - \lambda I_n| = 0$

$$\lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = 2 \text{ or } -1$$

For $\underline{\lambda = 2}$ ⇨ $(A - 2I_2)\mathbf{X} = \mathbf{0}$ ⇨ $\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$ ⇨ $\begin{aligned} -6x_1 - 6x_2 &= 0 \\ 3x_1 + 3x_2 &= 0 \end{aligned}$

$x_1 = -x_2$ ⇨ Let x_2=1 ⇨ x_1=-1

For $\lambda = -1$ ⇨ $\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$ ⇨ $\begin{aligned} -3x_1 - 6x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned}$ ⇨ $x_1 = -2x_2$

Let x_1=-2

x_2=1

$$X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$P = [X_1 \quad X_2] = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{adj(P)}{\det(p)} = \frac{\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}}{1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D$$

Example 2

Diagonalize $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, if possible

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = -\lambda^3 - 3\lambda^2 + 4$$

	-1	-3	0	4
1	↓	-1	-4	-4
	-1	-4	-4	0

$$(\lambda - 1)(-\lambda^2 - 4\lambda - 4) = -(\lambda - 1)(\lambda^2 + 4\lambda + 4)$$

$$= -(\lambda - 1)(\lambda + 2)^2$$

When $\lambda = 1 \Rightarrow [A - \lambda I_n]X = 0$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 3 & 3 & 0 \\ -3 & -6 & -3 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = -x_3, x_1 = -x_2 = x_3 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, k \neq 0$$

When $\lambda = -2$

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$x_3 = t, x_2 = s, x_1 = -s - t$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, t^2 + s^2 \neq 0$$

$$D = P^{-1} A P$$

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{adj(P)}{\det(p)}$$

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Homework

1. Let $\mathrm{A} = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$

a-Use the Eigen values method to compute $A^4 = P\begin{bmatrix} \lambda_1^4 & 0 \\ 0 & \lambda_2^4 \end{bmatrix}P^{-1}$, where P is the Eigen vectors of A.

b-Compute e^{A} and cos A

2. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 6 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$, Compute $\sin A$ *and* A^{-2}

3. If $B = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$, Show that $B^{-1} = \frac{1}{12}[B^2 - 8B + 19I]$

ملاحظة

$$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \ldots\ldots\ldots$$

$$\cos A = I - \frac{A^2}{2!} + \frac{A^4}{4!} - + \ldots\ldots\ldots$$

$$\sin A = A - \frac{A^3}{3!} + \frac{A5}{5!} - + \ldots\ldots\ldots$$

ENGINEERING ANALYSIS I

LECTURE 12

Double Integrals
(Cartesian Coordinates)

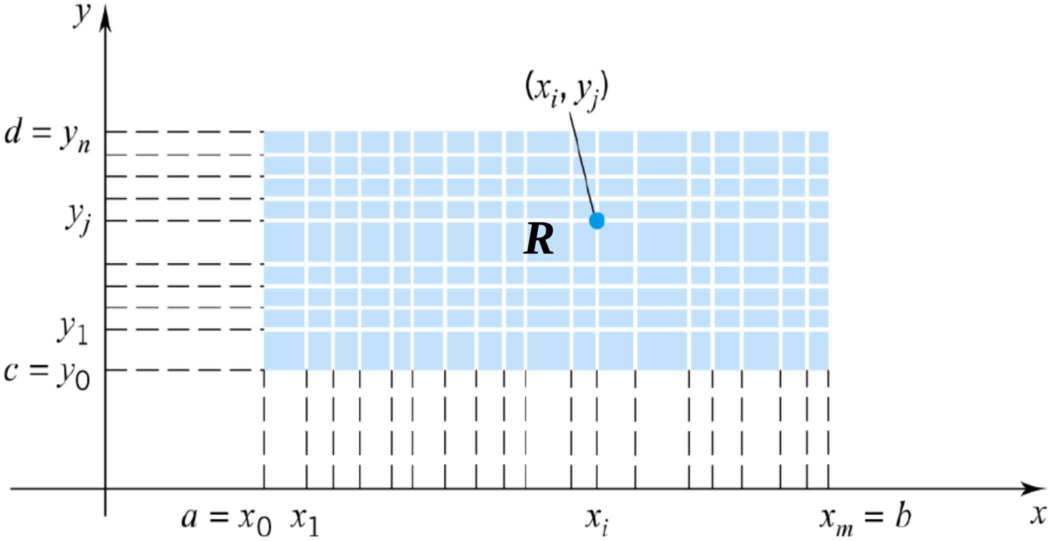
Prepared by: Abdurahman B. AYOUB

DOUBLE INTEGRAL

DEFINITION

Let ***f(x,y)*** be a function of two variables defined on a closed region ***R* .** Then the double integral of ***f*** over ***R*** is given by

$$\iint f(x,y)dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$



When ***f(x, y)*** **=** 1 on ***R*** then

$$\lim_{n\to\infty}\sum_{i=1}^{n}\Delta A_i$$ gives the area A

$$\boldsymbol{A} = \iint_R dA$$

when **z = f(x,y)** represents a surface the then the volume *V* of the solid above the region R and below the surface **z = f(x,y)** is given by*:*

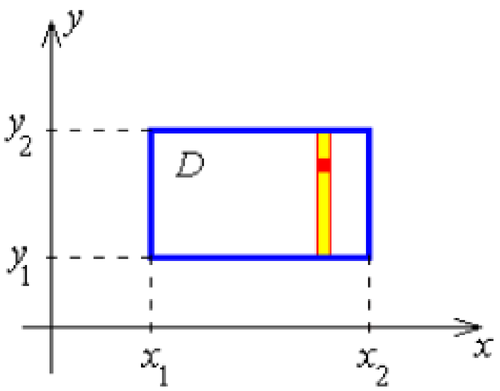
$$\boldsymbol{V} = \iint_R f(x,y)\, dA$$

Double Integrals (Cartesian Coordinates)

First case: The integration limits are constants

Example 1

Evaluate $\int_{-1}^{1}\int_{0}^{3}(x^2+y^2)\,dy\,dx$



$$I = \int_{-1}^{1}\int_{0}^{3}(x^2+y^2)\,dy\,dx = \int_{-1}^{1}\left(\int_{0}^{3} x^2+y^2\,dy\right)dx$$

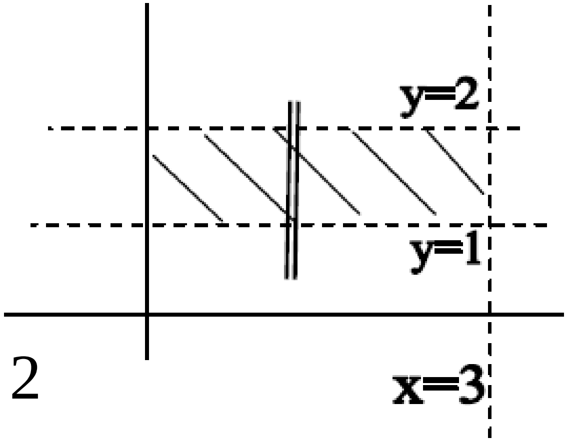
$$\mathbf{I} = \iint_R f(x,y)\,dA$$

$$= \int_{-1}^{1} x^2y+\frac{y^3}{3}\Bigg|_{0}^{3} dx = \int_{-1}^{1} 3x^2+9\,dx$$

$$= 3\frac{x^3}{3}+9x\Bigg|_{-1}^{1} = 20$$

Example 2:

Evaluate $\int_0^3 \int_1^2 (1 + 8xy)\,dy\,dx$



sketch: since dydx ⇒ vertical y =1 , y = 2

$$\int_0^3 \int_1^2 (1 + 8xy)\,dy\,dx = \int_0^3 \left(y + 8x\frac{y^2}{2}\right)\Big|_1^2 dx$$

$$= \int_0^3 \{[2 + 4x(4)] - [1 + 4x(1)]\}\,dx$$

$$= \int_0^3 \{[2 + 16x] - [1 + 4x]\}\,dx$$

$$= \int_0^3 \{1 + 12x\}\,dx$$

$$= \left(x + 12\frac{x^2}{2}\right)\Big|_0^3$$

$$= (3 + 6(9)) - (0) = (3 + 54) = \mathbf{57}$$

Second case: The integration limits are variables

1.If R is region of type one

Taking a vertical lamina means that we will integrate first with respect to y and in this case the integration limits will be a function of x , then integrate the result with respect to y which will be defined through a constant limits

$$I = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)\, dy\, dx$$

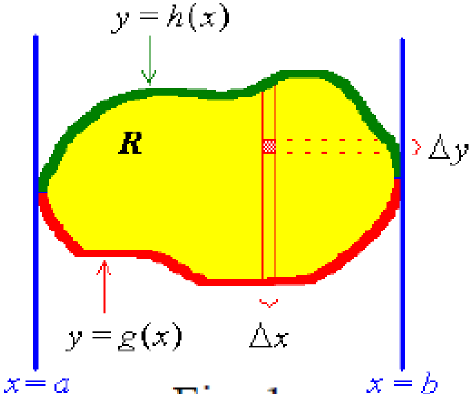


Fig. 1

2. If R is region of type Two

Taking a horizontal lamina means that we will integrate first with respect to x and in this case the integration limits will be a function of y , then integrate the result with respect to x which will be defined through a constant limits

$$I = \int_{y=c}^{y=d} \int_{x=p(y)}^{x=q(y)} f(x,y)\, dx\, dy$$

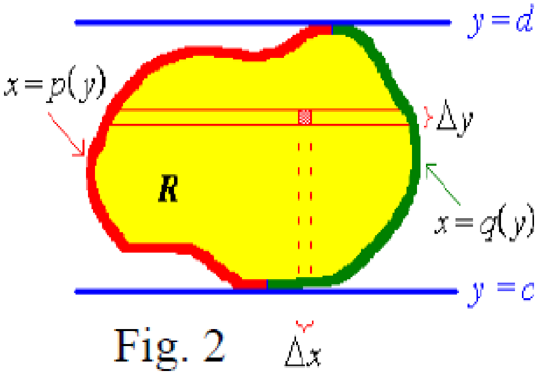


Fig. 2

Example1

Evaluate $\iint_R x^3 y^2 \, dA$ where R is the region bounded by y = x ; y = 1 and x=0

First Solution(figure.1)

$$\iint_R x^3 y^2 \, dA = \int_0^1 \int_{x=0}^{x=y} x^3 y^2 \, dx\,dy$$

$$= \int_0^1 y^2 \left.\frac{x^4}{4}\right|_0^y dy = \frac{1}{4}\int_0^1 y^6 \, dy = \left.\frac{y^7}{28}\right|_0^1 = \frac{1}{28}$$

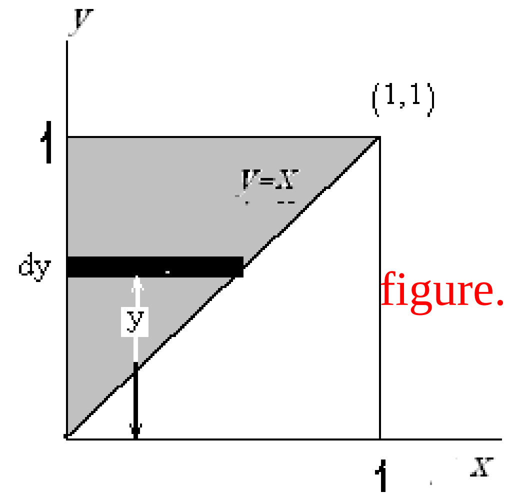


figure.1

Second solution (figure2)

$$\iint_R x^3 y^2 \, dA = \int_0^1 \int_x^1 x^3 y^2 \, dy\,dx = \int_0^1 x^3 \left.\frac{y^3}{3}\right|_x^1 dx$$

$$= \frac{1}{3}\int_0^1 x^3 (1 - x^3)\, dx = \frac{1}{3}\left(\frac{x^4}{4} - \frac{x^7}{7}\right)\Bigg|_0^1 = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right) = \frac{1}{28}$$

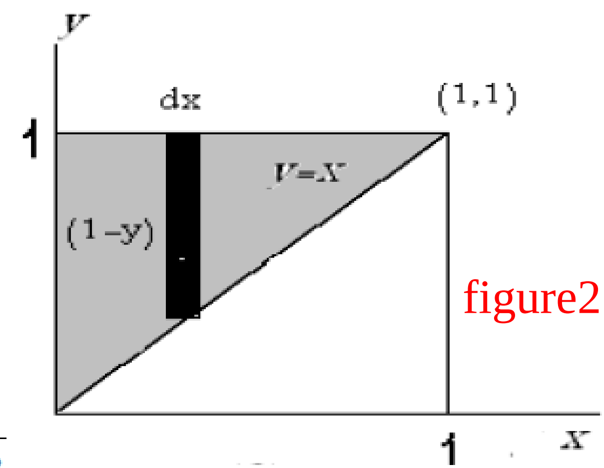
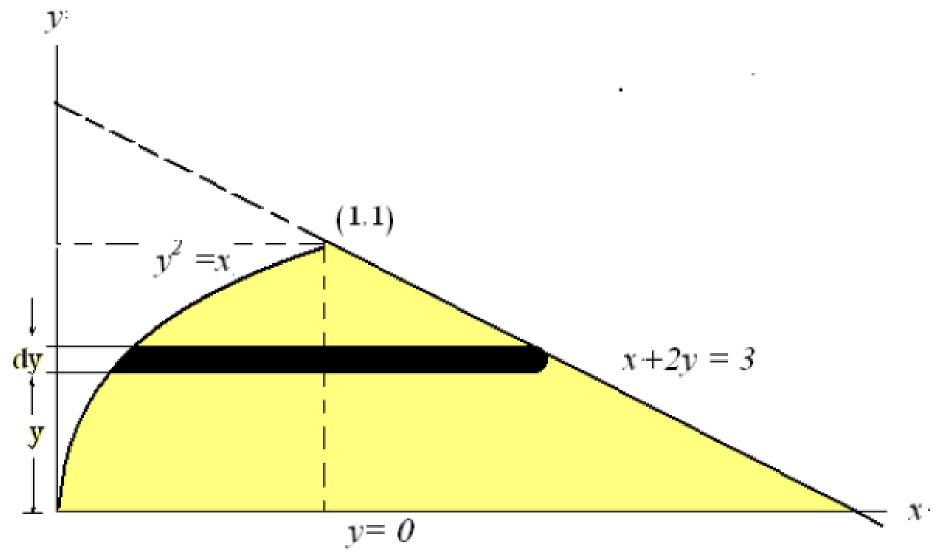


figure2

Example2

Evaluate $\iint_R x + y\, dxdy$ where R is the region bounded by $y^2 = x$; $x+2y = 3$ and y= 0 in the first quadrant



$$\iint_R x + y\, dxdy =$$

$$\int_0^1 \int_{y^2}^{3-2y} (x+y)dxdy = \int_0^1 \frac{x^2}{2} + yx \Big|_{y^2}^{3-2y} dy$$

$$= \int_0^1 \left\{ \frac{(3-2y)^2 - (y^2)^2}{2} + y\left[(3-2y) - y^2\right] \right\} dy$$

$$= \int_0^1 \left(\frac{9}{2} - 3y - y^3 - y^4\right) dy = 2.55$$

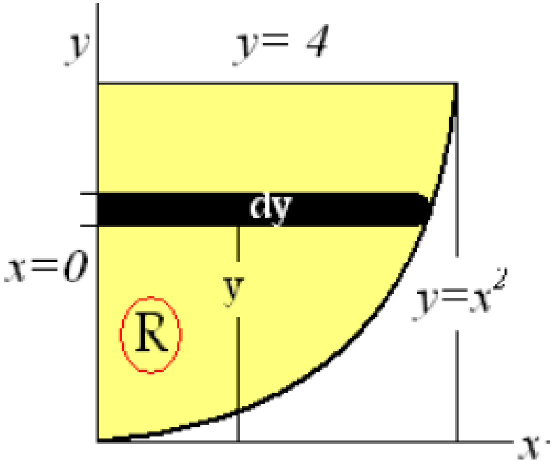
Example3

Evaluate $\iint_R x\, e^{y^2}\, dA$, where R is the region bounded by $y=x^2$; x=0 and y= 4

$$\iint_R x\, e^{y^2}\, dA = \int_0^4 \int_0^{\sqrt{y}} x\, e^{y^2}\, dx\,dy$$

$$= \int_0^4 \frac{x^2}{2}\Big|_0^{\sqrt{y}} e^{y^2}\, dy = \int_0^4 \frac{y}{2} e^{y^2}\, dy$$

$$= \frac{1}{4} e^{y^2}\Big|_0^4 = \frac{1}{4}(e^{16} - 1)$$



Reversing the order of Integration

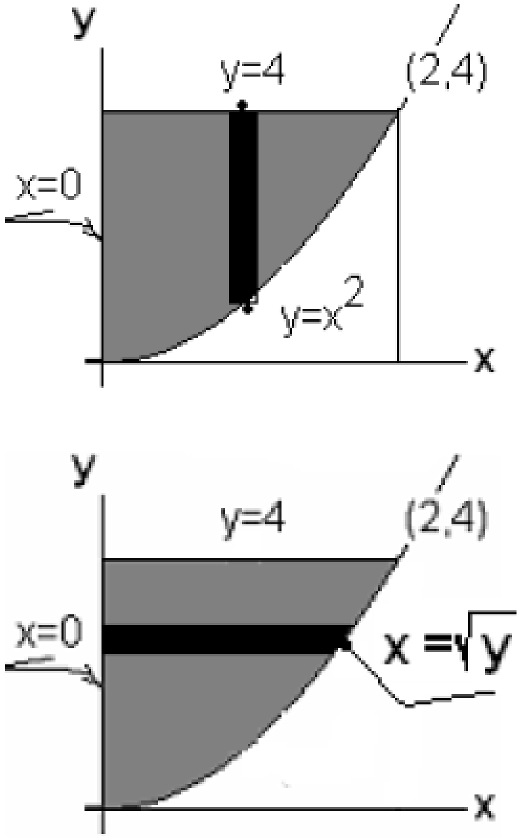
A problem may become easier when the order of integration is reversed or changed. Which means some integrals may be impossible to be evaluated with respect to one of the variables but can be done with respect to the other one

Example1

Evaluate $\int_0^2 \int_{x^2}^4 x\, e^{y^2}\, dA$ $= \int_0^2 \int_{x^2}^4 x\, e^{y^2}\, dy dx = \int_0^4 \int_0^{\sqrt{y}} x\, e^{y^2}\, dx dy$

$$= \int_0^4 \frac{x^2}{2}\Bigg|_0^{\sqrt{y}} e^{y^2}\, dy = \frac{1}{2}\int_0^4 y e^{y^2}\, dy$$

$$= \frac{1}{4}\int_0^4 e^{y^2}\, dy^2 = \frac{1}{4} e^{y^2}\Big|_0^4 = \frac{e^{16}-1}{4}$$

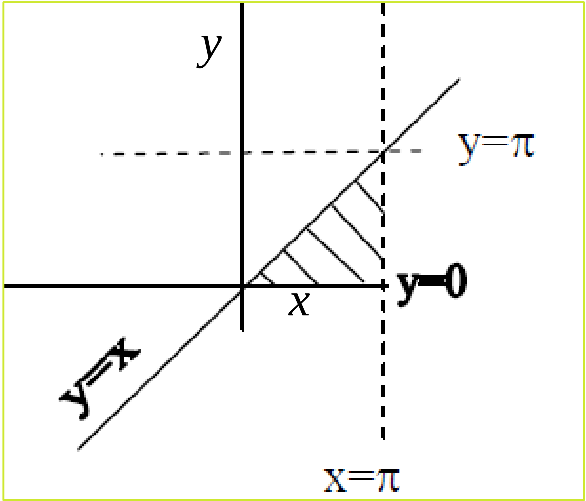


Example2

Evaluate $\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x} \, dx \, dy$

reverse the order

$$\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x} \, dx \, dy = \int_0^{\pi}\int_0^{x} \frac{\sin x}{x} \, dy \, dx$$



$$= \int_0^{\pi} \frac{\sin x}{x} \cdot y \Big|_0^x \, dx \quad = \int_0^{\pi} \frac{\sin x}{x} \cdot x \, dx$$

$$= \int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = -(-1-1) = \mathbf{2}$$

ENGINEERING ANALYSIS I

LECTURE 13

Double Integrals in Polar Coordinates

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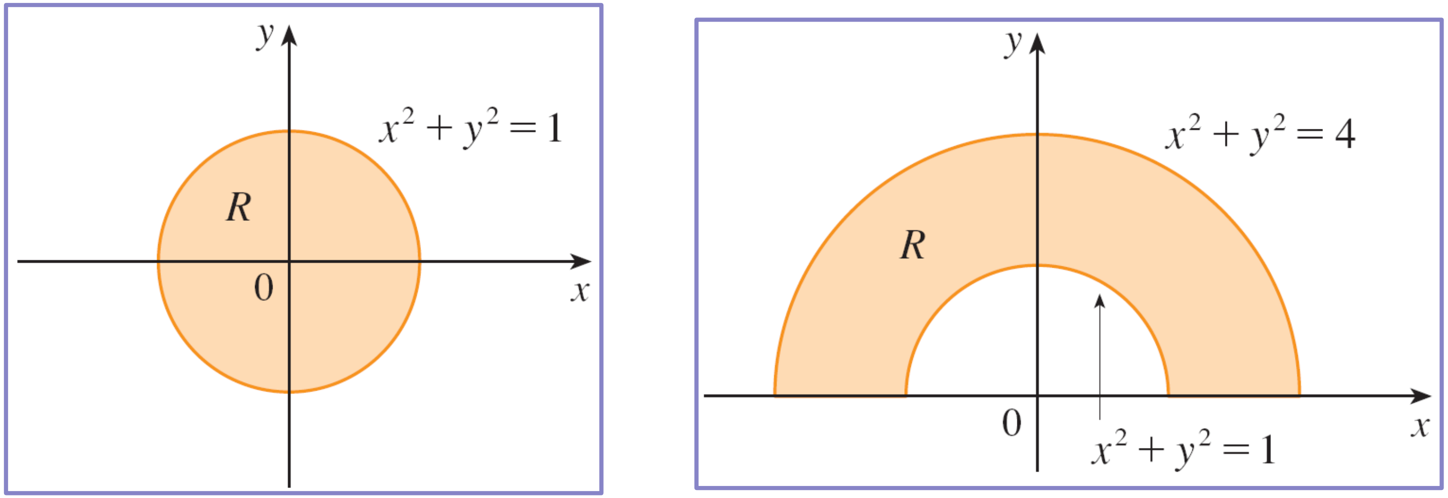
Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.



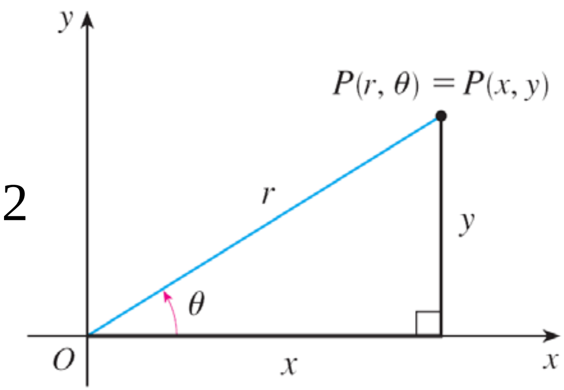
(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Figure 1

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

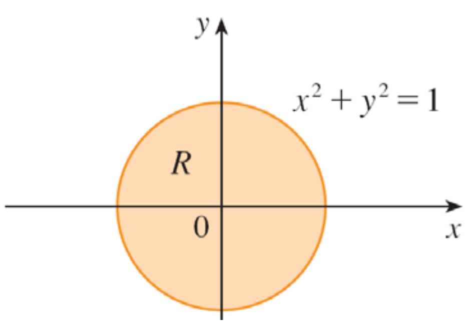
Figure 2



$$\theta = \tan^{-1}(y/x)$$

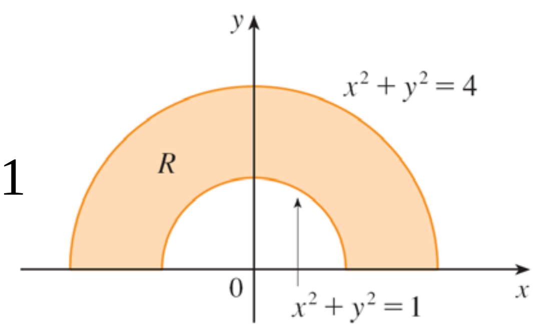
The regions in Figure 1 are special cases of a polar rectangle

$R = \{ (r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta \}$ which is shown in Figure 3.



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

Figure 1



(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Polar rectangle

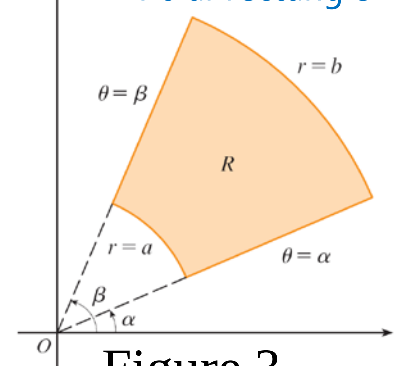


Figure 3

So changing to polar coordinates will transfer the segment area *dA (dx dy)* to another area in polar plane given by J dr dθ where J in the Jacobian and it is equal to r in the case of changing from Cartesian to Polar, so

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy$$

$$= \iint_D f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

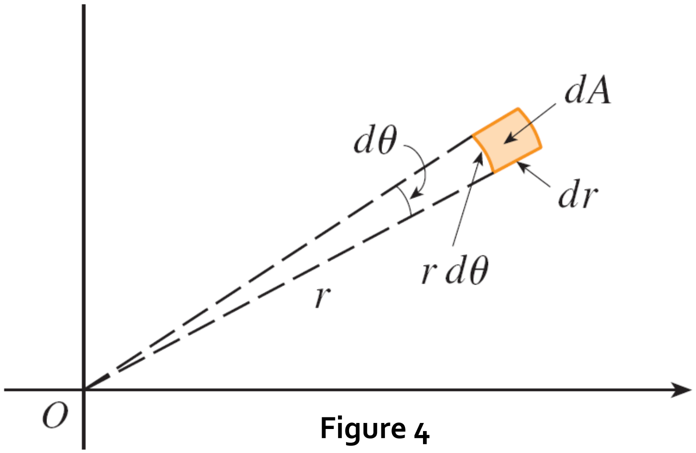


Figure 4

In general, in plane polar coordinates,

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

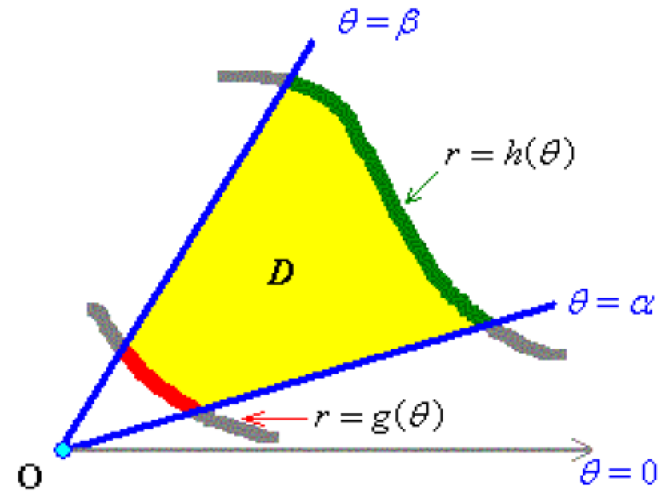


Figure 5

Example1

Evaluate $\int_1^2 \int_0^x \frac{1}{(x^2+y^2)^{3/2}}\, dy\, dx$

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/4} \int_{\sec\theta}^{2\sec\theta} \frac{1}{r^3} r\, dr\, d\theta$$

$$= \int_0^{\pi/4} -\frac{1}{r} \Big|_{\sec\theta}^{2\sec\theta} d\theta$$

$$= \int_0^{\pi/4} (\cos\theta - \frac{1}{2}\cos\theta) d\theta = \frac{\sqrt{2}}{4}$$

Example 2

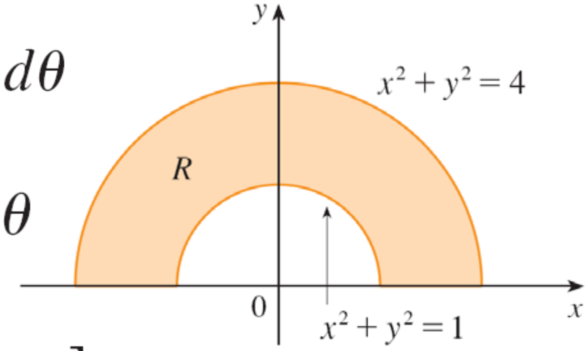
Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\iint_R (3x + 4y^2)\, dA = \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi} \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta$$

$$= \int_0^{\pi} \left[r^3\cos\theta + r^4\sin^2\theta\right]_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\, d\theta$$

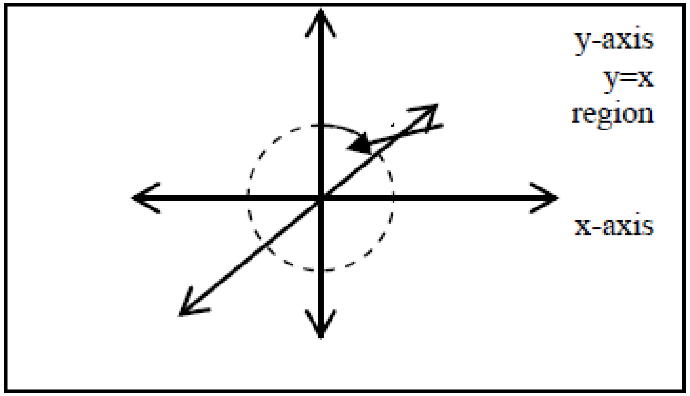


$$= \int_0^{\pi} \left[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta \Big]_0^{\pi} = \frac{15\pi}{2}$$

Example 3

Evaluate $\iint\limits_R \frac{1}{\left(x^2+y^2+1\right)^2}dA$ where R is the region in the first quadrant bounded by



the circle $x^2+y^2=9$ $x=0,\ and\ \ y=x$

$$\iint\limits_R \frac{1}{\left(x^2+y^2+1\right)^2}dA=\int_{\pi/4}^{\pi/2}\int_0^3 \frac{1}{\left(r^2+1\right)^2}rdrd\theta$$

$$=\int_{\pi/4}^{\pi/2}\int_0^3 \left(r^2+1\right)^{-2}rdrd\theta$$

$$=-\frac{1}{2}\int_{\pi/4}^{\pi/2}\left(r^2+1\right)^{-1}\Big|_0^3$$

$$=-\frac{1}{2}\left(\frac{1}{10}-1\right)\int_{\pi/4}^{\pi/2}d\theta=\frac{9}{20}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$=\frac{9\pi}{80}\approx 0.3534291735$$