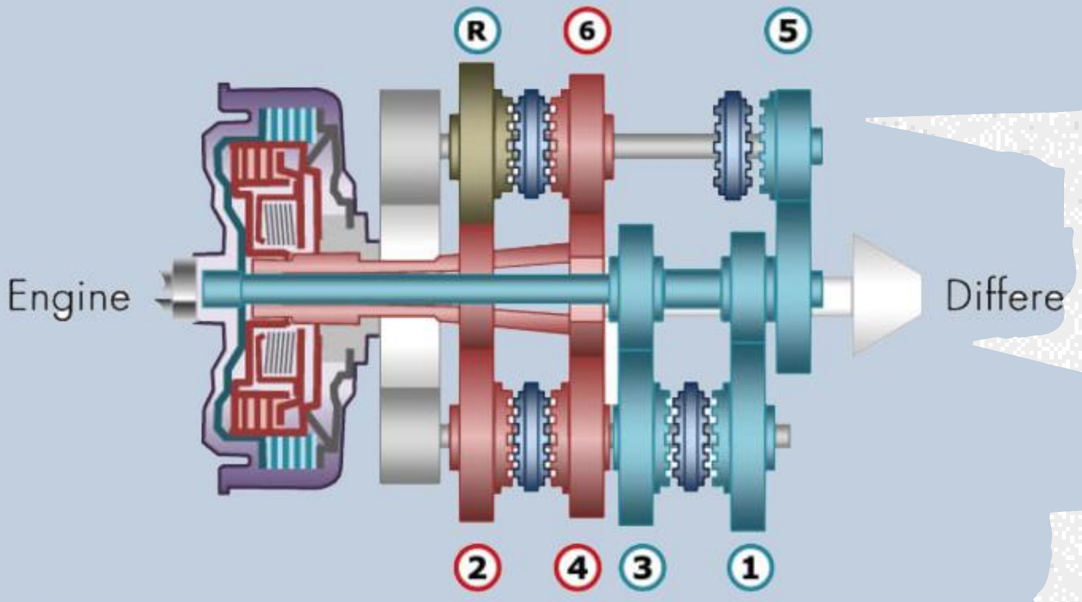
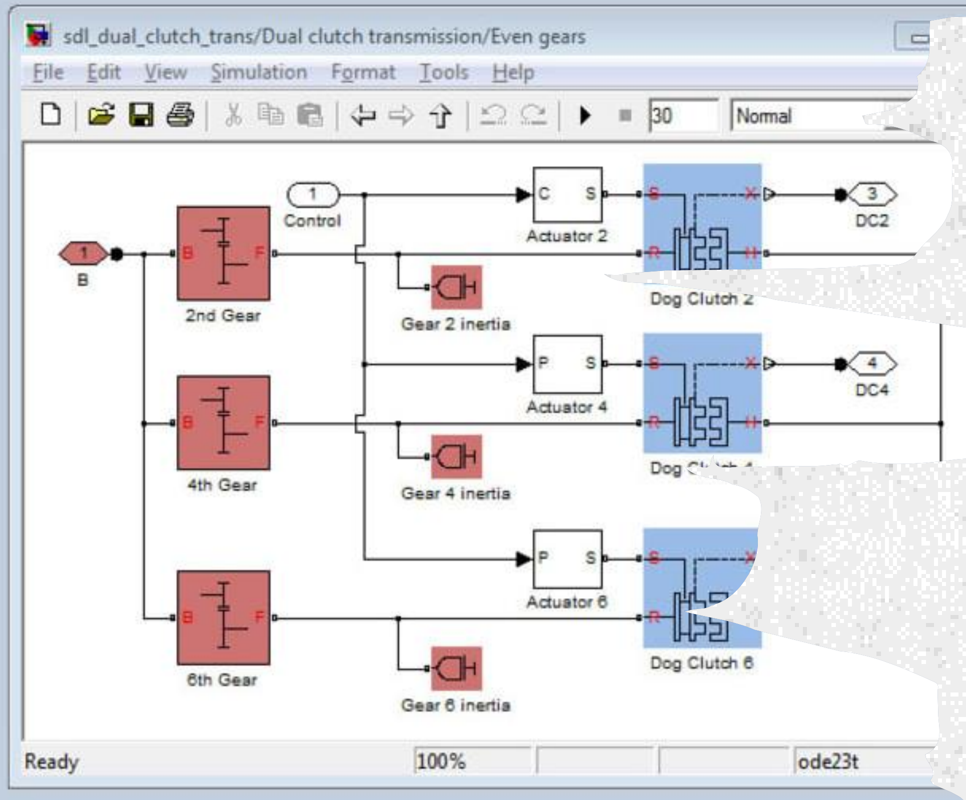




Ninevah University
Electronics Engineering College
Systems and control engineering department

SYSTEM MODELING

Lecture 1

Introduction to system dynamics

Instructor

Mohanad N. Noaman

COURSE DETAILS:

- Description

This course covers three main areas: modelling, simulation, and identification. It presents several modelling methodologies that can be used for control systems. This will cover mathematical and analytical work. Software tools, such as MATLAB/Simulink, may will be used to simulate the systems and analyze the responses. Also, an introduction to system identification will be provided. Modeling definition. Modeling of different physical systems (mechanical, fluid, thermal and electrical). Differential and Laplace equations. State-space representation. Computer simulation techniques (applications using MATLAB Program). System response and analysis.

- Course Learning Outcomes

Upon successful completion of this course, student should:

1. Understand fundamentals of system dynamics.
2. Study the Laplace, inverse Laplace transformation.
3. Obtain a mathematical Model of different physical systems (mechanical, fluid, thermal and electrical).

COURSE OUTLINES:

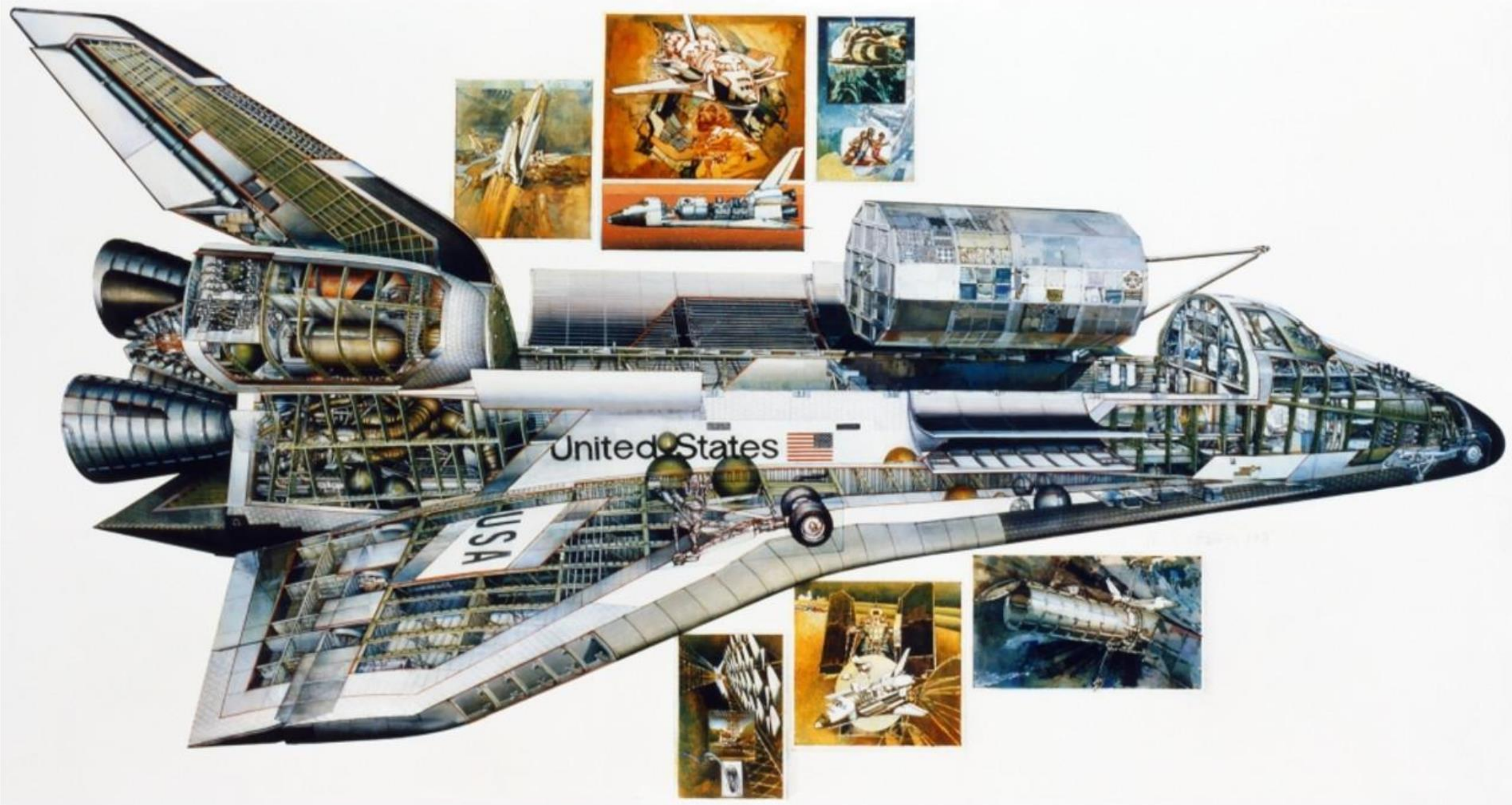
	Weeks
• **Introduction:** Engineering Systems Dynamics; Components and Systems; Principles of Modeling and Simulation; Modelling Categories	1
• **ODE and The Laplace Transform:** Concept; Model Formulation; Transfer Functions and Time Response; SISO and MISO Systems	1
• **Physical systems and Transfer function:** Mechanical, Electrical, Electromechanical, and fluid system.	4
• **State Space Analysis:** state space basics; state variable; state space system representation; electrical state space modeling; state space to transfer function; transfer function to state space form.	2
• **Transfer function to state space and vice versa.**	2
• **Linearization of Nonlinear Systems.**	2

ASSESSMENT GUIDANCE:

- Evaluation of the student performance during the semester (total final mark) will be conducted according to the following activities:

Quizzes	20%
Midterm Exam	10%
Seminar, Homework, Reports	20%
Final Exam	50%
TOTAL	100%

WHY MODELING AND SIMULATION?



NASA Space Shuttle!

WHY MODELING AND SIMULATION?

Nasa wants to build space shuttle will successfully land on MARS !

A single space shuttle* consists of 2.5 million parts including:

- 370 Km Long wires
- 1000 Valves
- 1400 Circuit breakers
- 25000 Insulating tiles

Space shuttle must accelerate from 0 Km/h to 28000 Km/h < 9 minutes !

Space shuttle must survive Temp as low as < -150 C and as high as > 1600 C !

*https://spaceflight.nasa.gov/

WHY MODELING AND SIMULATION?

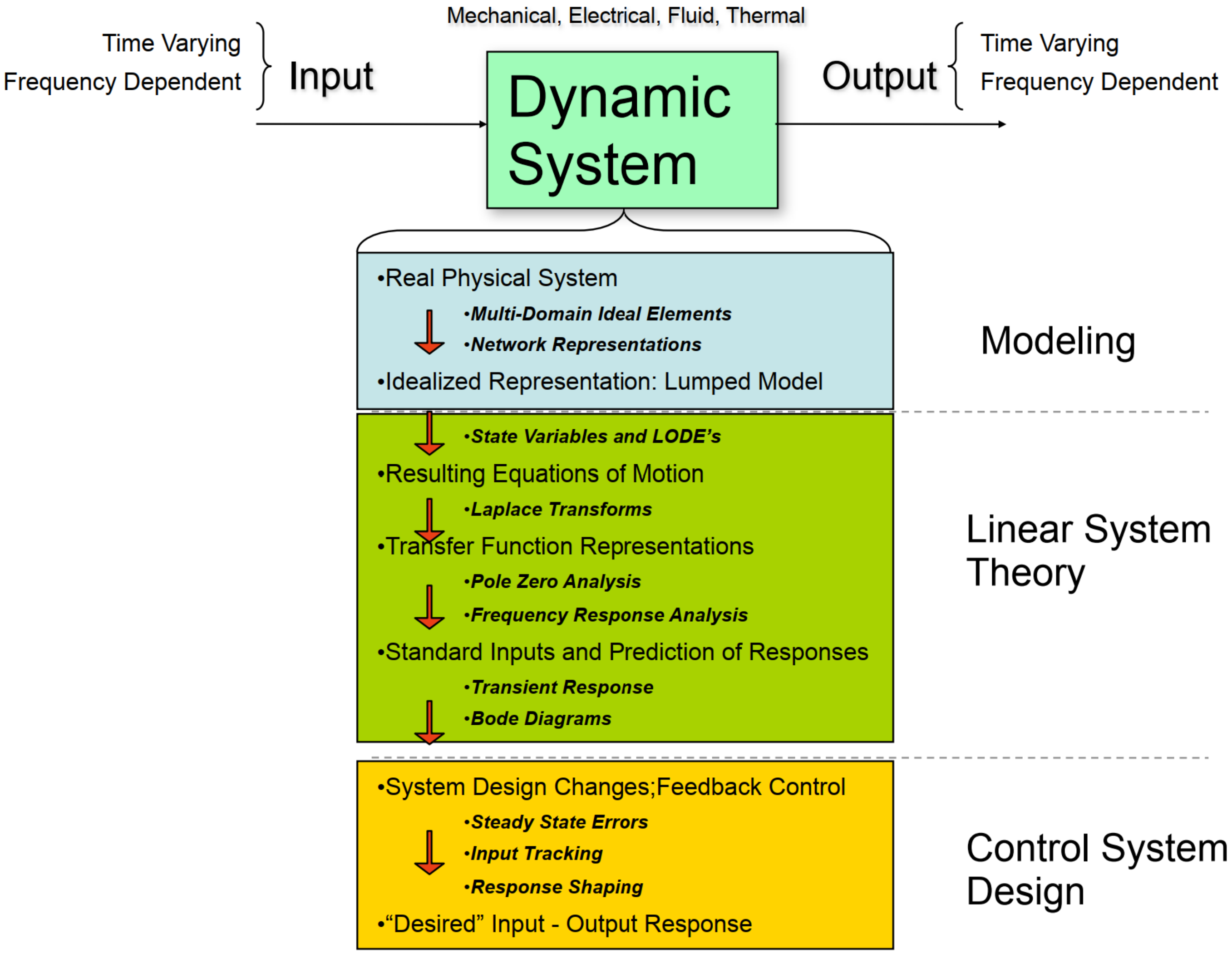
Cannot starting building space shuttle without Design Verification !

- Use Engineering knowledge to describe how each one of 2.5 million components would work (Modeling step) in the proposed design.

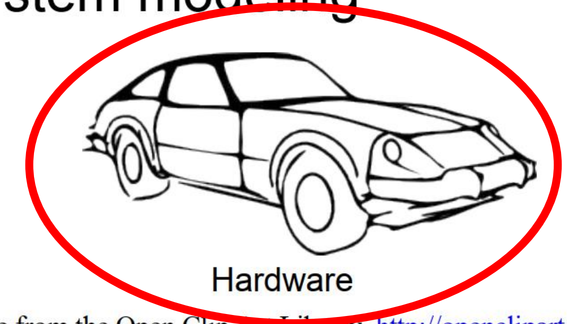
- Write computer program to see if those 2.5 million components would work as required (Simulation Step)

Once modeling and simulation verify the design, Nasa can either proceed to design refinement or building the space shuttle.

FRAMEWORK FOR SYSTEM CONTROL



- System modeling



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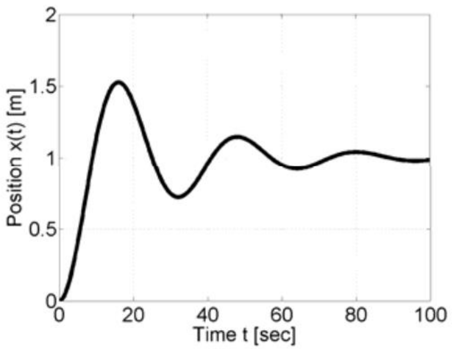
$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

Model: ordinary differential equation (ODE)
or other mathematical representation

- System dynamics

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

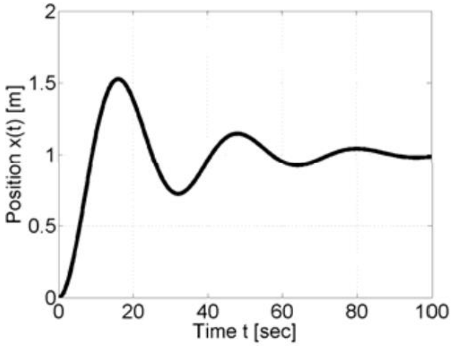
Model

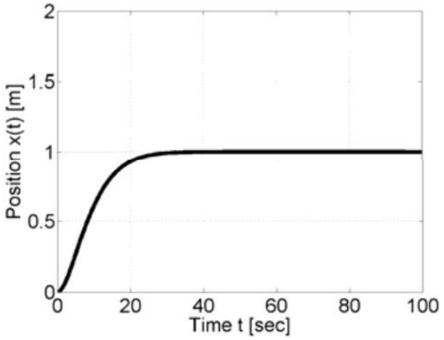


Response

- System control

Response





Desired
response

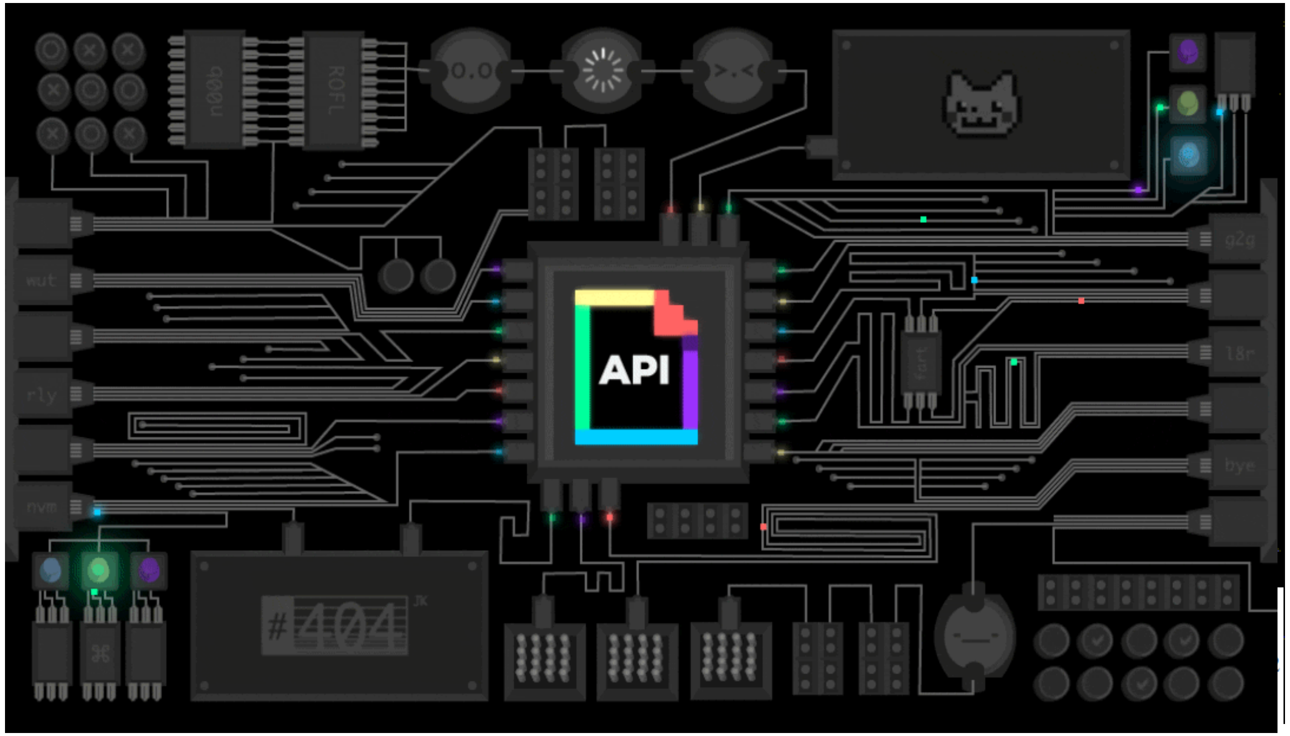
INTRODUCTION: WHAT IS THE SYSTEM?

It is a combination of elements intended to act together to accomplish an objective.

It is any collection of elements for which there are cause-and-effect relationships among the variables.

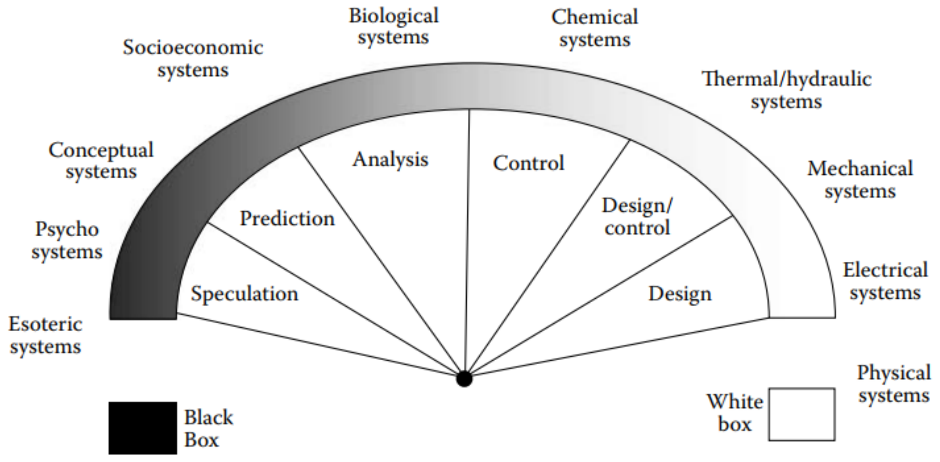
Some example of the systems :

- Medical/biological systems
- Socioeconomic systems
- Communication and information systems
- Planning systems
- Solar system
- Environmental systems
- Manufacturing systems
- Management systems
- Transportation systems
- Physical systems—electrical, mechanical, thermal, hydraulic systems, and combinations of them



INTRODUCTION: TYPES OF SYSTEMS

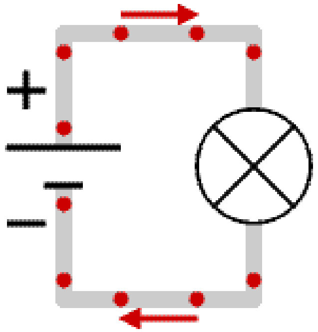
1. **According to the Time Frame**
   - Discrete
   - Continuous
   - Hybrid
2. **According to the Interactions:** Interactions may be unidirectional or bidirectional, crisp or fuzzy, static or dynamic, etc.
3. **According to the Nature and Type of Components**
   - Static or dynamic components
   - Linear or nonlinear components
   - Time-invariant or time-variant components
   - Deterministic or stochastic components
   - Continuous-time and discrete-time systems



4. **According to the Uncertainties Involved**
   - Deterministic—No uncertainty in any variables, for example, model of pendulum.
   - Stochastic - Some variables are random, for example, airplane in flight with random wind gusts

INTRODUCTION: TYPES OF SYSTEMS

1. **Static systems:** is one whose output at any given time depends only on the input at that time. For example, the current flowing through a resistor depends only on the present value of the applied voltage.



Mathematical example of static system:

$$y(t) = x(t)$$
$$y(t) = tx(t) + 2x(t)$$

2. **Dynamic Systems:** is one whose present output depends on past inputs. For example, the present position of a bike depends on what its velocity has been from the start.

Mathematical example of dynamic system:

$$y(t) = x(t + 1)$$
$$y(t) = tx(t) + x(t - 1)$$

INTRODUCTION: EXAMPLE

- **Example:** Check whether the following mathematical representation system is static or dynamic

a. $Y(t) = x(3t)$ *let t=1 so* $Y(1) = x(3)$ *System is Dynamic* $/Y(1)$ *depends on the future input* $x(3)$

b. $Y(t) = 5x(t)$ *let t=1 so* $Y(1) = 5x(1)$ *System is Static* $/Y(1)$ *depends on the present input* $x(1)$

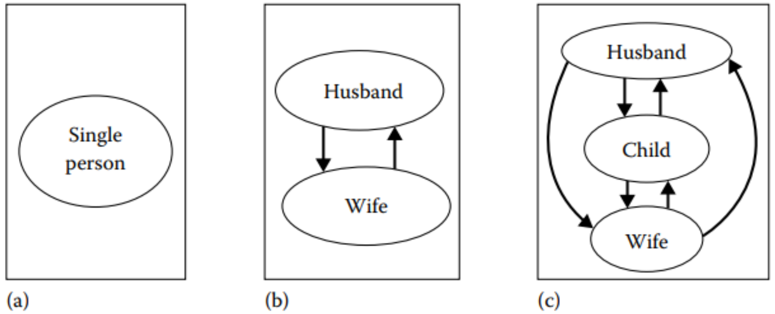
c. $Y(t) = x(-t)$ *let t=1 so* $Y(1) = x(-1)$ *System is Dynamic* $/Y(1)$ *depends on the past input* $x(-1)$

d. $Y(t) = x(cost)$ *let t=0 so* $Y(0) = x(cos0)$ *System is Dynamic* $/Y(0)$ *depends on the future input* $x(1)$

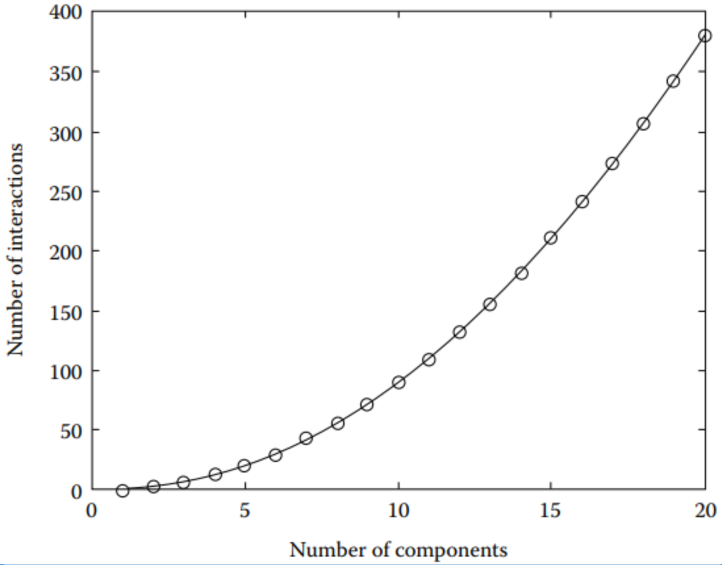
INTRODUCTION: COMPLEXITY OF SYSTEMS

Another basic issue is the complexity of a system. Complexity of a system depends on the following factors:

1. Number of interconnected components
2. Type/nature of component
3. Number of interactions
4. Strength of the interaction
5. Type/nature of interactions
    A. Static or dynamic
    B. Unidirectional or bidirectional
    C. Constrained or non-constraint interaction

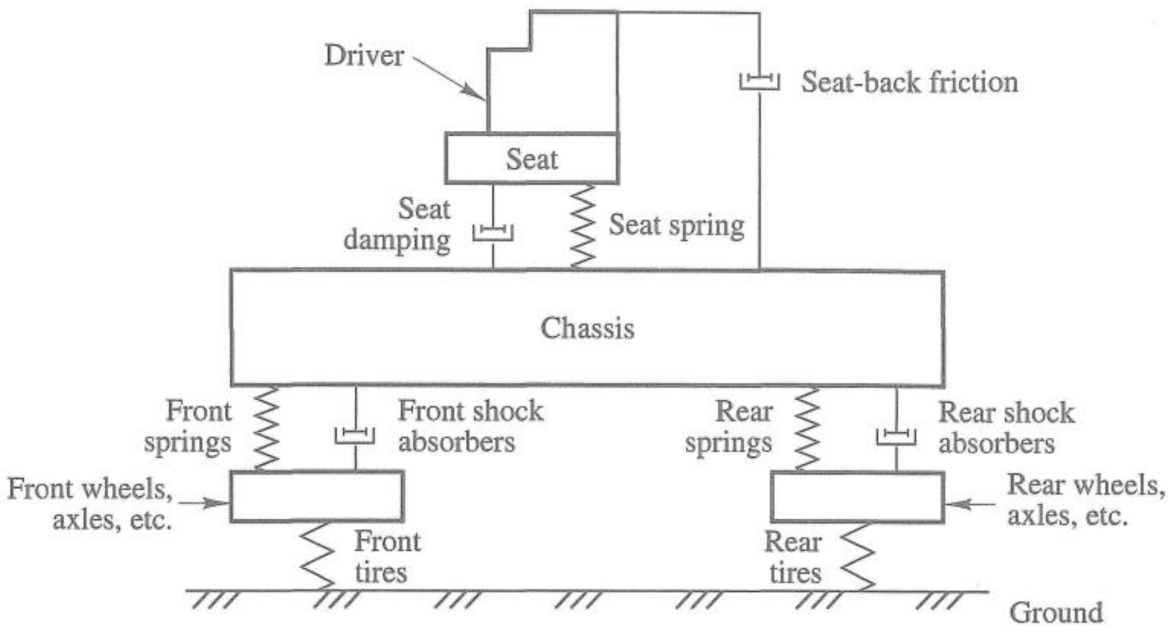


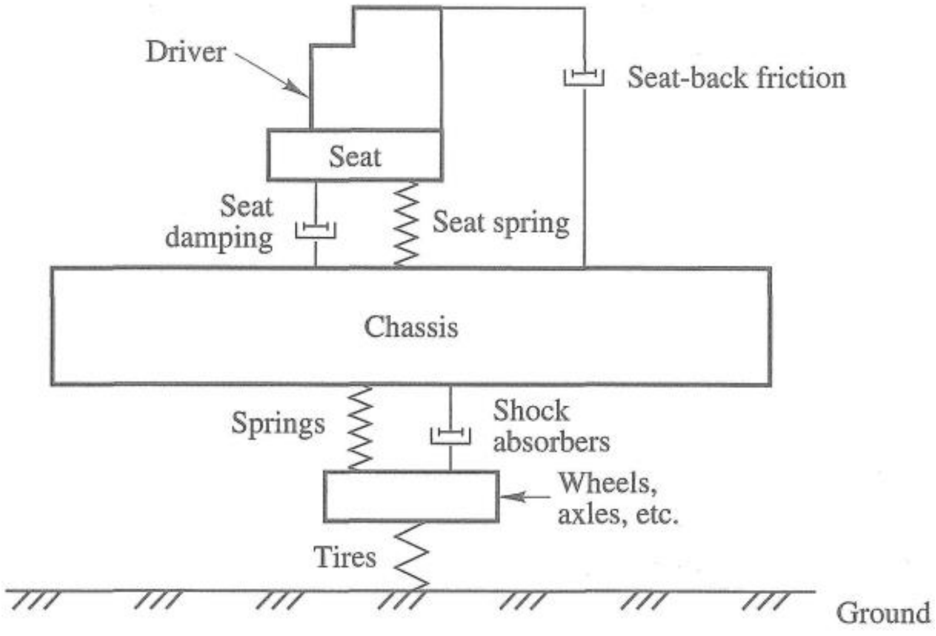
Number of components and number of interactions (a) 0, (b) 2, and (c) 6 in the family system.



INTRODUCTION: COMPLEXITY OF SYSTEMS

The complexity of a system model is sometimes measured by the number of independent energy-storing elements.





This figure is adapted from a drawing in Chapter 42 of *The Shock and Vibration Handbook,* third edition (1988), edited by Cyril M. Harris. It is used with the permission of the publisher, McGraw-Hill, Inc. Part (a) of the figure also appears in the fourth edition (1996) of that book

INTRODUCTION

Less Real, Less Complex, More Easily Solved

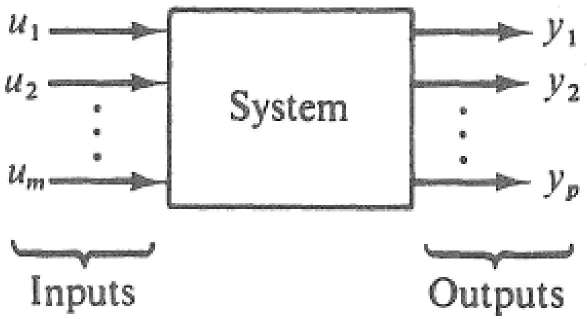
More Real, More Complex, Less Easily Solved

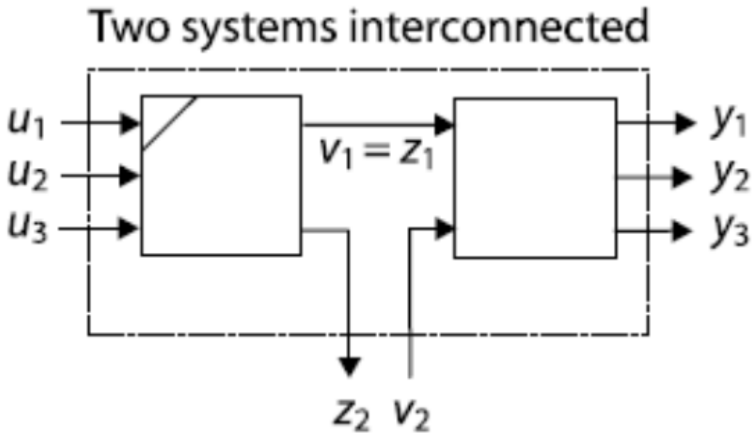
Hierarchy Of Models

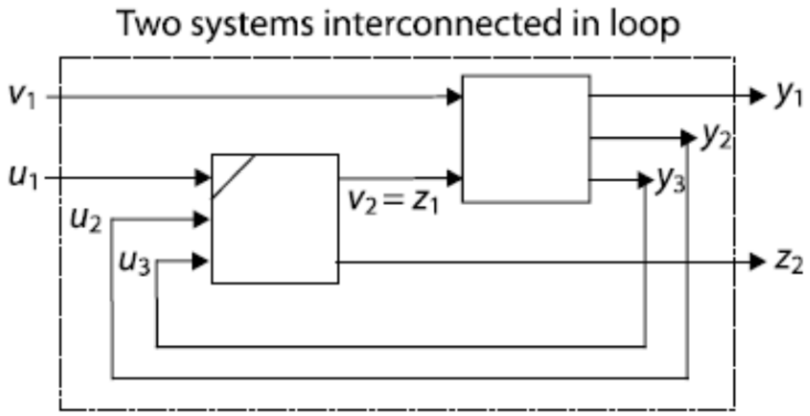
Always Ask: Why Am I Modeling?

INTRODUCTION: INPUT AND OUTPUT OF SYSTEM

- **Input:** is a variable that can affect the system's behavior.
- **Output:** is a variable that is to be calculated or measured.

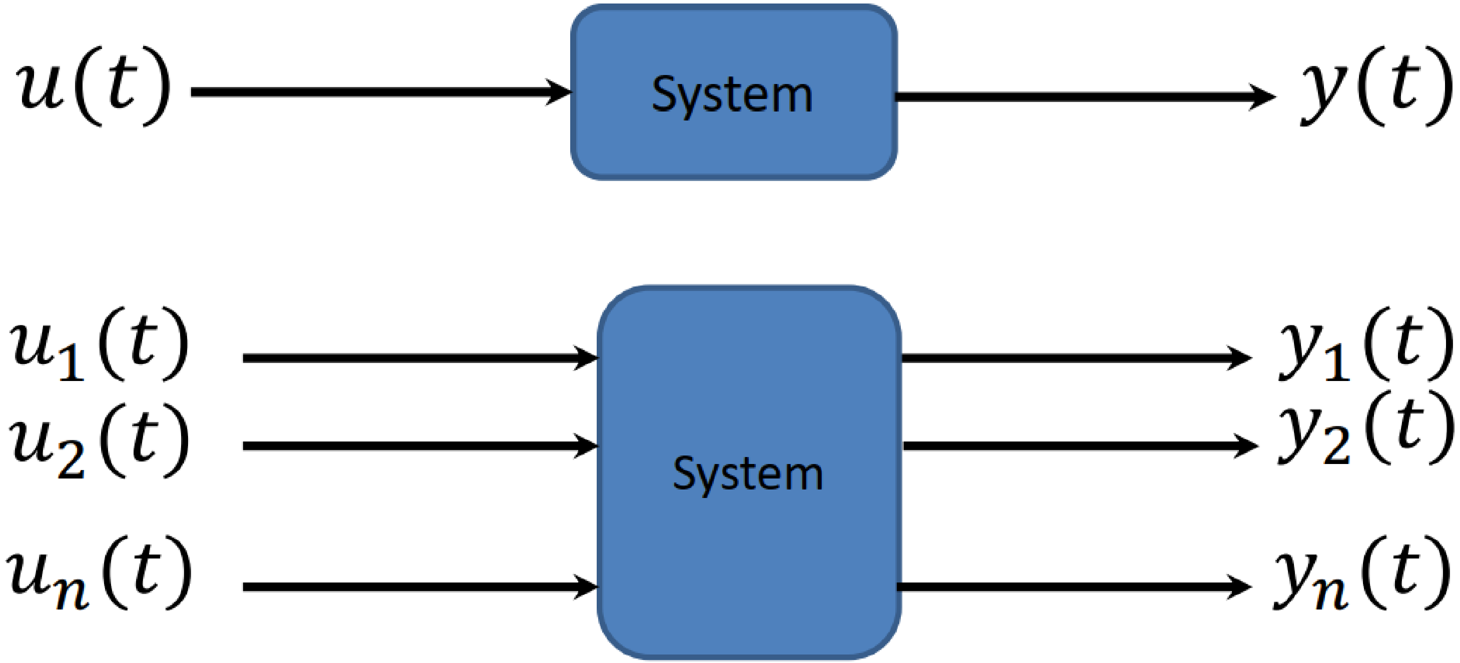




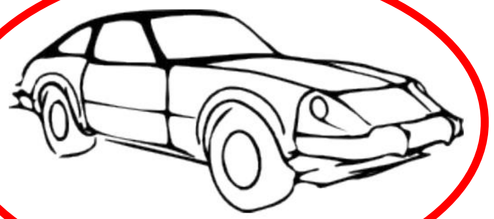


INTRODUCTION: INPUT AND OUTPUT OF SYSTEM

- Single-Input Single-Output (SISO)
- Multi-Input Multi-Output (MIMO)



- System modeling



Hardware

Image from the Open Clip Art Library, http://openclipart.org

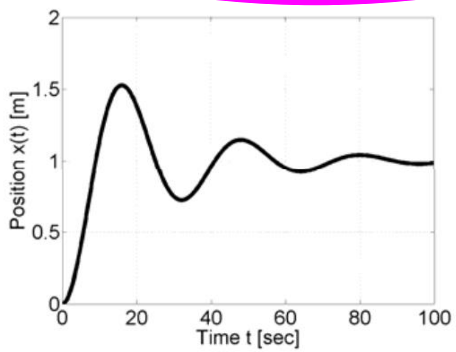
$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

Model: ordinary differential equation (ODE) or other mathematical representation

- System dynamics

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

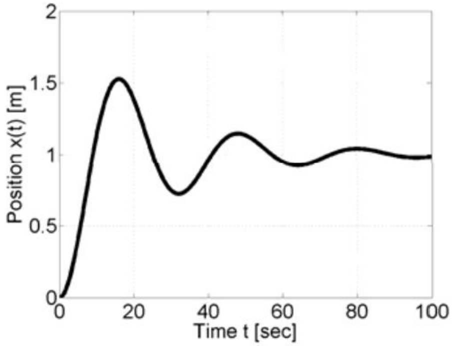
Model

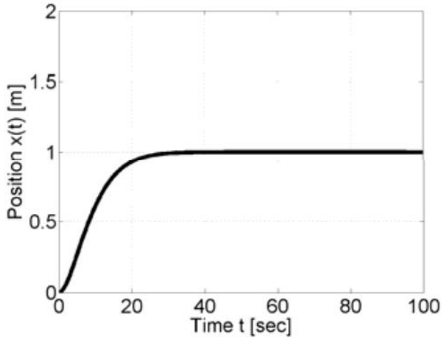


Response

- System control

Response



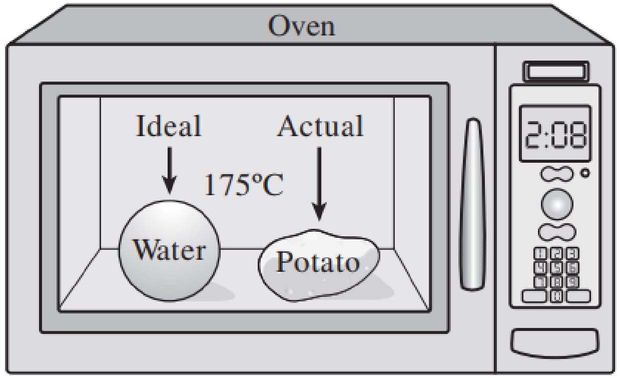


Desired response

INTRODUCTION: WHAT IS THE MODEL?

- **Model:** is a description of a system in terms of equations. The basis for constructing a model of a system are the physical laws (such as the conservation of energy and Newton's laws) that the system elements and their interconnections are known to obey.

- **Modelling:** Modeling is the art of obtaining a quantitative description of a system or one of its elements that is simple enough to be useful for making predictions and realistic enough to trust those predictions.

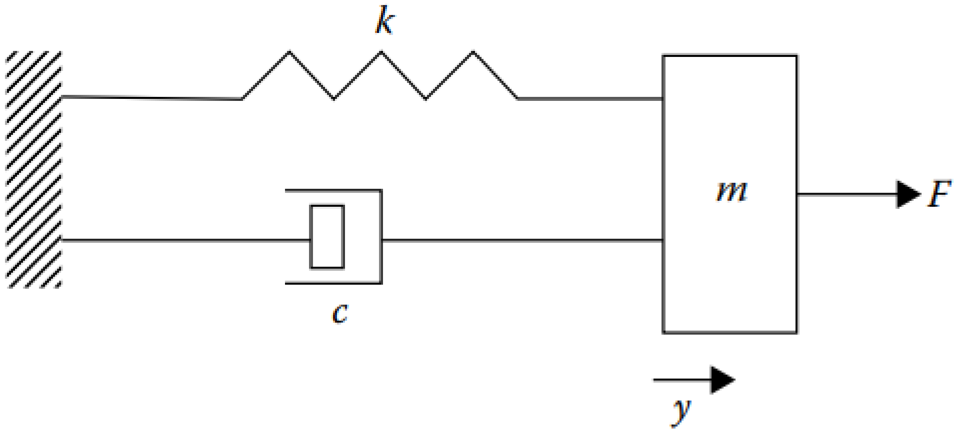


- **Mathematical Model:** A set of differential equations derived using Newton’s law, circuit law etc. that describe dynamic behavior of a physical system or process.
- There is no the model for a system. Many different models can be associated with the same system depending on what level of approximation we desire.

INTRODUCTION: WHAT IS THE MODEL?

- Applying Newton's second law, we can write the system equation as:

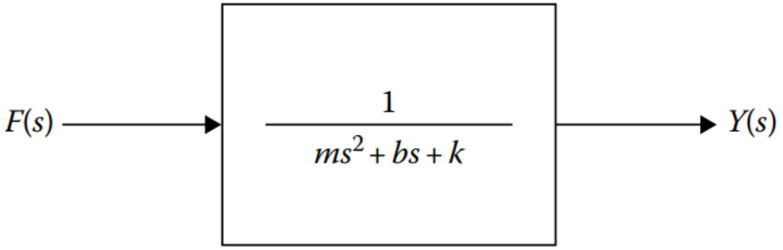
$$F = m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky$$



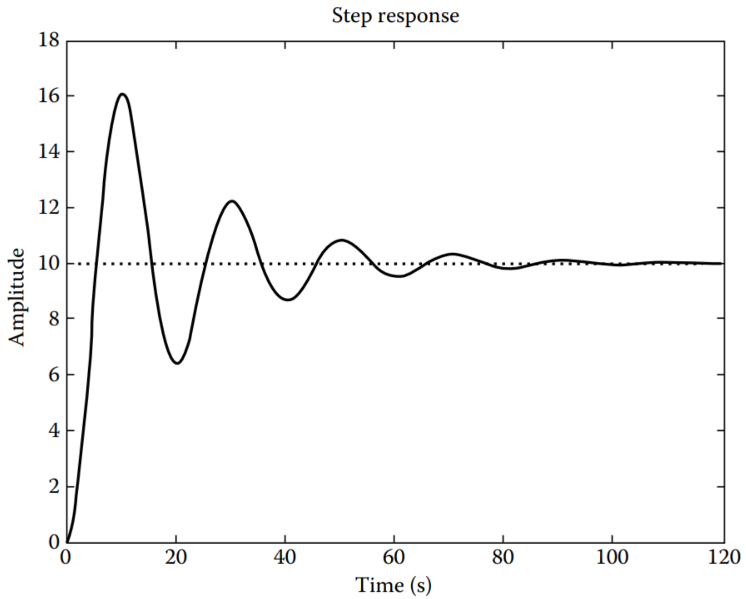
- Taking the Laplace transform of Equation above, the transfer function the system may be written as:

$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

- Block diagram of a simple mechanical system:



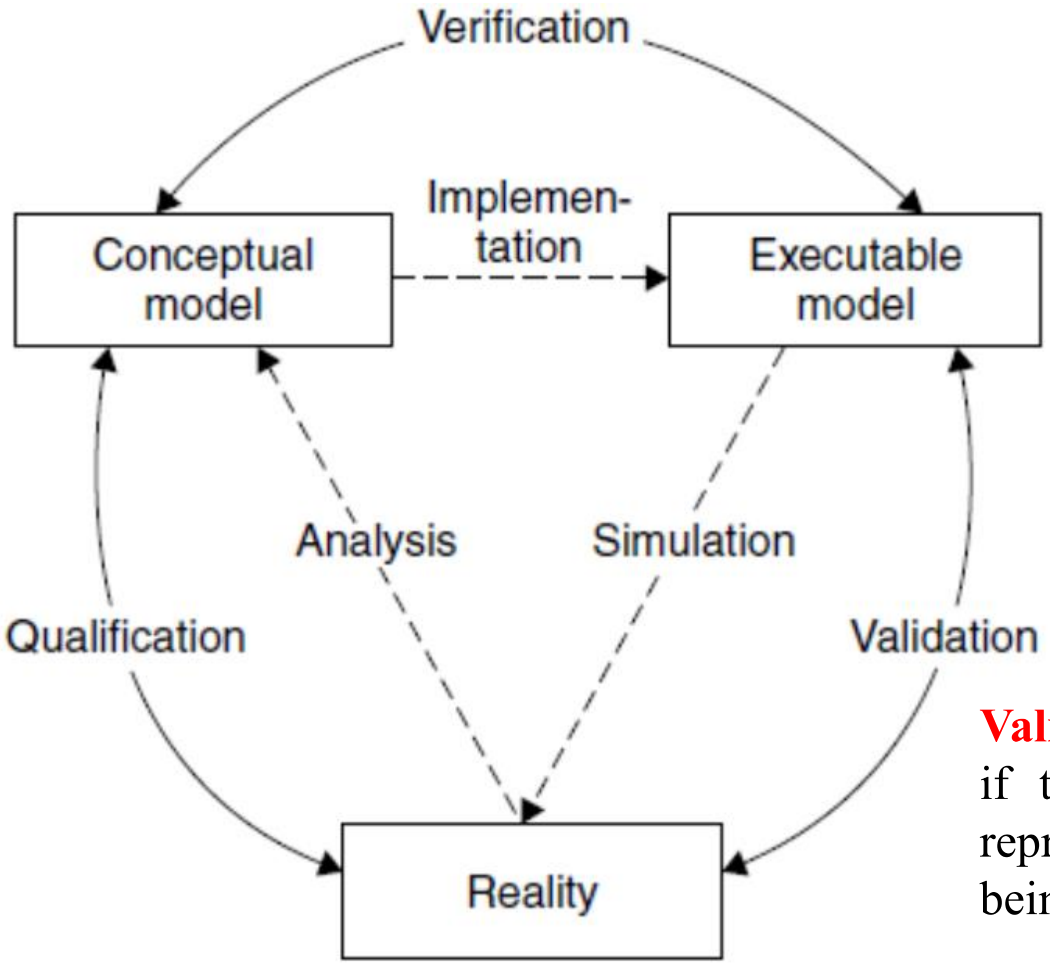
Results of step input to mechanical system



INTRODUCTION: MODELING CYCLE

Verification is the task of determining if the implementation of a model has been done correctly.

Conceptual modeling is the process of abstracting a model from a real or proposed system into a conceptual model.



An **executable model** is a model conforming to an executable modeling language and defines an aspect of the behavior of a system in sufficient detail such that the model can be executed.

Validation is the task of determining if the model constructed accurately represents the underlying real system being modeled.

MATHEMATICAL MODELING PROCESS

- Dynamic systems are modeled mathematically using ordinary differential equation.
- **There are two ways to obtain these models:**
  - Theoretical modeling based on first (physical) principles
  - Experimental modeling (identification) with measured input and output variables

System Formulation: For an unstable system, to design a state feedback controller, the system should be modelled mathematically and then formulated in state space form.

TYPES OF MATHEMATICAL MODELLING

1. **Simple Modeling:** In this modeling, the effect of an external influence in the performance of the dynamic system is not taken into consideration. The mathematical representation of the system is given by:

$$\frac{d_x}{d_y} = f(x)$$

Where $\boldsymbol{x} = [\boldsymbol{x1}, \boldsymbol{x2}, \dots \dots \boldsymbol{xn}]$ is the state vector of the system The differential equation is called autonomous system

Example: the following differential equation is used to model an electronic oscillator

$$\frac{d_{x1}}{d_t} = x_1 - x_1^3 - x_2$$

$$\frac{d_{x2}}{d_t} = x_1$$

Where x_1 and x_2 are the states of the system $\frac{d_{x1}}{d_t}, \frac{d_{x2}}{d_t}$ represent velocity of the state vector.

TYPES OF MATHEMATICAL MODELLING

2. **Reliable Modeling:** In this modeling type, the influence of external control input in the behavior of the dynamic system is taken into consideration. The differential equation used to model the system is given by:
$$\frac{d_x}{d_t} = f(x, u)$$

Where u is an external influence effect (control input). The system is called forced system. The rate of change of the state $\frac{d_x}{d_y}$ is influenced by the control input $u(t)$.

3. **Comprehensive Modeling:** This type of modeling can be used to represent the dynamic behavior of mechanical system influenced by an electrical system. In this modeling, the external control and sensors, which used to measure output of the system, are included.

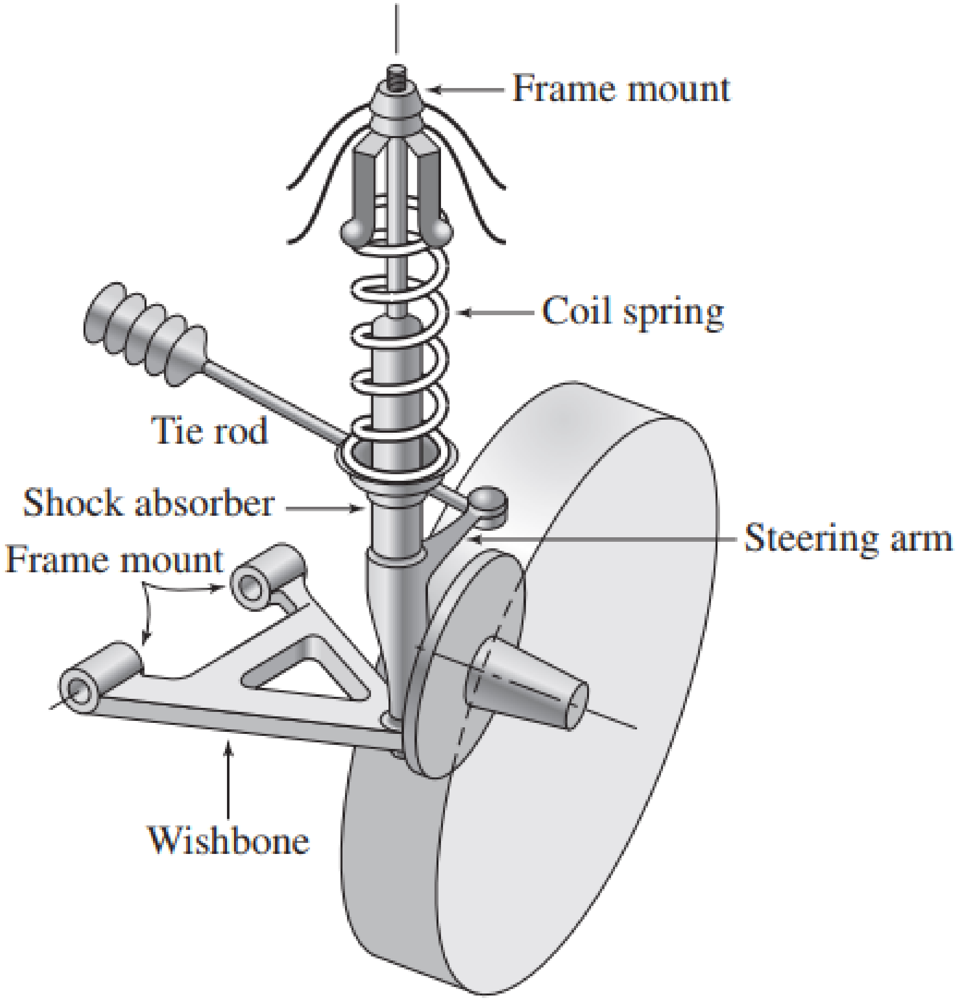
The model of the system is given by:

$$\frac{d_x}{d_t} = f(x, u)$$

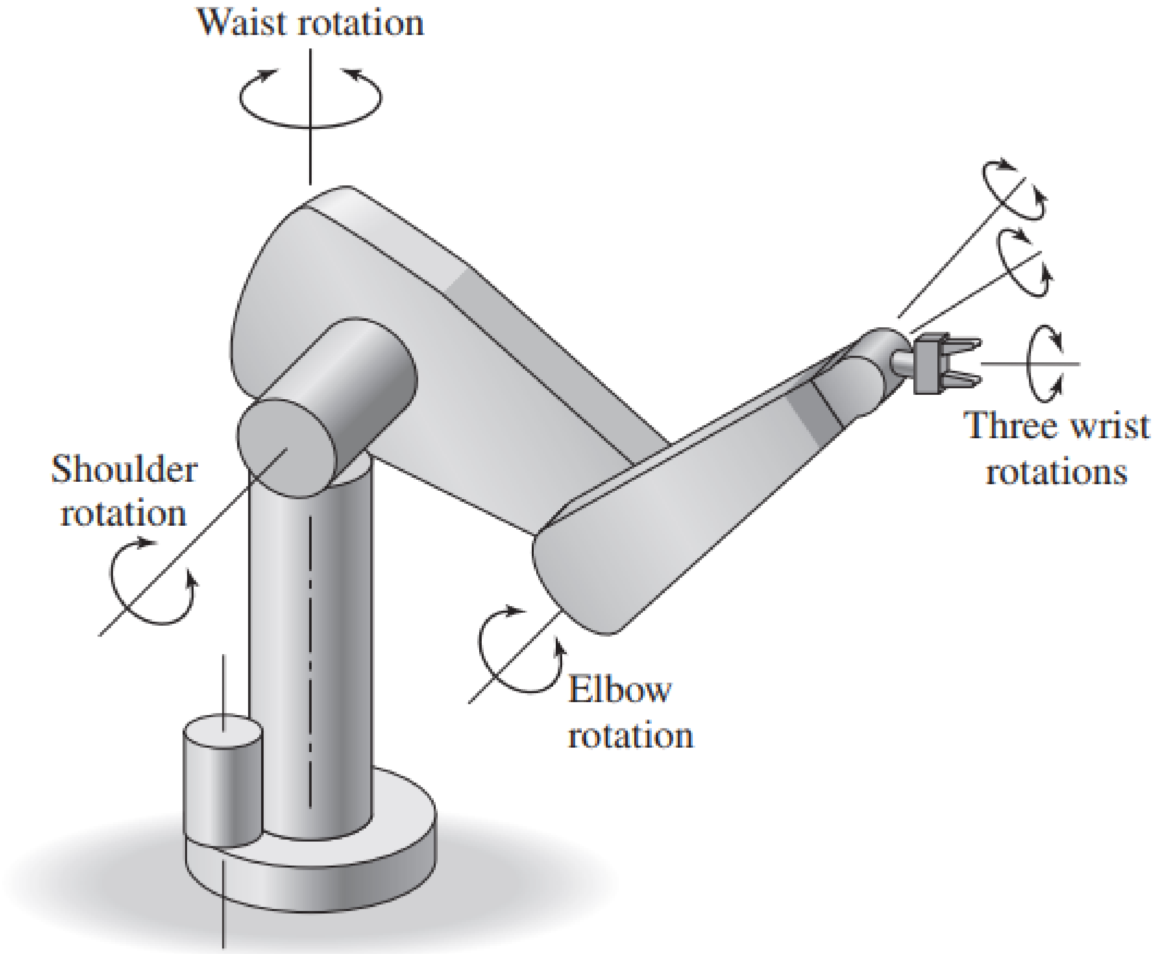
$$y = g(x, u)$$

Where $u(t)$ is a vector of control signal and y is a vector of system measurements. This model implies that the system output y is influenced by the control input signal $u(t)$.

APPLICATIONS IN MECHANICAL SYSTEMS

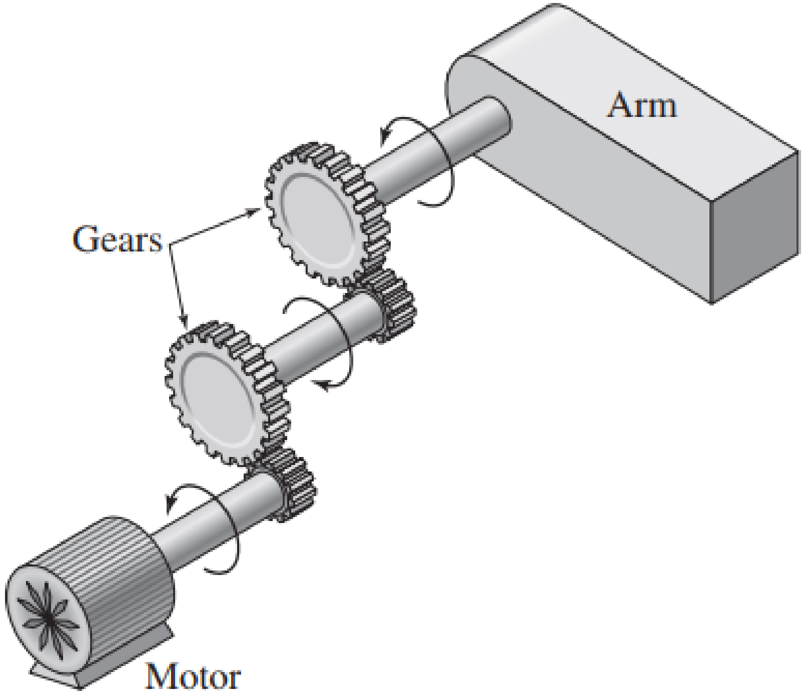


A vehicle suspension system.

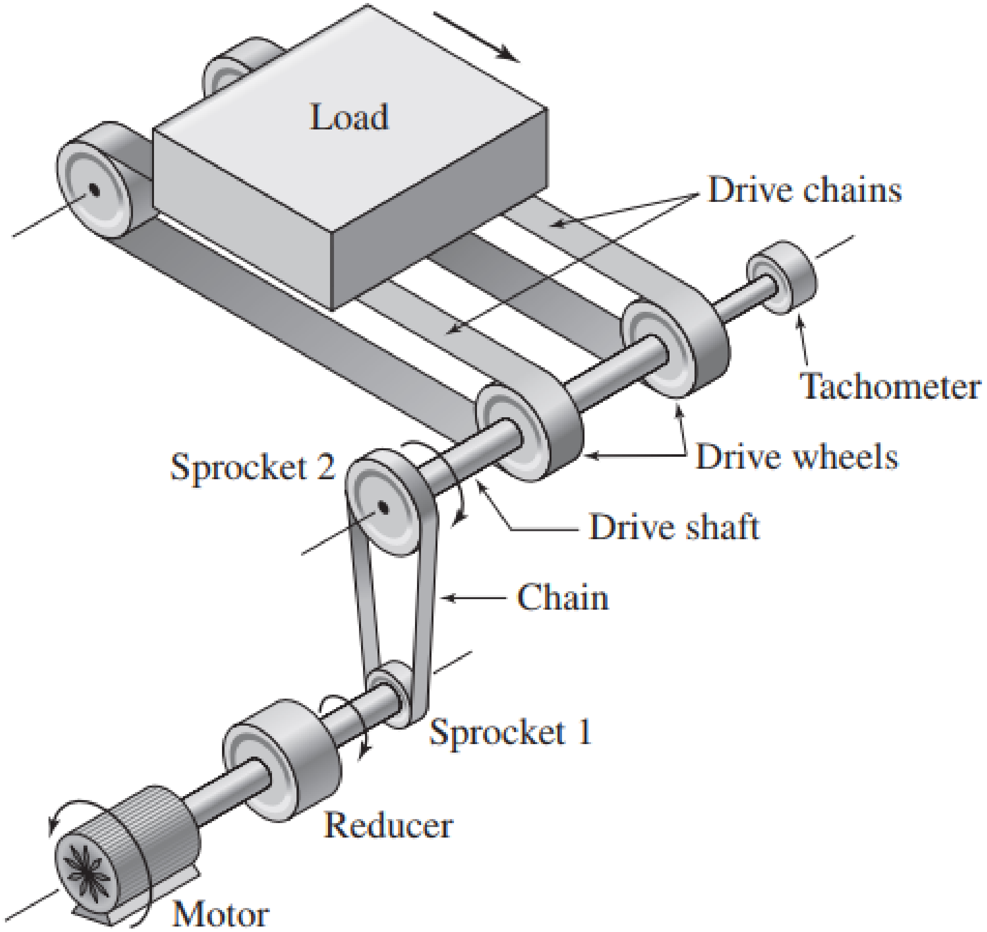


A robot arm.

APPLICATIONS IN ELECTRICAL AND ELECTROMECHANICAL SYSTEMS

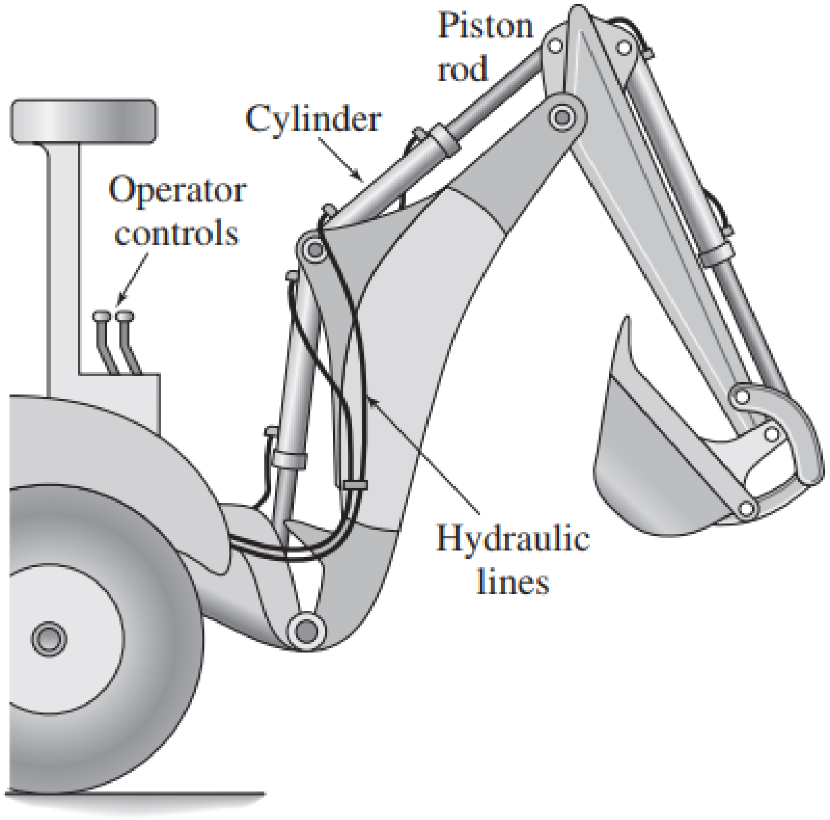


Mechanical drive for a robot arm joint.

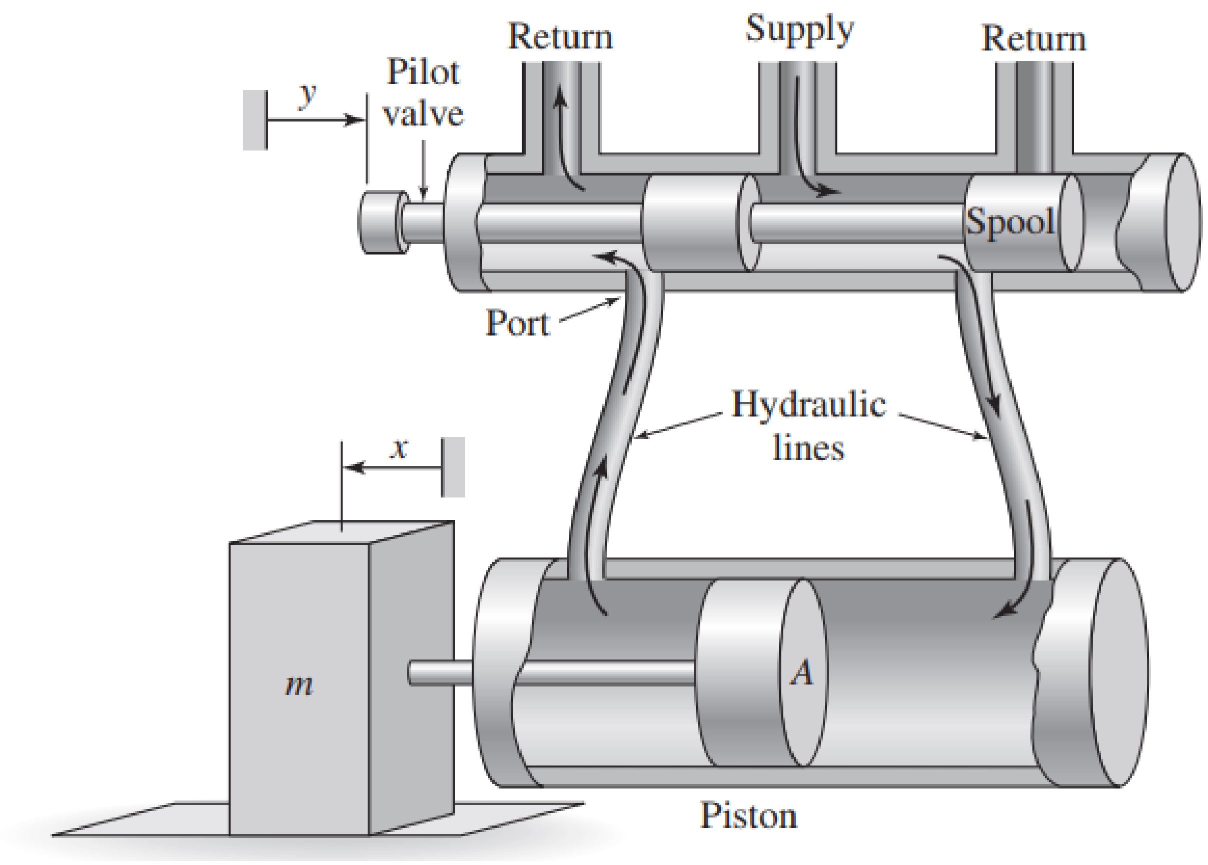


Mechanical drive for a conveyor system.

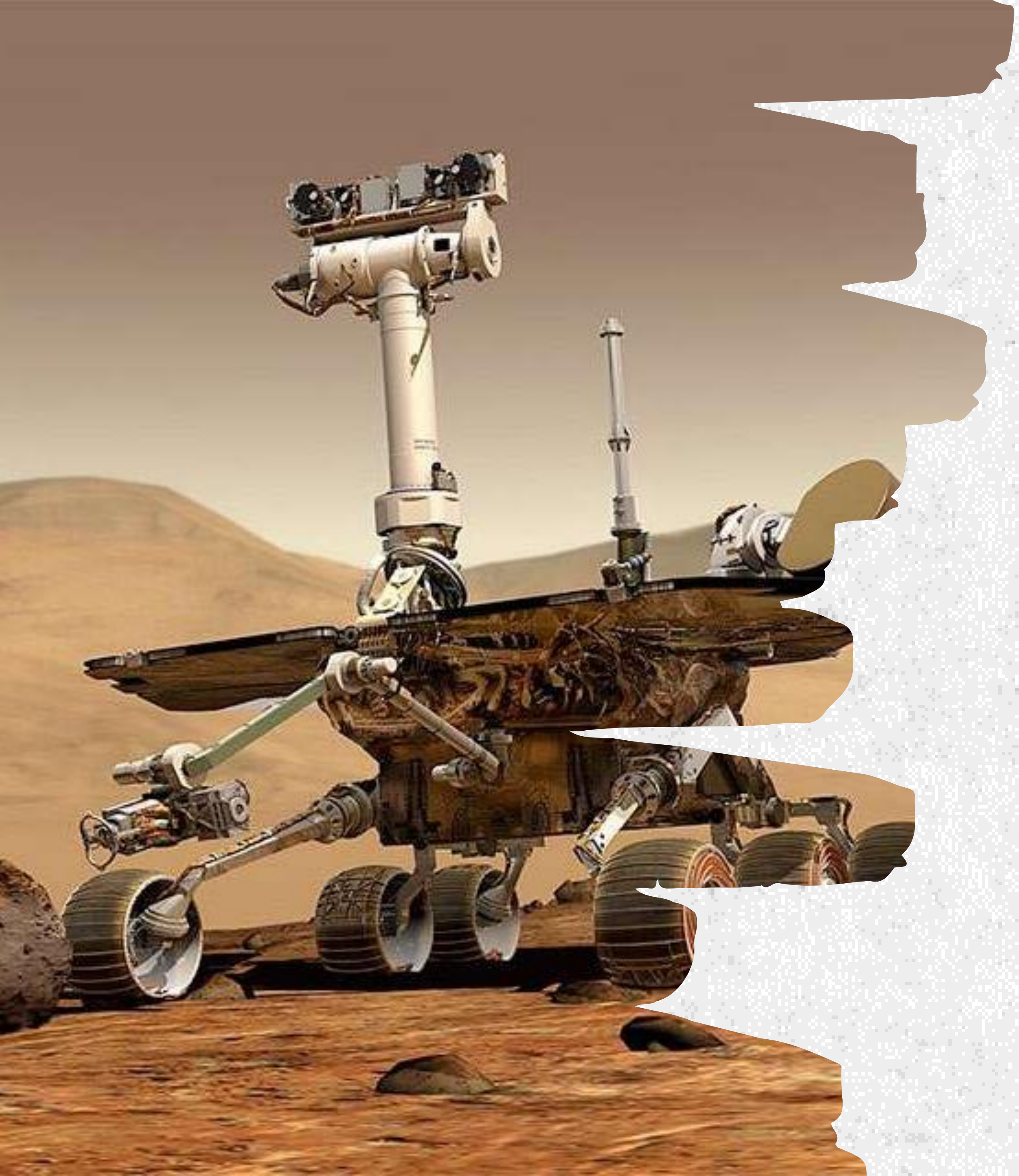
APPLICATIONS IN FLUID SYSTEMS



A backhoe



A hydraulic servomotor.



THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq

Ninevah University
Electronics Engineering College
Systems and control engineering department

SYSTEM MODELING

Lecture 2

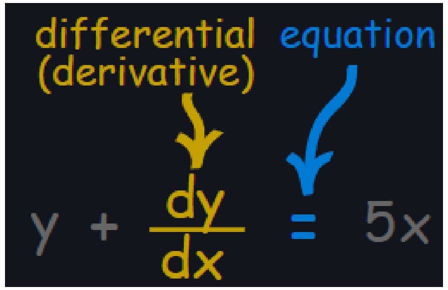
Differential Equations and Laplace Transform

Instructor
Mohanad N. Noaman

Differential Equations

- *Differential equation* is an equation involving an unknown function and one or more of its derivatives.

- Differential equations are divided into two categories: *ordinary differential equations (ODEs)* and *partial differential equations (PDEs).*



- The independent variable in our ODEs will be time $\boldsymbol{t}$ (System dynamics).

- We will often denote the time derivative with an over dot, as $\dot{x} = \frac{dx}{dt}$ $\quad \ddot{x} = \frac{d^2y}{dt^2}$

- The derivative of the highest order of the unknown function $x(t)$ with respect to t is the order of the ODE.

$$3\ddot{x} + 7\dot{x} + 2t^2x = 5 + sint$$

Second-order differential equation ($3\ddot{x}$)

Dependent variable (x) is called also the solution or the response

Input, or forcing function ($5 + sint$)

If this (input) is
- = 0, it is *homogeneous*
- ≠ 0, it is *nonhomogeneous*

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Differential equations can be divided into several types namely

1. **Ordinary Differential Equations**
2. **Partial Differential Equations**
3. **Linear Differential Equations**
4. **Nonlinear differential equations**
5. **Homogeneous Differential Equations**
6. **Nonhomogeneous Differential Equations**

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Differential equations can be divided into several types namely

1. **Ordinary Differential Equations (ODE):**
   are equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.

$$\frac{dy}{dt} = ky, \quad \text{(Exponential growth)}$$

$$\frac{dy}{dt} = k(A - y), \quad \text{(Newton's law of cooling)}$$

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t). \quad \text{(Mechanical vibrations)}$$

2. **Partial Differential Equations (PDE):**
   are equations that depend on partial derivatives of several variables. That is, there are several independent variables.

$$\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0, \quad \text{(Transport equation)}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{(Heat equation)}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \quad \text{(Wave equation in 2 dimensions)}$$

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

3. **Linear Differential Equations**

4. **Nonlinear differential equations**

How check whether your equation is linear or not?

There are four condition to check as the following:

1. The degree of dependent variable is 1.
2. The degree of differential is 1.
3. The dependent variable and its derivative are not multiplied.
4. Transcendental (sin, cos, tan, . . . etc) term does not contain dependent variable.

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

There are four condition to check as the following:

1. The degree of dependent variable is 1.
2. The degree of differential equation is 1.
3. The dependent variable and its derivative are not multiplied.
4. Transcendental (sin, cos, tan, . . . etc) term does not contain dependent variable.

$$\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = sinx$$

1. ✓ degree=1
2. ✓ degree of $(\frac{d^2y}{dx^2})^1$=1
3. ✗ $2y\frac{dy}{dx}$
4. ✓

Thus, the Equation is Non-linear

CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Check the following:

$$\dot{x} + 3x = 5 + t^2 \qquad \dot{x} + 3t^2x = 5 \qquad 3\ddot{x} + 7\dot{x} + 2t^2x = \sin t$$

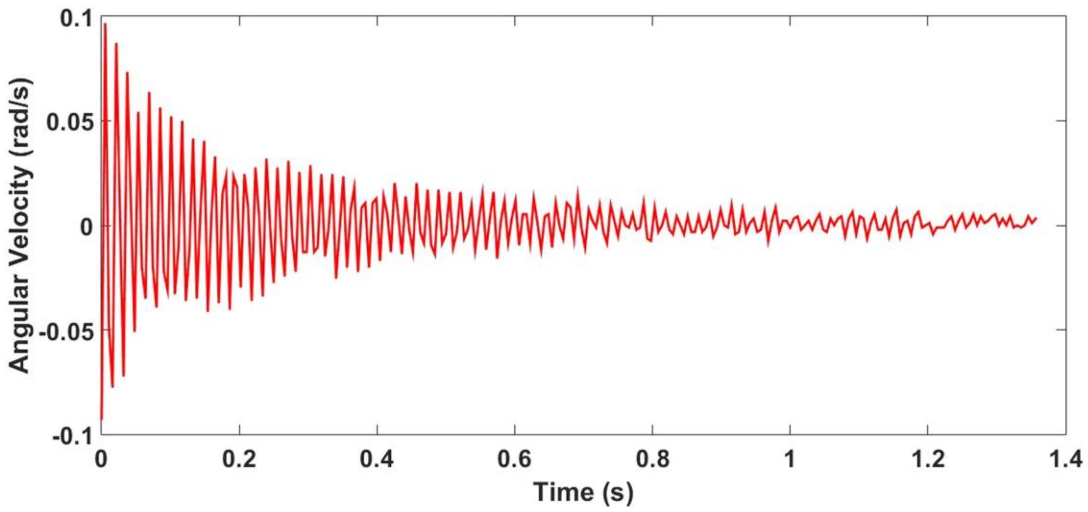
$$2\ddot{x} + 7\dot{x} + 6x^2 = 5 + t^2, \text{ because of } x^2$$

$$3\ddot{x} + 5\dot{x}^2 + 8x = 4, \text{ because of } \dot{x}^2$$

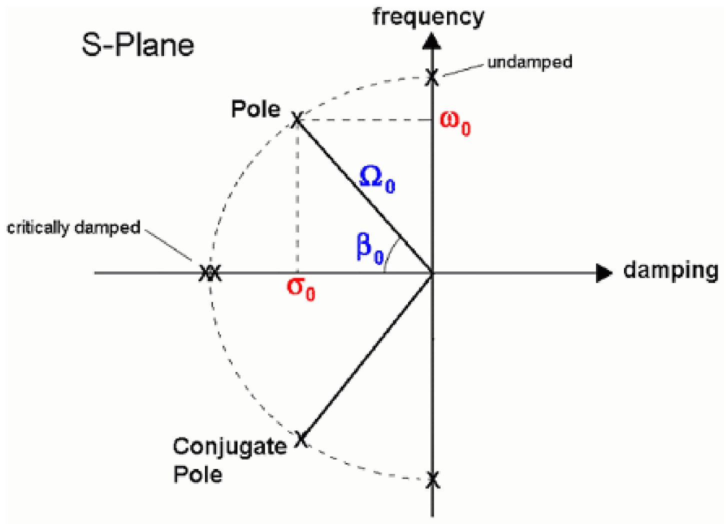
$$\ddot{x} + 4x\dot{x} + 3x = 10, \text{ because of } x\dot{x}$$

Laplace Transform: T-DOMAIN & S-DOMAIN

- Laplace transform converts time domain problems into functions of a complex variable, s, that is related to the frequency response of the system.



Convert to S-domain



Where S is the Laplace variable (Complex variable $S = \sigma + j\omega$) where σ and ω are the real and imaginary parts of S.

ICEBREAKING TIME !

Laplace Transform

- The *Laplace transform* provides a systematic and general method for solving linear ODEs and is especially useful either for nonhomogeneous equations whose right-hand side is a function of time or for sets of equations. Another advantage is that the transform converts linear differential equations into algebraic relations that can be handled easily.

$$\mathcal{L}[f(t)] = F(s)$$

Laplace transform operator

Input: $f(t)$ in t-domain

Output: $F(s)$ in S - domain

The defining equation for the Laplace Transform is

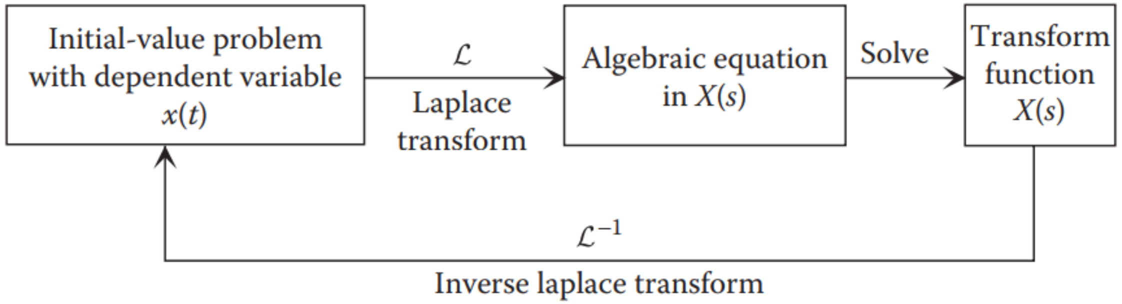
$$\mathcal{L}[f(t)] = F(s) = \lim_{A\to\infty}[\int_0^A e^{-st} f(t).dt]$$

Where S is the Laplace variable (Complex variable $S = \sigma + j\omega$) where σ and ω are the real and imaginary parts of S, respectively ($e^{-(\sigma+j\omega)t} = e^{-\sigma t}\, e^{-j\omega t}$)

Laplace Transform

Solution process for initial-value problems, using Laplace transformation.

Example 1: find Laplace transform when $f(t) = 1$ (unit step)



$$\mathcal{L}[f(t)] = F(s) = \lim_{A\to\infty}[\int_0^A e^{-st} f(t).dt]$$

$$\mathcal{L}[1] = F(s) = \lim_{A\to\infty}[\int_0^A e^{-st} 1.dt]$$

$$= \lim_{A\to\infty}[-\frac{1}{s}e^{-st}]_0^A = \lim_{A\to\infty}[-\frac{1}{s}e^{-sA} - (-\frac{1}{s})] = \frac{1}{s} \text{ (when } s > 0)$$

So $\mathcal{L}[1] = \frac{1}{s}$

Example 2: find Laplace transform when $f(t) = e^{at}$

$$\mathcal{L}[e^{at}] = F(s) = \lim_{A\to\infty}[\int_0^A e^{-st}e^{at}.dt]$$

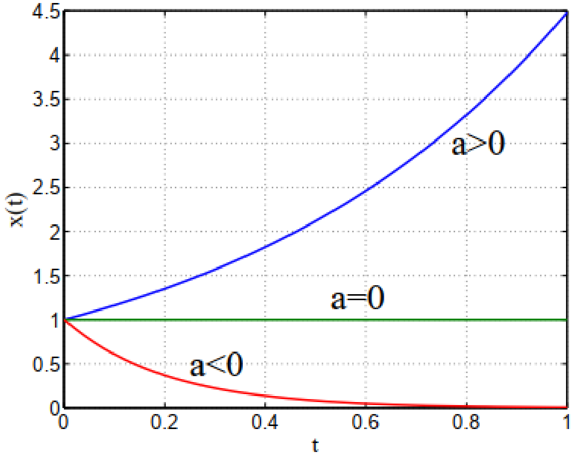
$$= \lim_{A\to\infty}[\int_0^A e^{(a-s)t}.dt]$$

$If\ a - s > 0, a > s\ no\ limit\ (diverage)$

$$e^{+\infty} = \infty$$

$If\ a - s < 0, s > a\ then \quad e^{-\infty} = 0$

$$= \lim_{A\to\infty}[\frac{1}{a-s}e^{(a-s)t}]_0^A = \frac{1}{a-s}[0 - 1] = -\frac{1}{a-s}$$



Laplace Transforms of Derivatives and Integrals

- The Laplace transform of the nth-order derivative of $x(t)$ is given by

$$\mathcal{L}\left\{x^{(n)}(t)\right\} = s^n X(s) - s^{n-1}x(0) - s^{n-2}\dot{x}(0) - \ \dots \ - x^{(n-1)}(0)$$

- In particular, for the first and second derivatives, the above-mentioned yields

$$\mathcal{L}\left\{\dot{x}(t)\right\} = sX(s) - x(0)$$

$$\mathcal{L}\left\{\ddot{x}(t)\right\} = s^2 X(s) - sx(0) - \dot{x}(0)$$

- The Laplace transform of the integral of a function $x(t)$ is given by

$$\mathcal{L}\left\{\int_0^t x(t)dt\right\} = \frac{1}{s}X(s)$$

THE INITIAL VALUE THEOREM (IVT)

- Sometimes we will need to find the value of the function $x(t)$ at $t = 0+$ (a time infinitesimally greater than 0), given the transform $X(s)$. The answer can be obtained with *the initial value theorem*, which states that

$$x(0+) = \lim_{t \to 0+} x(t) = \lim_{s \to \infty} [sX(s)]$$

- **Example 1:** let's find the initial value theorem for the following $X(s) = \frac{7s+2}{s(s+6)}$

Solution: the theorem gives

$$X(0) = \lim_{s \to \infty} [s.X(s)] = s.\frac{7s+2}{s(s+6)} = \frac{7s+2}{s+6}$$

Now, we will divide each term by the highest order in the numerator (the power of s in nominator).

$$= \frac{\frac{7s}{s} + \frac{2}{s}}{\frac{s}{s} + \frac{6}{s}} = \frac{7 + \frac{2}{\infty}}{1 + \frac{6}{\infty}} = \frac{7+0}{1+0} = 7$$

Homework: find the IVT for the following

$$X(s) = \frac{\frac{1}{3}s^2 + 1}{s^2(4s+3)}$$

Note: the order of numerator should be equal or higher than the order of denominator.

Final Value Theorem (FVT)

- Suppose $X(s)$ has no poles in the right half plane (RHP) or on the imaginary axis, except possibly a simple pole (multiplicity of 1) at the origin. Then, $x(t)$ has a definite steady-state value, and it is given by

$$x_{ss} = \lim_{s \to 0} \{sX(s)\}$$

- **Example 1:** let's find the final value theorem for the following $X(s) = \dfrac{2s+1}{s(s^2+4s+5)}$.

Solution: The poles of $X(s)$ are at 0 and − ± 2 j. The complex conjugate pair lies in the left half plane, and 0 is a simple pole (at the origin), all allowed by the FVT.

$$x_{ss} = \lim_{s \to 0}[s.X(s)] = \frac{2s+1}{s^2+4s+5}$$

Now, we will substitute each s be zero.

$$= \frac{2*0+1}{0^2+4*0+5} = \frac{1}{5}$$

Table of Laplace transform pairs

No.	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	1, unit step $u_s(t)$	$1/s$
3	t, unit ramp $u_r(t)$	$1/s^2$
4	$\delta(t-a)$	e^{-as}
5	$u(t-a)$	e^{-as}/s
6	$t^{n-1}, n=1,2,\ldots$	$(n-1)!/s^n$
7	$t^{a-1}, a>0$	$\Gamma(a)^*/s^a$
8	e^{-at}	$\frac{1}{s+a}$
9	te^{-at}	$\frac{1}{(s+a)^2}$
10	$t^n e^{-at}, n=1,2,\ldots$	$\frac{n!}{(s+a)^{n+1}}$
11	$\frac{1}{b-a}(e^{-at}-e^{-bt}), a\neq b$	$\frac{1}{(s+a)(s+b)}$
12	$\frac{1}{a-b}(ae^{-at}-be^{-bt}), a\neq b$	$\frac{s}{(s+a)(s+b)}$
13	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$
14	$\frac{1}{a^2}(-1+at+e^{-at})$	$\frac{1}{s^2(s+a)}$
15	$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
16	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
17	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
18	$e^{-\sigma t}\sin\omega t$	$\frac{\omega}{(s+\sigma)^2+\omega^2}$
19	$e^{-\sigma t}\cos\omega t$	$\frac{s+\sigma}{(s+\sigma)^2+\omega^2}$
20	$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
21	$\omega t-\sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
22	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
23	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$

Table of Laplace transform pairs

No.	$f(t)$	$F(s)$
24	$\frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$	$\frac{1}{(s^2+\omega^2)^2}$
25	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$
26	$\frac{1}{\omega_2^2-\omega_1^2}\left[\frac{1}{\omega_2}\sin\omega_2 t - \frac{1}{\omega_1}\sin\omega_1 t\right],\ \omega_1^2 \neq \omega_2^2$	$\frac{1}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
27	$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t),\ \omega_1^2 \neq \omega_2^2$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
28	$\sinh at$	$\frac{a}{s^2-a^2}$
29	$\cosh at$	$\frac{s}{s^2-a^2}$
30	$\frac{1}{a^2-b^2}\left[\frac{1}{a}\sinh at - \frac{1}{b}\sinh bt\right], a \neq b$	$\frac{1}{(s^2-a^2)(s^2-b^2)}$
31	$\frac{1}{a^2-b^2}[\cosh at - \cosh bt], a \neq b$	$\frac{s}{(s^2-a^2)(s^2-b^2)}$
32	$\frac{1}{3a^2}[e^{-at} + 2e^{\frac{1}{2}at}\sin(\frac{\sqrt{3}}{2}at - \frac{1}{6}\pi)]$	$\frac{1}{s^3+a^3}$
33	$\frac{1}{3a}[-e^{-at} + 2e^{\frac{1}{2}at}\sin(\frac{\sqrt{3}}{2}at + \frac{1}{6}\pi)]$	$\frac{s}{s^3+a^3}$
34	$\frac{1}{3a^2}[e^{at} - 2e^{-\frac{1}{2}at}\sin(\frac{\sqrt{3}}{2}at + \frac{1}{6}\pi)]$	$\frac{1}{s^3-a^3}$

No.	$f(t)$	$F(s)$
35	$\frac{1}{3a}[e^{-at} + 2e^{-\frac{1}{2}at}\sin(\frac{\sqrt{3}}{2}at - \frac{1}{6}\pi)]$	$\frac{s}{s^3-a^3}$
36	$\frac{1}{4a^3}[\cosh at\sin at - \sinh at\cos at]$	$\frac{1}{s^4+4a^4}$
37	$\frac{1}{2a^2}\sinh at\sin at$	$\frac{s}{s^4+4a^4}$
38	$\frac{1}{2a^3}(\sinh at - \sin at)$	$\frac{1}{s^4-a^4}$
39	$\frac{1}{2a^2}(\cosh at - \cos at)$	$\frac{s}{s^4-a^4}$

*Gamma function $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$.

MORE INFO

Refer to *(Laplace 1, Laplace 2)* Pdfs in the Classroom

MATLAB Exercises

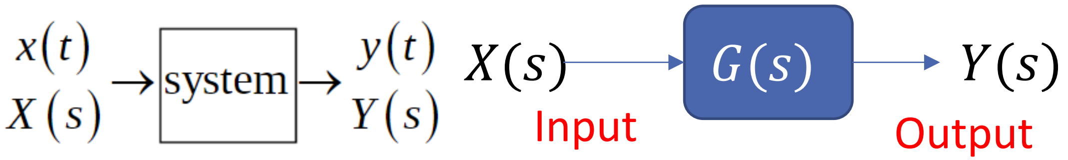
- Find the Laplace transform of the following in MATLAB:

1. $f(t) = e^{at}$
2. $f(t) = te^{-2t/3}$
3. $f(t) = \frac{\sin \omega t}{t}$

Transfer Function

- The concept of the transfer function is useful for analyzing the effects of the input. It represents the input-output relationship for a system and expressed in terms of s-domain when all the initial conditions are assumed to be zero.

$\boldsymbol{Y(s) = G(s).X(s)}$ Thus, the transfer function is $\boldsymbol{G(s) = \frac{Y(s)}{X(s)}}$



- Definitions:

1. Poles - roots of the denominator polynomial Values that cause transfer function magnitude to go to infinity.
2. Zeros - roots of the numerator polynomial Values that cause the transfer function to go to 0.
3. Eigenvalues - Characteristic responses of a system. Roots of the denominator polynomial. All eigenvalues must be negative for a system transient (natural response) to decay out.

$$G(s) = \frac{(s+z_1)\cdot(s+z_2)\cdot(s+z_3)\ldots\cdot(s+z_{n-1})\cdot(s+z_n)}{(s+p_1)\cdot(s+p_2)\cdot(s+p_3)\ldots\cdot(s+p_{n-1})\cdot(s+p_n)}$$

Zeros (numerator)

Poles (denominator)

STANDARD FORM OF TRANSFER FUNCTION

- The standard form of first order system is given by:

$$\frac{Y(s)}{X(s)} = \frac{\gamma}{1 + \tau s}$$

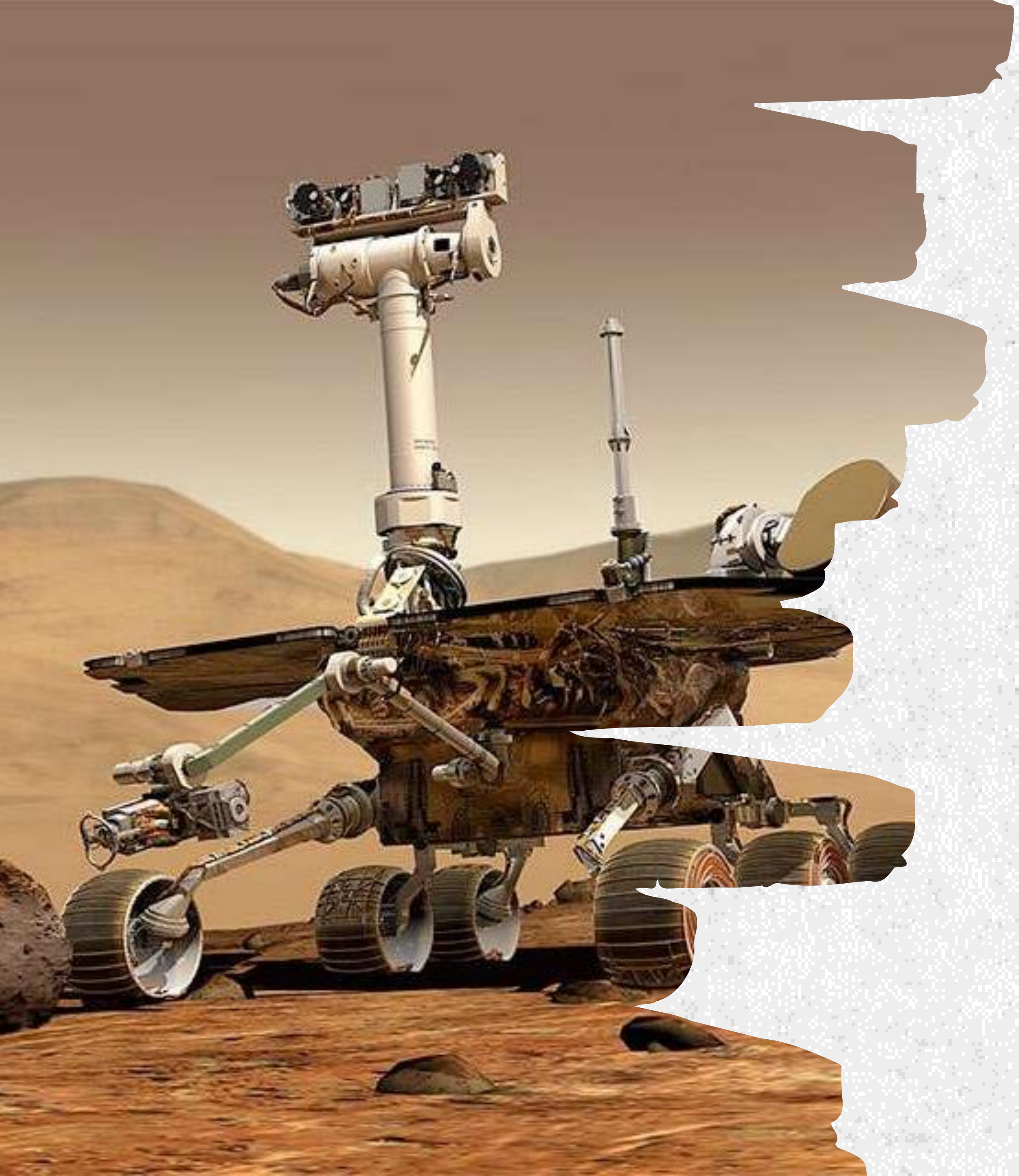
- Where, γ is the system gain and τ is the time constant of the response, which takes an indicate of the way system will response to the input.

- For second order system, the standard form as follows:

$$\frac{Y(s)}{X(s)} = \frac{\gamma}{1 + \frac{2\xi}{\omega_n} s + \frac{1}{\omega_n^2} s^2}$$

$$\frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Where ω_n is damping natural frequency [rad/s], ξ is damping ratio and γ is system gain When ξ=1, the system is in critical damping case, the system is just in the border of oscillation.



THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq

Ninevah University
Electronics Engineering College
Systems and control engineering department

SYSTEM MODELING

Lecture 3

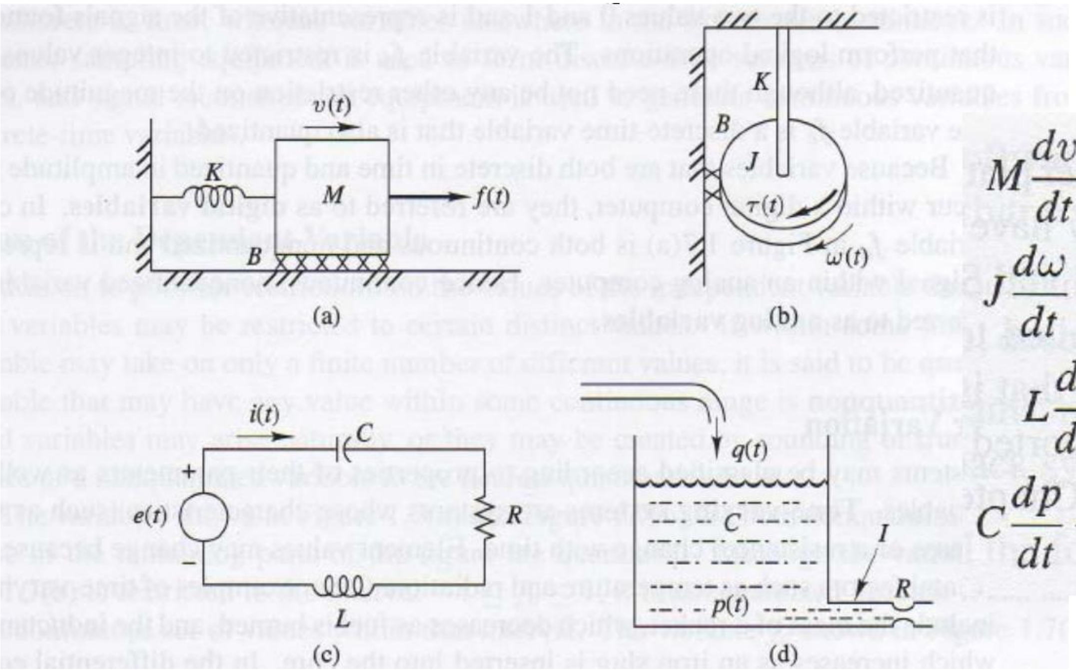
Basic System Models

Mechanical Systems

Instructor
Mohanad N. Noaman

ANALOGOUS SYSTEMS

- Electrical and mechanical systems possess fixed analogy and there exist similarity between the equilibrium equations of the two. This allows forming such electrical systems whose behavioral characteristics are similar to the given mechanical system.
- **Two systems are said to be analogous to each other if the following two conditions are satisfied:**
  - The two systems are physically different
  - Differential equation modelling of these two systems are same



$$M\frac{dv}{dt}+Bv(t)+K\int_0^t v(\lambda)d\lambda = f(t)$$

$$J\frac{d\omega}{dt}+B\omega(t)+K\int_0^t \omega(\lambda)d\lambda = \tau(t)$$

$$L\frac{di}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\lambda)d\lambda = e(t)$$

$$C\frac{dp}{dt}+\frac{1}{R}p(t)+\frac{1}{I}\int_0^t p(\lambda)d\lambda = q(t)$$

TRANSLATION MECHANICAL REVISION

Motion in mechanical systems can be:

- Translational
- Rotational, or
- Combination of above

Mechanical systems can be of two types:

- Translational systems
- Rotational systems

Variables that describe motion:

- Displacement, x
- Velocity, v
- Acceleration, a

TRANSLATION MECHANICAL REVISION

- The symbols for the basic variables used to describe the dynamic behavior of translational mechanical systems are:

- x, displacement in meters (m)
- v, in meters per second (m/s)
- a, acceleration in meters per second per second (m/s^2)
- f, force in newtons (N)
- w, energy in joules (J)
- p, power in watts (W)

Newton's first law states that a particle originally at rest, or moving in a straight line with a constant speed, will remain that way as long as it is not acted upon by an unbalanced external force.

Newton's second law states that the acceleration of a mass particle is proportional to the vector resultant force acting on it and is in the direction of this force.

Newton's third law states that the forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear.

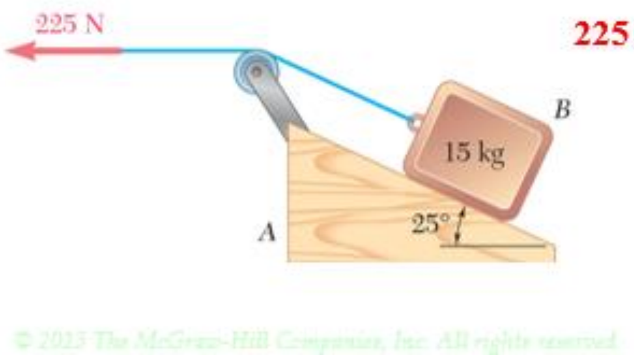
THE FREE BODY DIAGRAM

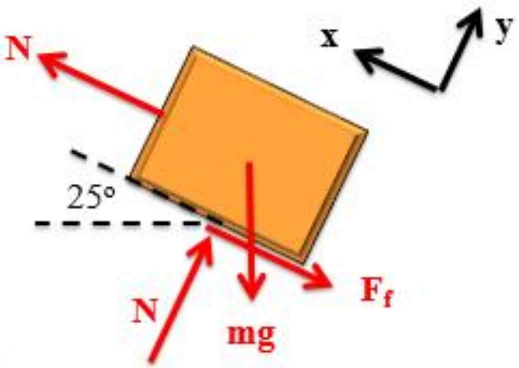
Is the same as you have done in statics; we will add the kinetic diagram in our dynamic analysis.

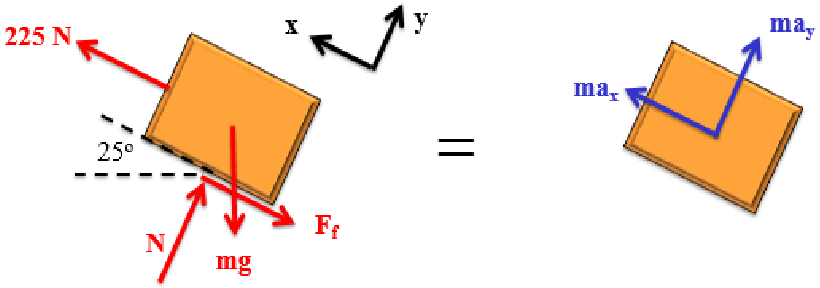
1. Isolate the body of interest (free body)
2. Draw your axis system (e.g., Cartesian, polar, path)
3. Add in applied forces (e.g., weight, 225 lb pulling force)
4. Replace supports with forces (e.g., normal force)
5. Draw appropriate dimensions (usually angles for particles)

Put the inertial terms for the body of interest on the kinetic diagram.

1. Isolate the body of interest (free body)
2. Draw in the mass times acceleration of the particle; if unknown, do this in the positive direction according to your chosen axes.







$$\Sigma F = ma$$

TRANSLATION & ROTATIONAL MECHANICAL REVISION

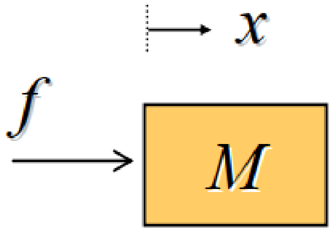
- It will be convenient to use the following abbreviated "dot" notation for time derivatives:

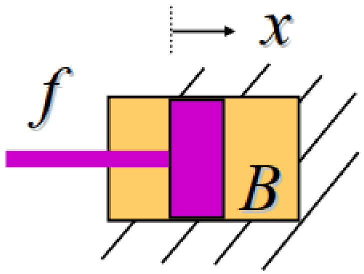
$$\dot{x}(t) = \frac{dx}{dt}, \qquad \ddot{x}(t) = \frac{d^2x}{dt^2}$$

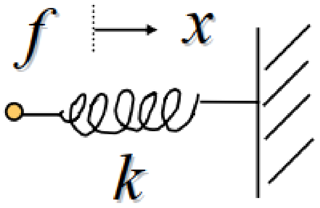
v ← $\dot{x}(t)$ $\qquad$ a ← $\ddot{x}(t)$

- Thus, we can express the scalar form of Newton's law as $f = m\ddot{x}(t)$

- **Power**: $p = fv$ or $p = \frac{dw}{dt}$ power is defined to be the rate at which energy is supplied or dissipated





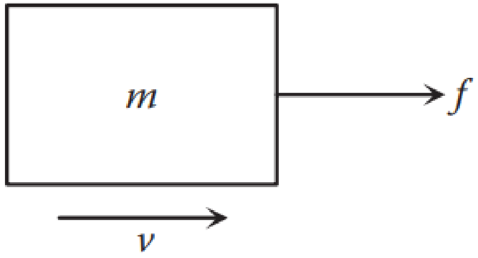


	force/velocity	force/position
Mass	$f = M\,dv/dt$	$f = M\,dx^2/dt^2$
Damper (Viscous friction)	$f = B\,v$	$f = B\,dx/dt$
Spring (Stiffness)	$f = k\int v\,dt$	$f = k\,x$

MECHANICAL ELEMENTS: TRANSLATION

- **Mass:** figure shows a mass m traveling with a velocity v. The basic variables used to describe the dynamic behavior of a translational mechanical system are the acceleration vector a, the velocity vector v, and the position vector x. They are related by the time derivatives:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$



Newton's second law is expressed as:

$$f = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma$$

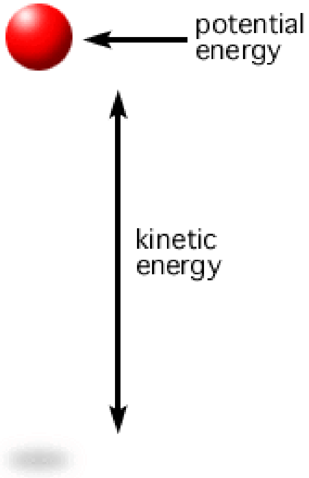
The kinetic energy is expressed as:

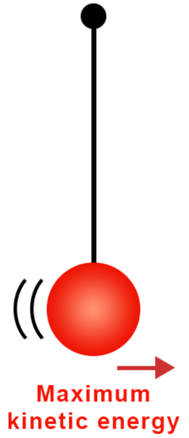
$$T \text{ or } KE = \frac{1}{2}mv^2$$

The energy stored in the mass is potential energy given by:

$$V_g \text{ or } PE = mgh$$

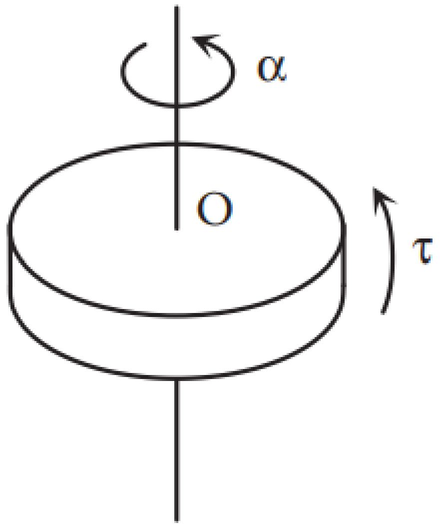
where:
g is the gravitational acceleration (9.81 m/s^2or 32.2 ft/s^2)
h is the height measured from the reference position or datum to the center of mass
Subscript g is used to denote that the potential energy is associated with gravity



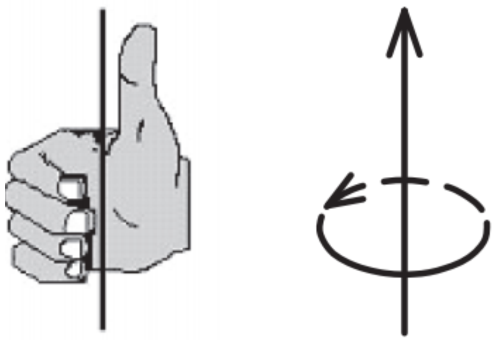


MECHANICAL ELEMENTS: ROTATIONAL

- **Mass:** For rotational mechanical systems, the basic variables used to describe system dynamics are the angular acceleration vector α, the angular velocity vector ω, and the angular position vector θ. The direction of an angular vector can be determined using the right-hand rule, as shown in Figure besides.



A disk rotating about an axis through a fixed-point O.



$$\alpha = \frac{dw}{dt} = \frac{d^2\theta}{dt^2}, \quad \alpha = \dot{\omega} = \ddot{\theta}$$

The torque τ or moment M about the fixed-point O:

$$\sum M_0 / \tau = I_o \alpha$$

The kinetic energy for a rotational mass about a fixed-point O is expressed as

$$T \ or \ KE = \frac{1}{2} I_o \omega^2$$

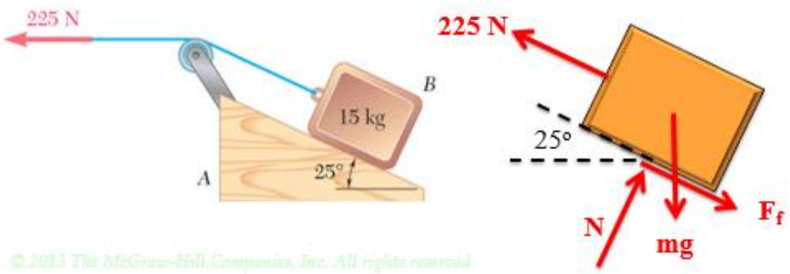
The potential energy for a rotational mass:

$$V_g \ or \ PE = mgh$$

- where:
  g is the gravitational acceleration (9.81 m/s^2or 32.2 ft/s^2)
  h is the height measured from the reference position or datum to the center of mass
  Subscript g is used to denote that the potential energy is associated with gravity

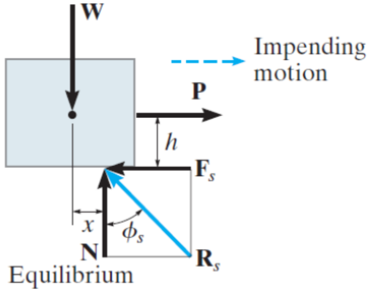
MECHANICAL ELEMENTS: TRANSLATION

- **Friction:** If a moving particle contacts a rough surface, it may be necessary to use the frictional equation, which relates the frictional and normal forces $\boldsymbol{F_s}$ and N acting at the surface of contact by using the coefficient of kinetic friction, i.e., $\boldsymbol{F_k} = \boldsymbol{\mu_k}$N. Remember that $\boldsymbol{F_k}$ always acts such that it opposes the motion of the particle relative to the surface it contacts. If the particle is on the verge of relative motion, then the coefficient of static friction should be used.



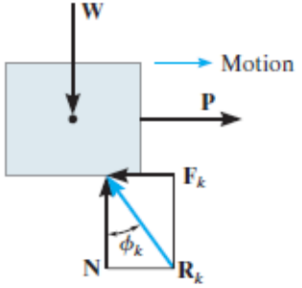
$$F_s = \mu_s N$$

- F_s: limiting static frictional
- μ_s: coefficient of static friction
- N: normal force



$$F_k = \mu_k N$$

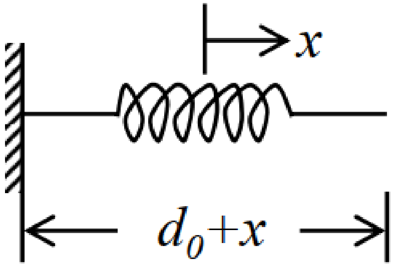
- F_k: kinetic frictional force
- μ_k: coefficient of kinetic friction
- N: normal force



MECHANICAL ELEMENTS: TRANSLATION

- **Spring:** Stiffness is the resistance of an elastic body to deflection or deformation by an applied force.

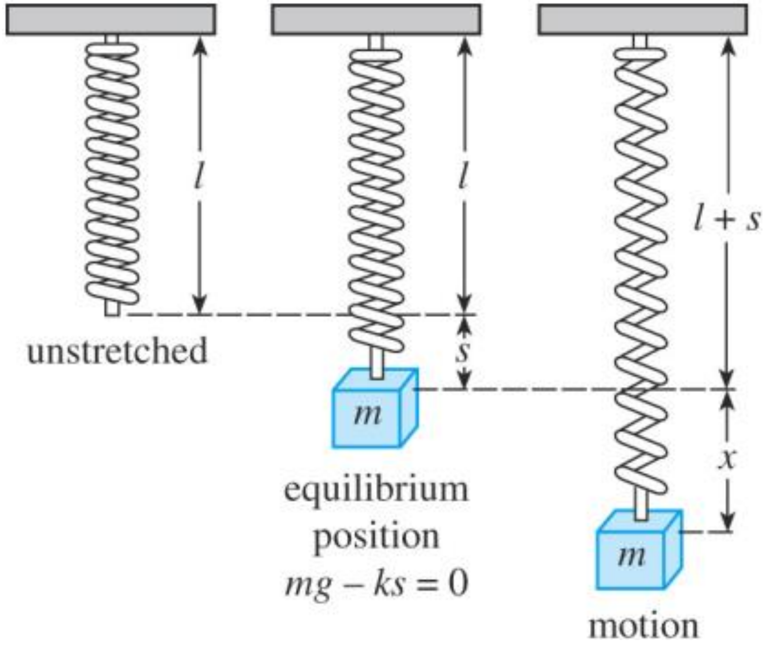
Most common: ideal spring



- $\boldsymbol{d_0}$ = Length of spring when no force applied
- $\boldsymbol{d_t}$ = Final total length of spring after applied force
- $\boldsymbol{x(t)}$ = Elongation caused by f

$$d(t) = d_0 + x(t) \quad \Rightarrow \quad x(t) = d(t) - d_0$$

$$f = kx$$

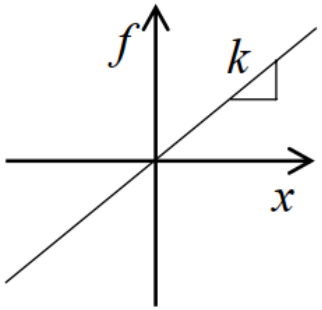


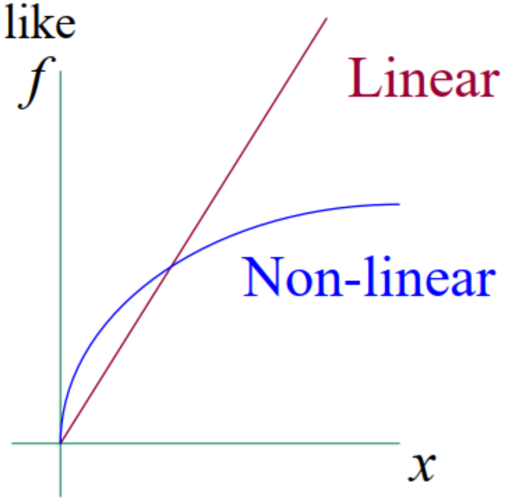
When a spring is stretched or compressed, potential energy is stored in the spring and is given by $\boldsymbol{V_e} = \frac{1}{2}\boldsymbol{kx^2}$

MECHANICAL ELEMENTS: TRANSLATION

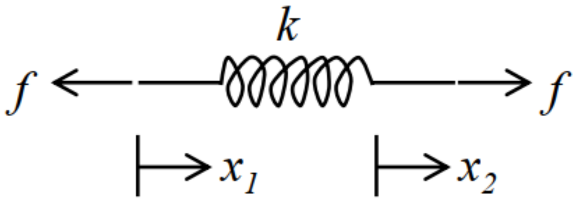
- **Spring:** Stiffness is the resistance of an elastic body to deflection or deformation by an applied force.

The linear spring is an approximation of something like





Multiple applied forces: $f = k(x_2 - x_1) = k\Delta x$

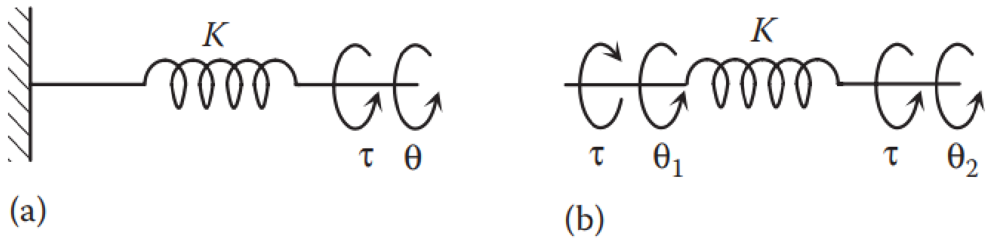


When a spring is stretched or compressed, potential energy is stored in the spring and is given by $\boldsymbol{V_e = \frac{1}{2}kx^2}$

MECHANICAL ELEMENTS: ROTATIONAL

- **Spring:** For a torsional spring, we have $\boldsymbol{\tau = k\Delta\theta}$

  where:
  τ is the applied torque
  K is the torsional spring stiffness in units of $N \cdot m/rad$ or $ft \cdot lb/rad$
  θ is the angular deformation of the spring



A torsional spring element with (a) one fixed end and (b) two free ends.

- Assume that θ_1 and θ_2 are the angular displacements of respective ends corresponding to the applied torque. If $\theta_2 > \theta_1 > 0$, then

$$\boldsymbol{\tau = k(\theta_2 - \theta_1)}$$

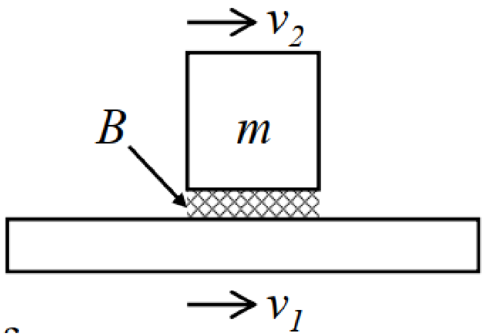
The potential energy stored in a torsional spring element is expressed as

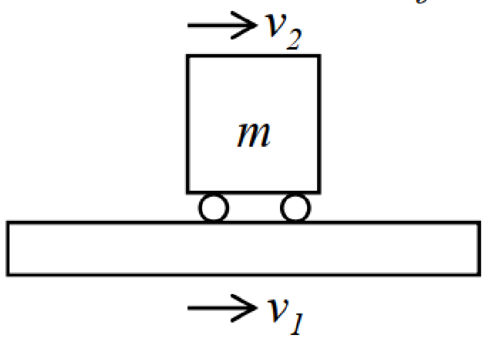
$$\boldsymbol{V_e = \frac{1}{2}k\theta^2}$$

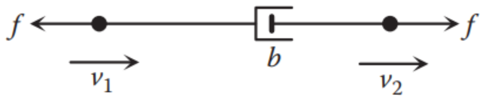
MECHANICAL ELEMENTS: TRANSLATION

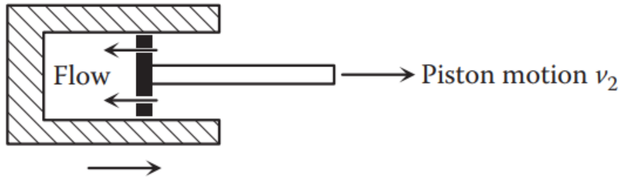
- **Damper:** Also known as viscous friction or linear friction. Friction is the force that opposes the relative motion or tendency of such motion of two surfaces in contact.

$f = B\Delta v$, where $\Delta v = v_2 - v_1$ and B = viscosity constant/coefficient







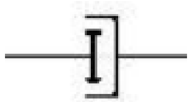


Above left:

- B is proportional to contact area and viscosity of oil.
- B is inversely proportional to the thickness of film.

Above right:

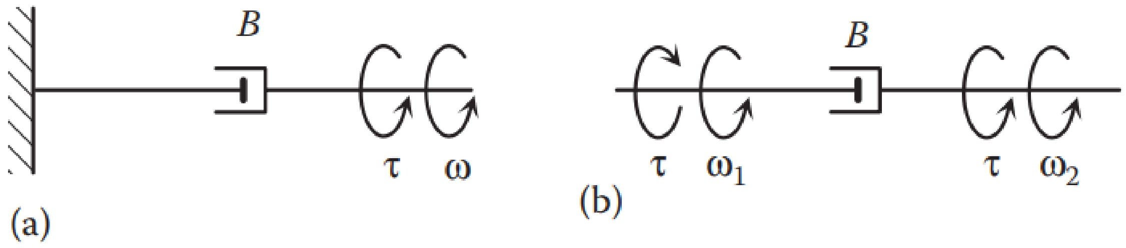
- B is small enough to be neglected (this is always an approximation.)
- Damping is used to model a dashpot (damper), e.g. shock absorbers on cars.



MECHANICAL ELEMENTS: ROTATIONAL

- **Damper:** For a torsional damper, the linear relationship between the externally applied torque and the angular velocity is given by:

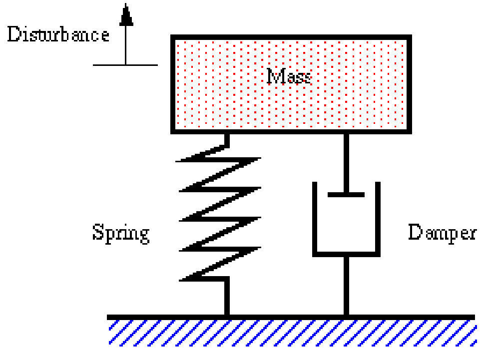
$$\tau = B\omega$$



A rotational viscous damper with (a) one fixed end and (b) two free ends

- If $\omega_2 > \omega_1 > 0$, the magnitude of the applied torque is

$$\tau = B(\omega_2 - \omega_1)$$



MECHANICAL ELEMENTS:

- Dashpot: A dashpot element is a form of damping and can be considered to be represented by a piston moving in a viscous medium in a cylinder. As the piston moves the liquid passes through the edges of the piston, damping the movement of the piston. The force *F* which moves the piston is proportional to the velocity of the piston movement and is given by

$$f = b\frac{dy}{dt}$$

Note: A dashpot does not store energy.

Door Stoppers

Vehicle Suspension

Bridge Suspension

Flyover Suspension

EQUIVALENCE

- In many mechanical systems, multiple springs or dampers are used. In such cases, an equivalent spring stiffness constant or damping coefficient can be obtained to represent the combined elements.
- **Springs in Parallel**
- to the two springs are f_1 and f_2, respectively. Because the system is in static equilibrium, the total force is given by

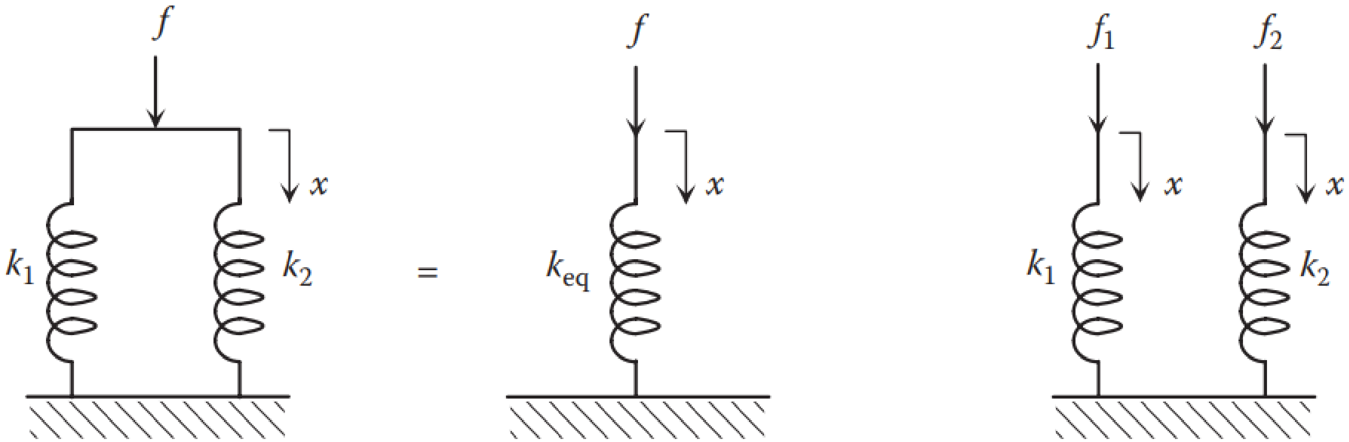
$$f = f_1 + f_2 = K_1 x + K_2 x = (K_1 + K_2)x = K_{eq} x$$

$$K_{eq} = K_1 + K_2$$

$$K_{eq} = K_1 + K_2 + \cdots + K_n$$

Note: for damping case, it is same:

$$C_{eq} = C_1 + C_2 + \cdots + C_n$$



EQUIVALENCE

- Springs in Series

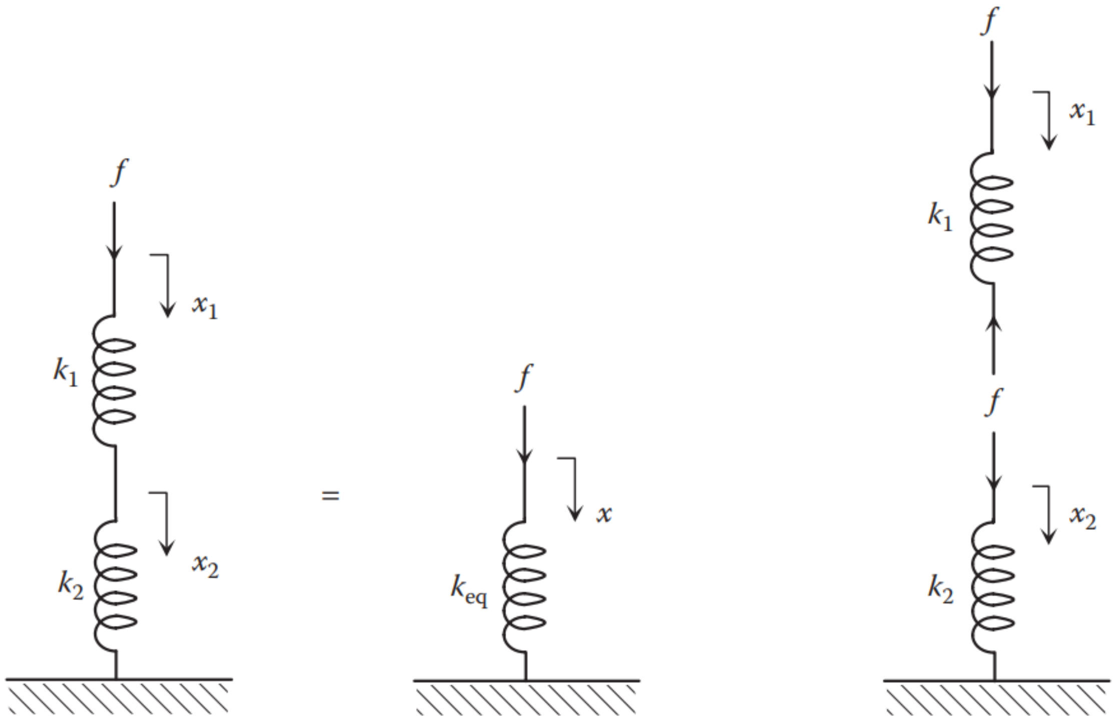
- Consider a system of two springs, k1 and k2, in series, as shown in below. the equivalent spring stiffness of the system is:

$$x = x_1 + x_2 = \frac{f}{K_1} + \frac{f}{K_2} = f\left(\frac{1}{K_1} + \frac{1}{K_2}\right) = \frac{f}{K_{eq}}$$

$$\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$$

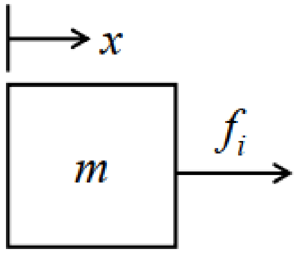
Note: for damping case, it is same: $C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$



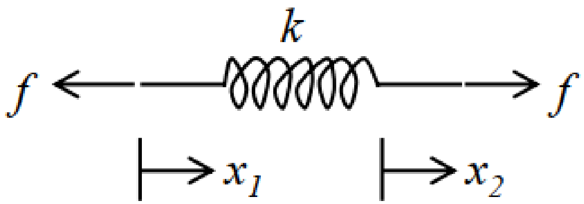
SUMMARY

Mass: Newton's 2nd Law

$$ma = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = \sum f_i$$

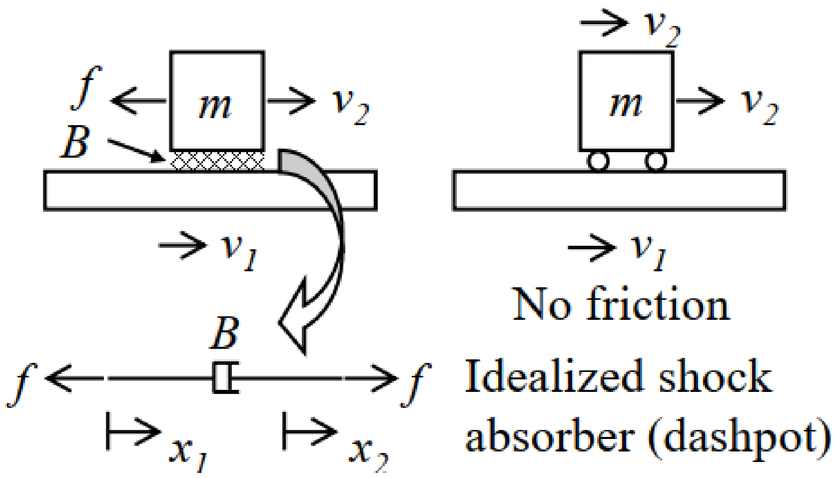


Stiffness (Spring): $f = k(x_2 - x_1)$



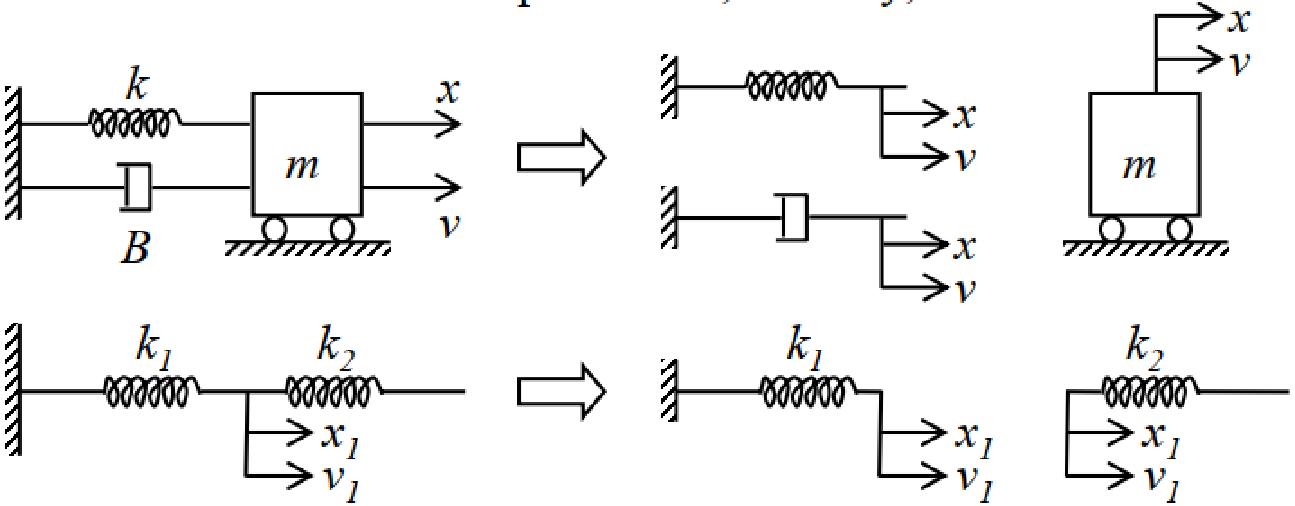
Friction (Damping): $f = B(v_2 - v_1) = B\Delta v$

B: viscosity constant, unit: N-s/m

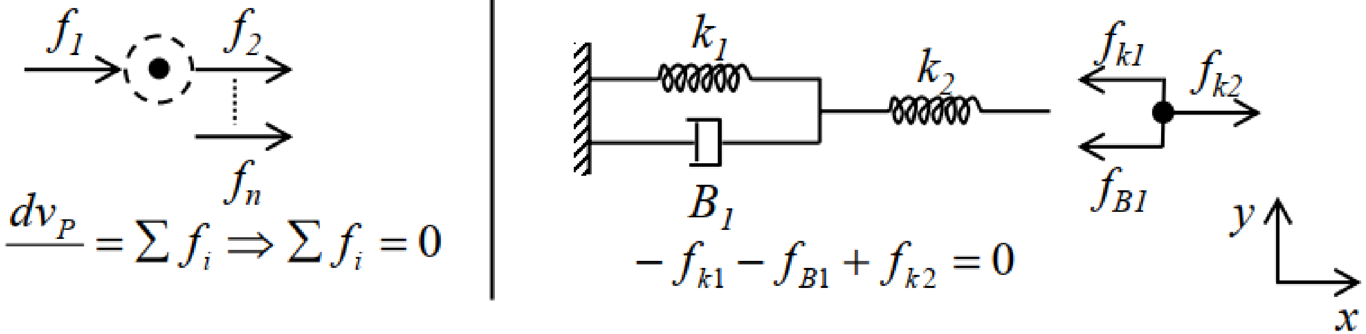


LAW OF DISPLACEMENTS

– If the ends of two elements are connected, these ends are forced to move with the *same* displacement, velocity, and acceleration.



– Newton's 2nd law at a point:
The sum of the forces at a connection between elements equals zero.

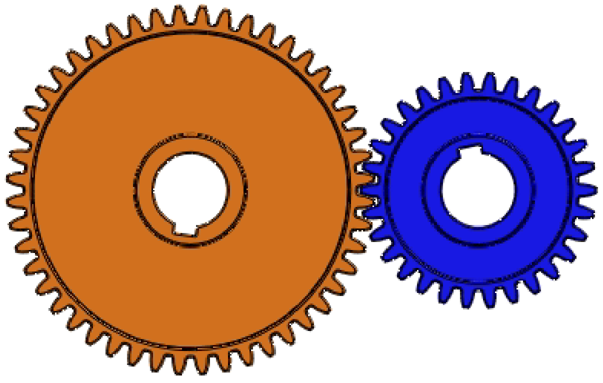


$$\overset{0}{\cancel{m}}_P \frac{dv_P}{} = \sum f_i \Rightarrow \sum f_i = 0$$

$$-f_{k1} - f_{B1} + f_{k2} = 0$$

MECHANICAL ELEMENTS:

- Gear train: A gear train is a mechanism used for transmitting rotary motion and torque through interlocking teeth.
- A gear train is made when two or more gears are meshed
- Driver gear causes motion then Driver gear causes motion
- The rpm of the larger gear is always slower than the rpm of the smaller gear.



- Variables to know

n = number of teeth
d = diameter
w = angular velocity (speed)
t = torque

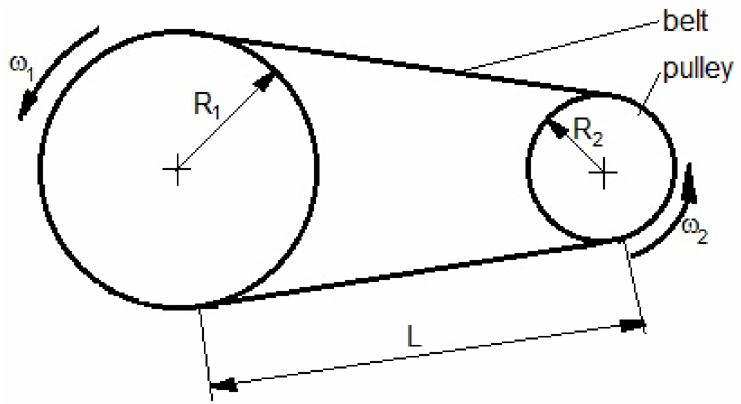
- Equations to know Gear Ratio

$$GR = \frac{N_G}{N_p} = \frac{D_G}{D_p} = \frac{\omega_p}{\omega_G} = \frac{T_G}{T_p}$$

MECHANICAL ELEMENTS:

- Pulley and Belt Systems:

- All power transmission belts are either friction drive or positive drive. Friction drive belts rely on the friction between the belt and pulley to transmit power. They require tension to maintain the right amount of friction. Flat belts are the purest form of friction drive while V-belts have a friction multiplying effect because of wedging action on the pulley.



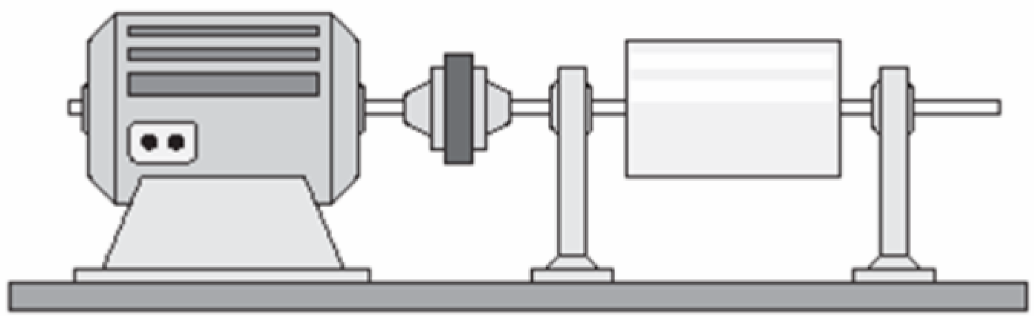
- Positive drive or synchronous belts rely on the engagement of teeth on the belt with grooves on the pulley. There is no slip with this belt except for ratcheting or tooth jumping.

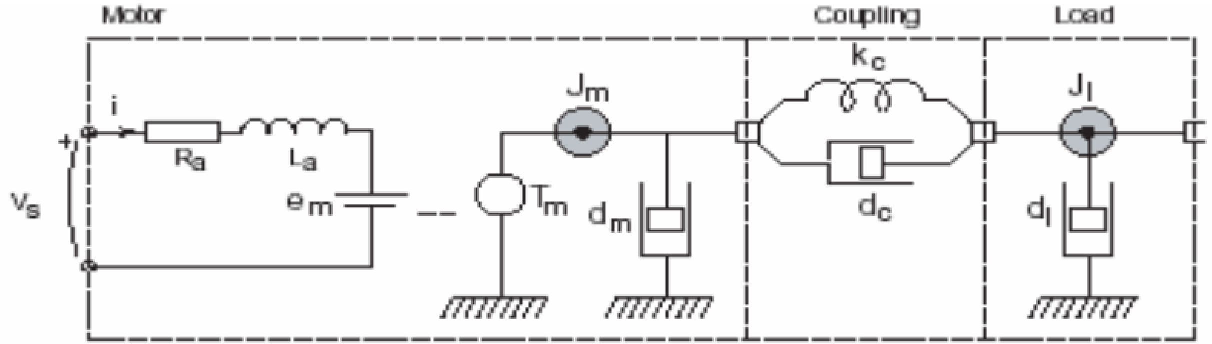
Equations:

$$\frac{D_1}{D_2} = \frac{\omega_2}{\omega_1} = \frac{T_1}{T_2}$$

PROCEDURE OF SYSTEM MODELING

- Divide the system into idealized components.
- Apply physical laws to the elements.
- Apply interconnection laws between elements.
- Combine the equations to obtain the model.





Apply physical laws

$$L_a \frac{di}{dt} = V_s - R_a i - K_m W_m$$

$$\frac{d\theta_m}{dt} = W_m$$

$$\frac{d\theta_l}{dt} = W_l$$

Apply interconnection laws

$$J_m \frac{dW_m}{dt} = K_m i - d_m W_m - K_c(\theta_m - \theta_l) - d_c(W_m - W_c)$$

$$J_l \frac{dW_l}{dt} = -d_l W_l - K_c(\theta_l - \theta_m) - d_c(W_l - W_m)$$

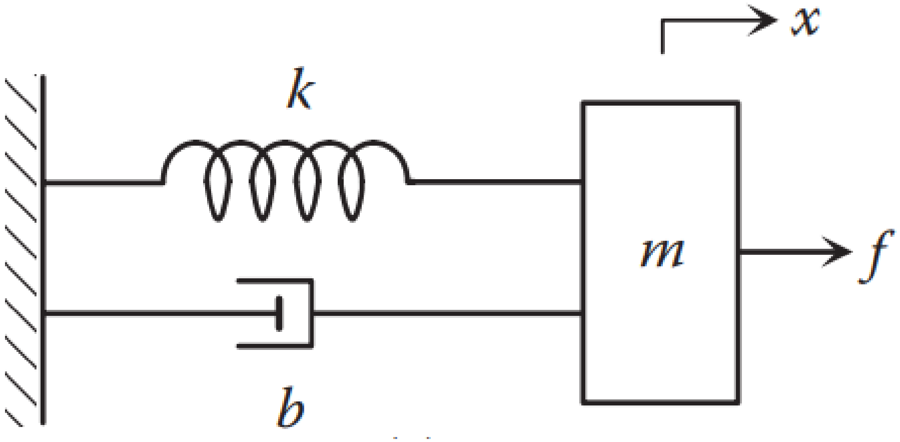
MECHANICAL APPLICATIONS:

Example: A Single-Degree-of-Freedom Mass–Spring–Damper System

Consider the simple mass–spring–damper system subjected to an input force f, as shown in figure:

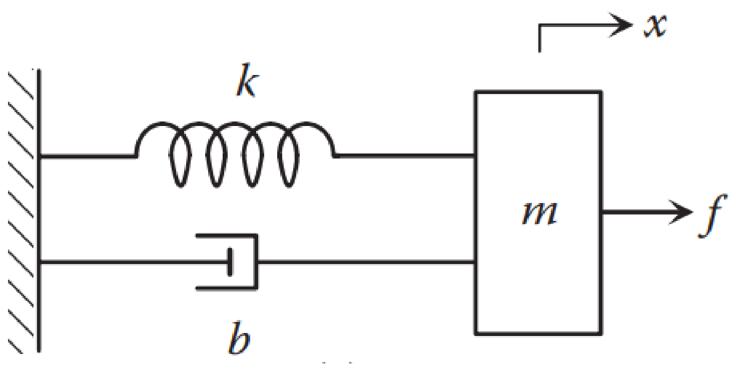
A. Apply Newton's second law to derive the differential equation of motion.

B. Determine the transfer function form. Assume that the system output is the displacement x and the initial conditions are $x(0) = 0$ and $\dot{x}(0) = 0$.



MECHANICAL APPLICATIONS:

- Solution:



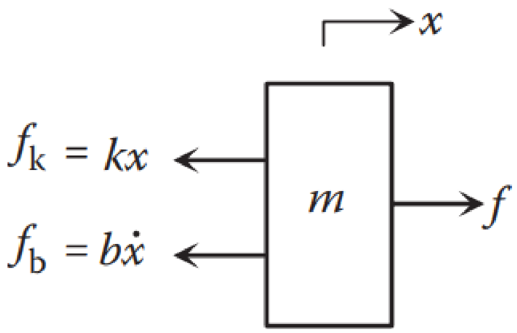
Let us choose the displacement of the mass as the coordinate x. The free-body diagram of the mass is shown below. Applying Newton's second law in the x direction gives

$$+\rightarrow x: \sum F_x = ma_x,$$

$$f(t) - kx - b\dot{x} = m\ddot{x},$$

which can be rearranged into the standard input–output differential equation form

$$m\ddot{x} + b\dot{x} + kx = f(t).$$



Taking the Laplace transform of both sides of the preceding equation with zero initial conditions results in

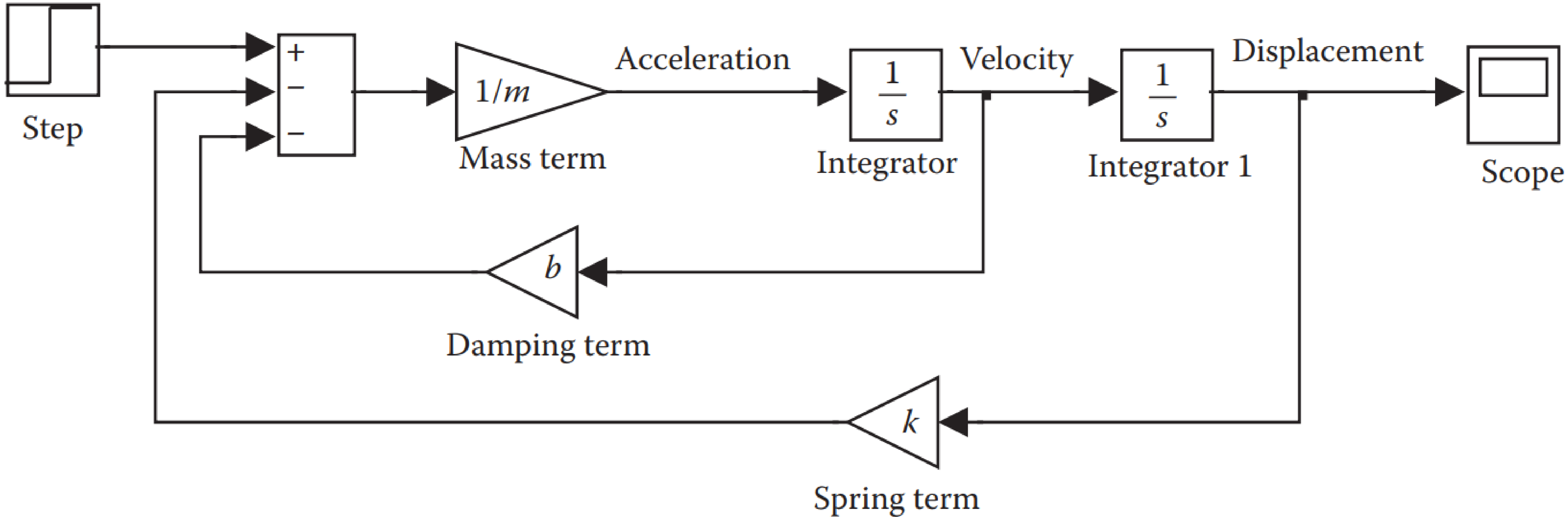
$$(ms^2 + bs + k)\,X(s) = F(s).$$

Thus, the transfer function relating the input $f(t)$ to the output $x(t)$ is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}.$$

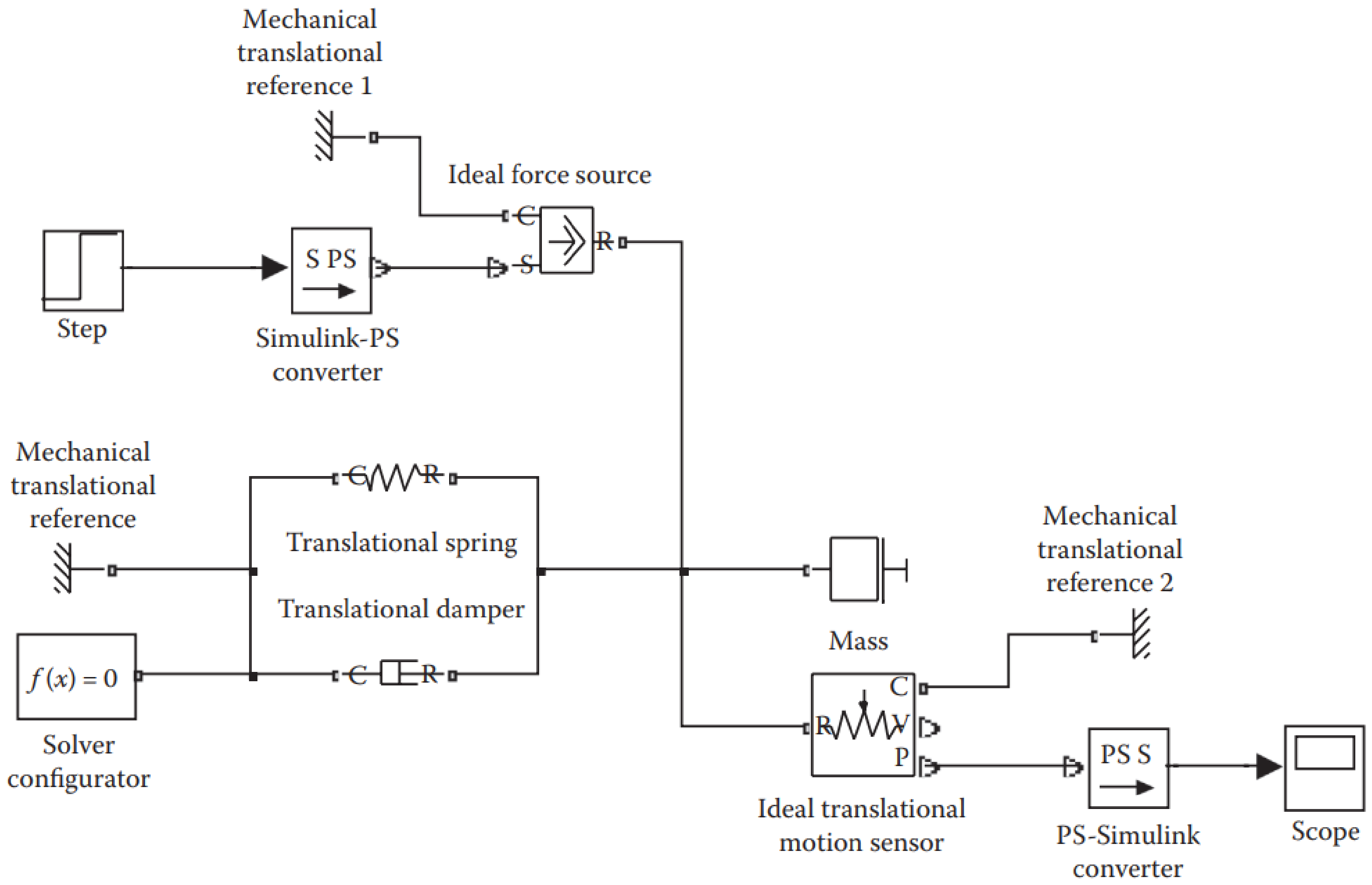
MECHANICAL APPLICATIONS:

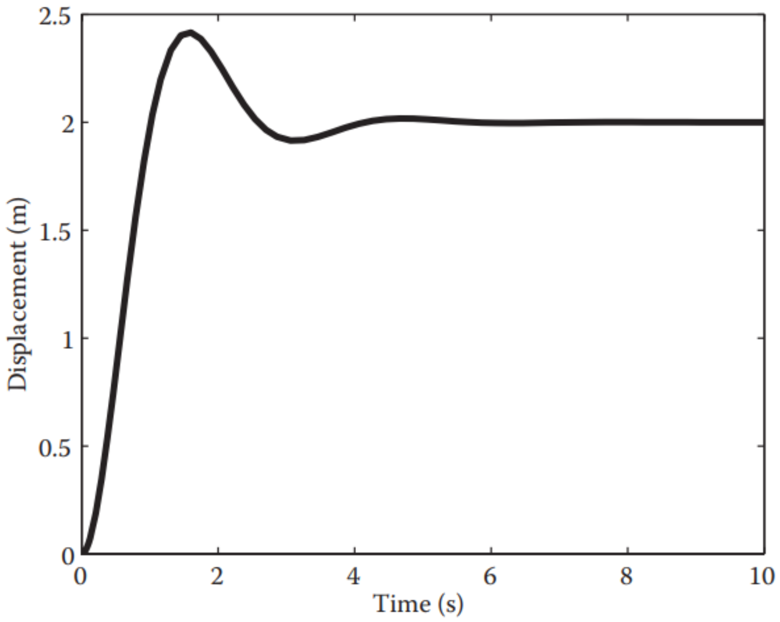
Simulink block diagram corresponding to Example. $(m = 5kg, b = 3N.s/m, K = 4N/m)$



MECHANICAL APPLICATIONS:

Simscape block diagram corresponding to Example.



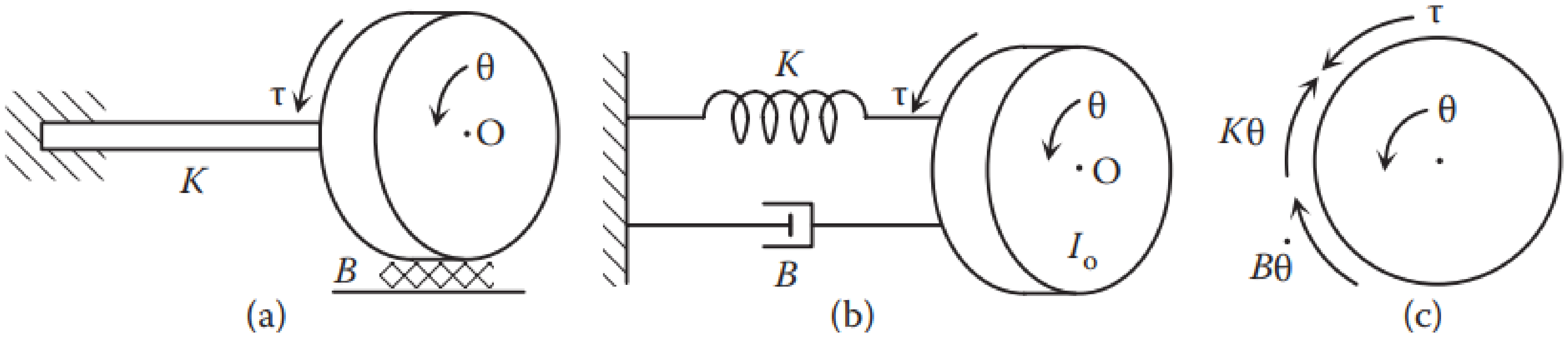


Displacement output $x(t)$ of the mechanical system in Example

MECHANICAL APPLICATIONS:

Example: A Single-Degree-of-Freedom Rotational Mass–Spring–Damper System

Consider a simple disk–shaft system shown in Figure a, in which the disk rotates about a fixed axis through point O. A single-degree-of-freedom torsional mass–spring–damper system in Figure b can be used to approximate the dynamic behavior of the disk–shaft system. I_o is the mass moment of inertia of the disk about point O, K represents the elasticity of the shaft, and B represents torsional viscous damping. Derive the differential equation of motion and find the transfer function.



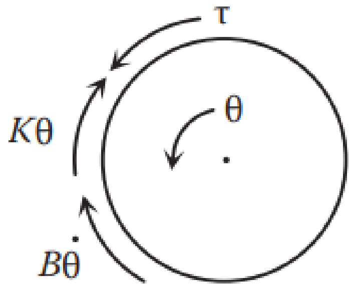
MECHANICAL APPLICATIONS:

Solution:

The free-body diagram of the disk is shown in Figure c. Because the disk rotates about a fixed axis, we can apply Rotational mass force equation about the fixed-point O. Assuming that counterclockwise is the positive direction, we have

$$+\curvearrowleft: \sum M_O = I_O\alpha,$$

$$\tau - K\theta - B\dot{\theta} = I_O\ddot{\theta}.$$



(c)

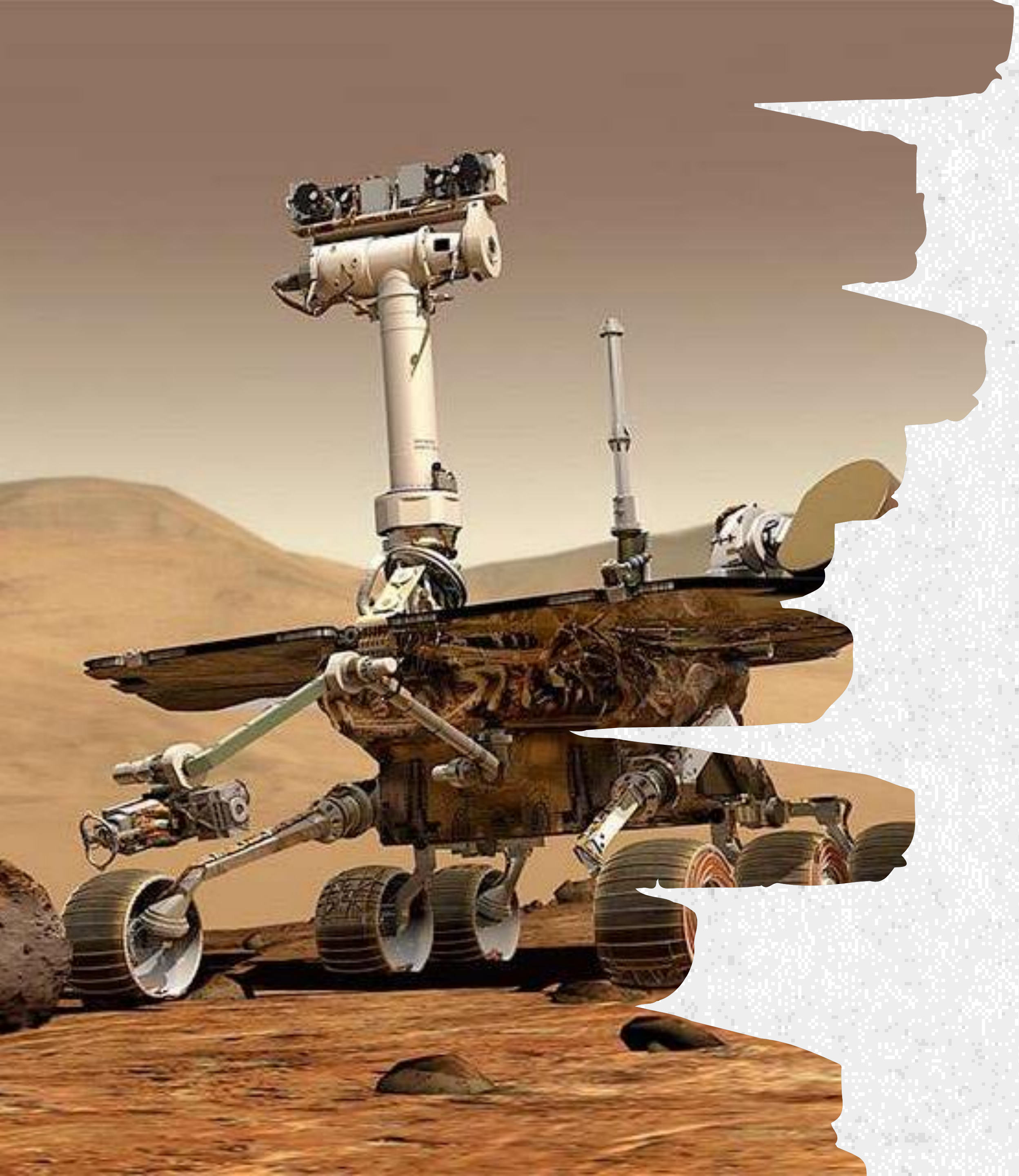
$$I_O\ddot{\theta} + B\dot{\theta} + K\theta = \tau.$$

By taking Laplace transform:

$$I_o s^2\theta_{(s)} + Bs\theta_{(s)} + K\theta_{(s)} = T_{(s)}$$

Then the TF is

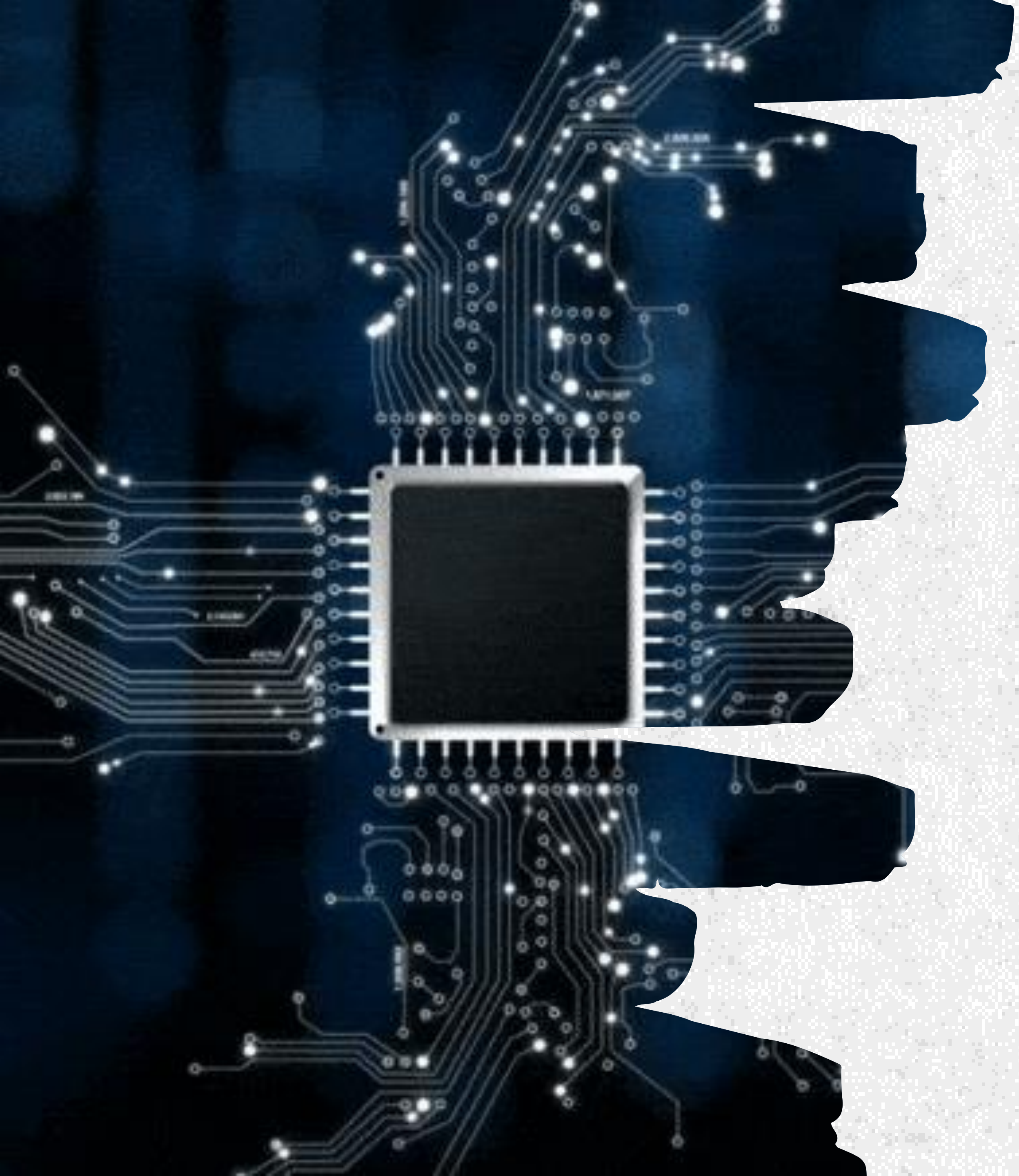
$$\frac{\theta_{(s)}}{T_{(s)}} = \frac{1}{I_o s^2 + Bs + K}$$



THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq



Ninevah University
Electronics Engineering College
Systems and control engineering department

SYSTEM MODELING

Lecture 4

Basic System Models
Electronic Systems

Instructor
Mohanad N. Noaman

ELECTRICAL ELEMENTS

- Voltage and current are the primary variables used to describe a circuit's behavior. ***Current*** is the flow of electrons. It is the time rate of change of electrons passing through a defined area, such as the cross section of a wire. The mathematical description of the relation between the number of electrons (called charge Q) and current i is

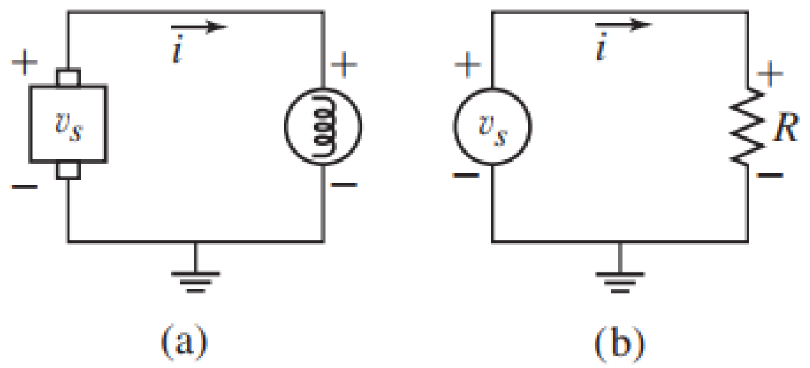
$$i = \frac{dQ}{dt} \qquad Q(t) = \int i\,dt$$

The unit of charge is the *coulomb* (C), and the unit of current is the *ampere* (A)

- **Energy** is required to move a charge between two points in a circuit. The work per unit charge required to do this is called voltage.

Most resistors are designed to have a linear relation between the current passing through them and the voltage difference across them. This linear relation is Ohm's law. It states that

$$v = iR$$



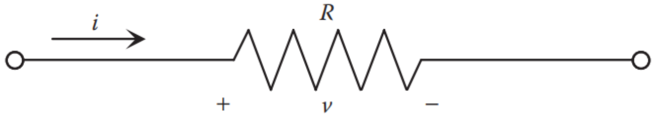
(a) A battery-lightbulb circuit.
(b) Circuit diagram representation of the battery-lightbulb circuit

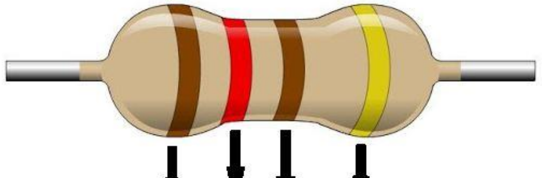
RESISTOR

- A **resistor** is an element for which there is an algebraic relationship between the voltage across its terminals and the current. it-that is, an element that can be described a curve of v versus i.

$$v = iR \qquad R = \frac{v}{i} \qquad P = Ri^2 = \frac{1}{R}v^2$$

- where R is the resistance in ohms (Ω).





CAPACITOR

- Capacitor stores electrons on 2 parallel plates separated by an insulating dielectric material in an electric field. For a linear the and voltage are Related by:

$$q = Cv$$

where C is the capacitance in farads (F).

- If the previous equation is differentiated the element law for a fixed linear capacitor becomes

$$i = C\frac{dv}{dt}$$

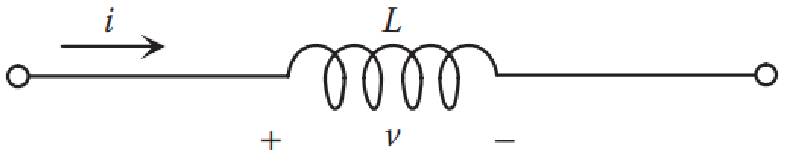
Energy stored in capacitor:

$$E = \frac{1}{2}Cv^2$$

INDUCTOR

- An inductor is an element for which there is an algebraic relationship between the voltage across its terminals and the derivative of the flux linkage. For a linear inductor,

$$v = L\frac{di}{dt}$$



where L is the inductance with units of Henries (H).

The energy supplied to an inductor is stored in its magnetic field $E = \frac{1}{2}Li^2$

IMPEDANCE

1. **RESISTOR** Z_R

2. **CAPACITOR** Z_C

3. **INDUCTOR** Z_L

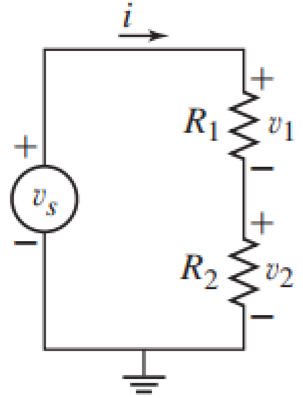
SERIES AND PARALLEL RESISTANCES

- Series

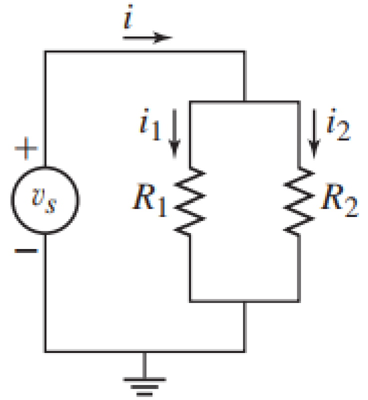
$$R_{eq} = R_1 + R_2 + \cdots + R_n$$

- Parallel

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$$



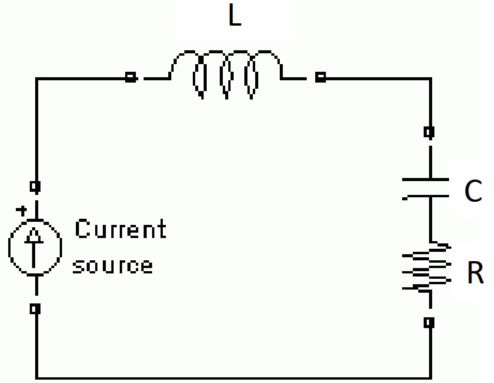
Series resistors.



Parallel resistors

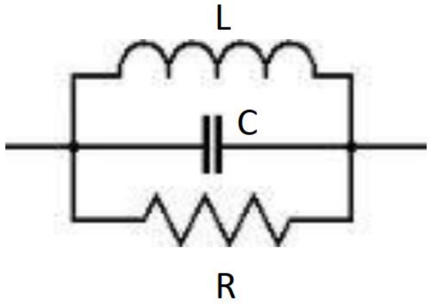
Equivalent Transform Impedance (Series)

$$\boldsymbol{Z_T = Z_R + Z_L + Z_C} \qquad \boldsymbol{Z_T = R + L(t) + \frac{1}{C(t)}}$$



Equivalent Transform Impedance (Parallel)

$$\boldsymbol{\frac{1}{z_T} = \frac{1}{z_R} + \frac{1}{z_L} + \frac{1}{z_C}} \qquad \boldsymbol{\frac{1}{z_T} = \frac{1}{R} + \frac{1}{L(t)} + \frac{1}{\frac{1}{C(t)}}}$$



LAPLACE IN ELECTRIC CIRCUITS

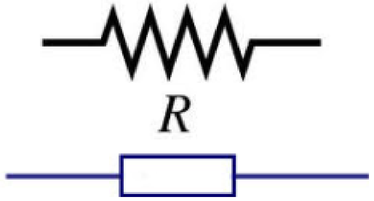
- For a resistor:

$$v(t) = R\, i(t)$$

$$\downarrow \mathcal{L}$$

$$V(s) = R\, I(s)$$

$$\mathcal{L}\{resistor\} = \frac{V(s)}{I(s)} = R$$



Impedance **Z**: instantaneous ratio of voltage difference to current

Remember that

$$Z_R = R$$

LAPLACE IN ELECTRIC CIRCUITS

- For a capacitor:

$$i(t) = C\frac{dv(t)}{dt}$$

$$\downarrow \mathcal{L}$$

$$I(s) = CsV(s)$$

$$\mathcal{L}\{capacitor\} = \frac{V(s)}{I(s)} = \frac{1}{sC}$$

Definition of C:

$$C \stackrel{\text{def}}{=} \frac{q(t)}{v(t)}$$

Remember that

$$Z_C = \frac{1}{j\omega C}$$

LAPLACE IN ELECTRIC CIRCUITS

- For an inductor:

$$v(t) = L\frac{di(t)}{dt}$$

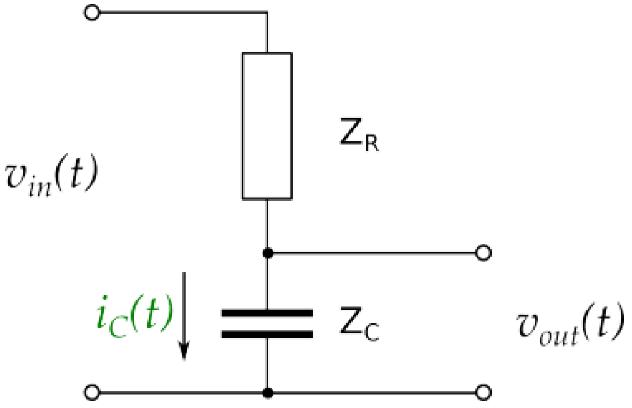
$$\downarrow \mathcal{L}$$

$$V(s) = LsI(s)$$

$$\mathcal{L}\{inductor\} = \frac{V(s)}{I(s)} = sL$$

Remember that
$Z_L = j\omega L$

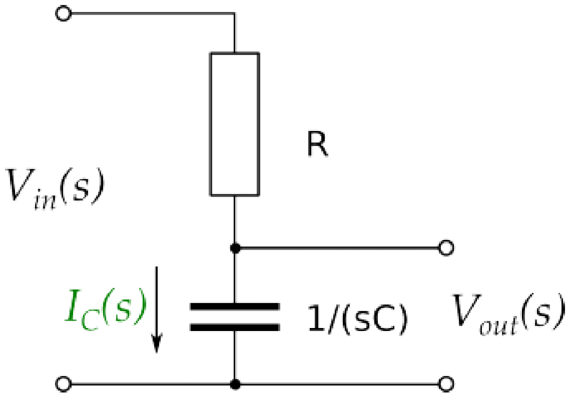
TF OF ELECTRIC CIRCUITS



$$i_C(t) = \frac{v_{in}(t)}{Z_R + Z_C}$$

$$v_{out}(t) = Z_C\, i_C(t)$$

$$v_{out}(t) = \frac{Z_C}{Z_R + Z_C} v_{in}(t)$$



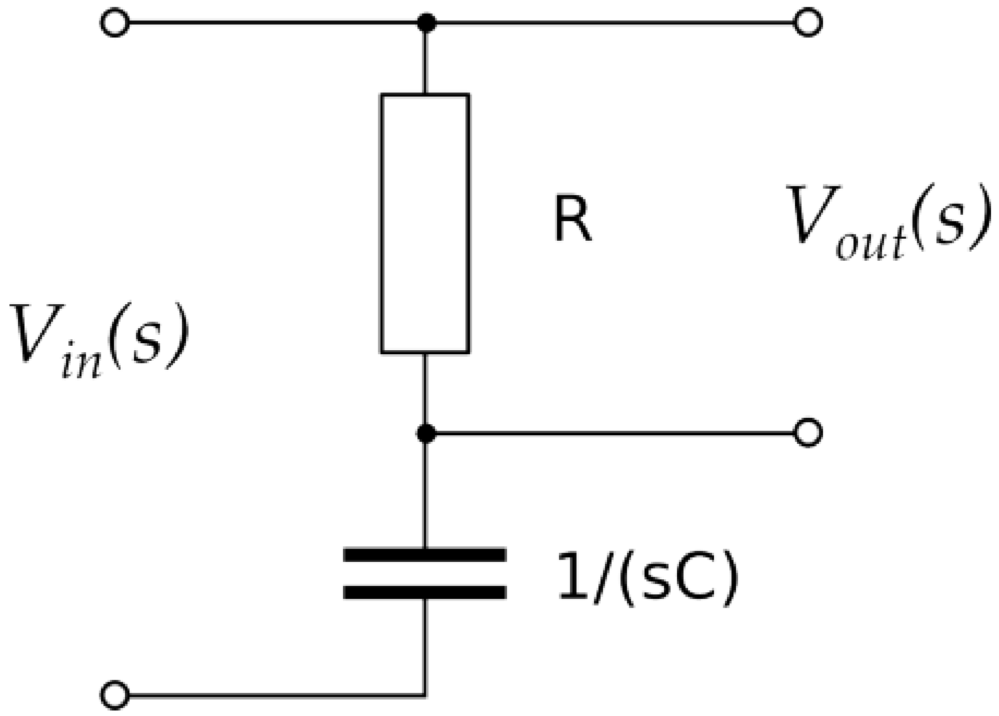
$$V_{out}(s) = \frac{Z_C}{Z_R + Z_C} V_{in}(s)$$

$$H(s) \stackrel{\text{def}}{=} \frac{V_{out}(s)}{V_{in}(s)} = \frac{Z_C}{Z_R + Z_C}$$

$$H(s) = \frac{1/(sC)}{R + 1/(sC)} = \frac{1}{sRC + 1}$$

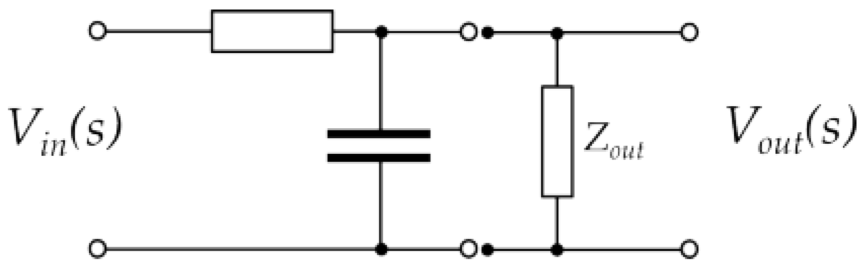
TF OF ELECTRIC CIRCUITS

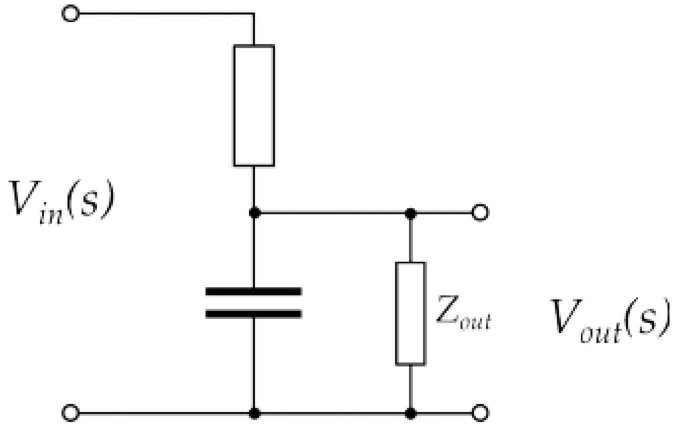
Calculate the TF of the following circuit:



EFFECT OF LOADING

when we connect an output (oscilloscope, speakers, …) we load the circuit:



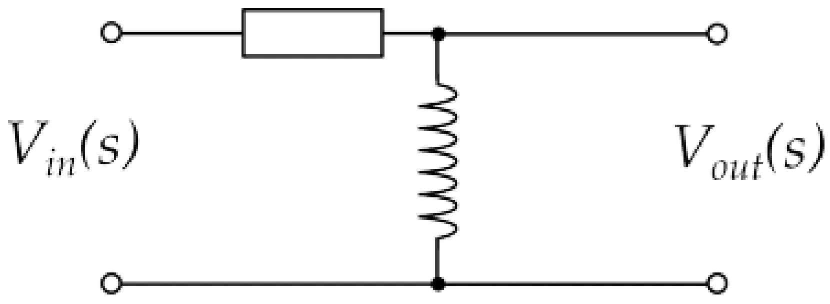


$$H(s) = \frac{1/(sC) \,||\, Z_{out}}{R + \left(1/(sC) \,||\, Z_{out}\right)}$$

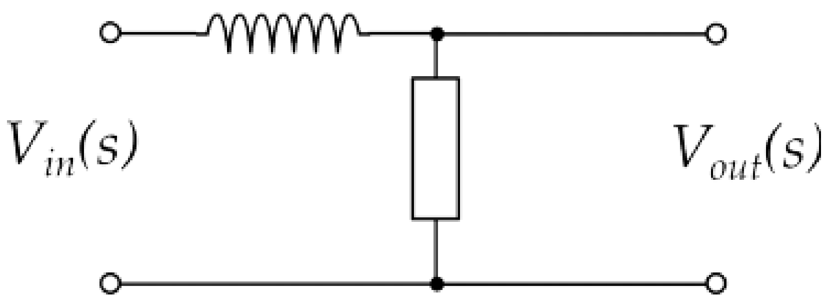
HOMEWORK: calculate TF when loading is a capacitor C

HOMEWORK: PRACTICE

Calculate the TF of the following circuits:



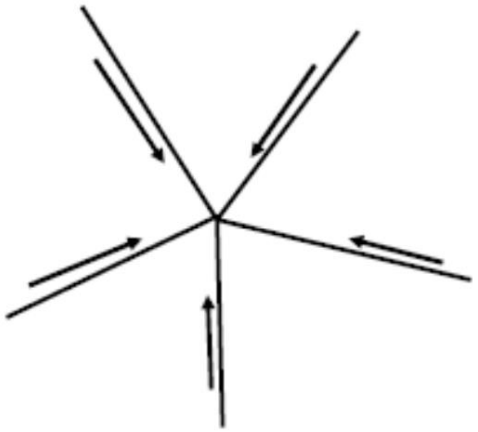
circuit A

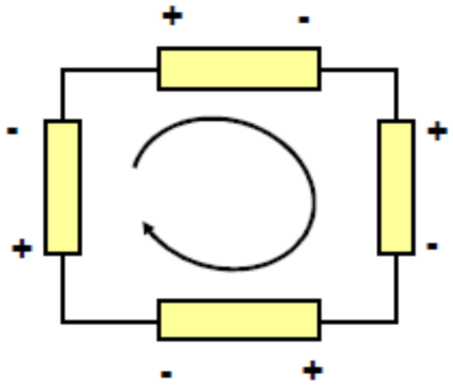


circuit B

KIRCHHOFF'S VOLTAGE LAW

- Kirchhoff's Current Law (KCL)
  - The sum of all currents (flow) entering a node is zero
- Kirchhoff's Voltage Law (KVL)
  - The oriented sum of all voltages (efforts) around any closed loop is zero


KCL



KVL

a. Applying Kirchhoff's voltage law to the single loop along the clockwise direction gives

$$-v_a + v_R + v_L + v_C = 0$$

For the series loop, the same current flows through each element. The expressions for v_R, v_L, and v_C are

$$v_R = iR$$

$$v_L = L\frac{di}{dt}$$

$$v_C = \frac{1}{C}\int i dt$$

We then have: $v_a = Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt$

KIRCHHOFF'S CURRENT LAW

- When the terminals of two or more circuit elements are connected together, the common junction is referred to as a node. For a node in a circuit, Kirchhoff's current law states that the sum of the currents entering the node must be equal to the sum of the currents leaving that node. The algebraic sum of the currents at the node must be zero,

$\sum_j i_j = 0$ where i_j is the current of the jth element at the node.

- Consider the parallel RLC circuit shown in Figure, in which an ideal current source supplies the desired current to the circuit.

We can apply Kirchhoff's law to either the ground or node 1. Applying Kirchhoff's current law to node 1 gives

$$i_a - i_R - i_L - i_C = 0$$

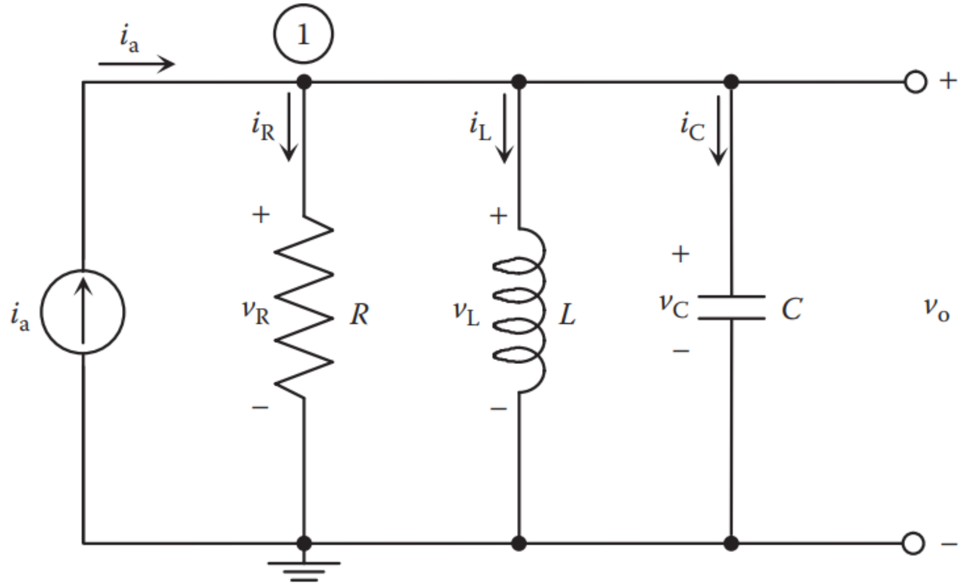
For the parallel connection, the voltages across all three elements are the same. The expressions for i_a, i_L, and i_C are

$$i_a = \frac{v_0}{R}$$

$$i_L = \frac{1}{L}\int v_o dt$$

$$i_C = C\frac{dv_o}{dt}$$

$$i_a = \frac{v_0}{R} + \frac{1}{L}\int v_o dt + C\frac{dv_o}{dt}$$



RECALL!

Recall both <u>Node and Loop analysis method.</u>

SUMMARY

- Voltage-current and energy relations for circuit elements.

Resistance:	$v = iR$	$P = Ri^2 = \dfrac{v^2}{R}$
Capacitance:	$v = \dfrac{1}{C}\displaystyle\int_0^t i\,dt$	$E = \dfrac{1}{2}Cv^2$
Inductance:	$v = L\dfrac{di}{dt}$	$E = \dfrac{1}{2}Li^2$

Relationship Between Electrical and Translational Elements

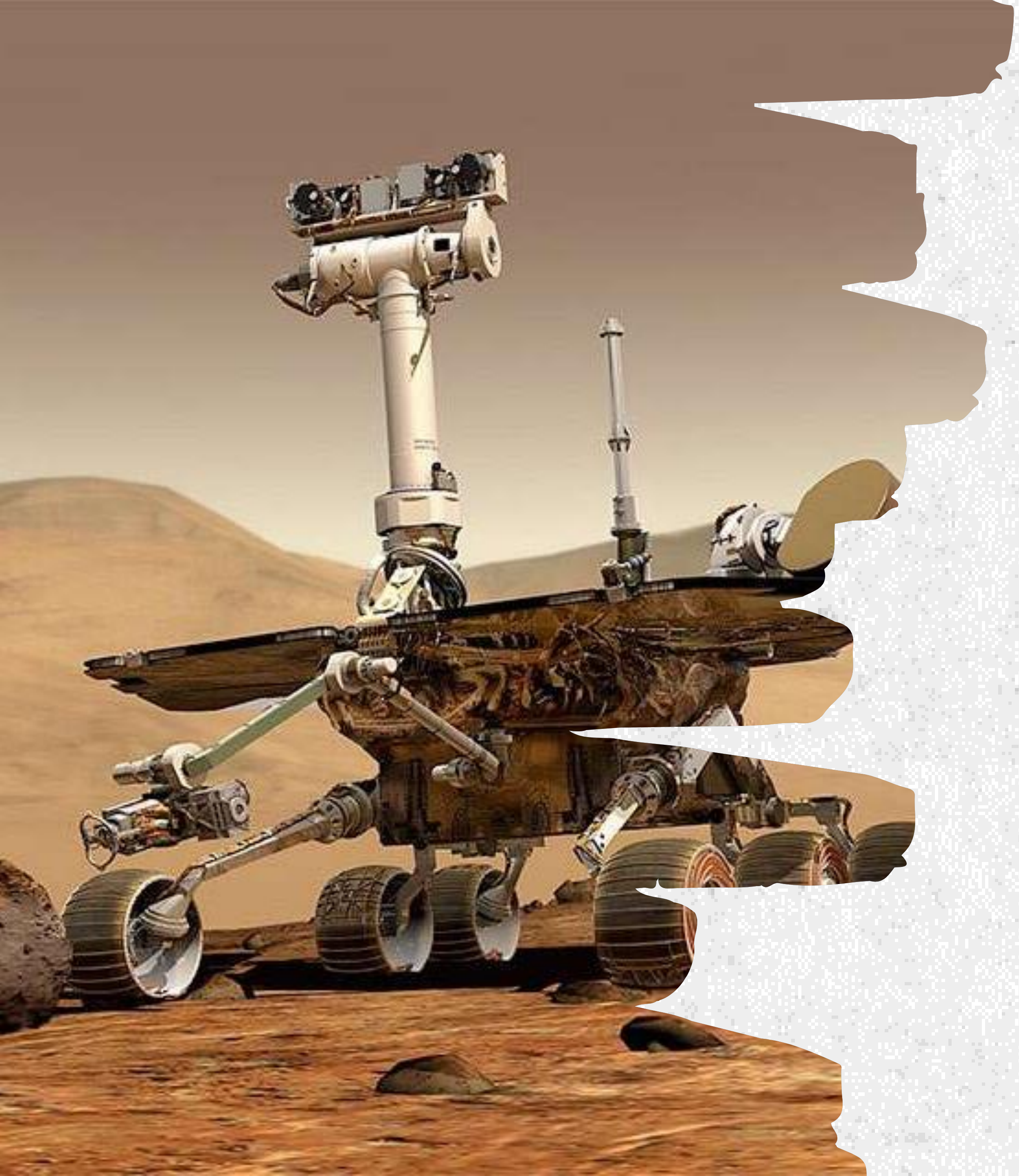
Notation	Variable	Notation	Variable
x	Position	q	Charge
$v(=\dot{x})$	Velocity	$i=\dot{q}$	Current
$a(=\dot{v}=\ddot{x})$	Acceleration	$\frac{di}{dt}=\ddot{q}$	Change in current
F	Force	e (or v)	Voltage

Relationship Between Electrical and Translational Elements

$F = Ma$	Newton's 2nd law	$e = L\frac{di}{dt}$	Inductor law
$F = Kx$	Spring law	$e(t) - e(0) = \frac{1}{C}\int_0^t i(u)du$ $e = \frac{1}{C}q$	Capacitor law
$F = Bv$	Damper law	$e = Ri$	Resistor law

Relationship Between Electrical and Translational Elements (cont'd)

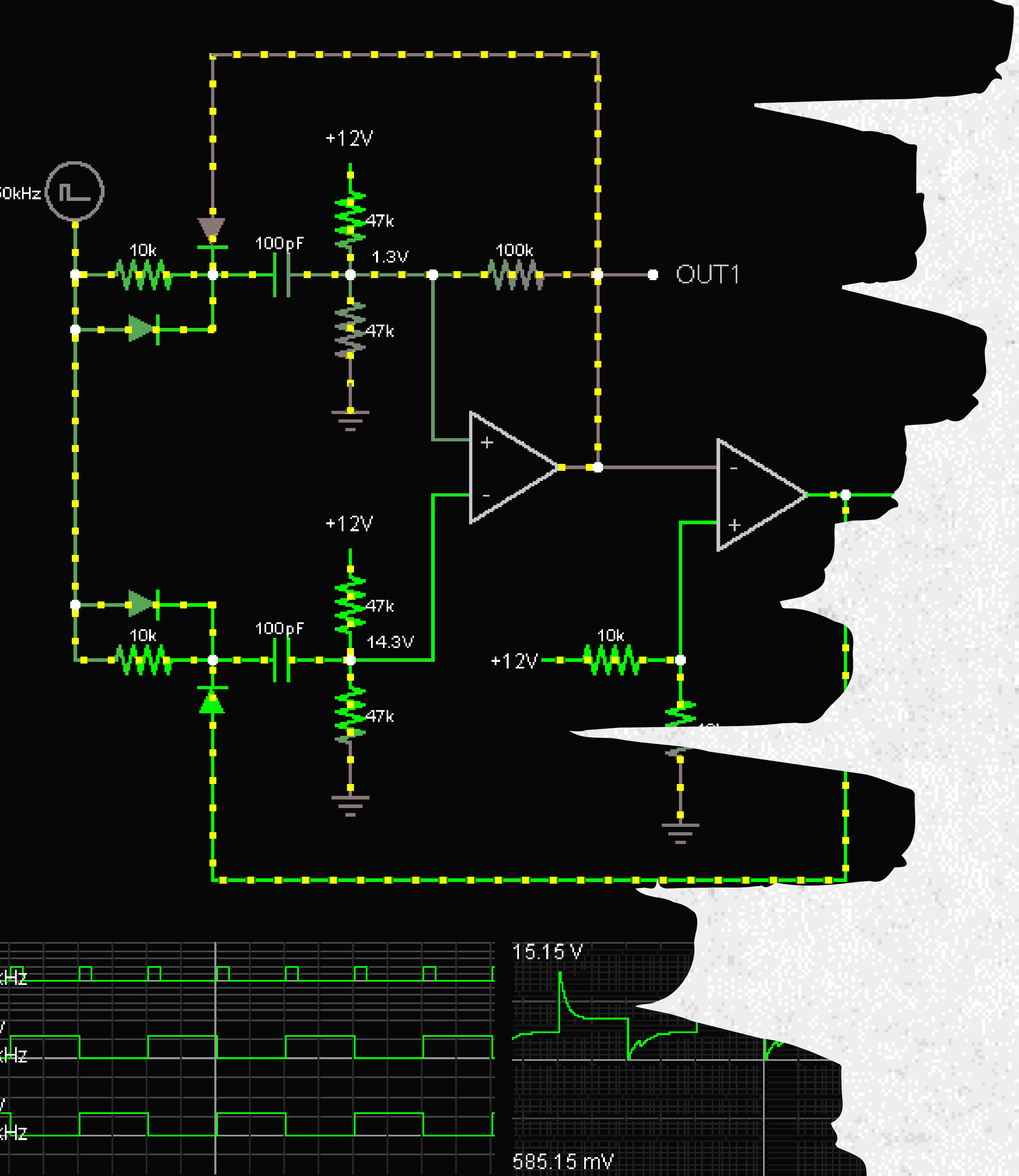
Notation	Variable	Notation	Variable
M	Mass	L	Inductance
K	Spring constant	$\frac{1}{C}$	1/Capacitance
B	Damping constant	R	Resistance



THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq



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SYSTEM MODELING

Lecture 4

Basic System Models

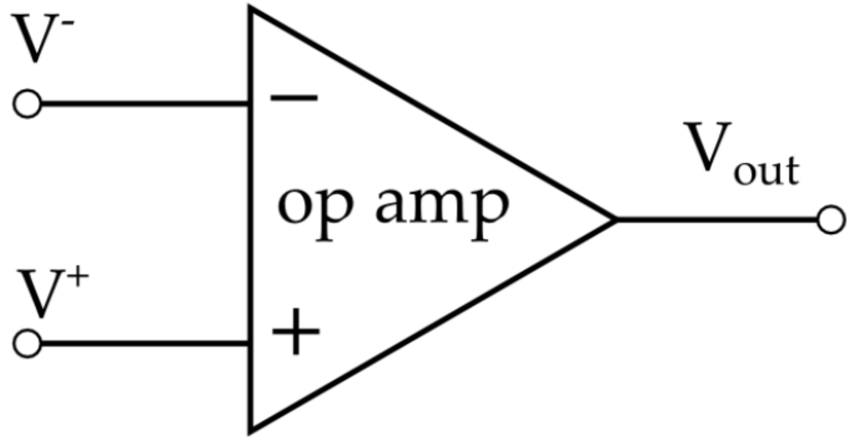
Op-Amps modelling

Instructor
Mohanad N. Noaman

OPERATIONAL AMPLIFIERS

- An ideal op amp has these characteristics:

1. Infinite input impedance
2. Infinite bandwidth
3. Infinite differential gain
4. Zero output impedance



Two “Golden Rules” follow:

Rule 1. The inputs draw no current.

Rule2. With negative feedback, output would do anything it takes to satisfy: $V^+ = V^-$.

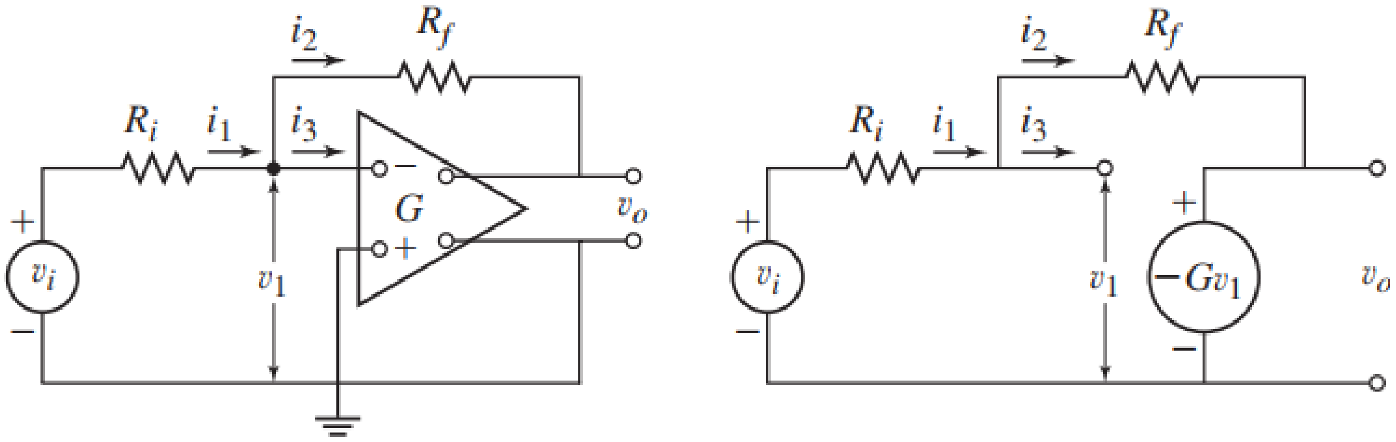
OPERATIONAL AMPLIFIERS

- Determine the relation between the input voltage v_i and the output voltage v_o of the op-amp circuit shown in Figure below. Assume that the op amp has the following properties:

1. The op-amp gain G is very large,

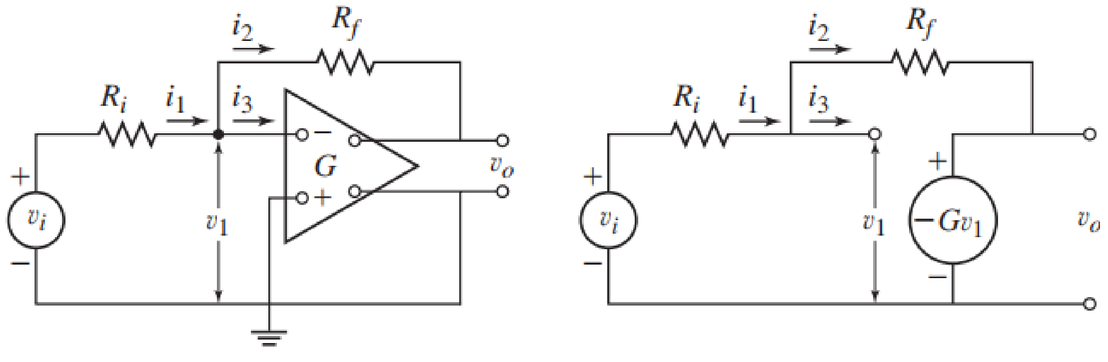
2. v_o = −Gv1; and

3. The op-amp input impedance is very large, and thus the current i_3 drawn by the op amp is very small



(a) (b)

OPERATIONAL AMPLIFIERS



(a) (b)

The voltage-current relation for each resistor gives

$$i_1 = \frac{v_i - v_1}{R_1}$$

and

$$i_2 = \frac{v_1 - v_o}{R_2}$$

From conservation of charge, $i_1 = i_2 + i_3$. However, from property 3, $i_3 \approx 0$, which implies that $i_1 \approx i_2$. Thus,

$$\frac{v_i - v_1}{R_1} = \frac{v_1 - v_o}{R_2}$$

From property 1, $v_1 = -v_o/G$. Substitute this into the preceeding equation:

$$\frac{v_i}{R_1} + \frac{v_o}{R_1 G} = -\frac{v_o}{R_1 G} - \frac{v_o}{R_2}$$

Because G is very large, the terms containing G in the denominator are very small, and we obtain

$$\frac{v_i}{R_1} = -\frac{v_o}{R_2}$$

Solve for v_o:

$$v_o = -\frac{R_2}{R_1} v_i \qquad (1)$$

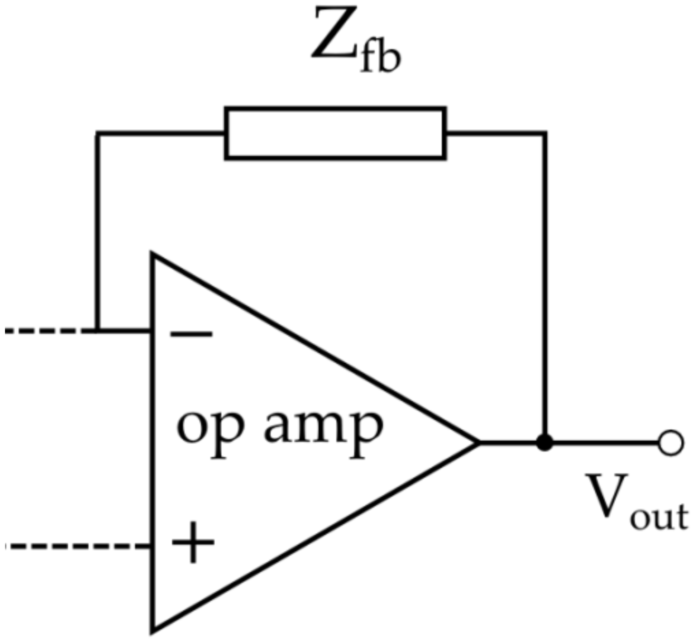
OPERATIONAL AMPLIFIERS

- Negative Voltage Feedback

Some of the voltage from the output is fed-back to the inverting (minus) terminal (negative feedback).

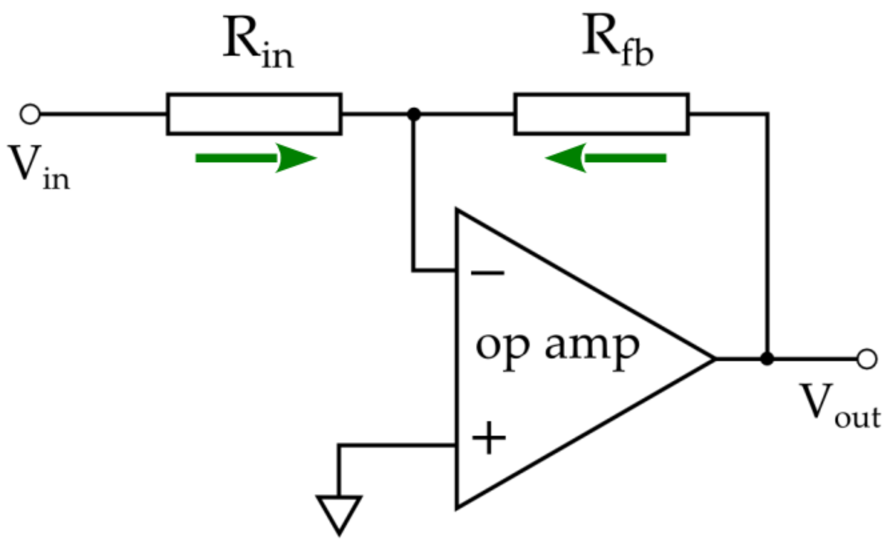
- If V+ increases, $(V^+ - V^-)$ increases
- op amp amplifies $(V^+ - V^-)$ to the output
- but output is connected to V^-
- so V^- also increases, and $(V^+ - V^-)$ is reduced

Result: with negative feedback, $V^+ \cong V^-$ (Rule 2)



OPERATIONAL AMPLIFIERS

- Inverting Configuration



- non-inverting input (+) is to ground
- inverting input is between two resistors that connect input to output

Ideal op amp:

- $V^+ - V^- = 0$, so $V^- = 0$
- infinite input impedance (so $I^- = 0$)

$$I_{in} = \frac{V_{in}}{R_{in}} \quad \text{and} \quad I_{fb} = \frac{V_{out}}{R_{fb}}$$

$V^+ - V^- = 0$, so $V^- = 0$

$$I_{in} + I_{fb} = 0$$

infinite input impedance (so $I^- = 0$)

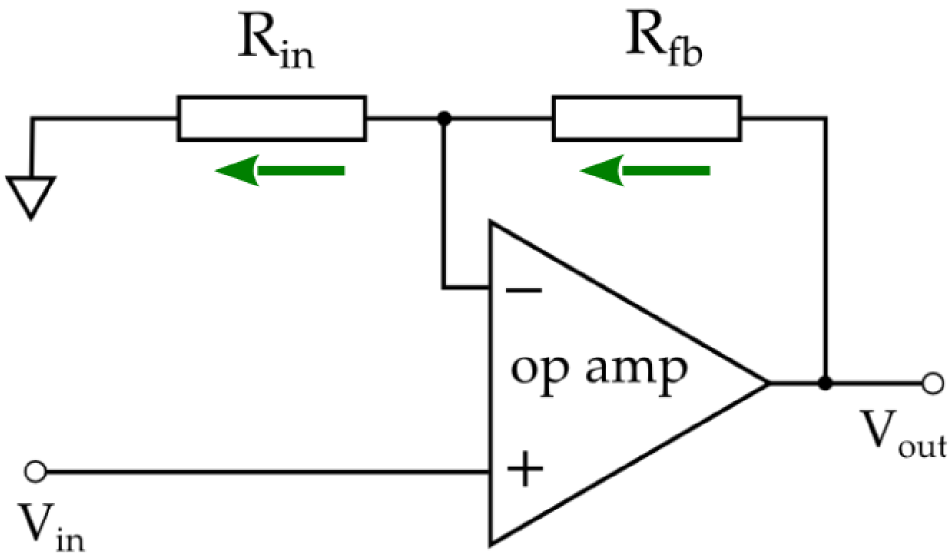
$$H \stackrel{\text{def}}{=} \frac{V_{out}}{V_{in}} = -\frac{R_{fb}}{R_{in}}$$

OPERATIONAL AMPLIFIERS

- Non-Inverting Configuration
- non-inverting input (+) is tied to input
- inverting input (-) is between two resistors that connect ground to output

Ideal op amp:

- $V^+ - V^- = 0$, so $V^- = V^+$
- infinite input impedance (so $I^- = 0$)



$$I_{in} = \frac{V_{in}}{R_{in}} \quad \text{and} \quad I_{fb} = \frac{V_{out} - V_{in}}{R_{fb}}$$

$V^+ - V^- = 0$, so $V^- = V^+$

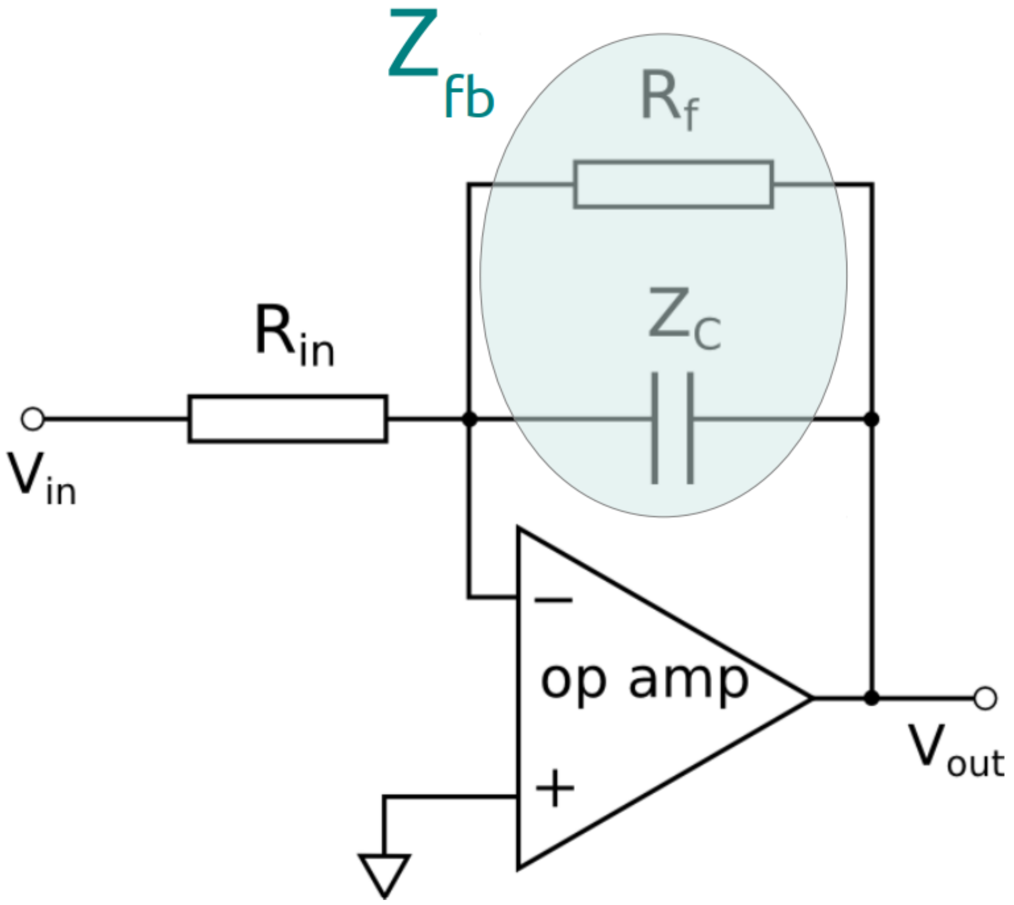
$$I_{in} = I_{fb}$$

infinite input impedance (so $I^- = 0$)

$$H \stackrel{\text{def}}{=} \frac{V_{out}}{V_{in}} = 1 + \frac{R_{fb}}{R_{in}}$$

OPERATIONAL AMPLIFIERS

- What is the transfer function of this circuit?



OPERATIONAL AMPLIFIERS

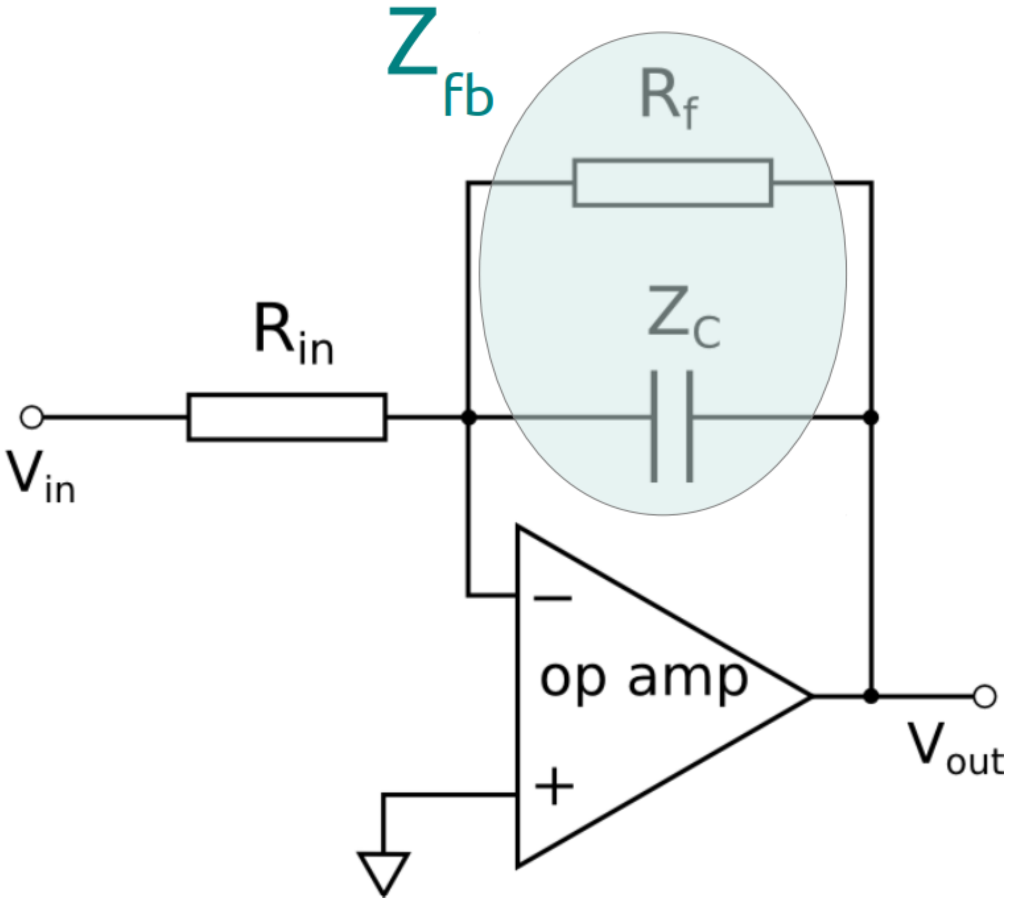
- What is the transfer function of this circuit?

$$H \overset{\text{def}}{=} \frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{Z_{\text{fb}}}{R_{\text{in}}}$$

There are two impedances in parallel:

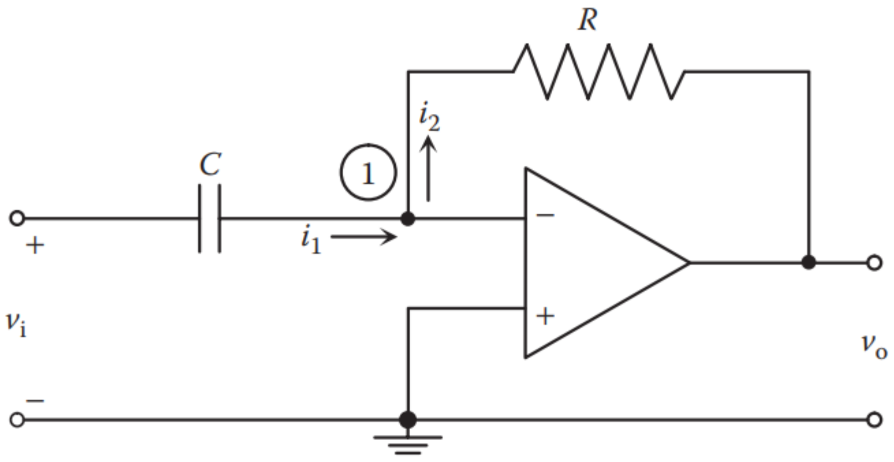
$$\frac{1}{Z_{fb}} = \frac{1}{Z_C} + \frac{1}{R_f} = sC + \frac{1}{R_f} = \frac{sR_fC+1}{R_f}$$

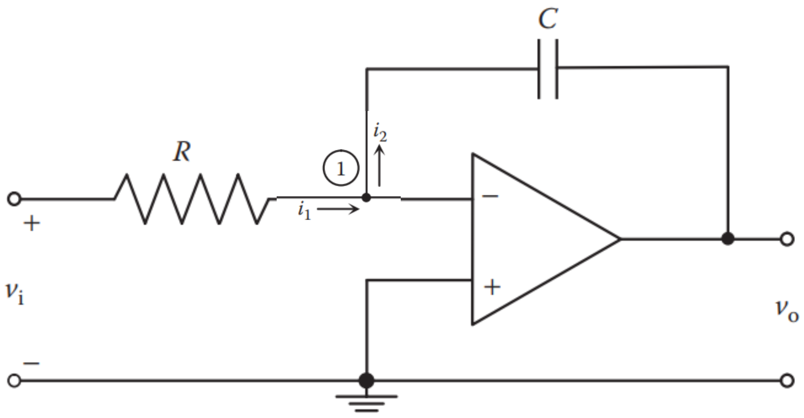
$$H(s) = -\frac{R_f/R_{\text{in}}}{sR_fC+1}$$



OPERATIONAL AMPLIFIERS

- An Op-Amp Differentiator & Integrator





$$i_1 = i_2$$

$$C\frac{d}{dt}(v_i - v_1) = \frac{v_1 - v_o}{R}$$

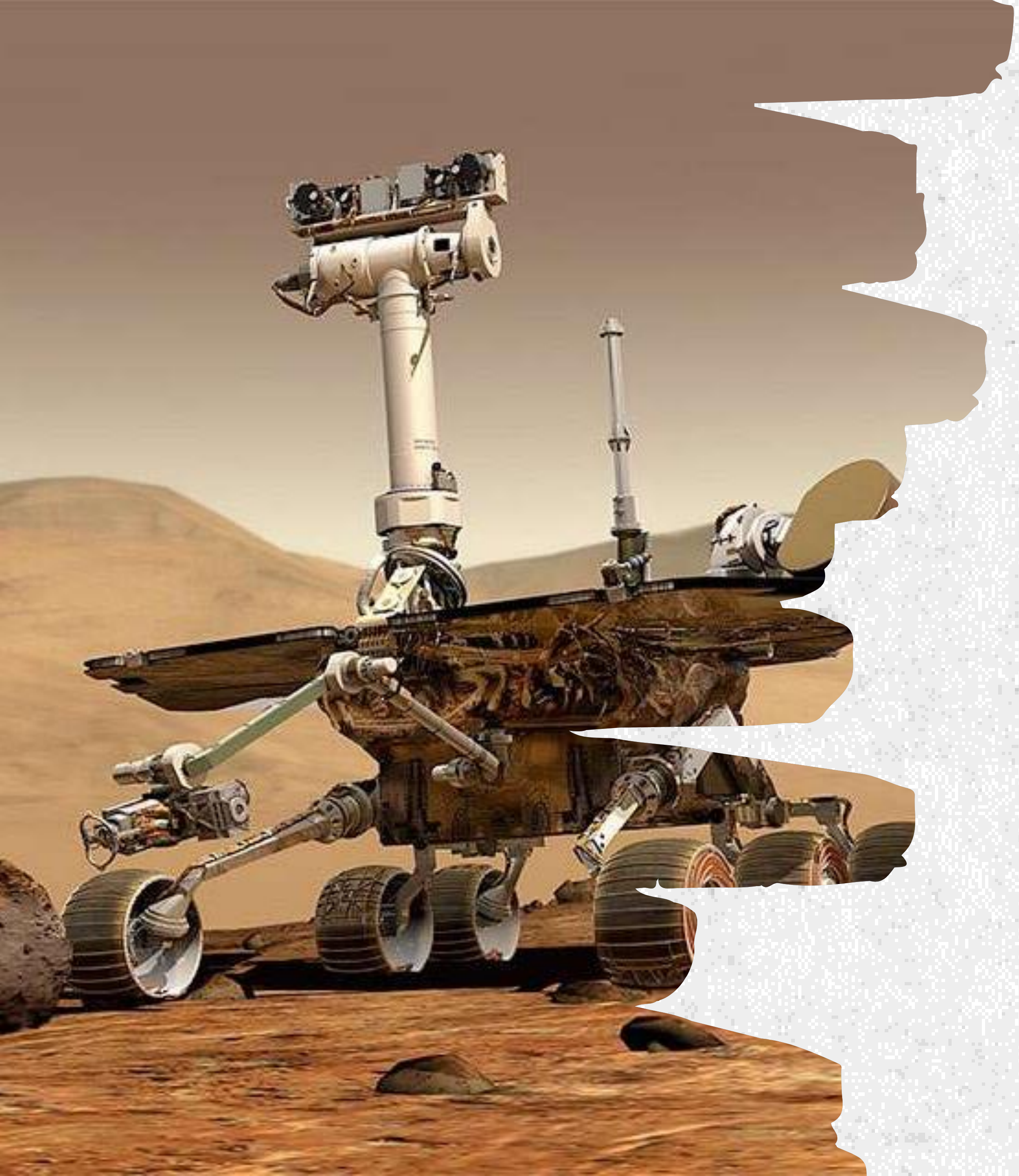
$$C\dot{v}_i = -\frac{v_o}{R}$$

$$v_o = -RC\dot{v}_i$$

$$i_1 = i_2$$

$$\frac{v_i - v_1}{R} = C\frac{d}{dt}(v_i - v_o)$$

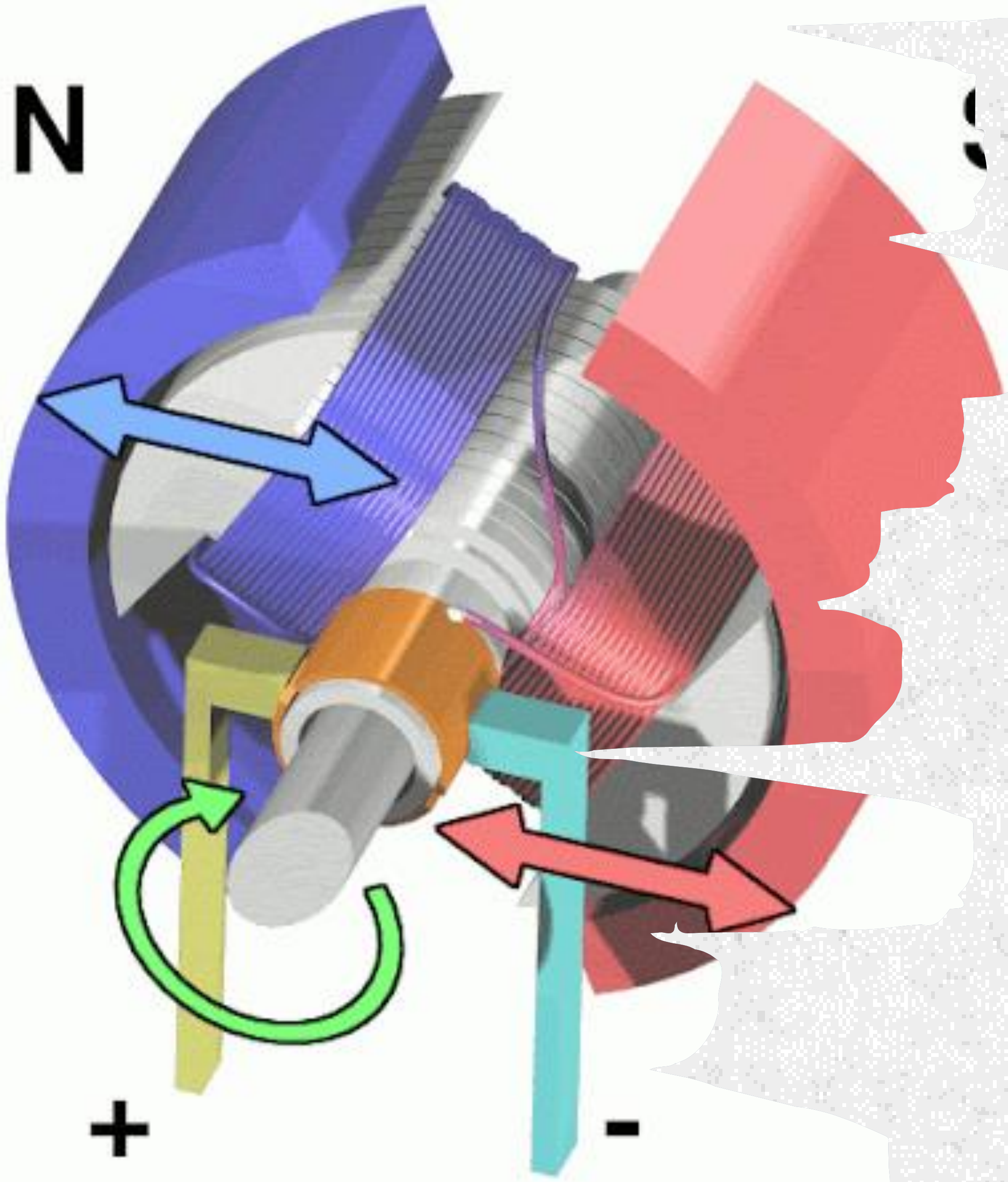
$$v_o = -\frac{1}{RC}\int v_i dt$$



THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq



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SYSTEM MODELING

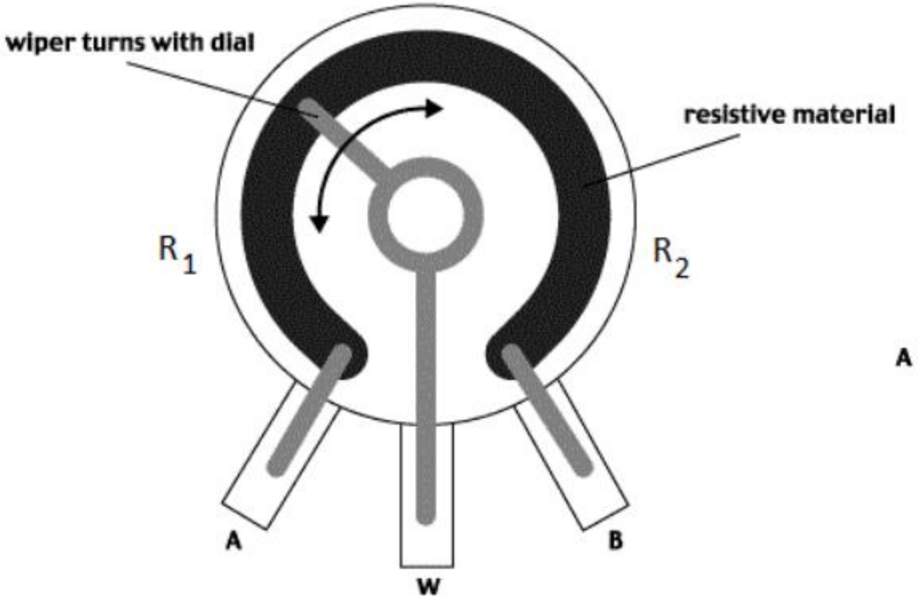
Lecture 6

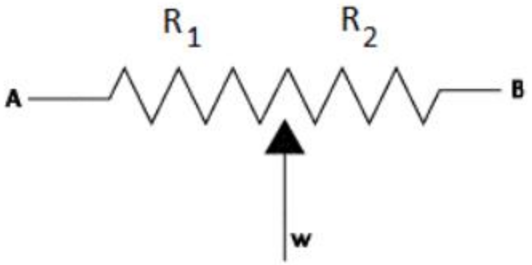
DC Motor Modeling

Instructor
Mohanad N. Noaman

POTENTIOMETERS MODELING

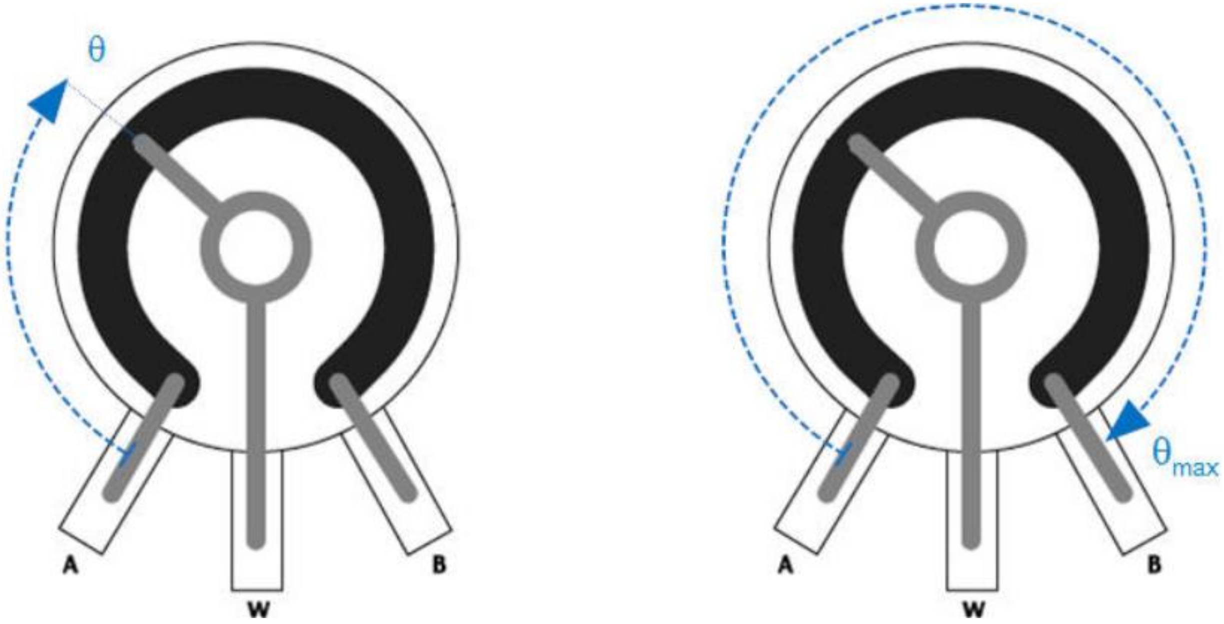
Perhaps the simplest coupled electromechanical system is the rotary potentiometer. The image below shows how a potentiometer works, followed by a photograph of an actual potentiometer.





POTENTIOMETERS MODELING

- This wiper can rotate about the center of the potentiometer while maintaining electrical contact with the terminal labeled "w.“
- The resistance between the wiper and "A" is labeled R1, the resistance between the wiper and "B" is labeled R2.
- The total resistance between "A" and "B" is constant, R1+R2= R_{tot}.
- If the potentiometer is turned to the extreme counterclockwise position such that the wiper is touching "A" we will call this θ=0; in this position R1=0 and R2=R_{tot}.
- If the wiper is in the extreme clockwise position such that it is touching "B" we will call this θ=θmax ; in this position R1=Rtot and R2=0.

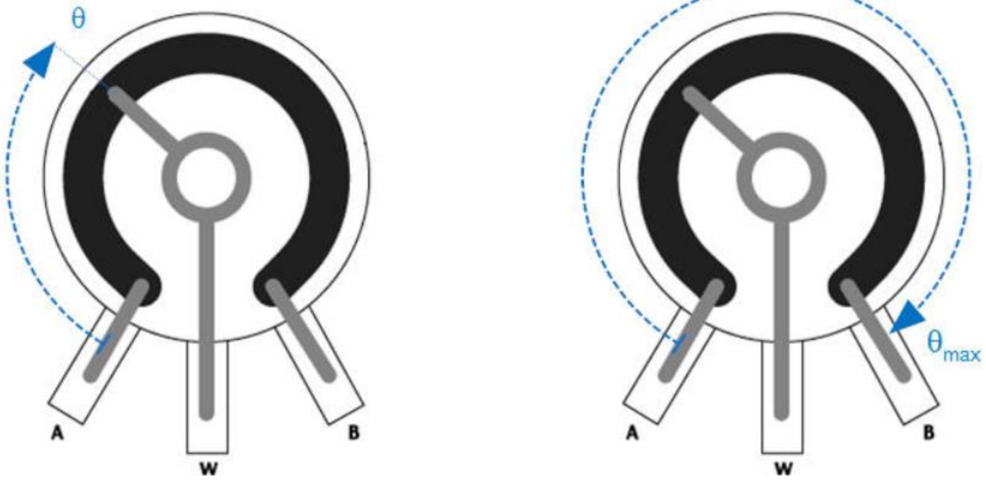


POTENTIOMETERS MODELING

R1 and R2 vary linearly with θ between the two extremes:

$$R_1 = \frac{\theta}{\theta_{max}} R_{tot}$$

$$R_2 = \frac{\theta_{max} - \theta}{\theta_{max}} R_{tot}$$

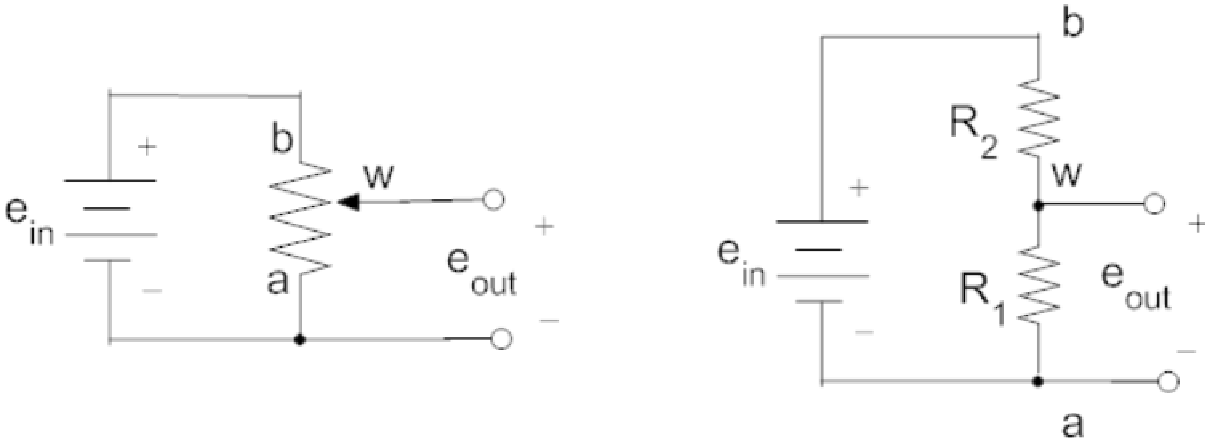


Using the voltage divider principle we can write:

$$e_{out} = \frac{R_1}{R_1 + R_2} e_{in} = \frac{R_1}{R_{tot}} e_{in}$$

We can now substitute the expression for R1 in terms of θ,

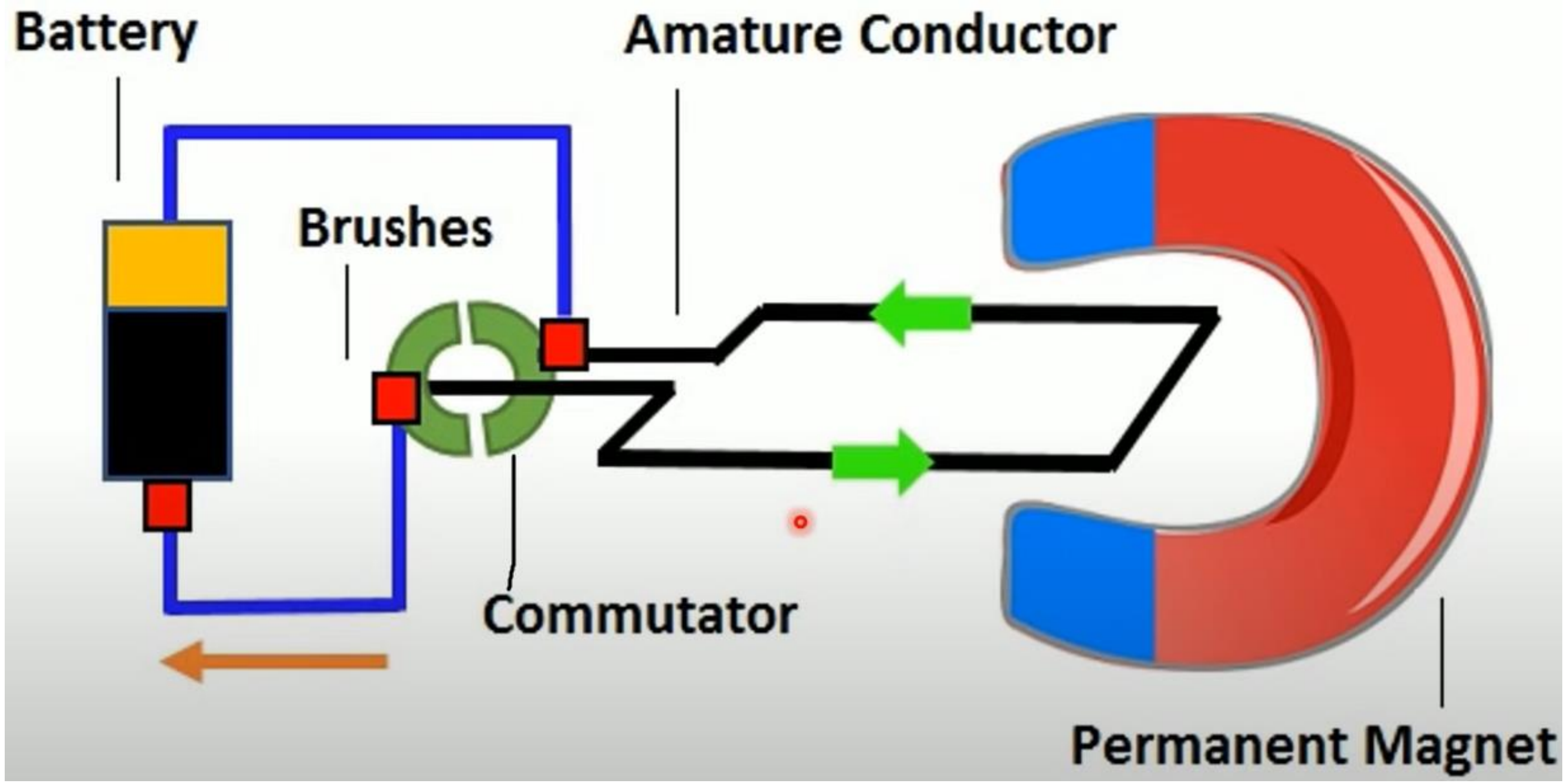
$$e_{out} = \frac{\frac{\theta}{\theta_{max}} R_{tot}}{R_{tot}} e_{in} = \frac{\theta}{\theta_{max}} e_{in}$$



In this expression you can see that e_{out} is directly proportional to θ, the rotation of the potentiometer.

MODELLING LINEAR SYSTEMS: MOTOR MODELING

- The DC motor is a common plant component in industrial systems – it directly provides rotary motion in response to an electrical input. In this section we shall derive the mathematical model of a simple DC motor and it's Transfer Function in the frequency domain.



DC MOTOR MODELING

The motor has a Voltage V applied to the motor's armature that is influenced by a fixed magnetic field. This results in an applied motor torque which rotates the motor in the direction of θ . The output of the system is the rotational angular rate ω of the shaft.

$$\omega = \frac{d\theta}{dt} = \dot{\theta} \quad (1)$$

Our model also assumes that the motor has simple viscous friction, that is the friction in the motor is directly proportional to the shaft angular rate but operating in the opposite direction (opposing rotation).

$$motor\ friction = -b\dot{\theta} \quad (2)$$

where b is the motor viscous friction constant.

DC MOTOR MODELING

The torque generated by a DC motor is proportional to the current flowing in the armature circuit and the strength of the magnetic field. In this analysis we'll assume the field is of fixed strength and thus that the motor torque T is related to the armature current i by a constant factor K_t as shown in equation 3

$$T = K_t i \qquad (3)$$

The back e.m.f. produced by the motor is directly proportional to the angular velocity of the shaft (and of opposite polarity to applied voltage) and is given by equation 4

$$e = K_e \dot{\theta} \qquad (4)$$

DC MOTOR MODELING

In SI units the motor torque and back e.m.f. constants are equal $K_t = K_e = K$.
By applying Newton's second law to the rotor we get

$$J\ddot{\theta} = T - b\dot{\theta} = Ki - b\dot{\theta} \qquad (5)$$

Applying Kirchoff's Voltage Law to the Armature circuit yields

$$L\frac{di}{dt} + Ri = V - e = V - K\dot{\theta} \qquad (6)$$

We can re-write these equations in terms of ω thus:

$$J\dot{\omega} + b\omega = Ki \qquad (7)$$

$$L\frac{di}{dt} + Ri = V - K\omega \qquad (8)$$

DC MOTOR MODELING

Thus 7 and 8 are the differential equations that represent the DC motor. In order to expressthe performance of the system as a transfer function relating the output ω to the input V expressed in terms of the Laplace variable s we need to take Laplace transforms of these equations:

$$JsW(s) + bW(s) = KI(s) \quad \text{\_\_\_\_\_\_\_\_\_\_(9)}$$

$$LsI(s) + RI(s) = V(s) - KW(s) \quad \text{\_\_\_\_(10)}$$

Where $W(s)$ and $I(s)$ are the Laplace transforms of $\omega(t)$ and $i(t)$.

By eliminating $I(s)$ from equations 9 and 10 we can see that the transfer function (in the s-domain) of the DC motor is given by

$$P(s) = \frac{W(s)}{V(s)} = \frac{K}{(Js+b)(Ls+R)+K^2} \quad \text{\_\_\_\_\_\_\_\_(11)}$$

DC MOTOR MODELING

Aside

If we rewrote 7 and 8 in terms of θ instead of ω we get

$$J\ddot{\theta} + b\dot{\theta} = Ki \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(12)$$

$$L\frac{di}{dt} + Ri = V - K\dot{\theta} \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(13)$$

Taking Laplace transforms:

$$Js^2\Theta(s) + bs\Theta(s) = KI(s) \quad \_\_\_\_\_\_\_\_\_\_(14)$$

$$LsI(s) + RI(s) = V(s) - Ks\Theta(s) \quad \_\_\_\_\_(15)$$

Re-arranging to eliminate $I(s)$ yields

$$\frac{\Theta(s)}{V(s)} = \frac{K}{\{(Js+b)(Ls+R)+K^2\}s} \quad \_\_\_\_\_\_\_\_\_\_(16)$$

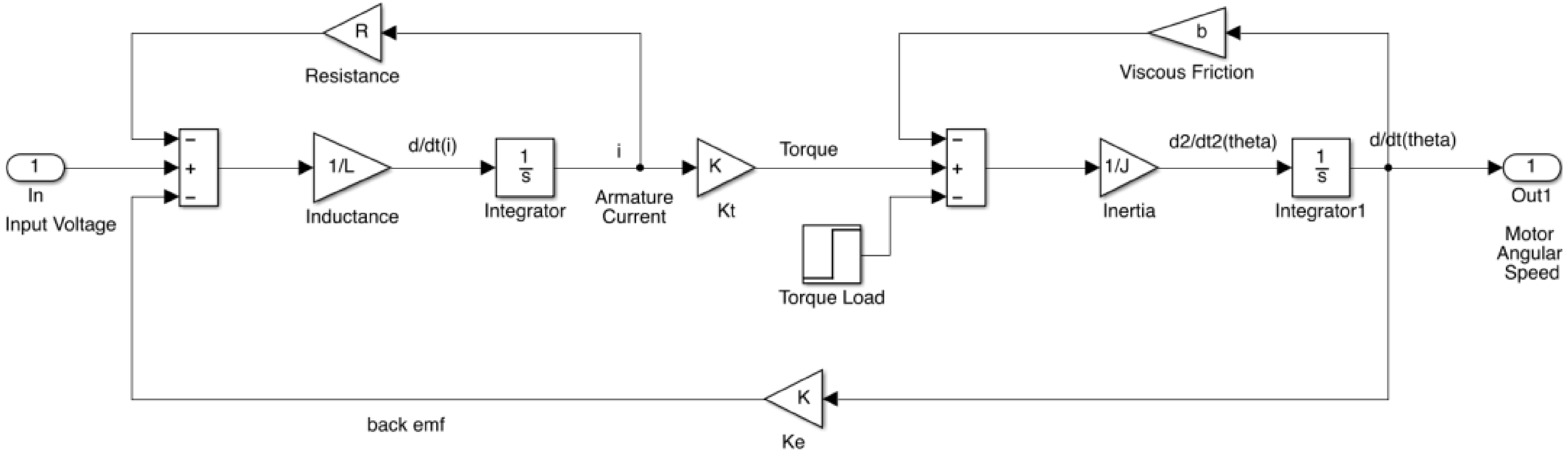
Thus

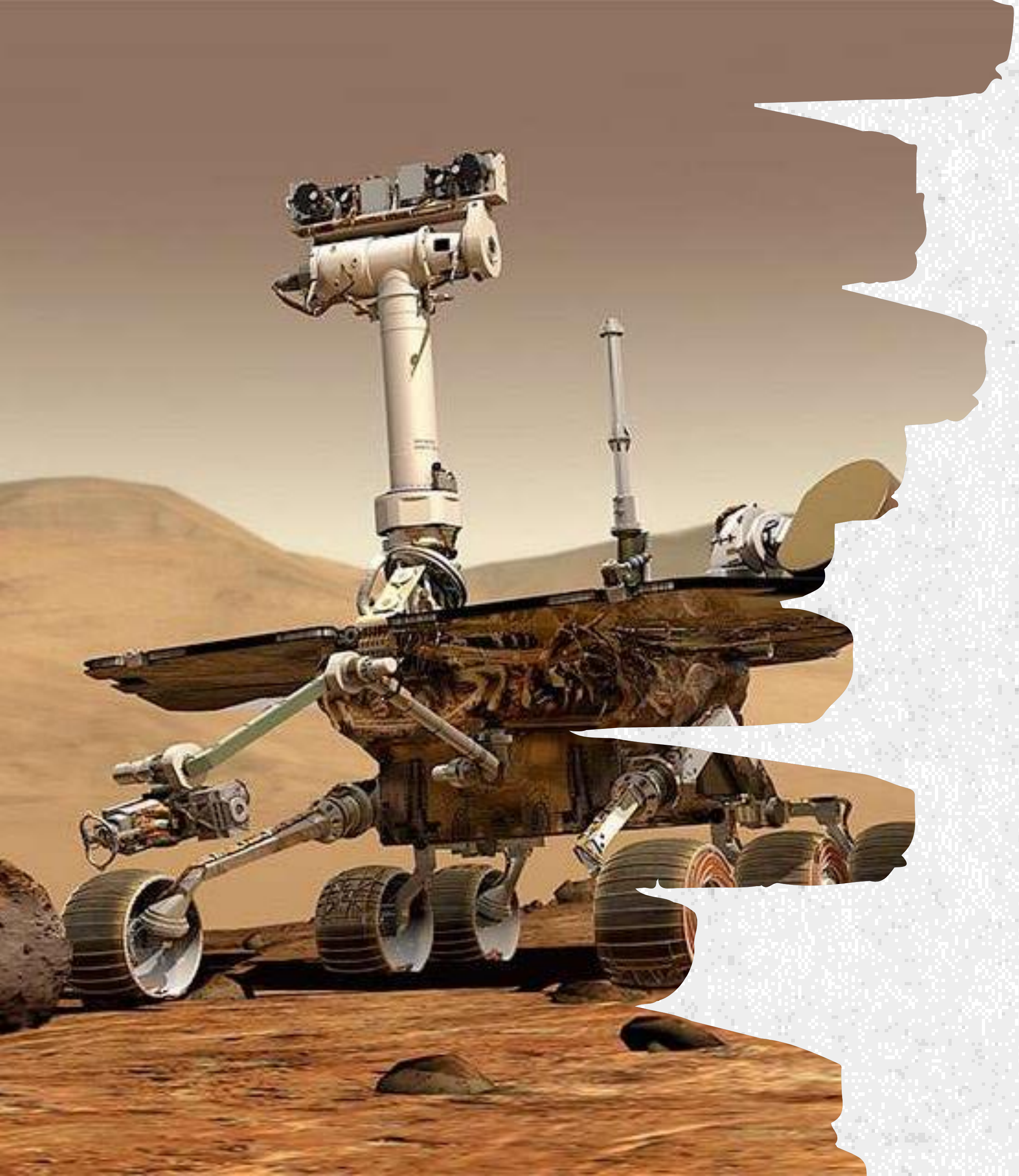
$$\frac{\Theta(s)}{V(s)} = \frac{1}{s}\frac{W(s)}{V(s)} \ or \ \Theta(s) = \frac{W(s)}{s} \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(17)$$

This is consistent with intuition and shows that in the s domain Θ is the integral of W. i.e. angular velocity is the derivative of angular displacement.

MOTOR MODELLING: DC MOTOR SPEED MODEL

For the simulations we are assuming the following motor parameters L=0.5, r=1, b=0.1, J=0.01, K=0.01

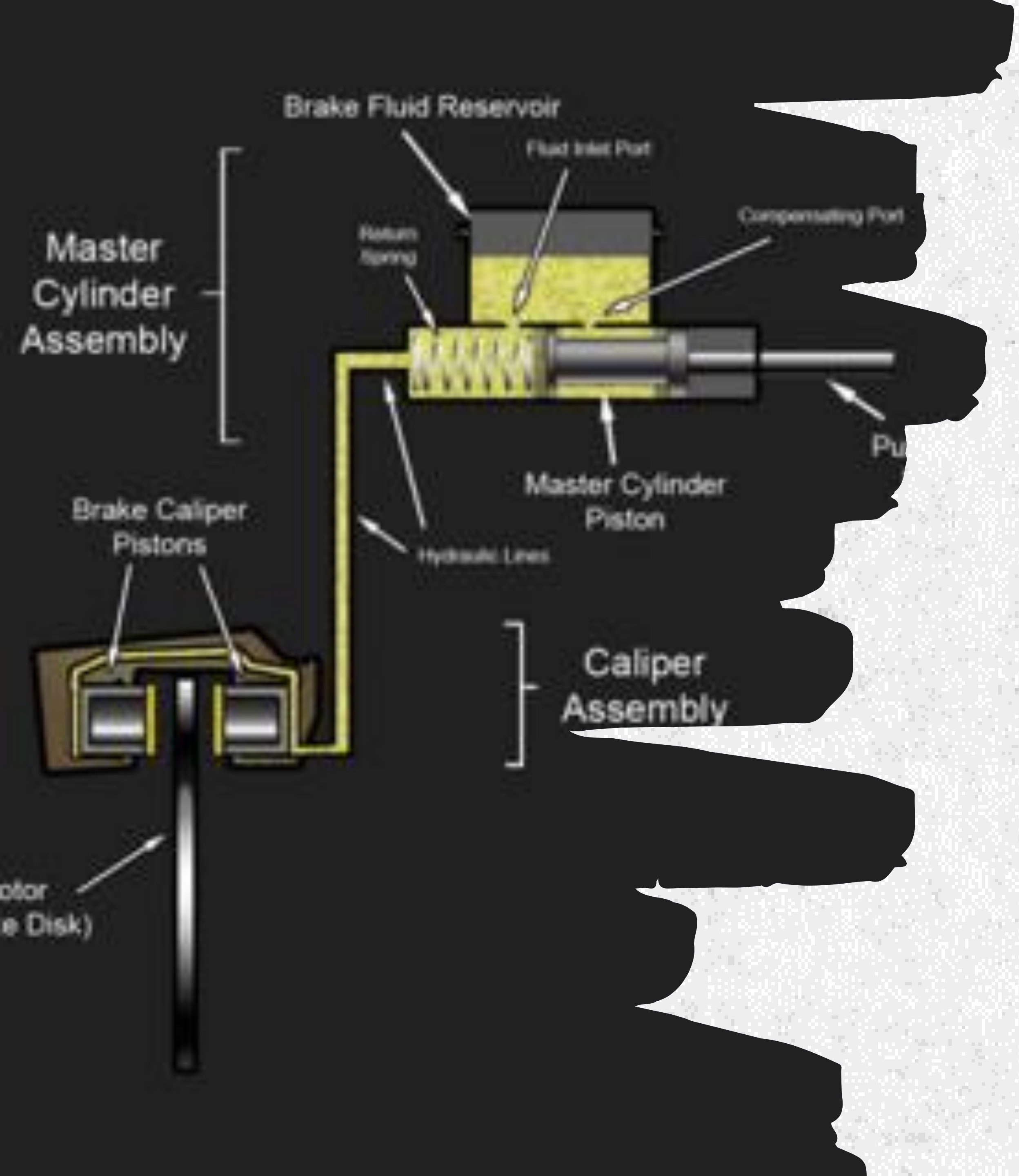




THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq



Ninevah University
Electronics Engineering College
Systems and control engineering department

SYSTEM MODELING

Lecture 7

Basic System Models
Fluid System Modeling

Instructor

Mohanad N. Noaman

MODELING OF FLUID SYSTEMS

Gases and **liquids** are collectively referred to as fluids. Fluid systems are used in many industrial as well as commercial applications. For example, liquid level control is a well-known application of liquid systems. Similarly, gas systems are used in robotics and in industrial movement control applications.

CONSERVATION OF MASS

For incompressible fluids, conservation of mass is equivalent to conservation of volume, because the fluid density is constant. If we know the mass density ρ and the volume flow rate, we can compute the mass flow rate. That is, $q_m = \rho q_v$, where q_m and q_v are the mass and volume flow rates.

- The weight density as $\gamma = \rho g$, where g is the acceleration due to gravity.
- The mass density of fresh water near room temperature is $1.94\ slug/ft^3, or\ 1000\ kg/m^3$.
- The mass density of air at sea level and near room temperature is approximately $0.0023\ slug/ft^3$ or $1.185\ kg/m^3$.
- Pressure is the force per unit area that is exerted by the fluid (P=F/A).
- For a liquid of density ρ, the absolute pressure p and the liquid height h are related by

$$p = p_a + \rho g h$$

MODELING OF FLUID SYSTEMS

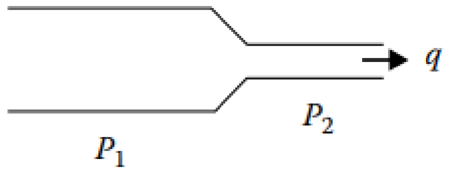
Gases and **liquids** are collectively referred to as fluids. Fluid systems are used in many industrial as well as commercial applications. For example, liquid level control is a well-known application of liquid systems. Similarly, gas systems are used in robotics and in industrial movement control applications.

HYDRAULIC SYSTEMS:

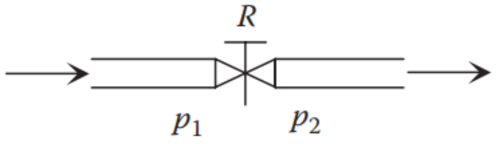
1- Hydraulic resistance

Hydraulic resistance is the resistance to flow which occurs as a result of a liquid flowing through valves or changes in a pipe diameter. If the pressures at either side of a hydraulic resistance are P1 and P2, then the hydraulic resistance R is defined as

$$R = \frac{\text{Pressure difference}}{\text{Change in flow rate, m}^3/\text{s}} = \frac{p_1 - p_2}{q}$$



$$P_1 - P_2 = Rq$$

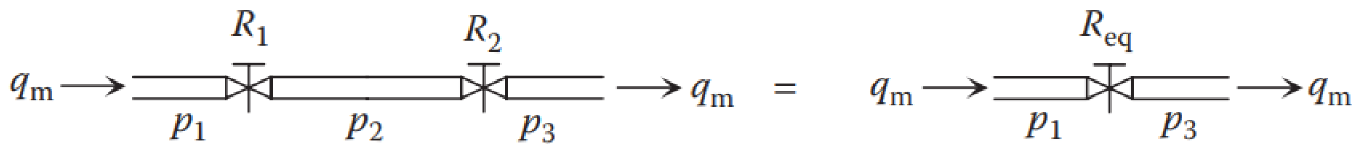


Hydraulic Resistance

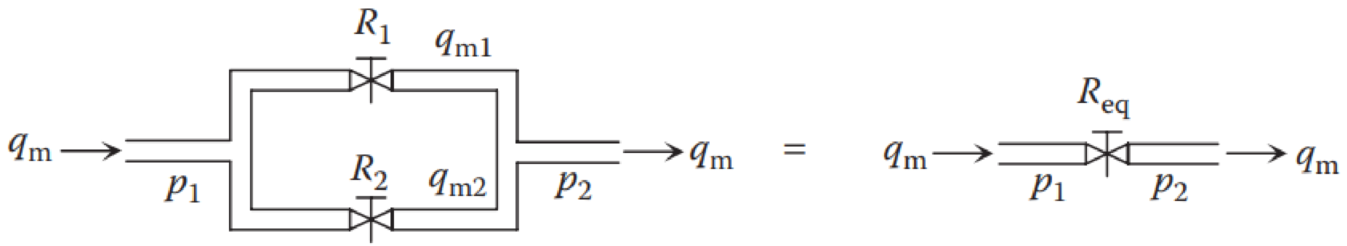
HYDRAULIC SYSTEMS:

1- Hydraulic resistance

Hydraulic resistance occurs whenever there is a pressure difference, such as liquid flowing from a pipe of one diameter to one of a different diameter. If the pressures at either side of a hydraulic resistance are P1 and P2, then the hydraulic resistance R is defined as



$$R = R_1 + R_2$$



$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

MODELING OF FLUID SYSTEMS

2- Hydraulic capacitance

Hydraulic capacitance is the term used to describe energy storage with a liquid where it is stored in the form of potential energy. Consider the tank shown in Figure 2.51b. If q_1 and q_2 are the inflow and outflow, respectively, and V is the volume of the fluid inside the tank, we can write

$$q_1 - q_2 = \frac{dV}{dt} = A\frac{dh}{dt}$$

Now, the pressure difference is given by

$$P_1 - P_2 = h\rho g = p \quad \text{or} \quad h = \frac{p}{\rho g}$$

Substituting first Equation in the second Equation , we obtain

$$q_1 - q_2 = \frac{A}{\rho g}\frac{dp}{dt}$$

$$q_1 - q_2 = C\frac{dp}{dt}$$

the hydraulic capacitance can be defined as

$$C = \frac{A}{\rho g}$$

$$C = \frac{\text{Change in liquid stored, m}^3}{\text{Change in head, m}}$$

$$p = \frac{1}{C}\int (q_1 - q_2)dt$$

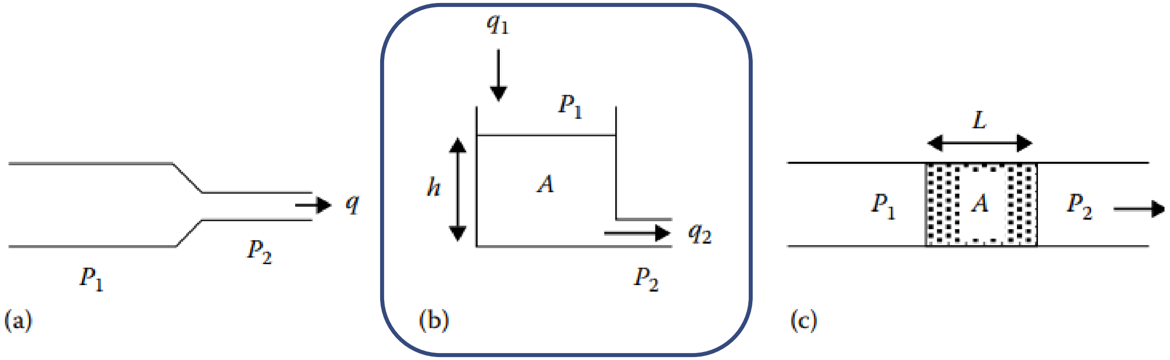


FIGURE 2.51
Hydraulic system elements. (a) Hydraulic resistance. (b) Hydraulic capacitance. (c) Hydraulic inertance.

MODELING OF FLUID SYSTEMS

3- Hydraulic inertance

Hydraulic inertance is similar to the inductance in electrical systems and is derived from the inertia force required to accelerate fluid in a pipe.

Let $P_1 - P_2$ be the pressure drop that we want to accelerate in a cross-sectional area A, where m is the fluid mass and v is the fluid velocity. Applying Newton's second law, we can write

$$m\frac{dv}{dt} = A(P_1 - P_2)$$

If the pipe length is L, then the mass is given by $m = L\rho A$

We can now write Equation above $L\rho A\frac{dv}{dt} = A(P_1 - P_2)$

$$(P_1 - P_2) = L\rho\frac{dv}{dt}$$

but the rate of flow is given by $q = Av$, so Equation above can be written as

$$(P_1 - P_2) = \frac{L\rho}{A}\frac{dq}{dt}$$

$$I = \frac{L\rho}{A}$$

$$(P_1 - P_2) = I\frac{dq}{dt}$$

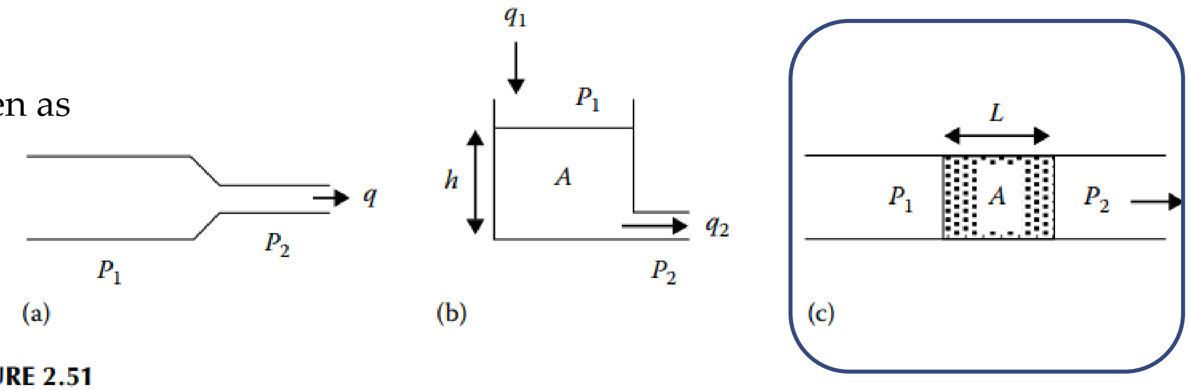


FIGURE 2.51
Hydraulic system elements. (a) Hydraulic resistance. (b) Hydraulic capacitance. (c) Hydraulic inertance.

MODELING OF FLUID SYSTEMS

Example: A Single-Tank Liquid-Level System with a Pump

Consider the single-tank liquid-level system shown in Figure below, where a pump is connected to the bottom of the tank through a valve of linear resistance R. The inlet to the pump is open to the atmosphere, and the pressure of the fluid increases by Δp when crossing the pump. Derive the differential equation relating the liquid height h and the volume flow rate q_0 at the outlet. The tank's cross-sectional area A is constant. The density ρ of the liquid is constant.

Solution

We begin by applying the law of conservation of V to the tank,

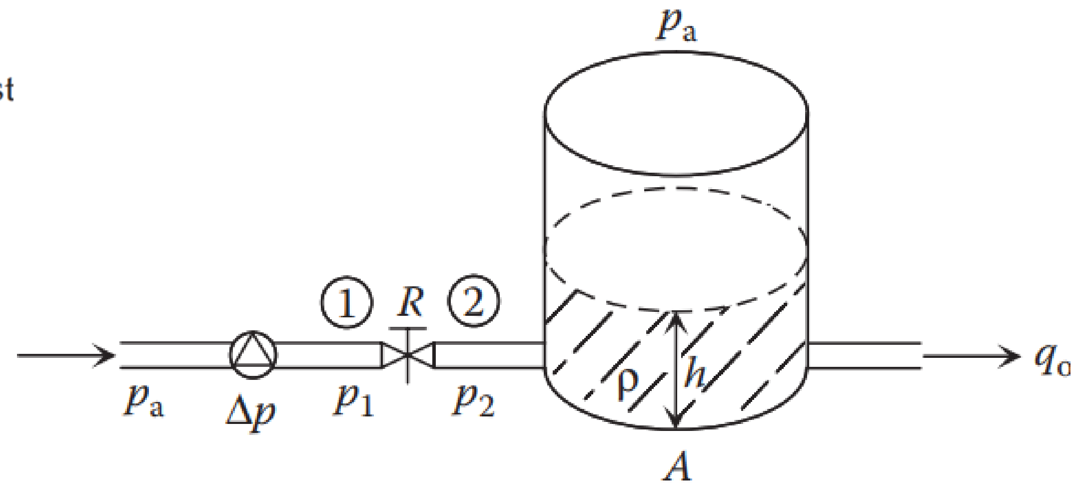
$$\frac{dV}{dt} = q_i - q_o$$

The fluid V inside the tank is Ah. For constant fluid density and const sectional area,

$$\frac{dV}{dt} = A\frac{dh}{dt}$$

The mass flow rate into the tank is

$$q_i = \frac{p_1 - p_2}{R}$$



MODELING OF FLUID SYSTEMS

where $p_1 = p_a + \Delta p$ and $p_2 = p_a + \rho g h$, which is equal to the hydrostatic pressure at the bottom of the tank. Thus,

$$q_{mi} = \frac{\Delta p - \rho g h}{R}$$

The mass flow rate out of the tank can be expressed in terms of the volume flow rate q_o, as

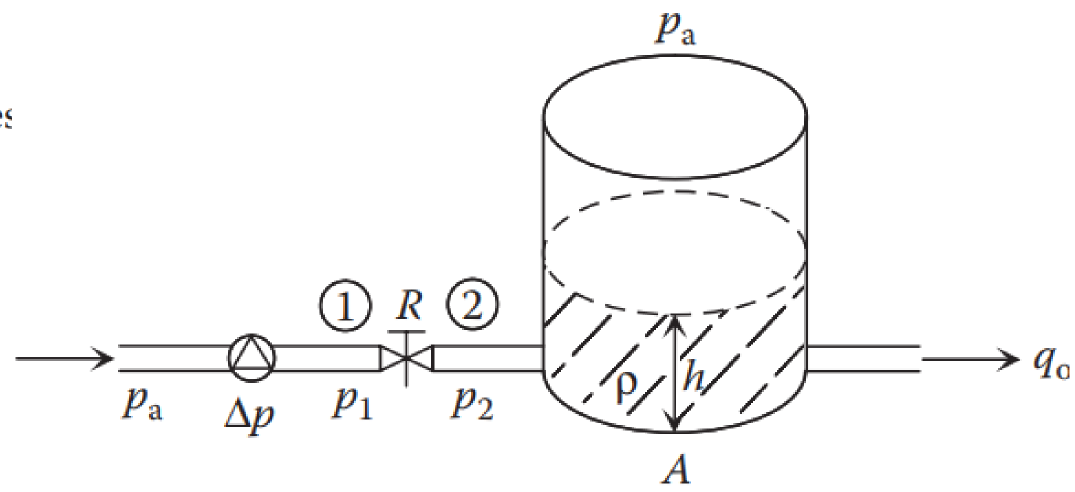
$$q_{mo} = q_o$$

Substituting these expressions into the law of conservation of mass gives

$$A\frac{dh}{dt} = \frac{\Delta p - \rho g h}{R} - q_o$$

Rearranging the equation gives

$$A\frac{dh}{dt} + \frac{\rho g}{R}h - \frac{\Delta p}{R} = - q_o$$



For a liquid-level system with two or more tanks, we apply the law of conservation of mass to each tank.

MODELING OF FLUID SYSTEMS

Example:
Figure below shows a liquid level system where liquid enters a tank at the rate of qi and leaves at the rate of q_0 through an orifice. Derive the mathematical model for the system, showing the relationship between the height h of the liquid and the input flow rate q_i.

Solution:

Recall $q_i - q_o = \frac{A}{\rho g}\frac{dp}{dt}$

Recalling that $p = h\rho g$,

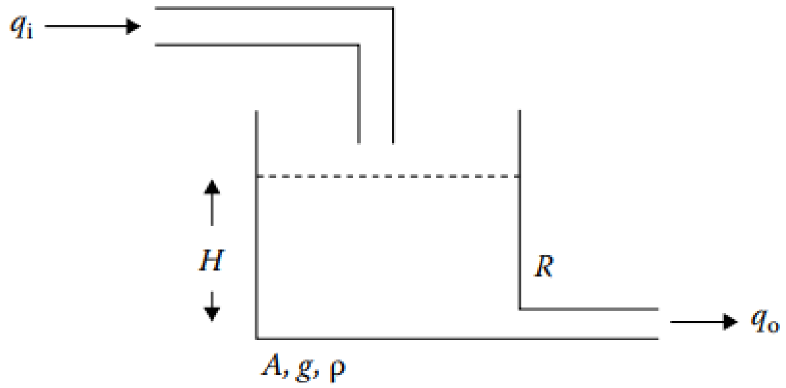
$$q_i = A\frac{dh}{dt} + q_o \qquad 1$$

Since $p_1 - p_2 = Rq_o$

$$q_o = \frac{p_1 - p_2}{R} = \frac{h\rho g}{R} \qquad 2$$

So that substituting in Equation 2 in 1 gives

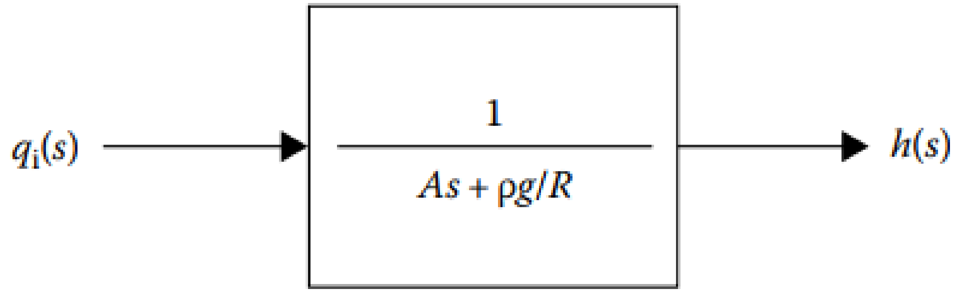
$$q_i = A\frac{dh}{dt} + \frac{\rho g}{R}h \qquad 3$$



MODELING OF FLUID SYSTEMS

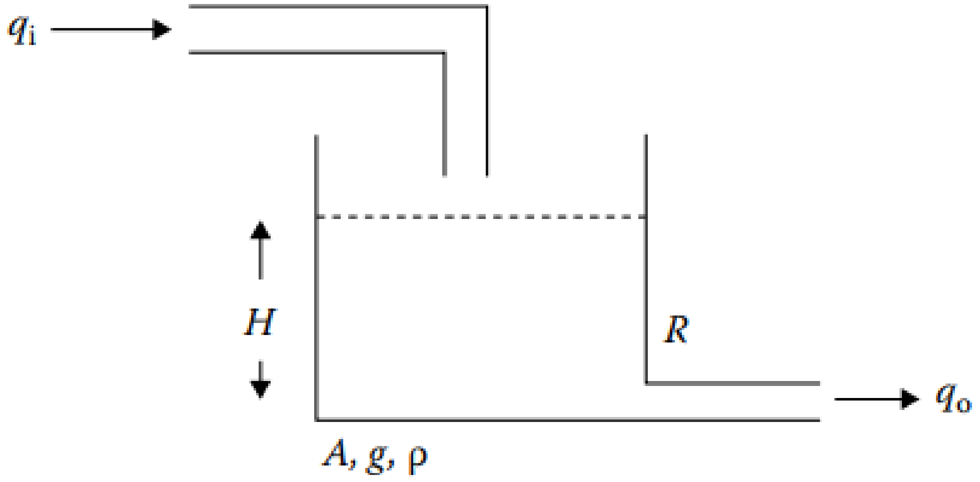
Equation 3 shows the variation of the height of the water with the inflow rate. If we take the Laplace transform of both sides, we obtain

$$q_i(s) = Ash(s) + \frac{\rho g}{R} h(s)$$



and the transfer function of the system can be written as

$$\frac{h(s)}{q_i(s)} = \frac{1}{As + \frac{\rho g}{R}}$$



HOMEWORK

A liquid level system is shown in Figure 2.69 where q_i and q_o are the inflow and outflow rates, respectively. The system has two fluid resistances, R_1 and R_2, in series. Derive an expression for the mathematical model for the system.

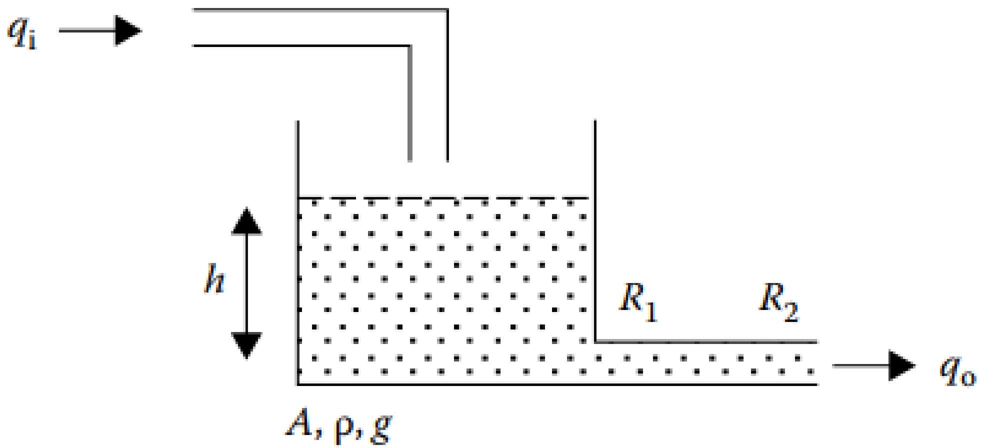
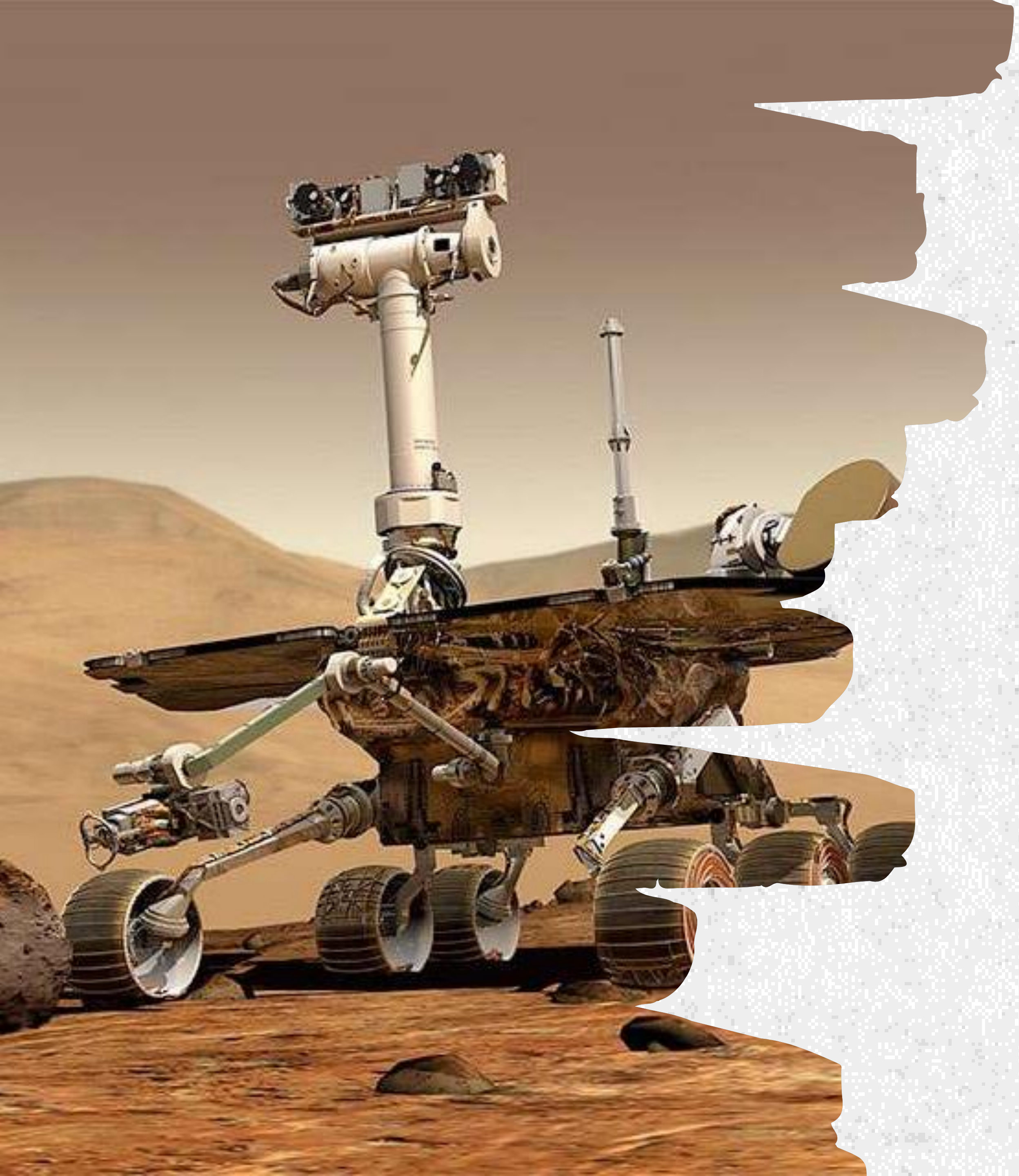


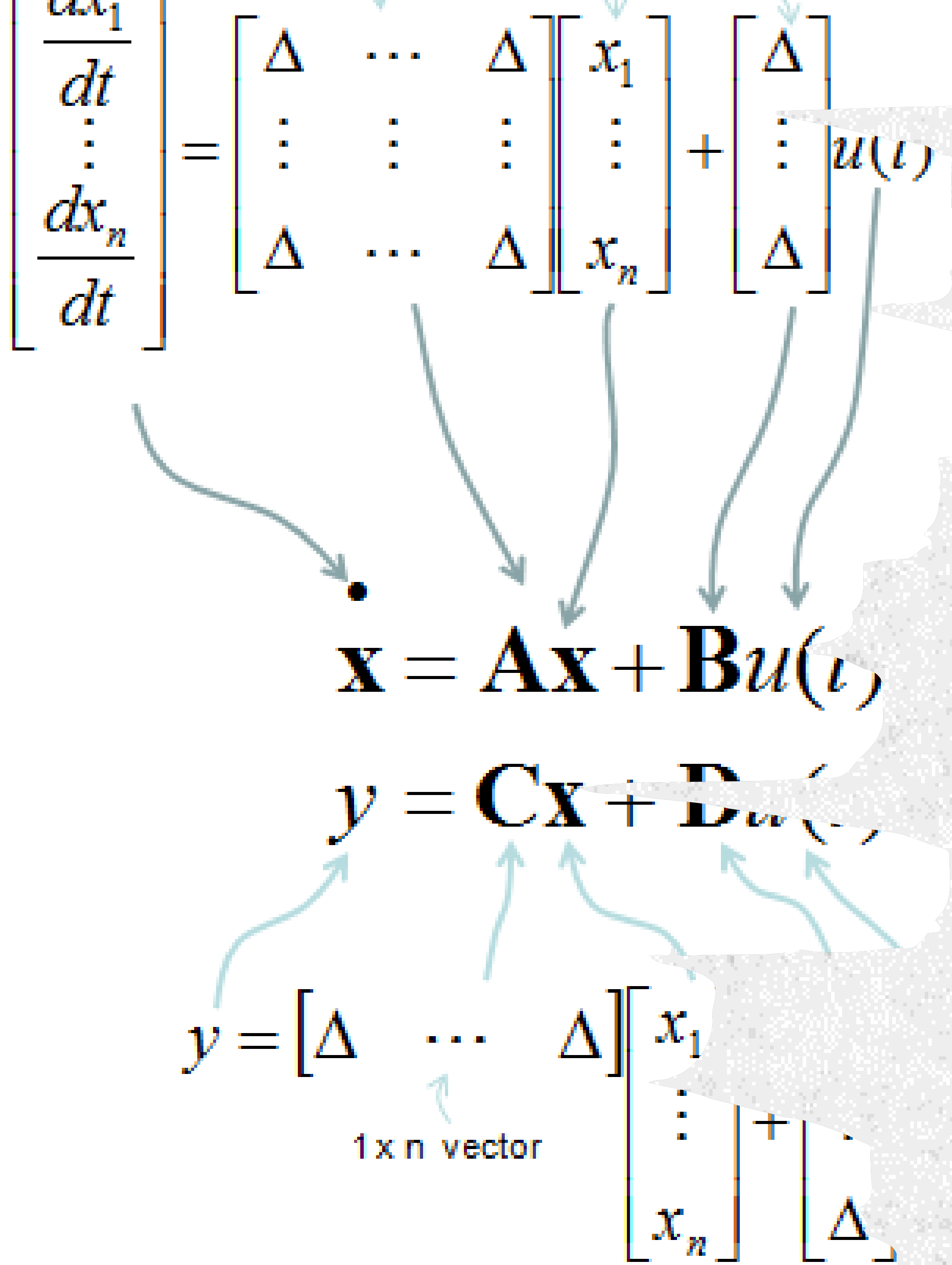
FIGURE 2.69
Liquid level system



THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq



Ninevah University
Electronics Engineering College
Systems and control engineering department

SYSTEM MODELING

Lecture 8

State-Space Representation

Instructor
Mohanad N. Noaman

OUTLINES

- How to find mathematical model, called a state- space representation, for a linear, time-invariant system
- How to convert between transfer function and state space models

MODELING METHODS

There are three methods used for modeling dynamic systems

1. Differential equations
2. Transfer functions
3. State space

- Uses matrices and vectors to represent the system parameters and variables
- In control engineering, a state space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations. To abstract from the number of inputs, outputs and states, the variables are expressed as vectors.

MOTIVATION FOR STATE-SPACE MODELING

Conventional Control Theory (root-locus and frequency response analysis and design) is applicable to linear, time invariant, single-input, single-output systems. This is a complex frequency-domain approach. The transfer function relates the input to output and does not show internal system behavior.

Modern Control Theory (state-space analysis and design) is applicable to linear or nonlinear, time-varying or time invariant, multiple-input, multiple-output systems. This is a time-domain approach. This state-space system description provides a complete internal description of the system, including the flow of internal energy.

MOTIVATION FOR STATE-SPACE MODELING

- Easier for computers to perform matrix algebra
  - e.g. MATLAB does all computations as matrix math
- Handles multiple inputs and multi output dynamic systems.
- Gives a more geometric understanding of dynamic systems.
- Forms the basic for much modern control theory.
- Provides more information about the system
  - Provides knowledge of internal variables (states)

STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION

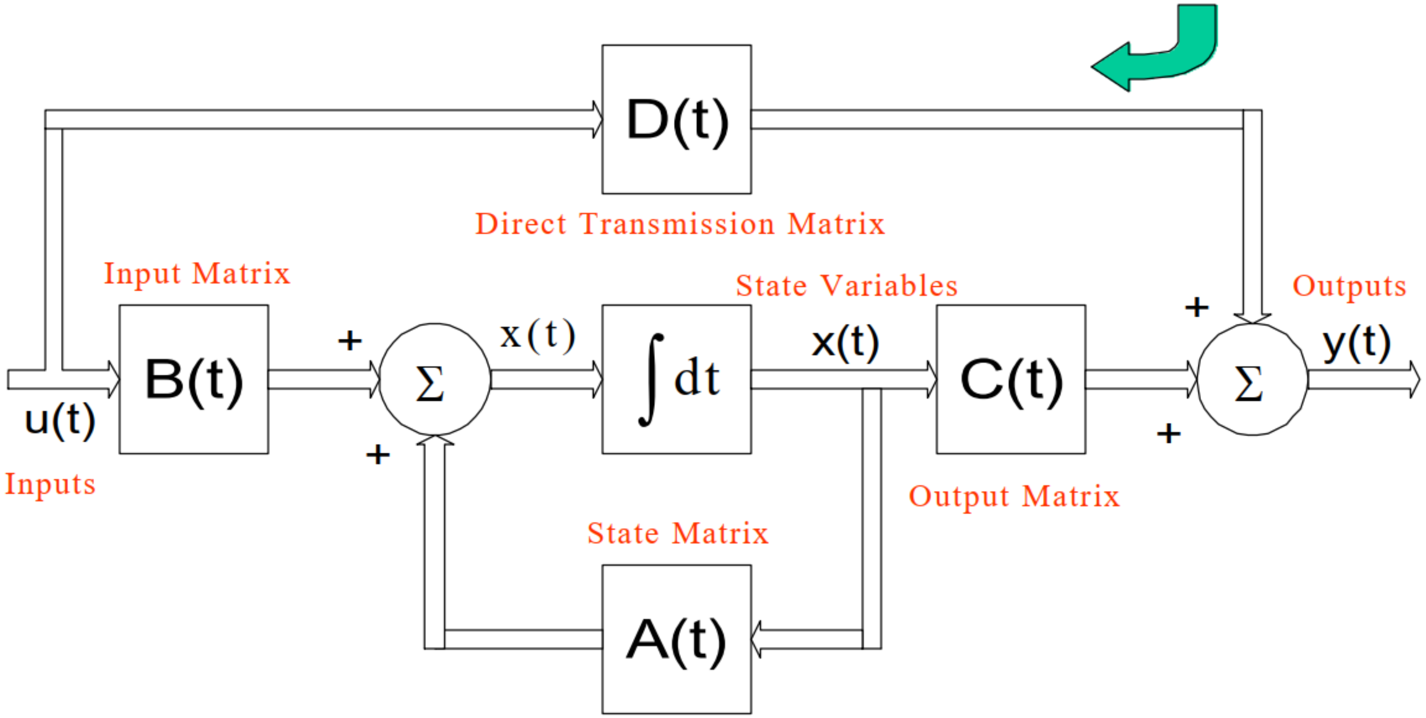
State-Space Form

$$\dot{\vec{x}}(t) = \vec{f}(\vec{x},\vec{u},t)$$
$$\vec{y}(t) = \vec{g}(\vec{x},\vec{u},t)$$

Non-Linear, Time-Varying

$$\begin{cases} \dot{\vec{x}}(t) = A(t)\vec{x}(t) + B(t)\vec{u}(t) \\ \vec{y}(t) = C(t)\vec{x}(t) + D(t)\vec{u}(t) \end{cases}$$

Linear, Time-Varying



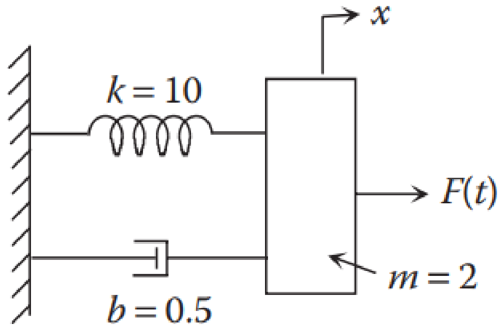
STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION

- The *state* is the minimum amount of information needed about the system at time t_0 such that its future behavior can be determined without reference to any input before t_0.
- *State variables* form the smallest set of independent variables that completely describe the state of a system.
- The *state variables* are independent variables capable of defining the state from which one can completely describe the system behavior. These variables completely describe the effect of the past history of the system on its response in the future.

- Given the mathematical model of a dynamic system, the state variables are determined as follows:

• The number of state variables is the same as the number of initial conditions needed to completely solve the system model.

STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION



State Variables

In the mechanical system shown in Figure aside, all parameter values are in consistent physical units. The model of this system is provided by its equation of motion

$$2\ddot{x} + 0.5\dot{x} + 10x = F(t)$$

Two initial conditions $x(0)$ and $\dot{x}(0)$

The state variables are selected as $x_1 = x$ and $x_2 = \dot{x}$

State Variables Equation:

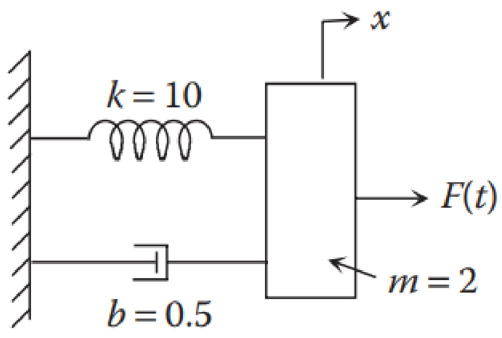
There are as many state-variable equations as there are state variables. Each state-variable equation is a first-order ODE whose left side is the first derivative of a state variable and whose right side is an algebraic function of the state variables, system inputs, and possibly time t.

STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION

<u>State Variables Equation:</u>

Suppose a dynamic system has n state variables $x_1, x_2, \ldots, x_n$ and m inputs $u_1, u_2, \ldots, u_m$. Then, the state-variable equations take the generic form

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \dot{x}_2 = f_2(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \\ \ldots \\ \dot{x}_n = f_n(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \end{cases}$$



$$2\ddot{x} + 0.5\dot{x} + 10x = F(t)$$

Where $f_1, f_2, \ldots, f_n$ are algebraic functions of the state variables and inputs and are generally nonlinear

The state-variable equations can be written as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{2}\left[-0.5x_2 - 10x_1 + F(t)\right] \end{cases}$$

STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION

State Equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix} = \text{state vector}, \quad \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{Bmatrix} = \text{input vector}$$

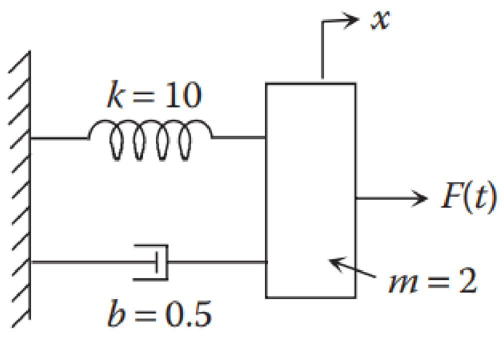
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \text{state matrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = \text{input matrix}$$

STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION

State Equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{2}\left[-0.5x_2 - 10x_1 + F(t)\right] \end{cases}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{1}{4} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} F(t)$$



$$2\ddot{x} + 0.5\dot{x} + 10x = F(t)$$

Therefore, the state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{1}{4} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \quad u = F(t)$$

Since there is only one input $F(t)$, input vector $\mathbf{u}$ is scalar and denoted by u.

STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION

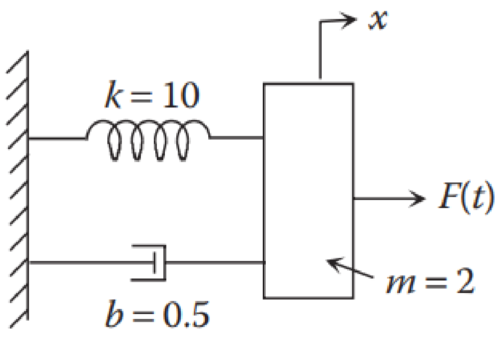
Output Equation and State-Space Form

Output Equation:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}_{p\times n} = \text{output matrix}$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}_{p\times m} = \text{direct transmission matrix}$$



$$2\ddot{x} + 0.5\dot{x} + 10x = F(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + 0 \cdot u$$

So that C = [0 1]and D = 0. Note that the direct transmission matrix D is 1 ×1 and hence denoted by D .

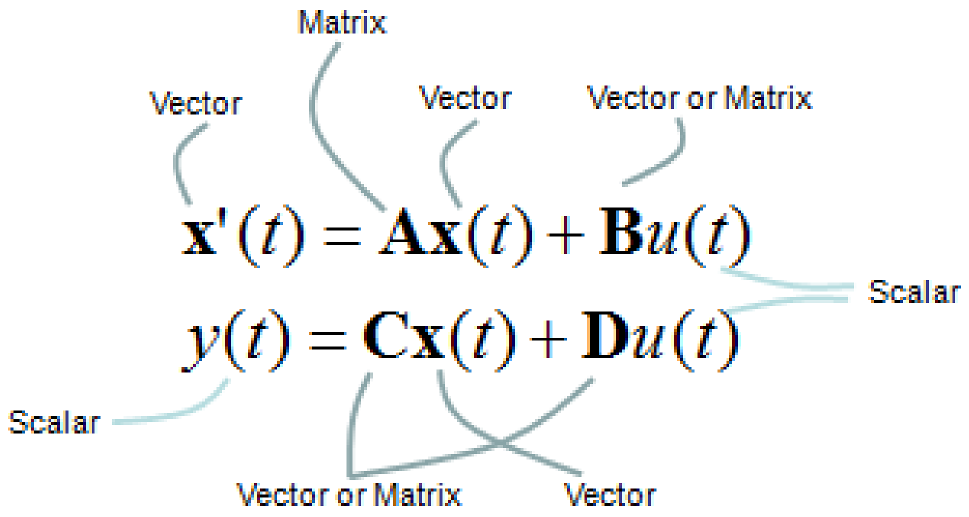
STATE VARIABLES, STATE-VARIABLE EQUATIONS, AND STATE EQUATION

Output Equation and State-Space Form

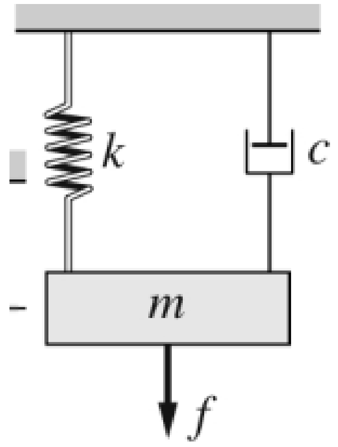
State-Space Form:
The combination of the state equation and the output equation is called the state-space form.

State Equation → $$\dot{\mathbf{x}} = \mathbf{A}_{n\times n}\mathbf{x}_{n\times 1} + \mathbf{B}_{n\times m}\mathbf{u}_{m\times 1}$$

Output Equation → $$\mathbf{y}_{p\times 1} = \mathbf{C}_{p\times n}\mathbf{x}_{n\times 1} + \mathbf{D}_{p\times m}\mathbf{u}_{m\times 1}$$



State-Space Example 1



system dynamics

$$m\ddot{x} + b\dot{x} + kx = f$$

input $u = f$

output $y = f - b\dot{x} - kx$

transfer function

$$\frac{Y(s)}{U(s)} = \frac{ms^2}{ms^2 + bs + k}$$

state-space modeling

state variables: $x_1 = x,\ x_2 = \dot{x}$ ← x_1, x_2: second-order system can completely describe the system's state

state-space dynamics

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k/m & -b/m\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{bmatrix}0\\ 1/m\end{bmatrix}u$$

← first-order vector differential equation

$$y = \begin{bmatrix}-k & -b\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + 1 \times u$$

state-space representation

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

STATE SPACE REPRESENTATION OF NTH-ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Consider the followin nth order system:

$$y^n + a_1 y^{n-1} + \quad \dots \dots \dots + a_{n-1}\dot{y} + a_n y = u$$

Where y is the system output and u is the system input. The system has nth integrators (state variables). Let us define n-state variables

$x_1 = y$

$x_2 = \dot{y}$

.

.

$x_n = y^{n-1}$

STATE SPACE REPRESENTATION OF NTH-ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

.

$x_{n-1} = x_n$

$\dot{x}_n = -a_n\, x_1 - \cdots \ldots\ldots -a_1 x_n + u$

Then the state space equation is given by:

$$\dot{x} = Ax + Bu$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

STATE SPACE REPRESENTATION OF NTH-ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Since, the output equation is:

$$y = x_1$$

The state space output equation is

$$y = Cx \qquad \text{where } C = [1\ 0\ 0\ \dots\ 0]$$

Note: No. of Order system equal to No. of State equation

STATE SPACE REPRESENTATION OF NTH-ORDER SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

General Procedure:

1- Solve ODE for highest order derivative $\left(\frac{d^n x(t)}{dt^n}\right)$

2- Define state vectors as $\bar{x}(t) = [x(t) \;\; \frac{dx(t)}{dt} \;\; \frac{d^2x(t)}{dt^2} \; \dots \; \frac{d^{n-1}x(t)}{dt^{n-1}}]^T$

3- Define Input/control vectors as $\bar{u}(t)=[u_1(t), u_2(t), u_{m(t)}]^T$

4- Write state equations:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

5- Write output equations

$$y(t) = Cx(t) + Du(t)$$

Example: $\dddot{y} + 3\ddot{y} + 7\dot{y} + 6y = f$

1- $\dddot{y}=f - 3\ddot{y} - 7\dot{y} - 6y$

2- $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}$

3- Input $u = f,$ output y

4- $\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = f - 3\ddot{y} - 7\dot{y} - 6y$

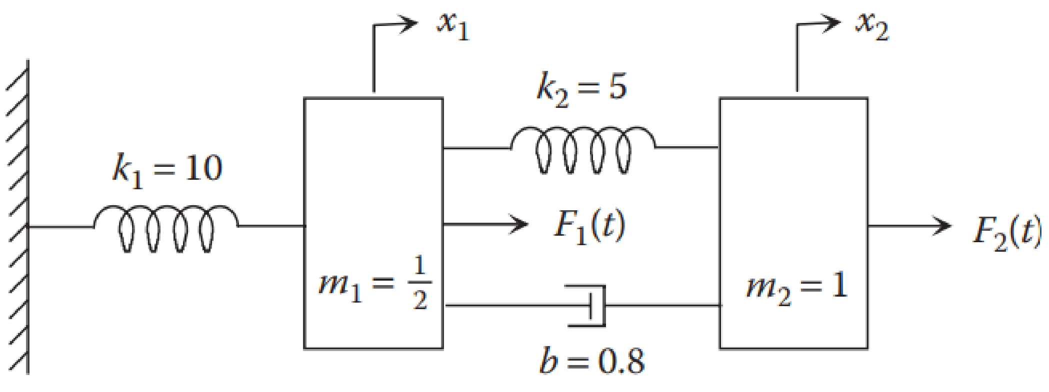
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -7 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

5- $y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

STATE-SPACE MODELING

The equations of motion are derived as

$$\begin{cases} \frac{1}{2}\ddot{x}_1 + 10x_1 - 5(x_2 - x_1) - 0.8(\dot{x}_2 - \dot{x}_1) = F_1 \\ \ddot{x}_2 + 5(x_2 - x_1) + 0.8(\dot{x}_2 - \dot{x}_1) = F_2 \end{cases}$$



The system model comprises two second-order ODEs; hence, a total of four initial conditions are needed for complete solution. There are therefore four state variables, selected as

$$x_1 = x_1$$
$$x_2 = x_2$$
$$x_3 = \dot{x}_1$$
$$x_4 = \dot{x}_2$$

The state-variable equations are then formed as

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -30x_1 + 10x_2 + 1.6x_4 - 1.6x_3 + 2F_1 \\ \dot{x}_4 = -5x_2 + 5x_1 - 0.8x_4 + 0.8x_3 + F_2 \end{cases}$$

STATE-SPACE MODELING

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}_{4\times1}, \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & 10 & -1.6 & 1.6 \\ 5 & -5 & 0.8 & -0.8 \end{bmatrix}_{4\times4}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}_{4\times2}, \quad \mathbf{u} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{2\times1}$$

Since the outputs are x_1 and $\dot{x}_1$, we have

$$\mathbf{y} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_3 \end{Bmatrix}$$

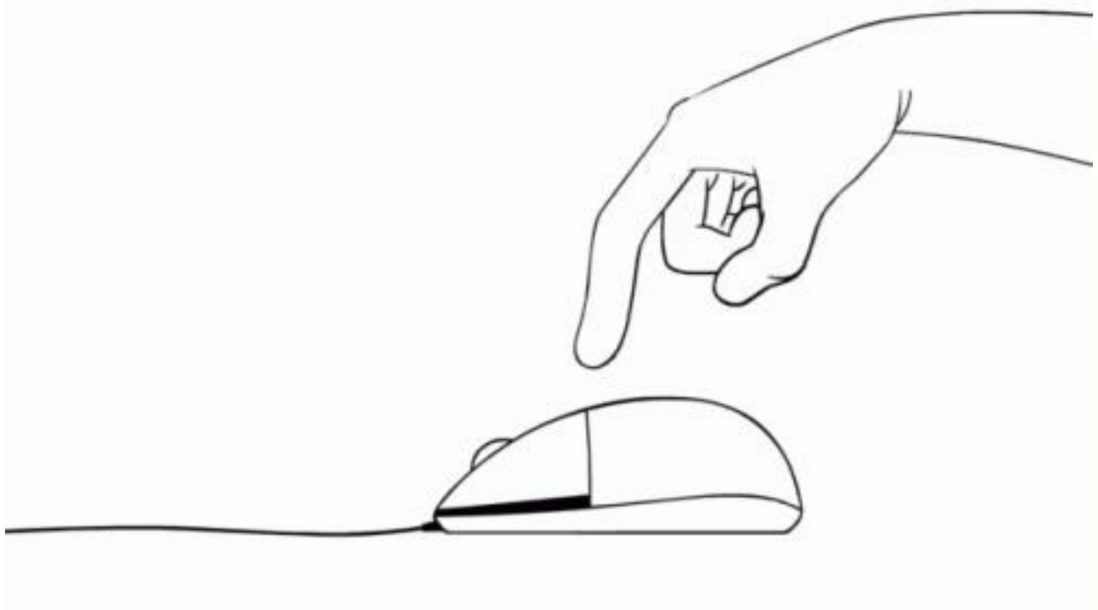
As a result, the output equation is

$$\mathbf{y}_{2\times1} = \mathbf{C}_{2\times4}\mathbf{x}_{4\times1} + \mathbf{D}_{2\times2}\mathbf{u}_{2\times1}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{2\times4}, \quad \mathbf{D} = \mathbf{0}_{2\times2}$$

FOR MORE INFORMATION



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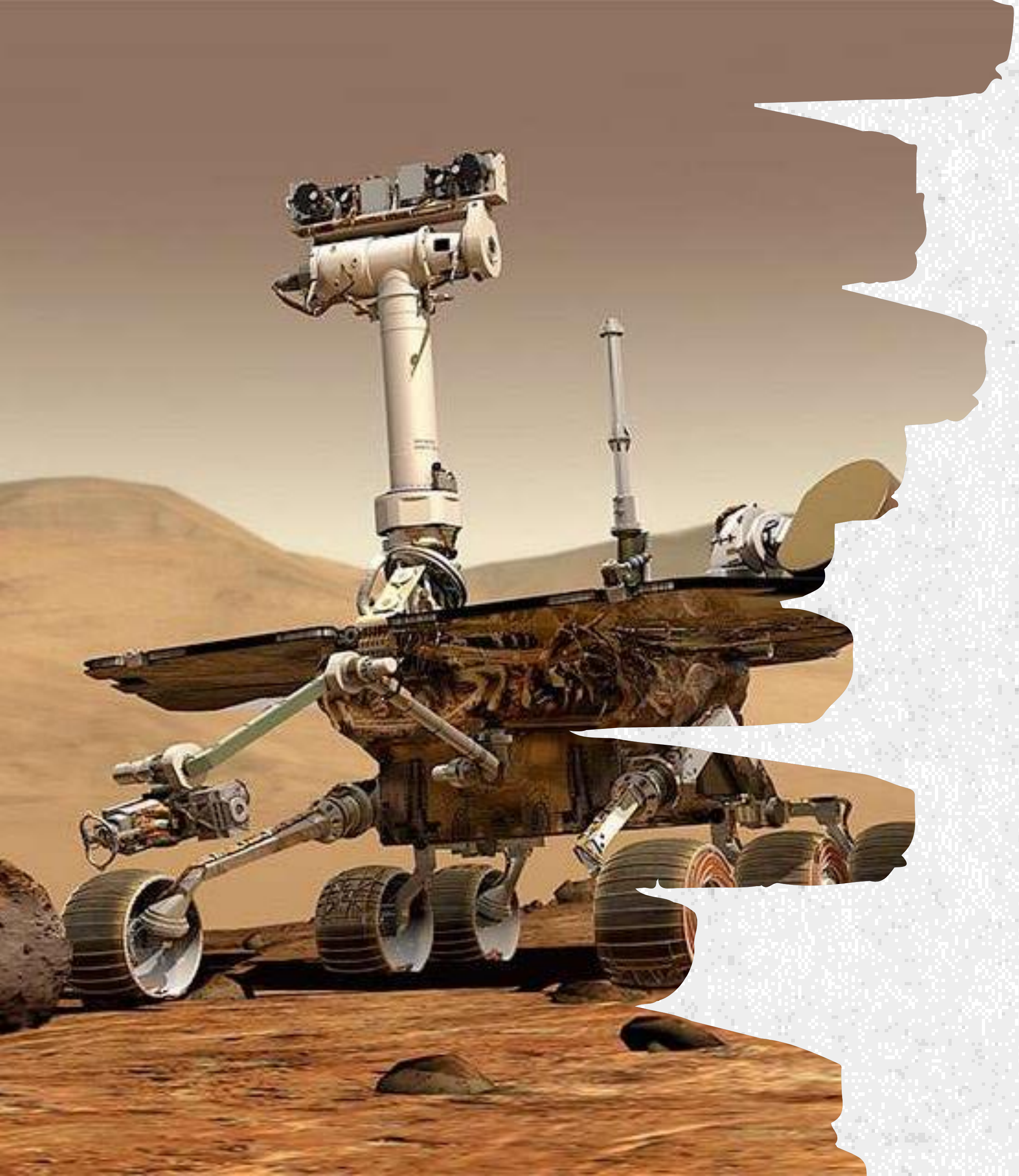
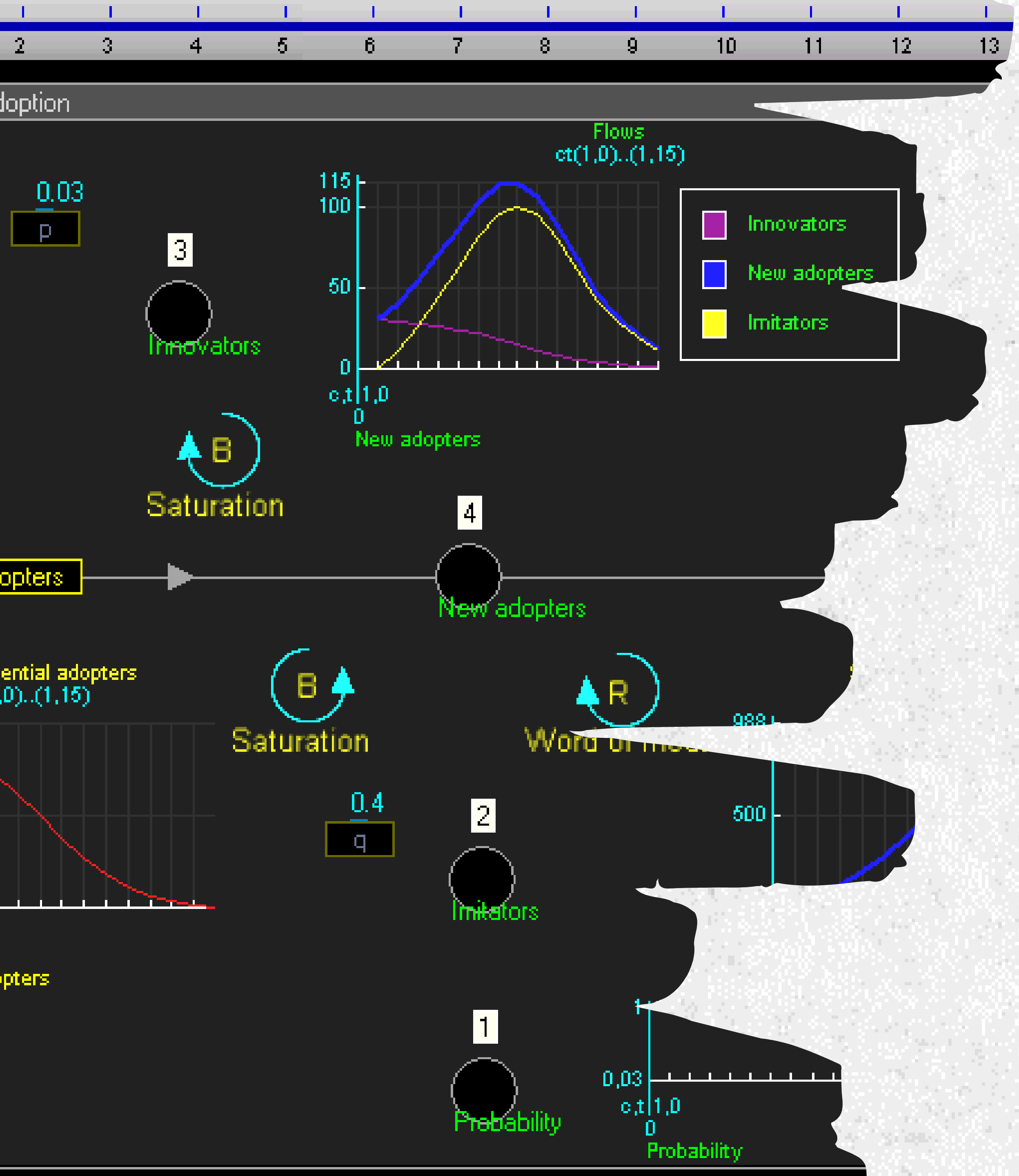


THANK YOU

Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq



Ninevah University
Electronics Engineering College
Systems and control engineering department

SYSTEM MODELING

Lecture 9

The relationship between State-Space and Transfer Function

Instructor
Mohanad N. Noaman

TRANSFER FUNCTION TO STATE SPACE

- Many techniques are available for obtaining state space representations of transfer functions

STATE SPACE REPRESENTATIONS IN CANONICAL FORMS

Consider a system defined by,

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1}\dot{u} + b_m u$$

where 'u' is the *input* and 'y' is the *output*. This equation can also be written as,

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

TRANSFER FUNCTION TO STATE SPACE

CANONICAL FORM I Controllable Canonical Form (CCF)

- We have, $\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$
- Let $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X(s)} \times \frac{X(s)}{U(s)}$
- Thus,

$$\frac{Y(s)}{X(s)} = b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

-

$$\therefore u(t) = \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x$$

$$\frac{d^n x}{dt^n} = u(t) - a_1 \frac{d^{n-1} x}{dt^{n-1}} - \cdots - a_{n-1} \frac{dx}{dt} - a_n x$$

TRANSFER FUNCTION TO STATE SPACE

Canonical Form I

Let $x_1 = x$; $x_2 = \frac{dx}{dt}$; $x_3 = \frac{d^2x}{dt^2} \cdots x_n = \frac{d^{n-1}x}{dt^{n-1}}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

- Case-1: If $m = n - 1[\because \frac{d^m x}{dt^m} = \frac{d^{n-1}x}{dt^{n-1}} = x_n]$

  We know that,

$$\begin{aligned} Y(s) &= X(s)(b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m) \\ \therefore y(t) &= b_0 \frac{d^{n-1}x}{dt^{n-1}} + b_1 \frac{d^{n-2}x}{dt^{n-2}} + \cdots + b_{n-1}\frac{dx}{dt} + b_n x \\ y(t) &= b_0 x_n + b_1 x_{n-1} + \cdots + b_n x_1 \end{aligned}$$

TRANSFER FUNCTION TO STATE SPACE

Canonical Form I

$$y = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [0]u$$

- Case-2: If $m = n$,

$$\begin{aligned} \frac{d^m x}{dt^m} &= u(t) - a_1 \frac{d^{n-1}x}{dt^{n-1}} - \cdots - a_{n-1}\frac{dx}{dt} - a_n x \\ y(t) &= b_0\left(u(t) - a_1 \frac{d^{n-1}x}{dt^{n-1}} - \cdots - a_n x\right) + b_1 \frac{d^{m-1}x}{dt^{m-1}} + \cdots + b_m x \\ y(t) &= (b_1 - a_1 b_0)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + (b_{n-1} a_{n-1} b_0)\frac{dx}{dt} + (b_n - a_n b_0) + b_0 u \end{aligned}$$

$$\therefore y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + [b_0]u$$

TRANSFER FUNCTION TO STATE SPACE

Example-1

Consider a transfer function, $G(s) = \frac{5s^2+7s+9}{s^3+8s^2+6s+2}$

Let $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X(s)} \times \frac{X(s)}{U(s)} = \frac{5s^2+7s+9}{s^3+8s^2+6s+2}$
Thus,

$$\frac{Y(s)}{X(s)} = 5s^2 + 7s + 9 \qquad (1)$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^3 + 8s^2 + 6s + 2} \qquad (2)$$

From eq(2), $U(s) = X(s)[s^3 + 8s^2 + 6s + 2]$

$$u(t) = \frac{d^3x}{dt^3} + 8\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 2x$$

$$\frac{d^3x}{dt^3} = u(t) - 8\frac{d^2x}{dt^2} - 6\frac{dx}{dt} - 2x$$

TRANSFER FUNCTION TO STATE SPACE

Examples on Canonical Form-I: Example-1

Let $x_1 = x$; $x_2 = \frac{dx}{dt}$; $x_3 = \frac{d^2x}{dt^2}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

From eq(1), $Y(s) = X(s)[5s^2 + 7s + 9]$

$$y(t) = 5\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 9x$$

$$y(t) = 5x_3 + 7x_2 + 9x_1$$

$$y = \begin{bmatrix} 9 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

TRANSFER FUNCTION TO STATE SPACE

Example-2

Consider a transfer function, $G(s) = \frac{5s^2+7s+9}{s^2+2s+15}$

Let

$$\frac{X(s)}{U(s)} = \frac{1}{s^2+2s+15}$$

$$\frac{d^2x}{dt^2} = -15x - 2\frac{dx}{dt} + u(t)$$

Let $x_1 = x$; $x_2 = \frac{dx}{dt}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -15 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

TRANSFER FUNCTION TO STATE SPACE

Example-2

$$
\begin{aligned}
\frac{Y(s)}{X(s)} &= 5s^2 + 7s + 9 \\
y(t) &= 5\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 9x \\
&= 5(-15x - 2\frac{dx}{dt} + u(t)) + 7\frac{dx}{dt} + 9x \\
&= -66x - 3\frac{dx}{dt} + 5u(t) \\
y &= \begin{bmatrix} -66 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [5]u
\end{aligned}
$$

STATE-SPACE TO TRANSFER FUNCTION

• The transfer function (for SISO systems) or transfer matrix (for MIMO systems) can be systematically derived from the state-space form.

- Consider the state-space form

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

First, we note that the Laplace transform of a vector such as x is handled as

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{Bmatrix} \xRightarrow{\text{Laplace transform}} \mathcal{L}\{\mathbf{x}\} = \mathbf{X}(s) = \begin{Bmatrix} \mathcal{L}\{x_1\} \\ \mathcal{L}\{x_2\} \\ \cdots \\ \mathcal{L}\{x_n\} \end{Bmatrix}$$

Assuming zero initial state vector, $x(0)$=$0_{n\times 1}$, Laplace transformation of the state-space form yields

$$\begin{cases} s\mathbf{X}(s) = \mathbf{AX}(s) + \mathbf{BU}(s) \\ \mathbf{Y}(s) = \mathbf{CX}(s) + \mathbf{DU}(s) \end{cases}$$

STATE-SPACE TO TRANSFER FUNCTION

- The first equation is manipulated as

$$(s\mathbf{I}-\mathbf{A})\mathbf{X}(s)=\mathbf{B}\mathbf{U}(s)\ \underset{(s\mathbf{I}-\mathbf{A})^{-1}}{\overset{\text{Pre-multiply by}}{\Rightarrow}}\ \mathbf{X}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

- Inserting this into the second equation results in

$$\mathbf{Y}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)+\mathbf{D}\mathbf{U}(s)$$

$$=\left[\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]\mathbf{U}(s)$$

- For a SISO system with input u and output y, the transfer function is $G(s)=Y(s)/U(s)$, so that $Y(s)=G(s)U(s)$.

$$\mathbf{G}(s)=\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

Where **I** is identity matrix

- To find the Inverse matrix:

$$\text{Let } E=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Downarrow$$

$$E^{-1}=\frac{adj(E)}{\det(E)}=\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-cb}$$

STATE-SPACE TO TRANSFER FUNCTION

- Example: Single-Input-Single-Output System

A system's state-space representation is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

Where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{2}{3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}, \quad D = 0, \quad u = u$$

Solution:

Since both u and y are 1×1, the system is SISO; hence, there is only one transfer function.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + \frac{2}{3}s + 1}\begin{bmatrix} s + \frac{2}{3} & 1 \\ -1 & s \end{bmatrix}$$

- With this, the transfer function is obtained as

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \frac{1}{s^2 + \frac{2}{3}s + 1}\begin{bmatrix} s + \frac{2}{3} & 1 \\ -1 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\frac{1}{2}s + 1}{s^2 + \frac{2}{3}s + 1}$$

STATE-SPACE TO TRANSFER FUNCTION

- Example: Single-Input-Single-Output System

A system's state-space representation is

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} + Du \end{cases}$$

Where

$$\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{2}{3} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix}, \quad D = 0, \quad u = u$$

Solution:

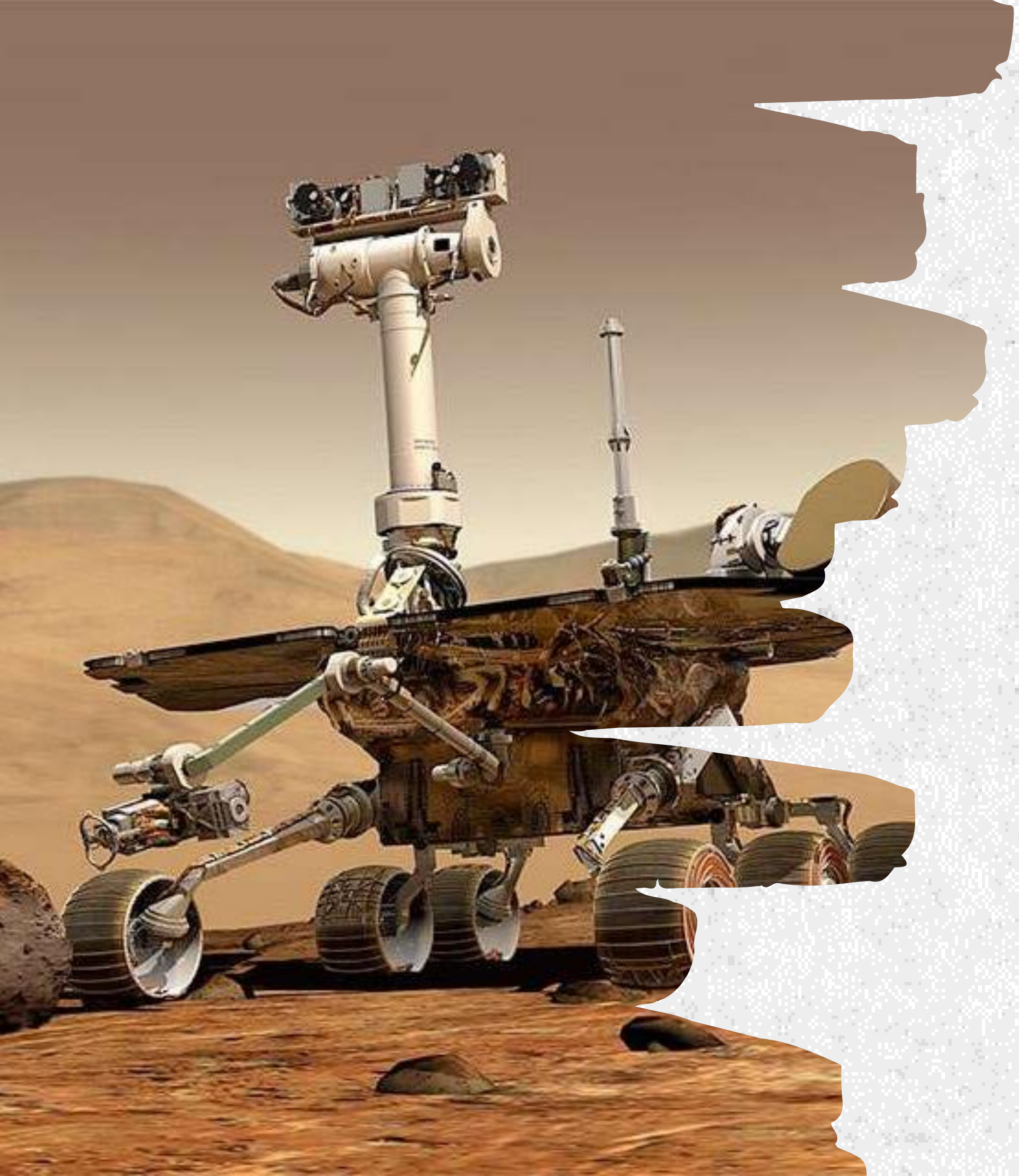
Since both u and y are 1×1, the system is SISO; hence, there is only one transfer function.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + \frac{2}{3}s + 1} \begin{bmatrix} s + \frac{2}{3} & 1 \\ -1 & s \end{bmatrix}$$

- With this, the transfer function is obtained as

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} 1 & \frac{1}{2} \end{bmatrix} \frac{1}{s^2 + \frac{2}{3}s + 1} \begin{bmatrix} s + \frac{2}{3} & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\frac{1}{2}s + 1}{s^2 + \frac{2}{3}s + 1}$$



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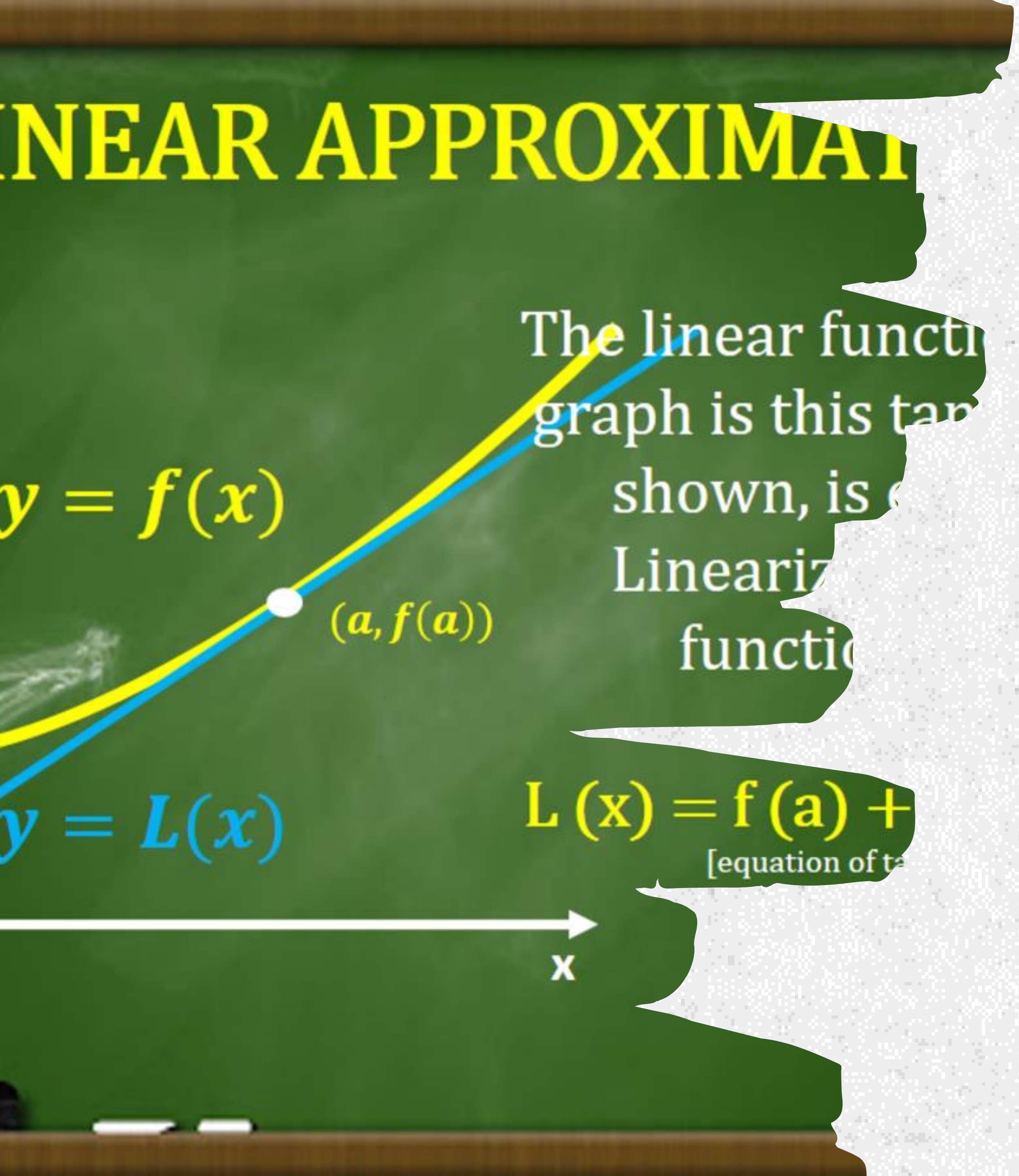
Mohanad N. Noaman

Mohanad.noaman@uoninevah.edu.iq



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SYSTEM MODELING

Lecture 10

Linearization

Instructor
Mohanad N. Noaman

LINEARIZATION

- The object of linearization is to derive a linear model whose response will agree closely with that of the nonlinear model.

- The systems that we have studied so far have primarily been assumed linear, making their analysis somewhat straightforward.

- Many dynamic systems contain elements that are inherently nonlinear, which cannot be treated as linear, except for a restricted range of operating conditions.

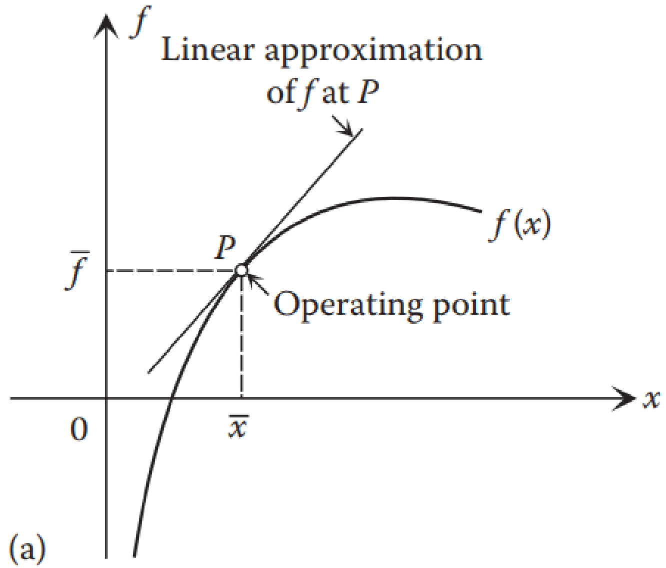
- The most direct way of dealing with a nonlinear model is to linearize it about a reference equilibrium.

LINEARIZATION

- Consider again the general, nonlinear system in Equation below:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)),$$
$$\mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t))$$

- Often have a nonlinear set of dynamics given by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- where x is once gain the state vector, u is the vector of inputs, and f (·, ·) is a nonlinear vector function that describes the dynamics.
- First step is to define the point about which the linearization will be performed typically about ***equilibrium points*** – a point for which if the system starts there it will remain there for all future time.
- So in theory, if the state gets to an equilibrium, it will perfectly stay at the equilibrium forever.



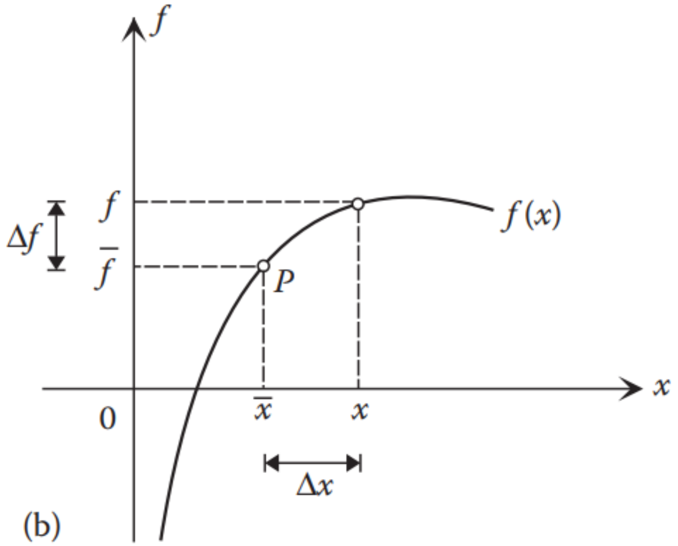
LINEARIZATION

- In practice, because of noise, disturbances, etc., the state never stays perfectly at $\bar{x}$, but may rather stay close to it. This motivates the study of nonlinear systems near their equilibria, as we see next.

- Let

$$\Delta x = x(t) - \bar{x}$$
$$\Delta u = u(t) - \bar{u}$$

- How to find equilibrium point or operating point?
  Characterized by setting the state derivative to zero: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) = 0$
  We can solve for $\bar{x}, \bar{u}$, if the input is known! Then these are equilibrium points or operation points.



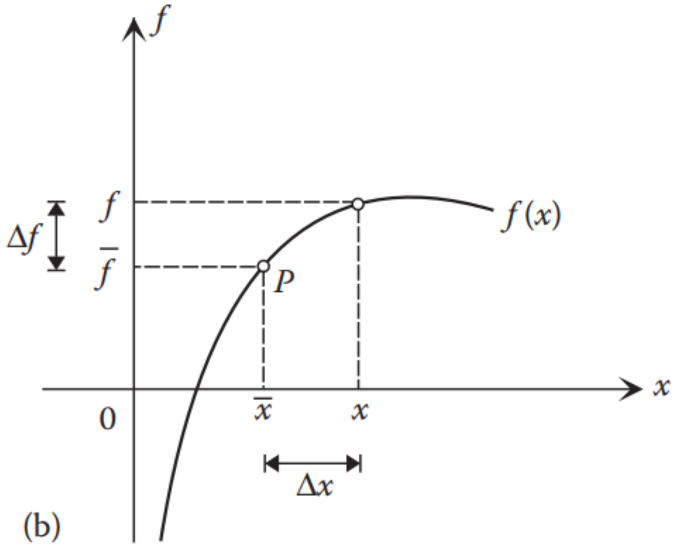
Analytically, we can use the Taylor series expansion of $f(x)$ about the operating point, as

$$f(x) = \underbrace{\boxed{f(\bar{x}) + \left.\frac{df}{dx}\right|_P (x - \bar{x})}}_{\text{Linear terms}} + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_P (x - \bar{x})^2 + \ \ldots + H.O.T$$

LINEARIZATION

Assuming $\Delta x = x(t) - \bar{x}$ is small, the linear approximation of $f(x)$ is achieved by retaining the first two terms, while neglecting the remaining terms containing higher powers of $x(t) - \bar{x}$:

$$f(x) \cong f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} \Delta x$$



Therefore, the closer $x(t)$ is to $\bar{x}$ of the operating point, the better the linear approximation will be at.

Recall the vector equation $\dot{x}= f(x, u)$, each equation of which can be expanded as

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)),$$
$$\mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t))$$

Linearization

$$\dot{\boldsymbol{x}} \approx \boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}|_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}|_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}(\boldsymbol{u} - \bar{\boldsymbol{u}})$$

$$\boldsymbol{y} \approx \boldsymbol{h}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}|_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}|_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})}(\boldsymbol{u} - \bar{\boldsymbol{u}})$$

LINEARIZATION

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)),$$
$$\mathbf{y}(t) = h(\mathbf{x}(t), \mathbf{u}(t))$$

Linearization →

$$\dot{\boldsymbol{x}} \approx \boldsymbol{f}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}(\boldsymbol{x} - \overline{\boldsymbol{x}}) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}(\boldsymbol{u} - \overline{\boldsymbol{u}})$$

$$\boldsymbol{y} \approx \boldsymbol{h}(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}}) + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}(\boldsymbol{x} - \overline{\boldsymbol{x}}) + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}(\boldsymbol{u} - \overline{\boldsymbol{u}})$$

$\Delta x = x(t) - \bar{x}$ $\Delta u = u(t) - \bar{u}$ $y - h(\bar{x}, \bar{u}) = \Delta y$

Then $\Delta\dot{x} = \dot{x} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}\Delta\boldsymbol{x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}\Delta\boldsymbol{u}$ since $\bar{x}$ is constant and $f(\bar{x}, \bar{u})$=0 at operating points. $\frac{\partial f_i}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_i}{\partial x_1} & \cdots & \frac{\partial f_i}{\partial x_n} \end{bmatrix}$

$$\Delta y \approx \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}\Delta\boldsymbol{x} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}|_{(\overline{\boldsymbol{x}}, \overline{\boldsymbol{u}})}\Delta\boldsymbol{u}$$

LINEARIZATION

$$\Delta \dot{x} \approx \underbrace{\boxed{\frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{u})}}}_{A} \Delta x + \underbrace{\boxed{\frac{\partial f}{\partial u}\Big|_{(\bar{x},\bar{u})}}}_{B} \Delta u$$

$$\Delta y \approx \underbrace{\boxed{\frac{\partial h}{\partial x}\Big|_{(\bar{x},\bar{u})}}}_{C} \Delta x + \underbrace{\boxed{\frac{\partial h}{\partial u}\Big|_{(\bar{x},\bar{u})}}}_{D} \Delta u$$

LINEARIZATION

$$\Delta\dot{x} \approx \frac{\partial f}{\partial x}\Big|_{(\bar{x},\bar{u})}\Delta x + \frac{\partial f}{\partial u}\Big|_{(\bar{x},\bar{u})}\Delta u \qquad \Delta y \approx \frac{\partial h}{\partial x}\Big|_{(\bar{x},\bar{u})}\Delta x + \frac{\partial h}{\partial u}\Big|_{(\bar{x},\bar{u})}\Delta u$$

$$A(t) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & & \vdots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(\bar{x},\bar{u})} \qquad B(t) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ & & \vdots & \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{(\bar{x},\bar{u})}$$

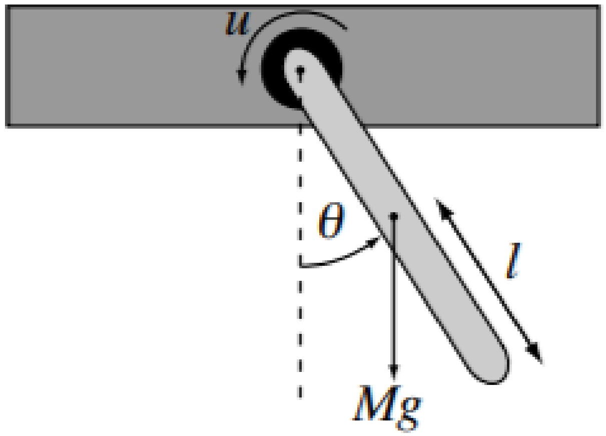
$$C(t) = \left[\frac{\partial h}{\partial x_1} \dots \dots \dots \dots \frac{\partial h}{\partial x_n}\right] \qquad D(t) = \frac{\partial h}{\partial u}$$

Typically drop the "Δ" as they are rather cumbersome, and (abusing notation) we write the state equations as:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t) \end{aligned}$$

EXAMPLE:

(A simple pendulum). Consider the dynamics of the pendulum depicted below, where u denotes an input torque provided by a DC motor.

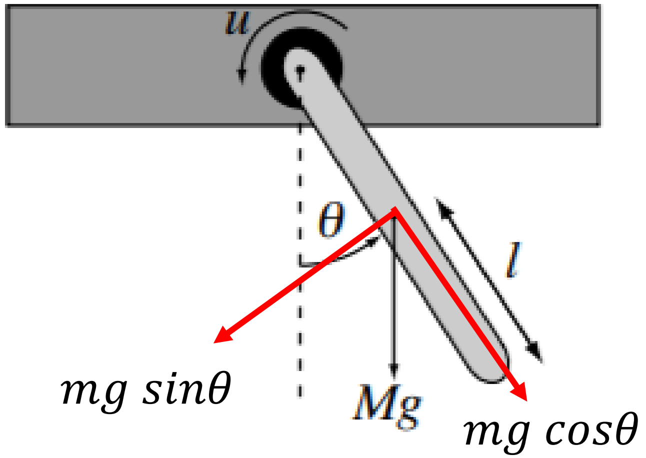


SOLUTION:

The equation of motion for this system is

$$I\frac{d^2\theta}{dt^2} + Mgl\sin\theta = u$$

$$y = \theta,$$



Solve for higher order:

$$\ddot{\theta} = -\frac{mgl}{I}sin\theta + \frac{1}{I}u$$

State variables are $x_1 = \theta\ , x_2 = \dot{\theta}$

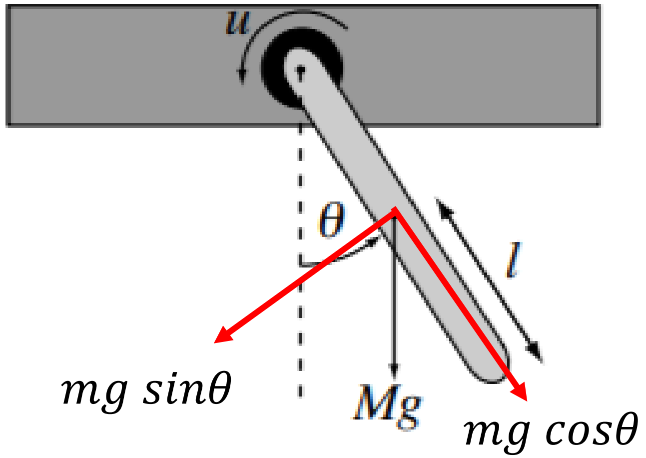
State variable equations are $\dot{x}_1 = x_2\ ,\ \dot{x}_2 = -\frac{mgl}{I}\boxed{sinx_1} + \frac{1}{I}u$ output $y = x_1$

Nonlinear term

So, we need linearization!

SOLUTION:

$$\mathbf{f}(\mathbf{x}, u) = \begin{bmatrix} x_2 \\ -\frac{Mgl}{I}\sin x_1 + \frac{u}{I} \end{bmatrix}, \quad h(\mathbf{x}, u) = x_1.$$



$$f_1 = \dot{x}_1 = x_2, f_2 = \dot{x}_2 = -\frac{mgl}{I} sinx_1 + \frac{1}{I}u$$

Calculate the equilibrium point when $f(x, u) = 0$

$x_2 = 0\ , -\frac{mgl}{I} sinx_1 + \frac{1}{I}u = 0$ then $sin\bar{x} = \frac{\bar{u}}{mgl}$

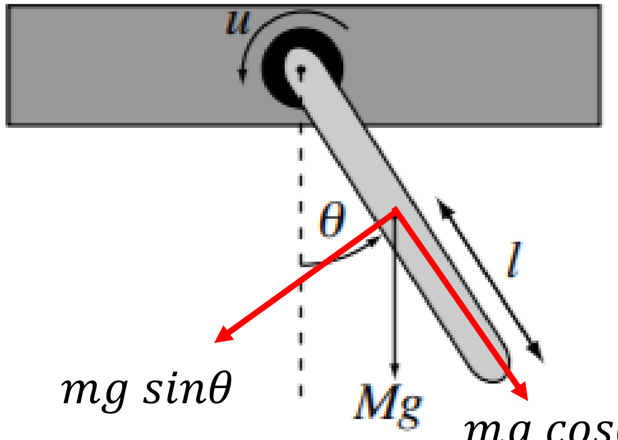
suppose we turn off the DC motor, that is, we set $\bar{u} = 0$.

$$sin\bar{x} = 0, \bar{x} = [0\ , k\pi]\ where\ k = 0, \pm 1, \pm 2, \ldots \ldots$$

SOLUTION:

$$\mathbf{f}(\mathbf{x},u)=\begin{bmatrix} x_2 \\ -\frac{Mgl}{I}\sin x_1+\frac{u}{I}\end{bmatrix},\qquad h(\mathbf{x},u)=x_1.$$

$$f_1=\dot{x}_1=x_2,\ f_2=\dot{x}_2=-\frac{mgl}{I}sinx_1+\frac{1}{I}u$$



Calculate the ***equilibrium point*** when $f(x,u)=0$

$x_2=0\ ,-\frac{mgl}{I}sinx_1+\frac{1}{I}u=0$ then $sin\bar{x}=\frac{\bar{u}}{mgl}$

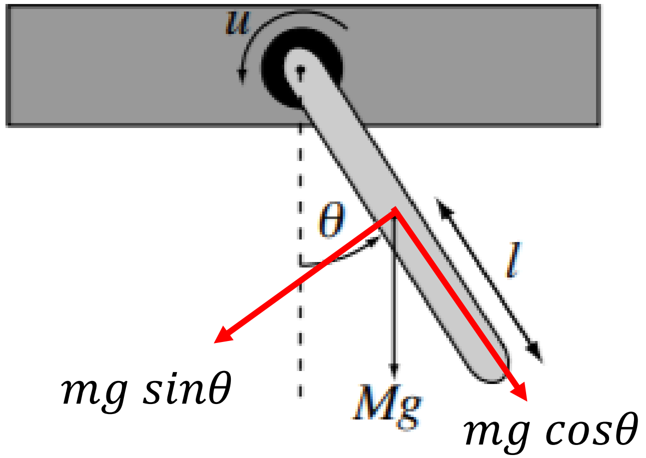
suppose we turn off the DC motor, that is, we set $\bar{u}=0$.

$$sin\bar{x}=0,\bar{x}=[0\ ,k\pi]\ \ where\ k=0,\pm 1,\pm 2,\dots\dots$$

SOLUTION:

Linearization

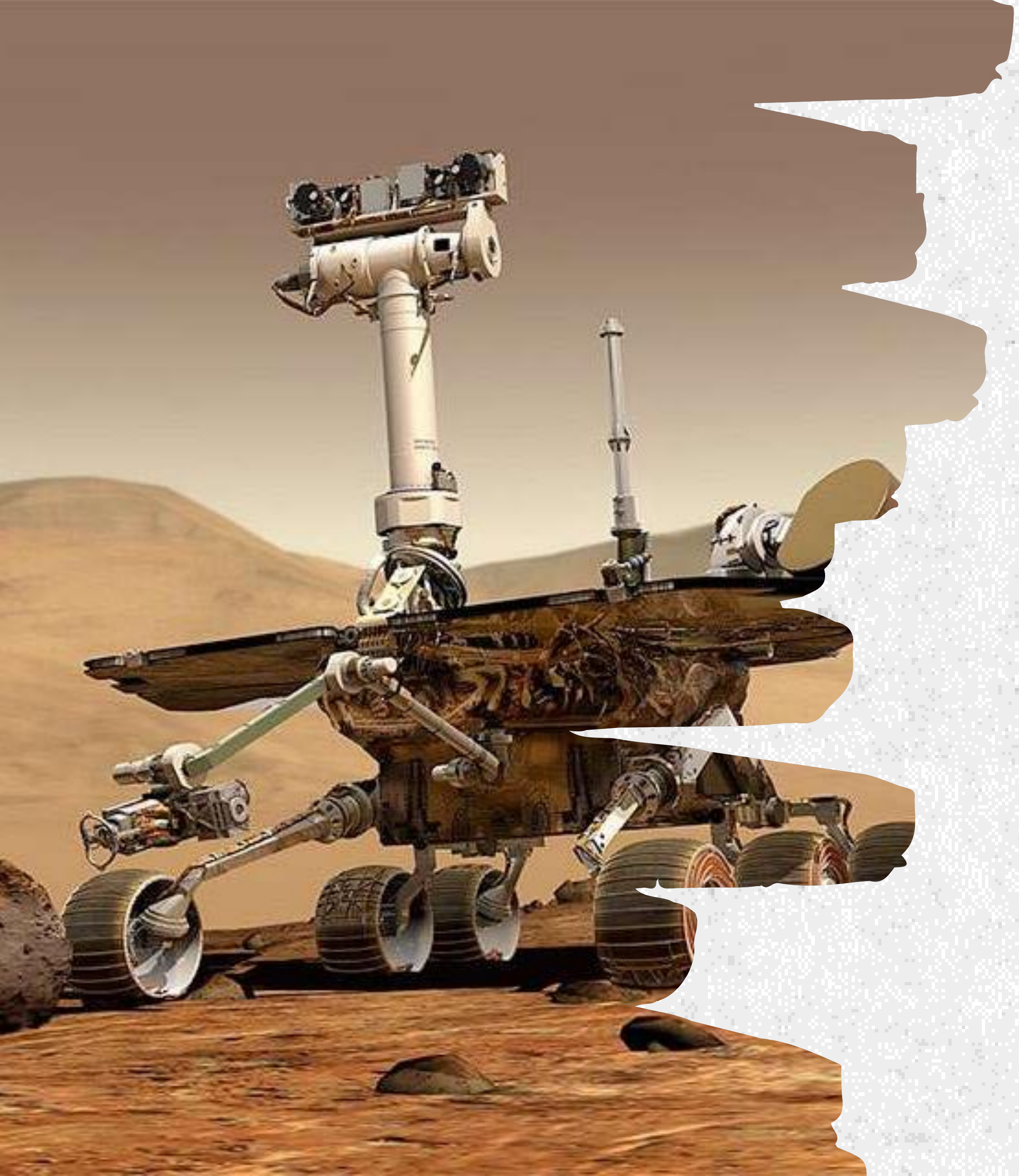
$$A = \begin{bmatrix} 0 & 1 \\ -\frac{Mgl}{I}\cos x_1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$



So at equilibrium point $(\bar{x}, \bar{u}) = (\pi\ , 0)$

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{I} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} \Delta u$$

$$\Delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \Delta x_1.$$



THANK YOU

Mohanad N. Noaman
Mohanad.noaman@uoninevah.edu.iq