College of Electronics Eng. Ninevah University

Systems and Control Eng. Department

Lecture_1

Subject: Linear Quadratic Gaussian (LQG) Control

Lecturer: Dr. Ibrahim Khalaf Mohammed

LQG Control

In full state feedback optimal control system as shown below, the system is assumed noiseless and its variables are available for feedback process.

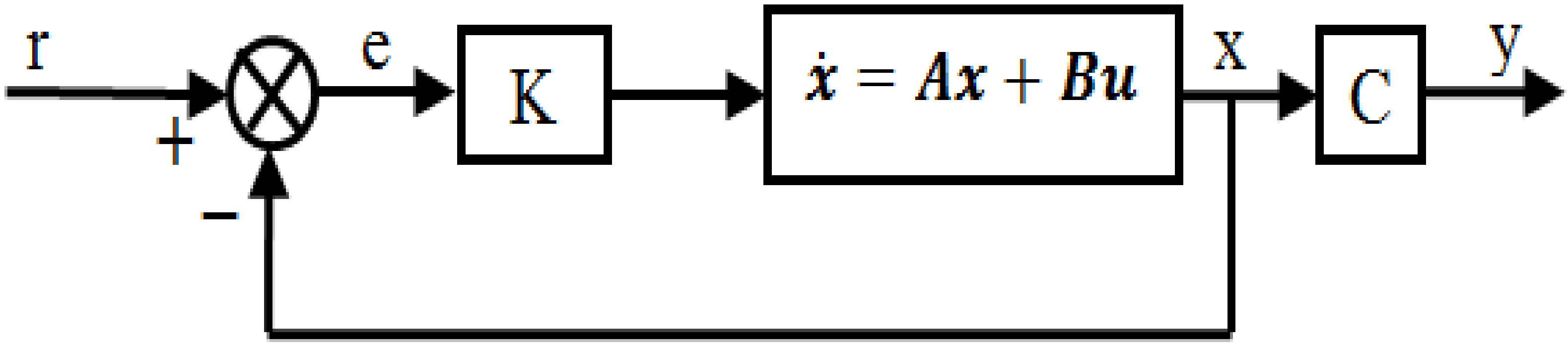

Figure (1). Full state feedback control system.

Where the state vector x=$[x1\ x2\ \ldots.\ xn]^T$ K=[k1 k2 ... Kn]

However, this assumption may be not hold in practice as it is subject to noises and not all the system states are physically measurable. In order to apply state feedback controller technique a Kalman filter must be included to the system.

LQG Control

LQG controller is a modern state space technique for designing optimal dynamic regulators. It is used to reject process (*w*) and mesurement (*v*) noises and for state estimation for non measurable systems.

LQG is a combination of an optimal LQR controller with Linear Quadratic Estimator (LQE) (Kalman Filter) as shown in figure (2).

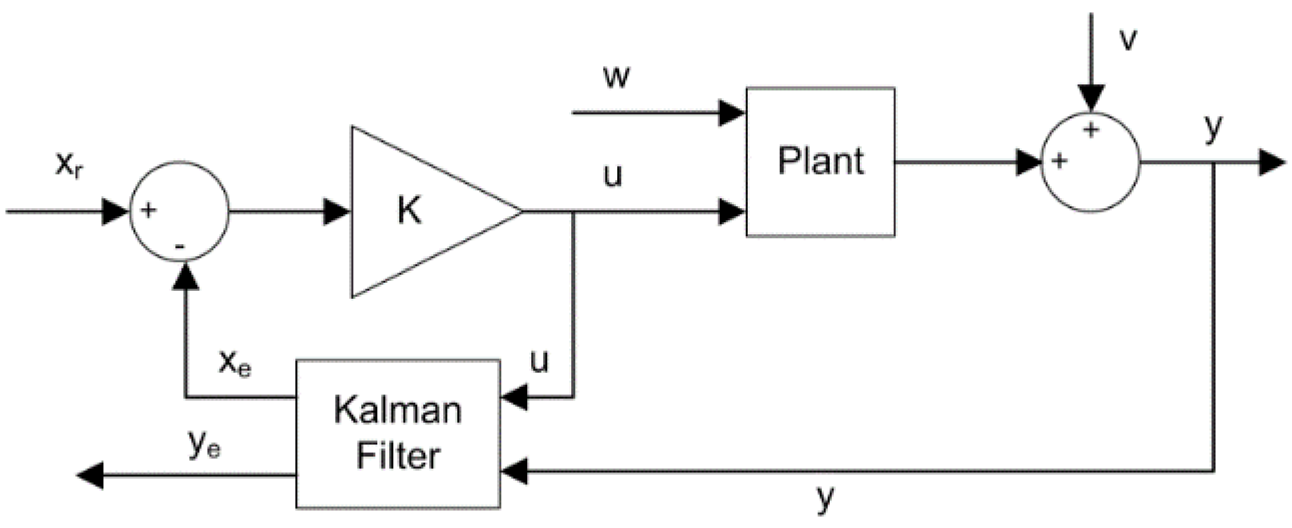

Figure (2). LQG control system

LQG Control

Consider a plant that subject to a Gaussian disturbance *w*(t) (process noise) and measurement noise *v*(t) as shown in figure (3)

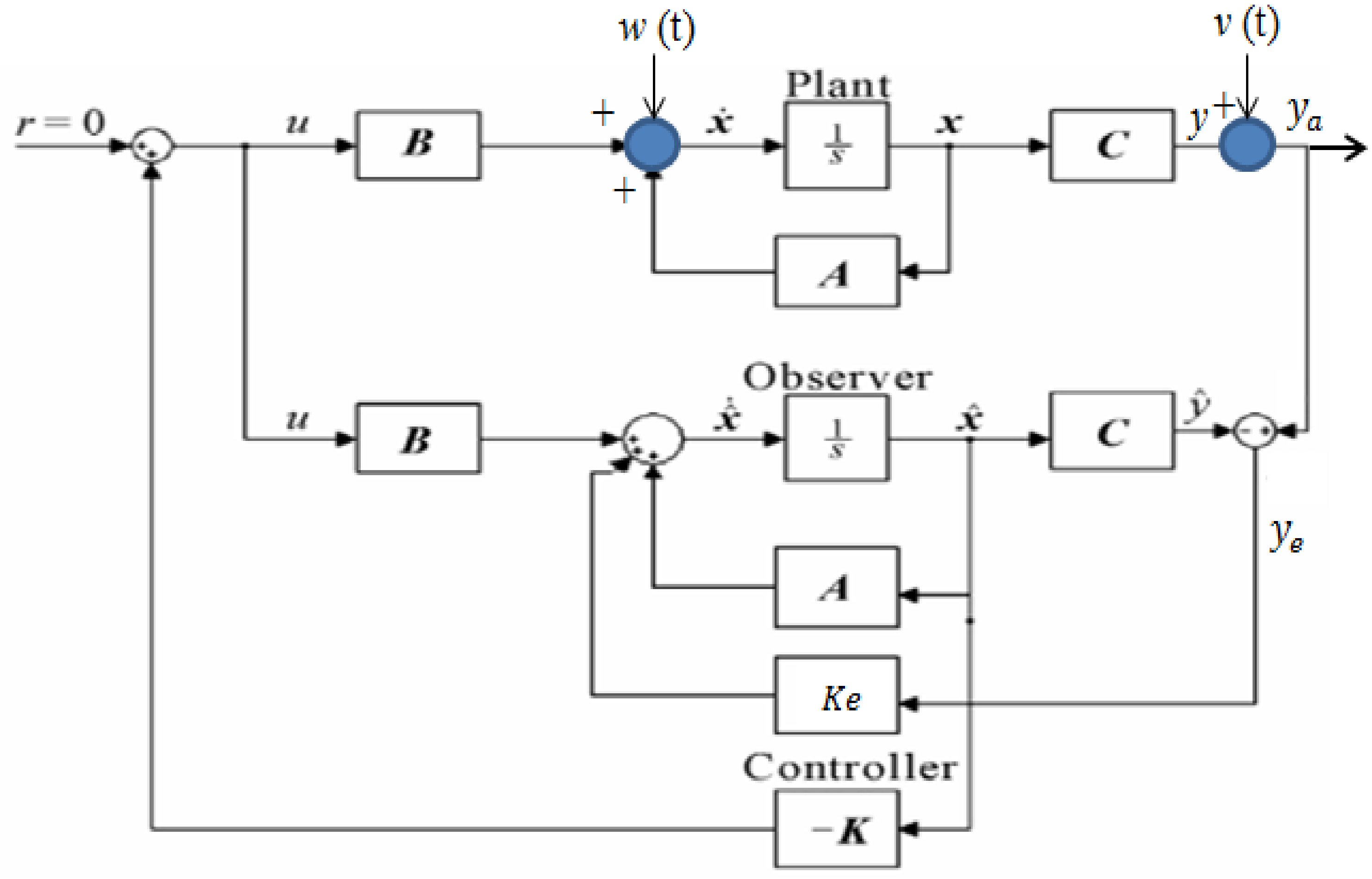

Figure (3). LQG controller structure.

LQG Control

The state and output equations of the system are as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \quad (1)$$

$$y(t) = Cx(t) + v(t) \quad (2)$$

While the state and output equations of the estimator are given below:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_e y_e \quad (3)$$

$$\hat{\boldsymbol{y}}(t) = C\hat{x}(t) \quad (4)$$

Where $\hat{x}(t)$ is the estimated value of the state $x(t)$.

LQG Control

$$\dot{\hat{x}}(\mathrm{t}) = A\hat{x}(t) + Bu(t) + K_e(y_a - \hat{y})$$

Based on (4), the above equation becomes

$$\dot{\hat{x}}(\mathrm{t}) = A\hat{x}(t) + Bu(t) + K_e(y_a - \mathrm{C}\hat{x}(\mathrm{t}))$$

$$\dot{\hat{x}}(\mathrm{t}) = (A - K_e C)\hat{x}(t) + Bu(t) + K_e y + K_e V(t) \qquad (5)$$

Where K_e is the estimator gain matrix. The estimation error is geven by:

$$e(t) = x(t) - \hat{x}(t) \qquad (6)$$

$$\dot{e}(\mathrm{t}) = \dot{x}(\mathrm{t}) - \dot{\hat{x}}(\mathrm{t}) \qquad (7)$$

Using (1) and (5), (7) can be written as follows:

$$\dot{e}(\mathrm{t}) = Ax(t) + Bu(t) + W(t) - (A - K_e C)\hat{x}(t) - Bu(t) - K_e y - K_e V(t)$$

LQG Control

$$\dot{e}(\mathrm{t}) = Ax(t) + W(t) - (A - K_e C)\hat{x}(t) - K_e C x(t) - K_e V(t)$$

$$\dot{e}(\mathrm{t}) = (A - K_e C)x(t) - (A - K_e C)\hat{x}(t) + W(t) - K_e V(t)$$

$$\dot{e}(\mathrm{t}) = (A - K_e C)\big(x(t) - \hat{x}(t)\big) + W(t) - K_e V(t)$$

$$\dot{e}(\mathrm{t}) = (A - K_e C)e(t) + W(t) - K_e V(t) \qquad (8)$$

Based on (6), the control effort is given by:

$$\mathrm{u(t)} = -\mathrm{K}\hat{x}(t) = -\mathrm{K}\ (\mathrm{x(t)} - \mathrm{e(t)}) = -\mathrm{Kx(t)} + \mathrm{Ke(t)} \ \ldots\ldots.(9)$$

LQG Control

Using (9), (1) becomes:

$$\dot{x}(t) = Ax(t) + B\big(-Kx(t) + Ke(t)\big) + W(t)$$

$$\dot{x}(t) = (A - BK)x(t) + BKe(t) + W(t) \qquad (10)$$

Based on (8) and (10), the state equation of the LQG controller is as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BK & BKe \\ 0 & A - K_e C \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & -K_e \end{bmatrix} \begin{bmatrix} W(t) \\ V(t) \end{bmatrix} \quad (11)$$

Stability of the LQG controller is governed by the closed loop poles of the LQG's state matrix. Dynamic behavior of the observer depands on the eigenvalues of the matrix $(A - K_e C)$. If $(A - K_e C)$ is stable then $\hat{x}(t) \rightarrow x(t) as\ t \rightarrow \infty$.

LQG Control

The transient response of the observer is faster than that of the system itself. While the stability of the LQR controller depends on the closed-loop poles of the matrix (A-BK).

It is worth considering that if the noises *w*(t) and *v*(t) have small values then their effect on the system can be neglected.

To design a stable state observor, the system must be completely observable. The observability of the system is examined by the following observability matix N which is given by: $N = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$

LQG Control

If the rank of N equals the order of the system (n), then the system is completely observable. If the produced N matrix is an identity matrix, then the system is completely observable.

It can also check the observability of the system easly using Matlab command “**obsv(A,C)**”

Observability

Example: Compute the obsevalility of a system with the following state and output matrices

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, C = [1\ 0]$$

Solution: order of the system (n) is 2

$$N = \begin{bmatrix} C \\ CA \end{bmatrix}$$

CA=[0 1],

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

N is a second identity matrix then the system is observable.

College of Electronics Eng. Ninevah University

Systems and Control Eng. Department

Lecture_2

Subject: Linear Quadratic Gaussian (LQG) Control

Lecturer: Dr. Ibrahim Khalaf Mohammed

Estimation Kinds

Types of estimators

1. Full-order state estimator

In this type, the observer estimates all the state variables of the system.

2. Reduced-order state estimator

In this type, the observer estimates fewer than n state variables, where n is the dimension of the system state vector.

In this course, design techniques of full-order state observer will be taken into consideration

Estimation Kinds

Full-order Observer Design Methods

1- Direct Comparison Method

2-Observable Canonical Form Method

3-Ackermann's Formula Method

States Estimation Techniques

Direct Comparison Method

Using the state equation:

$$\dot{x}(t)= Ax(t) + Bu(t) + W(t)$$

sX(s)=AX(s)+BU(s)

sX(s)-AX(s)=0

(s – A)X(s)=0

The characteristic equation of the open loop system (orignal system) is:

|sI- A|=0

$$s^n+a_{n-1}s^{n-1}+\ldots\ldots+a_1 s + a_0=0$$

Design of the estimator depends on the locations of the closed-loop poles of the observer which should be selected properly

States Estimation Techniques

Direct Comparison Method

The estimator 's poles ($\mu_1, \mu_2 \ldots \ldots \ldots \mu_n$) are determined from closed-loop observer matrix.

$$s_1 = \mu_1,\ s_2 = \mu_2,\ \ldots\ldots\ldots\ldots\ldots\ s_n = \mu_n$$

Based on (8):

$$\dot{e}(\mathrm{t}) = (A - K_e C)e(t)$$

$$sE(s) = (A - K_e C)E(s)$$

$$(s - A + K_e C)E(s) = 0$$

Based on the above equation, the characteristics equation of the observer is as follows:

$$|sI - A + K_e C| = 0$$

$$|sI - A + K_e C| = (s - \mu_1)(s - \mu_2)\ldots\ldots\ldots\ldots(s - \mu_n)$$

$$= s^n + \alpha_{n-1} s^{n-1} + \ldots\ldots + \alpha_1 s + \alpha_0 = 0$$

States Estimation Techniques

Observable Canonical Form Method

Define the following transformation matrix (Q) which uses to transform the state equation of the system into the observable canonical form:

$$\boldsymbol{Q} = (\boldsymbol{MN})^{-1}$$

Where M is the given by:

$$M = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & & 1 & 0 \\ \vdots & & \ddots & \vdots & \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{bmatrix}, N = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

States Estimation Techniques

State and output equations of the system in controallable canonical form are given below

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} == \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \mathbf{0} & 0 & & 0 & 0 \\ \cdot \;\vdots & \cdot & \ddots & \vdots & \cdot \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \mathrm{u(t)}$$

$$y(t) = Cx(t) + Du(t)$$

$$y = [c_0 \quad c_1 \quad \cdot \quad c_{n-1}] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

States Estimation Techniques

The observable form of the state equation is as follows

$$\dot{x}(t) = A_o x(t) + B_o u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ \vdots & & \ddots & \vdots & \\ . & . & & . & . \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} c_0 \\ c_1 \\ . \\ . \\ c_{n-1} \end{bmatrix} u(t)$$

Note: the state matrix (A_o) of the observable canonical form is the transport of the controllable canonical form (A^T) and the input matrix (B_o) of the observable canonical form is the transport of the output matrix for controllable canonical form (C^{T}): $A_O = A^T, B_O = C^T$

States Estimation Techniques

The expression used to find the observer gain matrix is given below:

$$K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \\ \cdot \\ \cdot \\ K_{en} \end{bmatrix} = (\boldsymbol{MN})^{-1} \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \\ \cdot \\ \cdot \\ \alpha_{n-1} - a_{n-1} \end{bmatrix} \quad , N(obs.matrix) = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

Where a_0..... a_{n-1} are the solution polynomails of the open loop characteristics equation for the system $|sI - A|$. While α_0 ..... α_{n-1} are the solution polynomails of the desired characteristics equation for the observer. For example, equation of the second order observer may have the following form:

$$s^2 + 2\xi w_n s + {w_n}^2 = 0$$

$$(s - \mu_1)(s - \mu_2) = 0$$

Where μ_1 and μ_2 are the desired closed loop ploes of the observer.

Solution of the above equations is given by:

$$s^n + \alpha_{n-1} s^{n-1} + ... + \alpha_1 s + \alpha_0 = 0$$

States Estimation Techniques

Ackemann's Formula Method

The solution polynomails of the open loop characteristics equation for the system $|sI - A|$ is as follows:

$$s^n + a_{n-1}s^{n-1} + \ldots\ldots + as + a = 0$$

The solution of the desired characteristics equation $\emptyset(s)$ of the observer is as follows:

$$\emptyset(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots\ldots + \alpha_1 s + \alpha_0 = 0$$
$$\emptyset(A) = A^n + \alpha_{n-1}A^{n-1} + \ldots\ldots + \alpha_1 A + \alpha_0 = 0$$

States Estimation Techniques

Ackemann's Formula Method

The gain matrix of the observer is given by:

$$K_e = \emptyset(A)\begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ CA^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ 1 \end{bmatrix}$$

Systems and Control Eng. Dept.

Subject: Optimal Control II

Lecture_3: Estimator Design Methods

Asst. Prof. Dr. Ibrahim K. Mohammed

Example: Consider the system defined by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, B= $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, C= $[1 \quad 0]$

Design a full-order state observer that has an undamping frequency of 10 rad/s and a damping ratio of 0.5.

Solution:

To design a state estimator, we have to check the observability of the system,

The order of the system n = 2

$$N = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

$$CA = [1 \quad 0]\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = [0 \quad 1]$$

$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the rank of N is 2, rank(N) = n,

Then the system is completley observable.

To find poles of open-loop system:

|sI-A| = 0,

$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right|$=0, $\begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}$=0

s(s+3)+2=0

$s^2 + 3s + 2 = 0$

Comparing the above equation with the below standard equation:

$s^2 + a_1 s + a_0 = 0$ -----→ a_0=2, a_1=3

$s^2 + 3s + 2 = 0$

(s+1)(s+2) = 0

s_1 = -1

s_2 = -2

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The characteristics equation of the desired estimator is as follows:

$s^2 + 2\xi w_n s + {w_n}^2=0$

$s^2 + 2 * 0.5 * 10s + (10)^2 = 0$

$s^2 + 10s + 100 = 0$

But: $s^2 + \alpha_1 s + \alpha_0 = 0$

$\alpha_0 = 100,\ \alpha_1 = 10$

Poles of estimator: s_1 = -5+j8.66, s_2 = -5-j8.66

<u>1. Observer design using Direct Comparison method:</u>

The charactersitics equation of the observer is given by

$|sI - A + K_e C| = s^2 + \alpha_1 s + \alpha_0$

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} K_{e1} \\ K_{e2} \end{bmatrix} [1 \quad 0]\right| = s^2 + 10s + 100$$

$$\left|\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} + \begin{bmatrix} K_{e1} & 0 \\ K_{e2} & 0 \end{bmatrix}\right| = s^2 + 10s + 100$$

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$$s^2 + (3 + K_{e1})s + (3K_{e1} + K_{e2} + 2) = s^2 + 10s + 100$$

$$3 + K_{e1} = 10 \qquad ---\rightarrow \quad K_{e1} = 7$$

$$3K_{e1} + K_{e2} + 2 = 100 \; ---\rightarrow \; K_{e2} = 77$$

The observer gain matrix $K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \end{bmatrix} = \begin{bmatrix} 7 \\ 77 \end{bmatrix}$

2. Observer design using Observable Canonical Form method

$$K_e = \begin{bmatrix} K_{e1} \\ K_{e2} \\ \cdot \\ \cdot \\ K_{ne} \end{bmatrix} = (MN)^{-1} \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \\ \cdot \\ \cdot \\ \alpha_{n-1} - a_{n-1} \end{bmatrix}$$

$$K_e = (MN)^{-1} \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \end{bmatrix}$$

$$M = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

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$$MN = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q = (MN)^{-1} = \frac{\begin{bmatrix} 0 & -1 \\ -1 & 3 \end{bmatrix}}{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$K_e = Q \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 100 - 2 \\ 10 - 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 98 \\ 7 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 7 \\ 77 \end{bmatrix}$$

<u>3. Observer design using Ackermann's Formula method</u>

$$K_e = \emptyset(A) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = [A^2 + \alpha_1 A + \alpha_0 I] \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \left[\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^2 + 10 \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + 100 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \left[\begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 10 \\ -20 & -30 \end{bmatrix} + \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}\right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 98 & 7 \\ -14 & 77 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 98 & 7 \\ -14 & 77 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 7 \\ 77 \end{bmatrix}$$

Systems and Control Eng. Dept.

Subject: Optimal Control II

Lecture: 4

Lecture Title: **LQG System Design**

Asst. Prof. Dr. Ibrahim K. Mohammed

LQG Design for Noiseless System

The state space representation of dynamic system that subject to process (W) and measurement (V) noises is given below:

$$\dot{x}(t) = Ax(t) + Bu(t) + GW(t) + 0\ V(t)$$
$$y(t) = Cx(t) + Du(t) + 0W(t) + HV(t)$$

If the process (W) and measurement (V) noises are very small so that it can neglect their effect on the system, then the state and output equations are given below:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

The estimator gain matrix can be calculated using **direct comparison method, observable canonical method, and Akermann's formula method.**

LQG Design for Noisless System

While the **LQR Controller gain matrix** can be calculated using the following equation

$$K = R^{-1}B^{T}P$$

Where **P** is the solution of the following Riccati equation:

$$PA + A^{T}P - PBR^{-1}B^{T}P + Q = 0$$

LQG Design for Noisless System

In matlab the LQR gain matrix is calculated as below:

$$K = lqr(A, B, Q, R)$$

Q is the state weighting matrix of the system

R is the control (input) weighting matrix of the system

Example: Consider the system defined by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Where $A = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$, B= $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = [1 \quad 0]$,

If the state weighting matrix is Q= $\begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$, the input weighting matrix is $R = 0.01$ and LQR controller Riccati matrix P= $\begin{bmatrix} 0.99 & 0.012 \\ 0.012 & 0.126 \end{bmatrix}$, design a LQG controller for the following if the desired eigenvalues of the full-order observer are $\mu_1 = -9$, $\mu_2 = -10$.

LQG Design for Noisless System

Solution

The LQG controller is a combination of LQR controller with an estimator:

To design a state estimator, we have to check the observability of the system,

The order of the system n = 2

$$N = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

$$N = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$N = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, the rank of N is 2, rank(N) = n,

Then the system is completley observable.

To find poles of open-loop system:

|sI - A| = 0,

$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} \right|$=0, $\begin{vmatrix} s+1 & -1 \\ -2 & s+4 \end{vmatrix}$=0

(s+1)(s+4)-2=0

$s^2 + 5s + 2 = 0$

Comparing the above equation with the below standard equation:

$s^2 + a_1 s + a_0 = 0$ -----→ a_0 = 2, a_1 = 5

$s^2 + 5s + 2 = 0$

s_1 = - 0.4384

s_2 = - 4.56

The plant of the system is stable.

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The characteristics equation of the desired estimator is as follows

$(s-\mu_1)(s-\mu_2) = 0$

$(s+9)(s+10) = 0$

$s^2 + 19s + 90 = 0$

But: $s^2 + \alpha_1 s + \alpha_0 = 0$

$\alpha_0 = 90,\ \alpha_1 = 19$

1. Observer design using Direct Comparison method:

The charactersitics equation of the observer is given by

$|sI - A + K_e C| = s^2 + \alpha_1 s + \alpha_0$

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right| = s^2 + 19s + 90$$

$$\left|\begin{bmatrix} s+1 & -1 \\ -2 & s+4 \end{bmatrix} - \begin{bmatrix} k_{e1} & 0 \\ k_{e2} & 0 \end{bmatrix}\right| = s^2 + 19s + 90$$

$$s^2 + (5 + k_{e1})s + (4k_{e1} + k_{e2} + 2) = s^2 + 19s + 90$$

$5 + k_{e1} = 19 \quad ----\rightarrow \; k_{e1} = 14$

$4k_{e1} + k_{e2} + 2 = 90 \; ----\rightarrow \; k_{e2} = 32$

The observer gain matrix $K_e = \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$

2. Observer design using Observable Canonical Form method

$$K_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ \cdot \\ \cdot \\ k_{ne} \end{bmatrix} = (MN)^{-1} \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \\ \cdot \\ \cdot \\ \alpha_{n-1} - a_{n-1} \end{bmatrix}$$

$$K_e = (MN)^{-1} \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \end{bmatrix}$$

$$M = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

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$$MN = \begin{bmatrix}5 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix} = \begin{bmatrix}4 & 1\\1 & 0\end{bmatrix}$$

$$Q = (MN)^{-1} = \frac{\begin{bmatrix}0 & -1\\-1 & 4\end{bmatrix}}{-1} = \begin{bmatrix}0 & 1\\1 & -4\end{bmatrix}$$

$$K_e = Q\begin{bmatrix}\alpha_0 - a_0\\\alpha_1 - a_1\end{bmatrix}$$

$$K_e = \begin{bmatrix}0 & 1\\1 & -4\end{bmatrix}\begin{bmatrix}90-2\\19-5\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & -4\end{bmatrix}\begin{bmatrix}88\\14\end{bmatrix}$$

$$K_e = \begin{bmatrix}14\\32\end{bmatrix}$$

3. Observer design using Ackermann’s Formula method

$$K_e = \emptyset(A)\begin{bmatrix}C\\CA\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

$$K_e = [A^2 + \alpha_1 A + \alpha_0 I]\begin{bmatrix}C\\CA\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

$$K_e = \left[\begin{bmatrix}-1 & 1\\2 & -4\end{bmatrix}^2 + 19\begin{bmatrix}-1 & 1\\2 & -4\end{bmatrix} + 90\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right]\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

.

$$K_e = \left[\begin{bmatrix} 3 & -5 \\ -10 & 18 \end{bmatrix} + \begin{bmatrix} -19 & 19 \\ 38 & -76 \end{bmatrix} + \begin{bmatrix} 90 & 0 \\ 0 & 90 \end{bmatrix}\right] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 74 & 14 \\ 28 & 32 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 74 & 14 \\ 28 & 32 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$$

It can calculate the estimator gain matrix based on pole placement technique using the Matlab command "**place**" as follows:

Ke=place(A',C',p)'

Where **p** is the desired ploes vector of the estimator

p = [-9 , -10]

LQR controller design

$$K = R^{-1}B^{T}P$$

$$K = [0.01]^{-1}\begin{bmatrix}1\\0\end{bmatrix}^{T}\begin{bmatrix}0.99 & 0.012\\0.012 & 0.126\end{bmatrix}$$

$$K = [99 \quad 1.194]$$

It can determine the **LQR gain matrix K** using Matlab command as follows:

K = lqr (A,B,Q,R) $= [99 \quad 1.194]$

[K,P,Eigvalues] = *lqr*(A,B,Q,R) $= [99 \quad 1.194]$

LQG Design for Noisy System

If the process (W) and measurement (V) noises are included to the system, then the state and output equations are given below:

$$\dot{x}(t) = Ax(t) + Bu(t) + GW(t) + 0\ V(t)$$
$$y(t) = Cx(t) + Du(t) + 0W(t) + HV(t)$$

$$Q_e = E(WW^T)$$
$$R_e = E(VV^T)$$

Inputs of the system is *u(t)*, *W(t)* and *V(t)*

Where G (n x 1) and H (1 x 1) are noise matrices, Q_e is the state weighting matrix of the estimator based on process noise and R_e is the control weighting matrix of the estimator based on measurement noise.

Q_e and R_e like Q and R matrices are designed by the designer.

LQG Design for Noisy System

The estimator gain matrix based on process and measurement noises is calculated using the following expression :

$$\boldsymbol{K_e = P_e C^T R_e^{-1}}$$

Where P_e is the solution of the following Riccati equation of the estimator:

$$AP_e + P_e A^T - P_e C^T R_e^{-1} C P_e + G Q_e G^T = 0$$

LQG Design for Noisy System

In Matlab, the estimator gain matrix of the noisy system can be calculated as follows:

Sys=ss(A,[B G],C,[D H])

$$K_e = kalman(Sys, Q_e, R_e)$$

$$Q_e = E(WW^T)$$

$$R_e = E(VV^T)$$

LQR Controller

The gain matrix of the LQR controller is:

$$K = R^{-1}B^T P$$

P is the solution of the following Riccati equation:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

LQG Design for Noisy System

In matlab the LQR gain matrix is calculated as below:

$$K = lqr(A, B, Q, R)$$

Q is the state weighting matrix

R is the control (input) weighting matrix

Example: Consider the system defined by:

$$\dot{x}(t) = Ax(t) + Bu(t) + GW(t)$$
$$y(t) = Cx(t) + Du(t) + 0W(t) + HV(t)$$

Where $A = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = [1 \quad 0]$, $G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $H = 0$.

Design a LQG controller so that for LQR controller , the state weighting matrix, input weighting matrix and Riccati matrix are given by:

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix},\ R = 0.01,\ P = \begin{bmatrix} 0.99 & 0.012 \\ 0.012 & 0.126 \end{bmatrix}$$

The Riccati matrix, state weighting matrix and control weighting matrix of the estimator are given by:

$$P_e = \begin{bmatrix} 0.2687 & 0.0845 \\ 0.0845 & 0.0363 \end{bmatrix}, Q_e = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.35 \end{bmatrix}, \ R_e = 0.15$$

LQG Design for Noisy System

Solution

To design a state estimator, we have to check the observability of the system,

The order of the system n = 2

$$N = \begin{bmatrix} C \\ CA \\ . \\ . \\ CA^{n-1} \end{bmatrix}$$

$$N = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$N = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, the rank of N is 2, rank(N) = n,

Then the system is completley observable.

LQR controller design

$$K = R^{-1}B^{T}P$$

$$K = [0.01]^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{T}\begin{bmatrix} 0.99 & 0.012 \\ 0.012 & 0.126 \end{bmatrix}$$

$$K = \begin{bmatrix} 99 & 1.194 \end{bmatrix}$$

It can determine gain matrix K using Matlab commands as follows:

[Ke,P,Eigvalues]=lqr(A,B,Q,R) $= \begin{bmatrix} 99 & 1.194 \end{bmatrix}$

Observer design

$$K_e = P_e C^T {R_e}^{-1} = \begin{bmatrix} 0.2687 & 0.0845 \\ 0.0845 & 0.0363 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}^T (0.15)^{-1} = \begin{bmatrix} 1.7916 \\ 0.5632 \end{bmatrix}$$

.

Observer design using Matlab Commands:

Sys=ss(A,[B G],C,[D H])

$$K_e = kalman(Sys, Q_e, R_e)$$

Adaptive Control

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Systems and Control Engineering Department

Lecture 5

Asst. Prof. Dr. Ibrahim Khalaf Mohammed

Text Books

- P. A. Ioannou and B. Fidan, **Adaptive Control Tutorial**, SIAM, 2006.

- P. Ioannou and J. Sun, **Robust Adaptive Control**, Prentice Hall, 1996.

- K. J. Astrom and B. Wittenmark, **Adaptive Control**, 2nd Edition, Addison-Wesley, 1995.

History of Adaptive Control

1950’s

- **Autopilots for high-performance**
- **Aircrafts** operating over a wide range of **speeds** and **altitudes**.

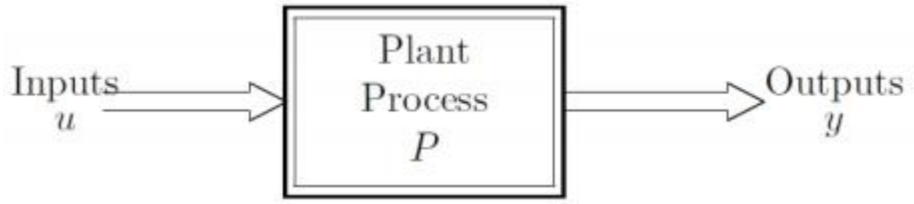
1960’s

- Space state and stability theory.

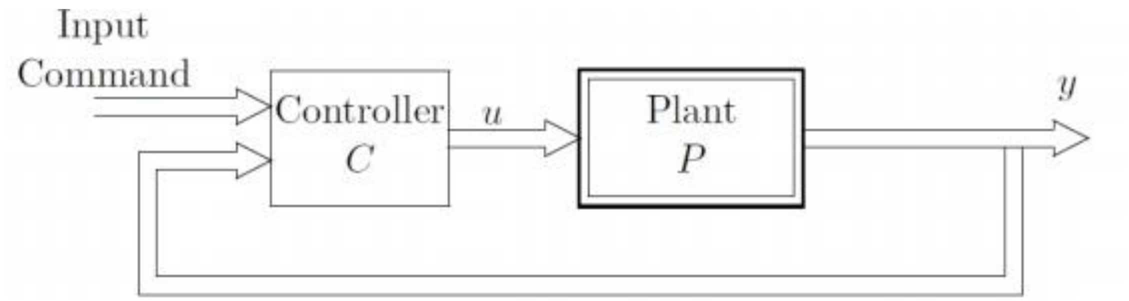
1970’s-1980’s

- Proof for stability of adaptive control systems.

Control System

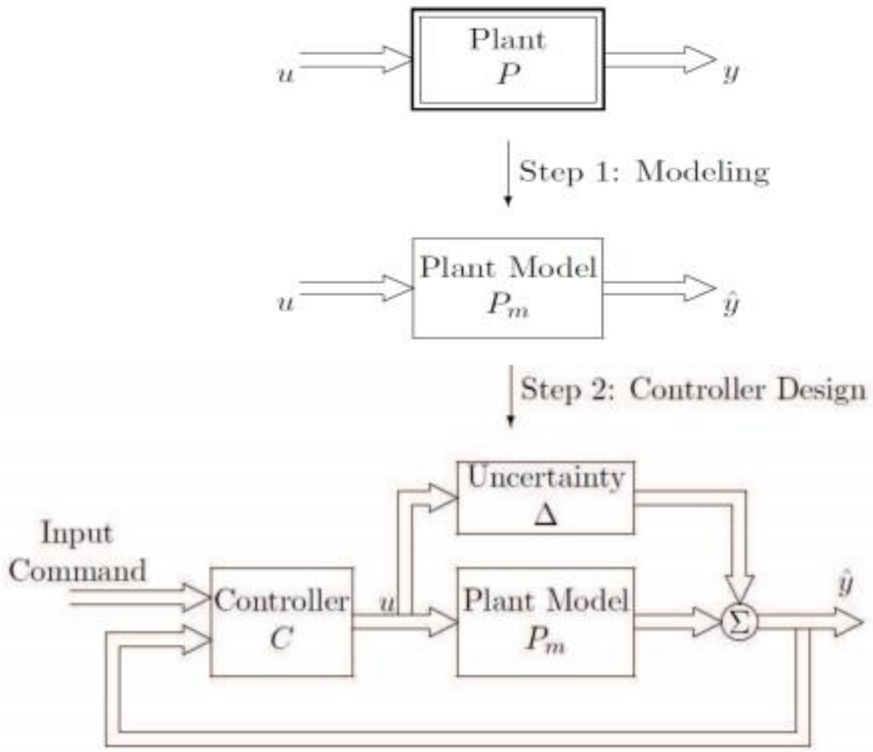


Plant Representation

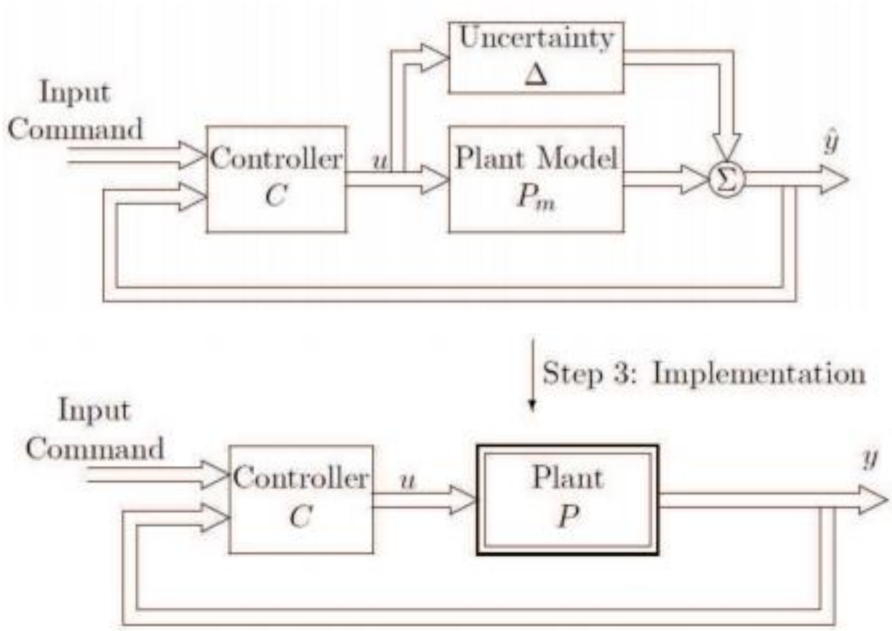


Control system

Control System Design



Control System Design (cont.)



Problems

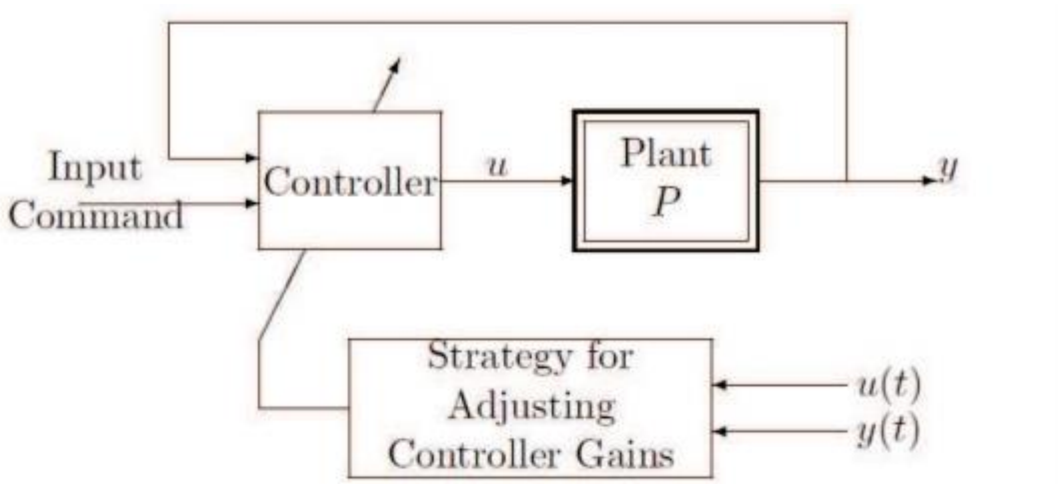
- Unknown plant model or parameters
- Plant parameters can vary with time!
- Unknown disturbance characteristics

- **Adaptive controller**: adapt to changes

- **To adapt:** to change a behaviour to conform to new circumstances.

Adaptive Controller

- A controller that adjusts its gain parameters to adapt to changes in the system plant and process which occur with time.



Adaptive Control Design Methods

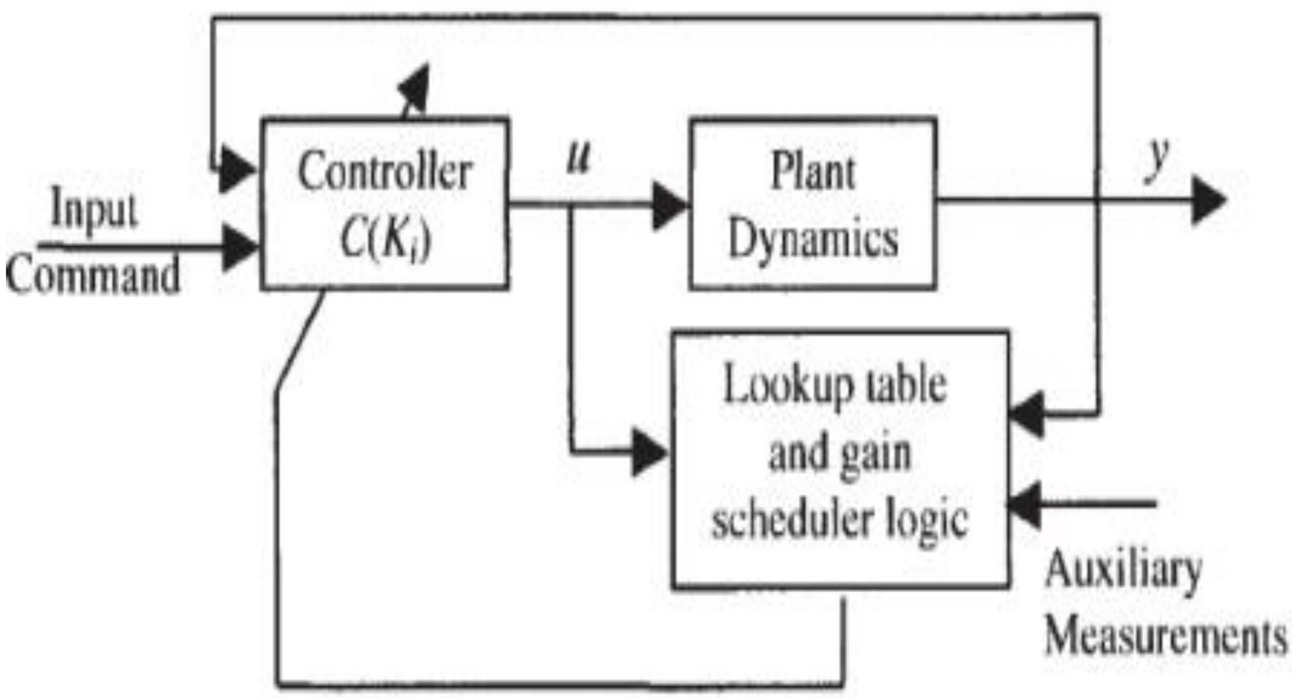
- Identifier-based Adaptive Control

- Non-identifier-based Adaptive Control

Non-identifier-based Adaptive Control

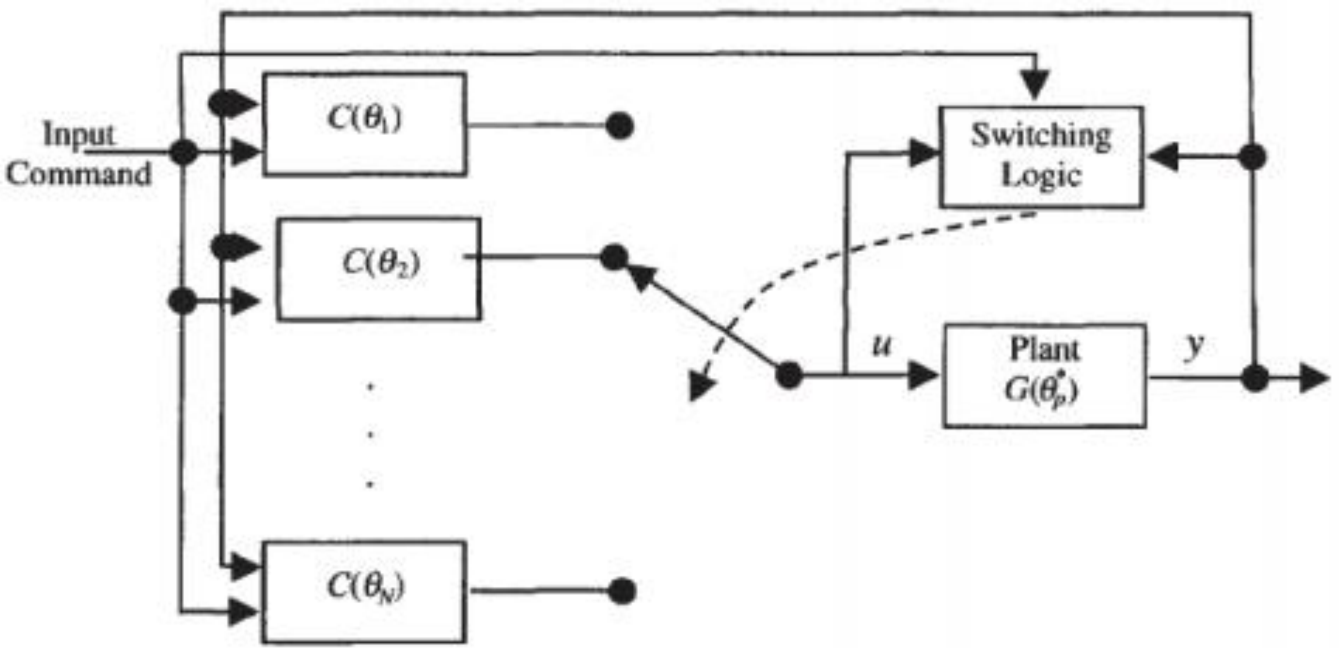
- Gain Scheduling
- Switching Control
- Multiple Model Control

Gain Scheduling



Switching Control

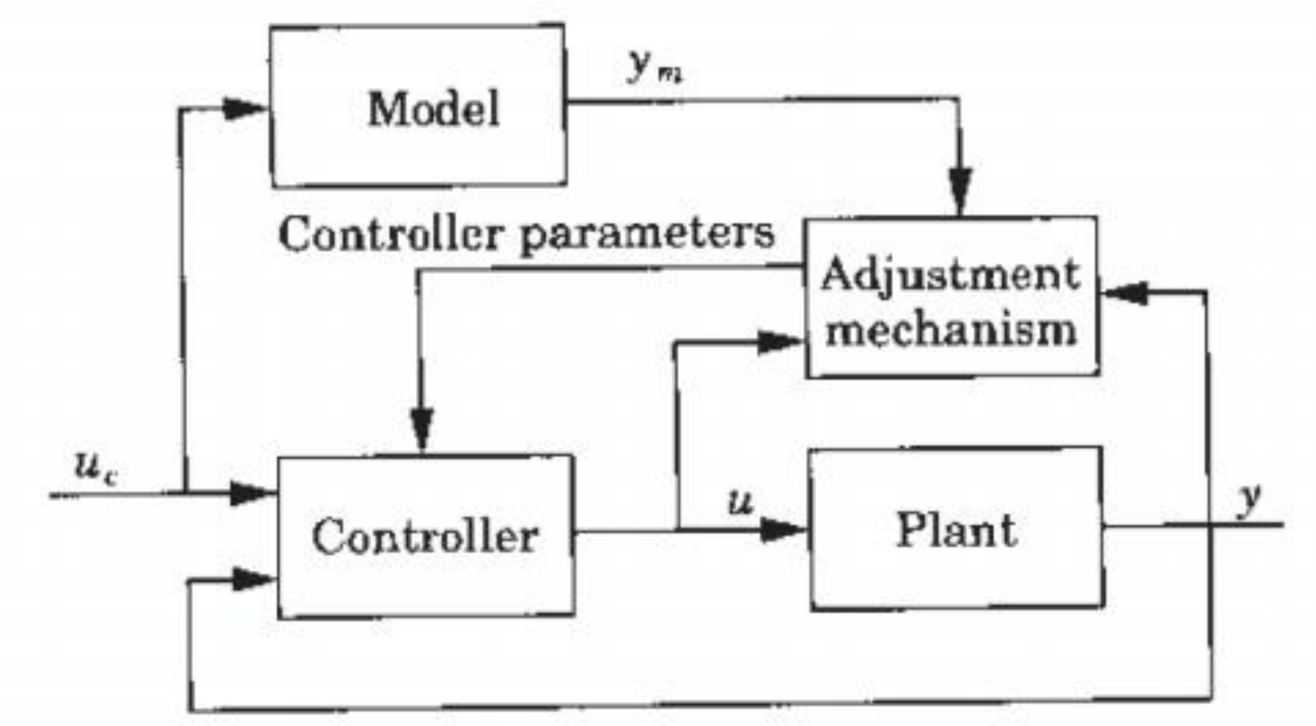
- Switching between multiple models



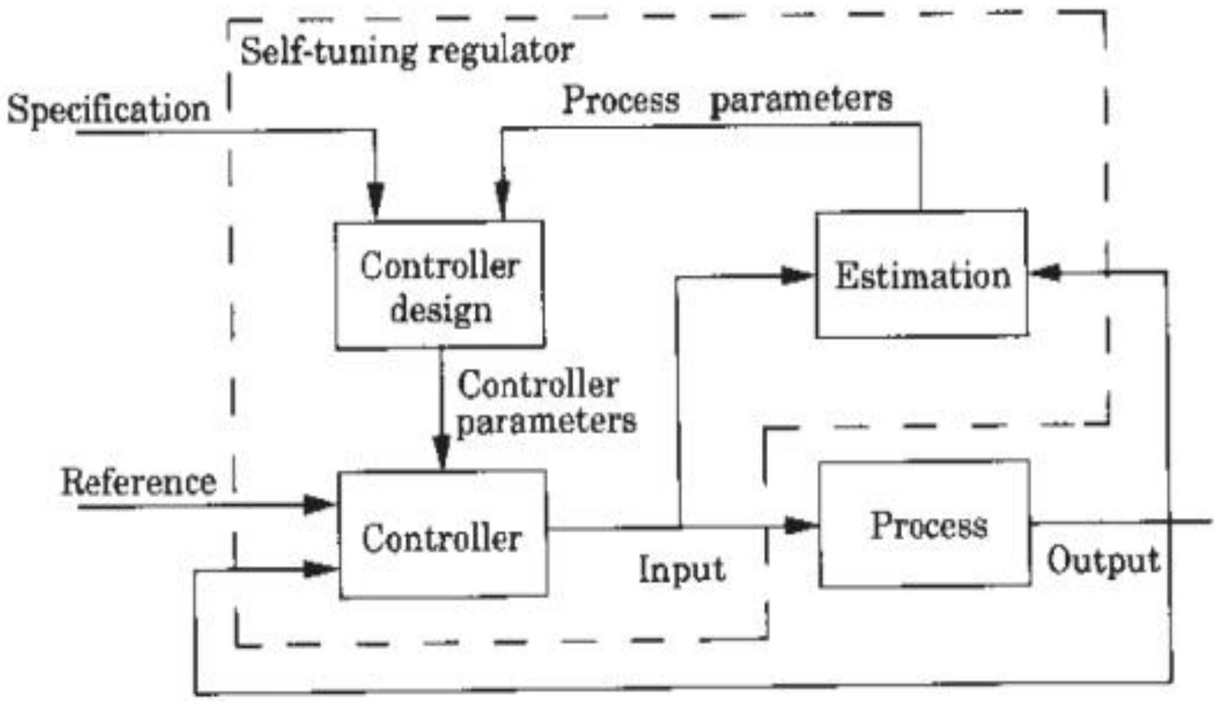
Identifier-based Adaptive Control

- Model Reference Adaptive Control
- Self-Tuning Regulator
- PID control

Model Reference



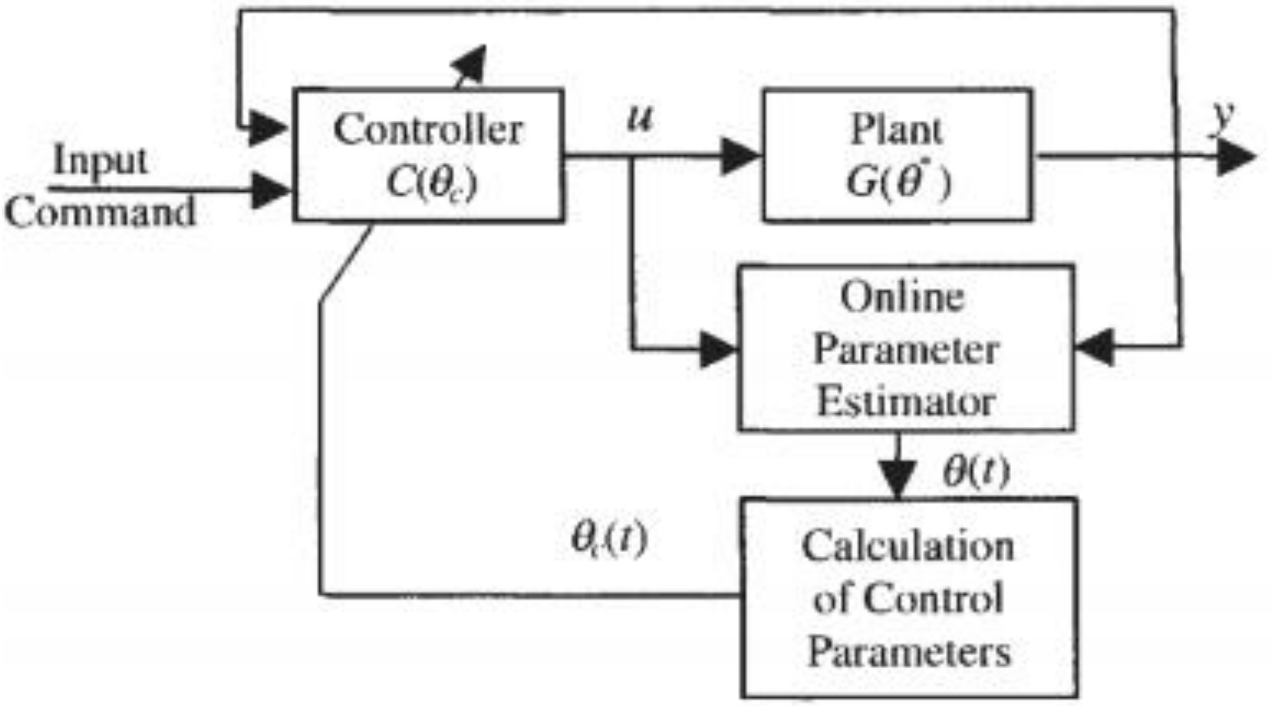
Self-Tuning Regulator



Adaptive Control Strategies

- Indirect Adaptive control
- Direct Adaptive Control

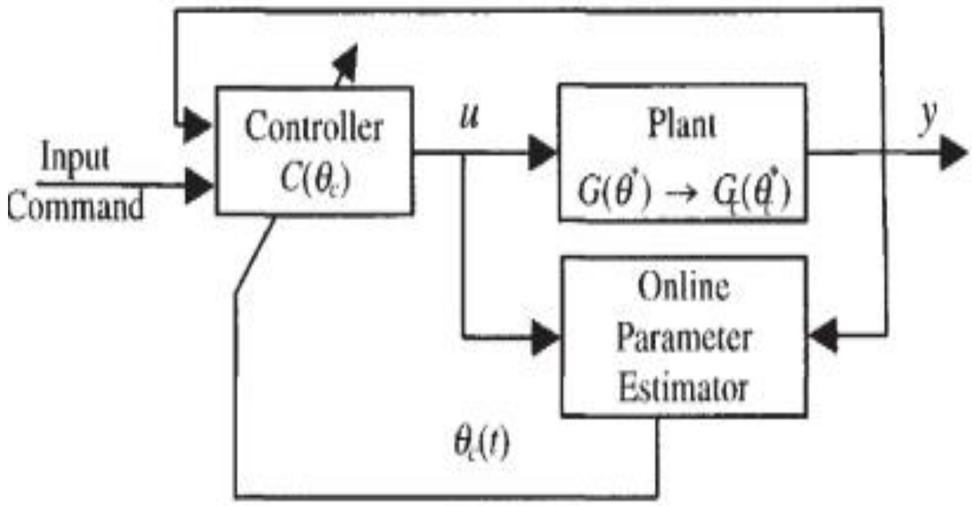
Indirect Adaptive Control



Function of the indirect adaptive control

- Estimatation of plant parameters
- Computing of controller parameters

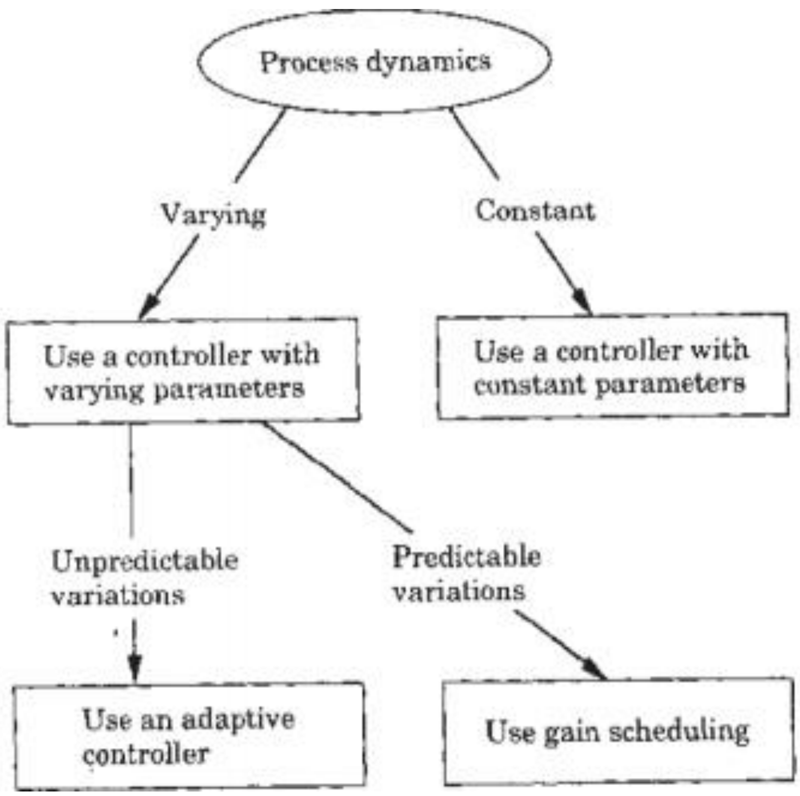
Direct Adaptive Control



- No plant parametes estimation

- Estimation for controller parameters only

What kind of controller?



Identification of System Parameters

Adaptive Control: It is feedback control for systems with uncertain parameters.

Types of Plantes

1. Non - linear time - varying (NLTV) with unknown paramerters θ

$\dot{x}(\mathrm{t}) = f(x, u, \theta, t)$ state equation

$y = h(x, u, \theta, t)$ output equation

2. Linear time - varying (LTV) with unknown paramerters θ

$\dot{x}(\mathrm{t}) = A(\theta, t)x + B(\theta, t)u$ state equation

$y = C(\theta, t)x + D(\theta, t)u$ output equation

3. Linear time-invariant (LTI) with unknown paramerters θ

$\dot{x}(\mathrm{t}) = A(\theta)x + B(\theta)u$ state equation

$y = C(\theta)x + D(\theta)u$ output equation

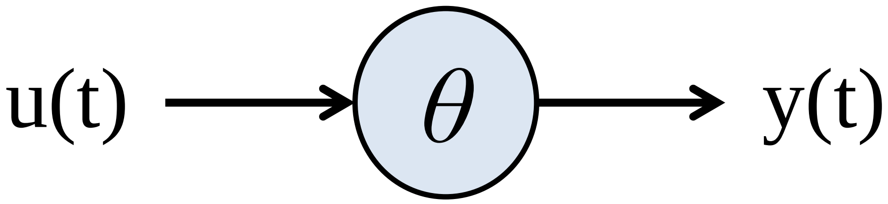
4. Linear time-invariant (LTI)

$\dot{x}(\mathrm{t}) = Ax(t) + Bu(\mathrm{t})$ state equation

$y = Cx(t) + Du(t)$ output equation

Identification of plant single parameter

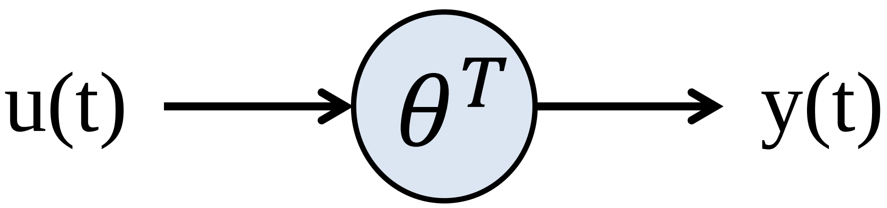
Let a system with an input of u(t) and output of y(t)



Where θ is unknown scalar parameter of the plant and identified as $\theta^{\wedge}$ using measurements of u(t) and y(t) at every instant.

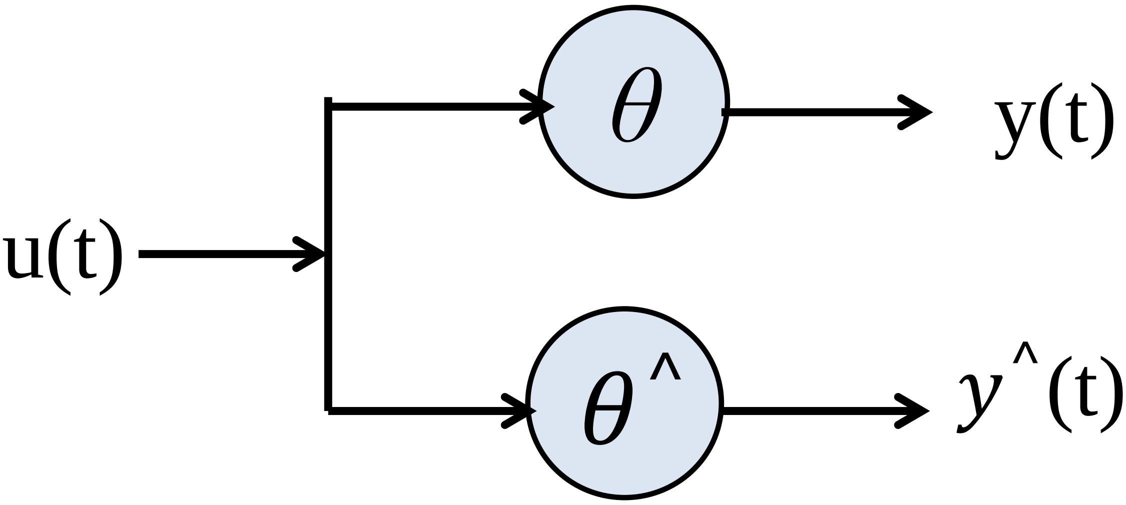
$$y(t) = \theta \; u(t)$$

Identification of plant vector parameter

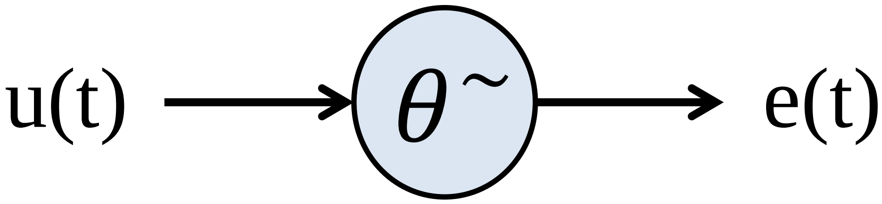


Where θ^T is unknown vector parameter of the plant and identified as $\theta^{T^{\wedge}}$ using measurements of u(t) and y(t) at every instant.

Error Model



Where $\theta^{\wedge}$ is identified of θ

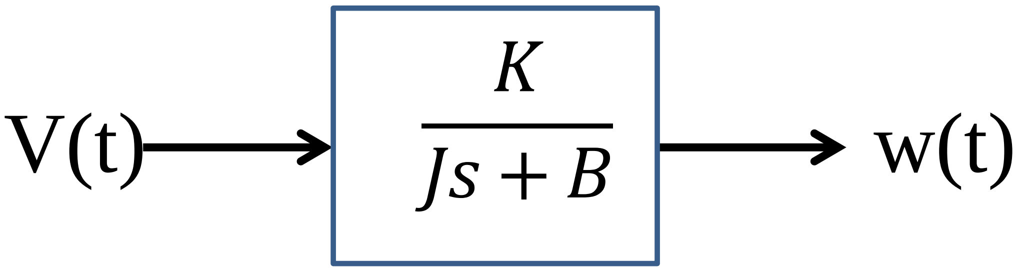


$$\theta^{\sim} = \theta^{\wedge} - \theta$$

Where $\theta^{\sim}$ is the error between the identified and real value of the plant parameter. $\theta^{\sim}$ is identified using measurements of u(t) and e(t) at every instant.

Identification of a parameter in a dynamic system

Let a simple transfer function of a DC motor



Where V is the input voltage of the motor, W is the angular velocity output, K J B are the physical parameters of the DC motor .

The open-loop transfer function of the motor is given by:

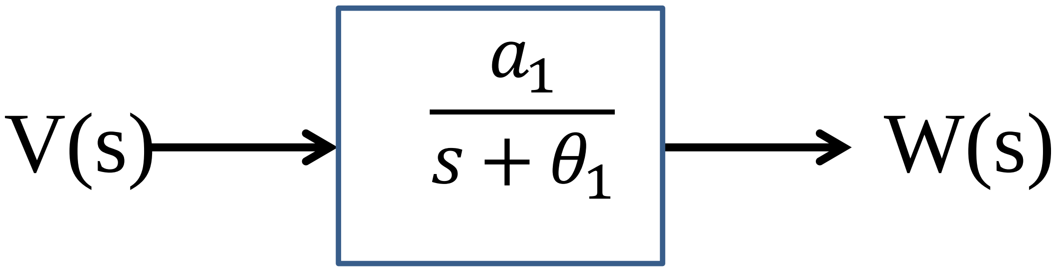
$$G(s) = \frac{K}{Js+B}$$

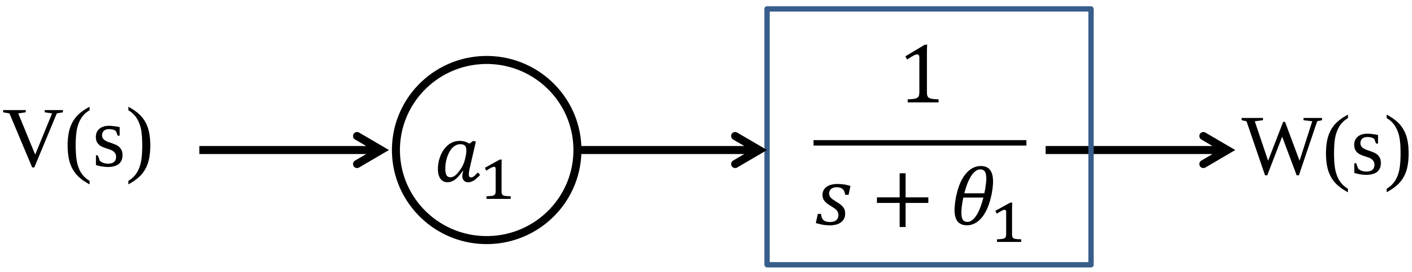
$$G(s) = \frac{\frac{K}{J}}{s+\frac{B}{J}} = \frac{a_1}{s+\theta_1}$$

Where $a_1 = \frac{K}{J}$ and $\theta_1 = \frac{B}{J}$ are unkown parameters

Identification of a parameter in a dynamic system

Let a simple transfer function of a motor





Assume that a_1 is known

$$\mathrm{W(s)} = \frac{a_1}{s+\theta_1}\mathrm{V(s)}$$

$$sW(s) + \theta_1 W(s) = a_1 V(s)$$

Let $U(s) = a_1 V(s)$, $(u(t) = a_1 V(t))$, then

$$sW(s) + \theta_1 W(s) = U(s)$$

Identification of a parameter in a Dynamic System

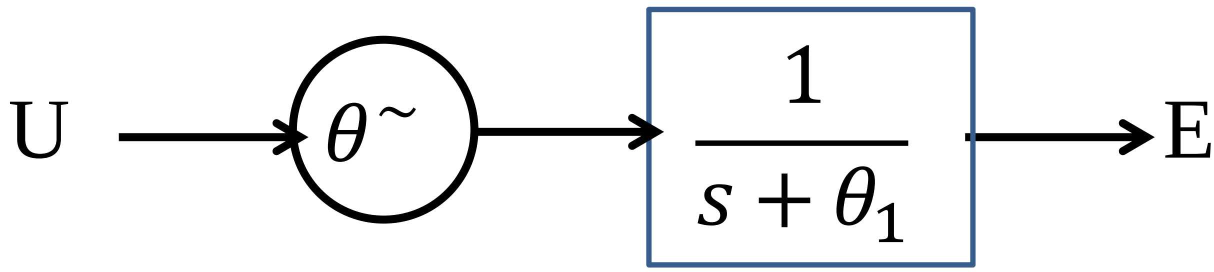
Laplace inverse of the following expression yields:

$$sW(s) + \theta_1 W(s) = U(s)$$

$$\dot{W}(t) = -\theta_1 W(t) + u(t)$$

The error model of the system is as follows:

$$E = \theta^{\sim} W$$



$$\mathrm{E} = \frac{\theta^{\sim} U}{s + \theta_1}$$

$$sE + E\theta_1 = \theta^{\sim} U$$

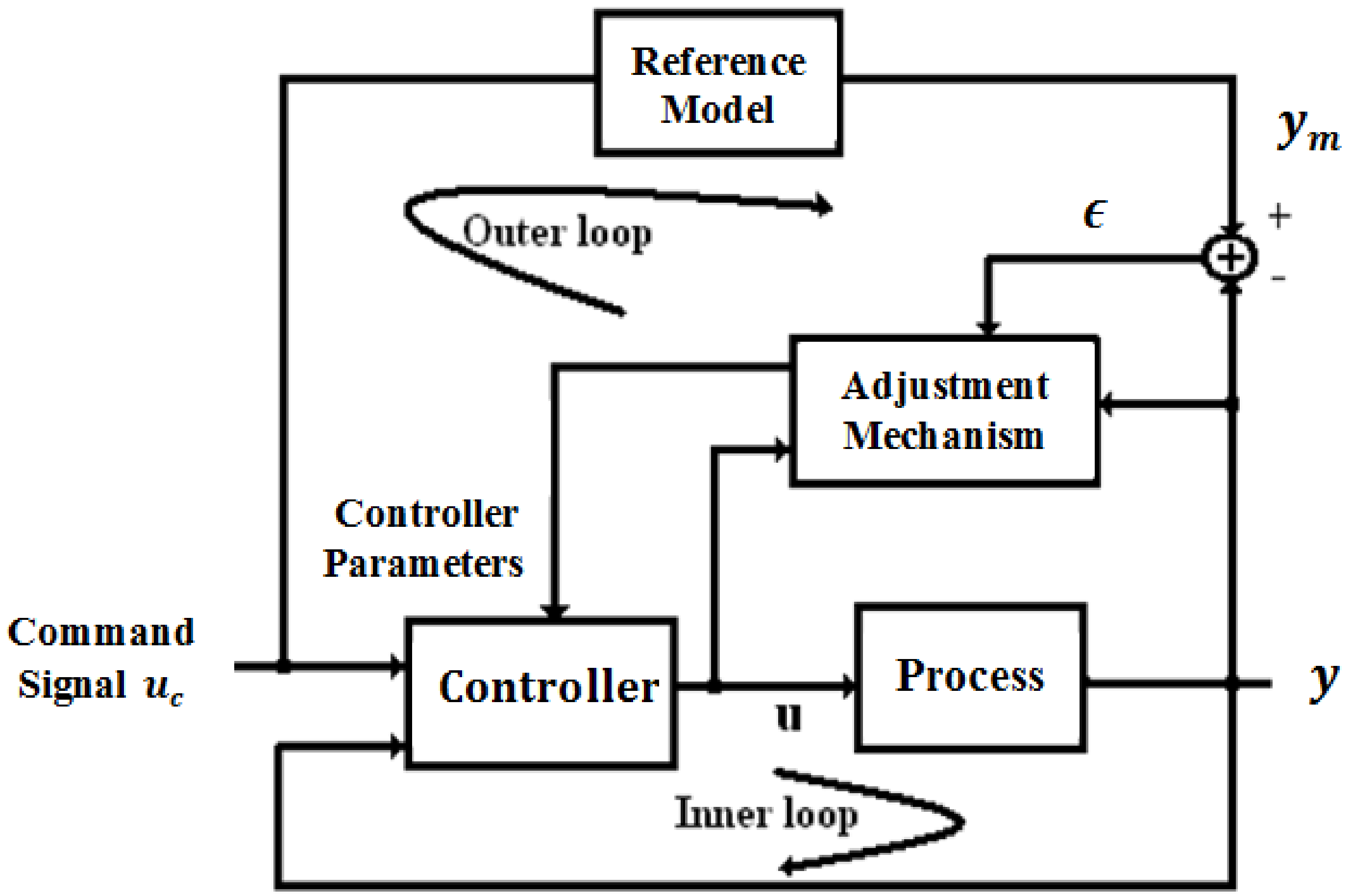
$$sE = -\theta_1 E + \theta^{\sim} U$$

Laplace inverrse of the following expression gives:

$$\dot{e}(t) = -\theta_1 e(t) + \theta^{\sim} u(t)$$

Model Reference Adaptive Controller (MRAC)

MRAC is one of the main techniques of adaptive control . The basic block diagram of the controller is given by:



The desired performance is expressed in terms of a reference model which gives desired response (ym). There are two feedback loops:

1. Inner loop. Its a ordinary feedback loop.
2. Outer loop. It adjusts the parameters on the inner loop

Model Reference Adaptive Controller (MRAC)

In MRAC technique the parameters of the controller are adjusted based on the error between the reference model and the system plant.

MRAC Design

There are three methods used to design the MRAC which are:

1. MIT rule
2. Lyapunov funtions
3. Passivity theory

In this course MIT rule approach will be considered in design MRAC.

Thank You!

1

Adaptive Control

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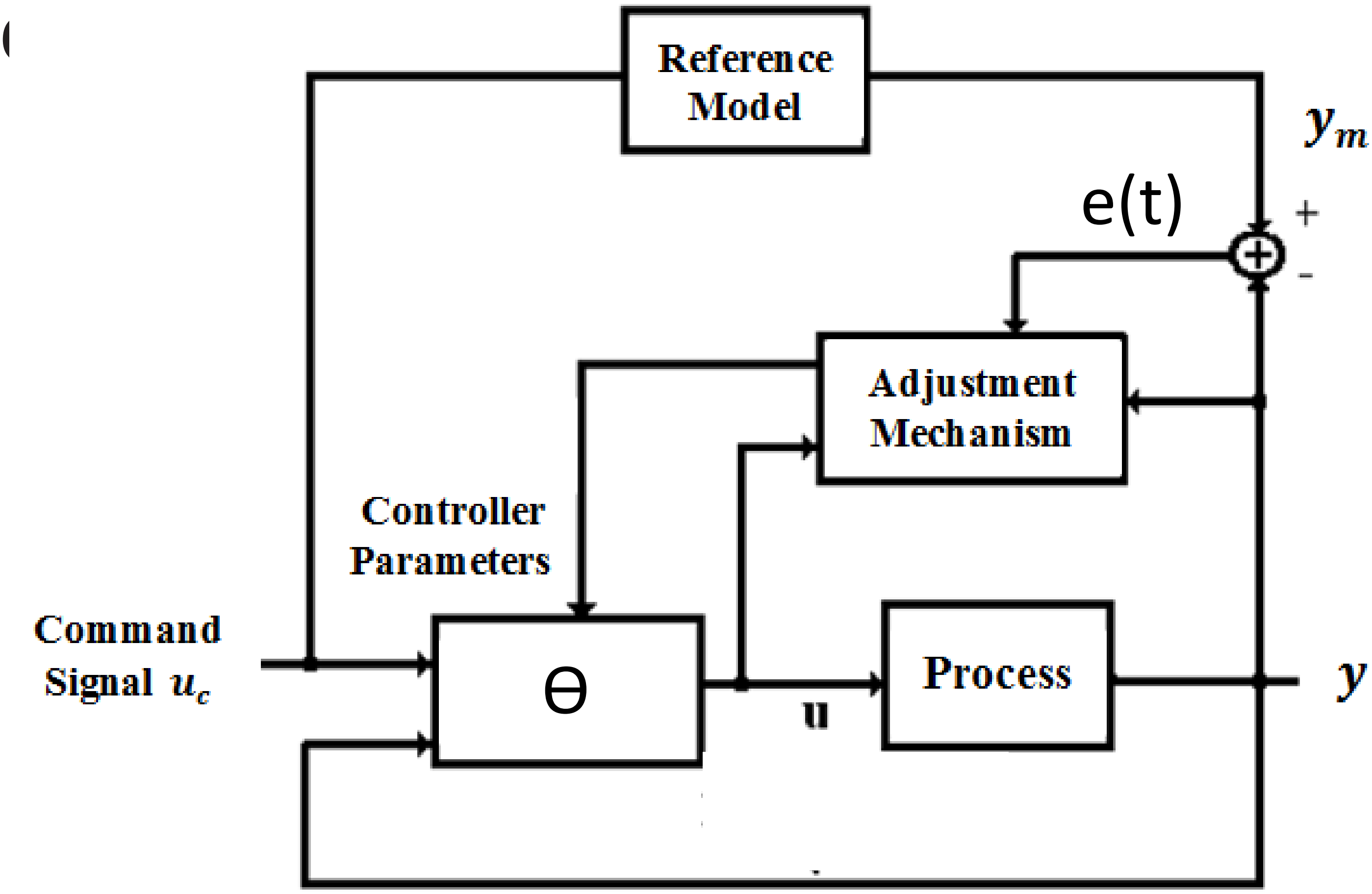
Lecture 6

Asst. Prof. Dr. Ibrahim Khalaf Mohammed

Model Reference Adaptive Controller (MRAC)

MIT Design Method

It is a scalar parameter adjustment law was proposed in 1961 for the MRAC system



If the transfer function of the reference model and plant (process) model are $G_m(s)$ and G(s) respectively. The controller gain θ based on MIT rule can be computed by using the following formula:

$$\frac{d\theta}{dt} = -\gamma\, e(t)\ \frac{de(t)}{d\theta}$$

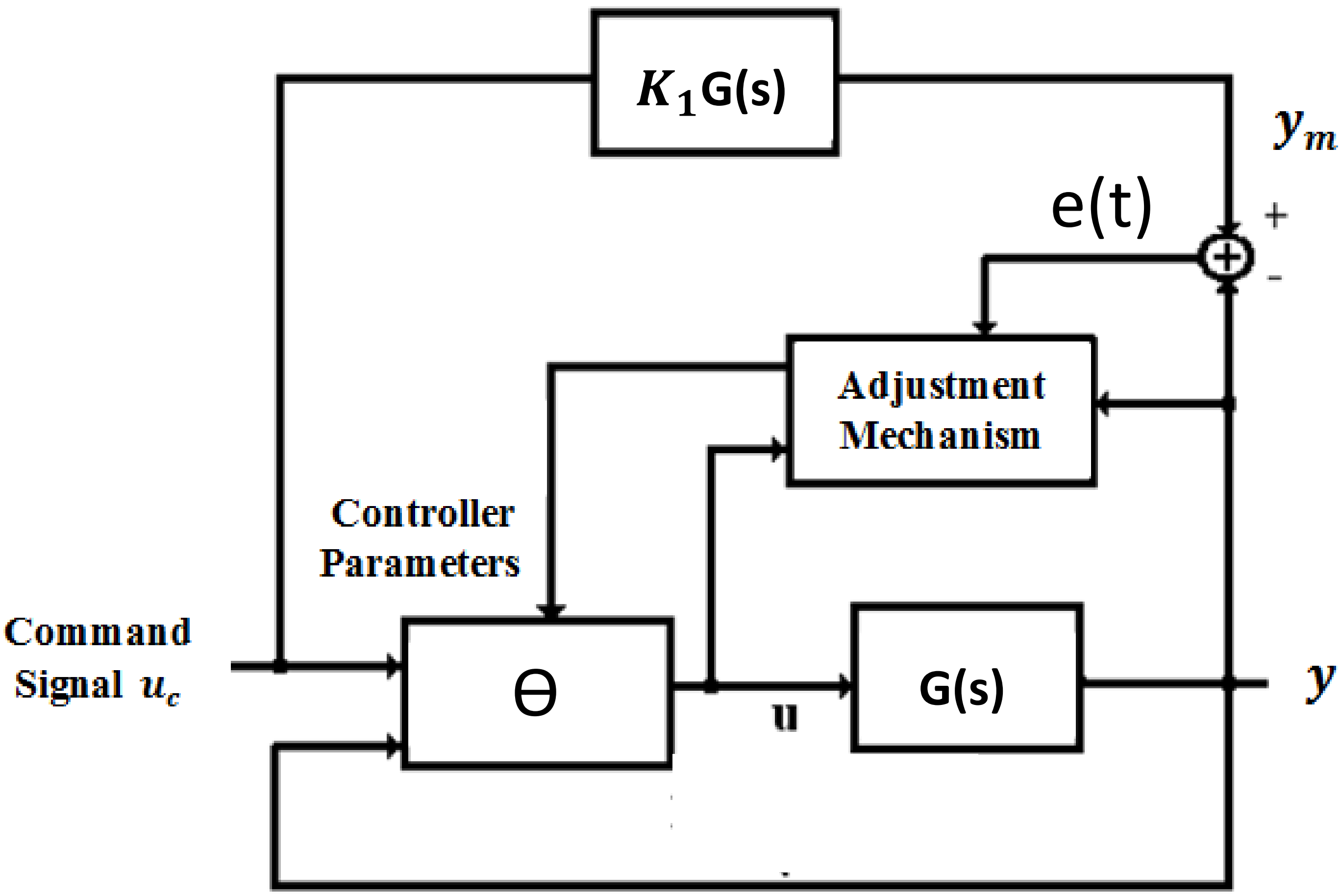
Where $e(t) = y(t) - y_m(t)$

Model Reference Adaptive Controller (MRAC)

θ depends on the system design. It can be a scalar unknown gain parameter θ_1 or vector of unknown gain parameters $[\theta_1, \theta_2, \theta_3 \ldots . \theta_n]^T$. Where n is number of unknown parameters . While γ is a gain constant.

$\frac{de}{d\theta}$ is the sensitivity derivative of the system, which is derived from adaptation error e(t).

Example: An MRAC system with the following block diagram. If K_1 is a known parameter and θ is the controller parameter.



Model Reference Adaptive Controller (MRAC)

Derive the adjustment mechanism using MIT rule method

Solution:

$$e = y - y_m \tag{1}$$

$$y = G(s)u = G(s)\theta u_c \tag{2}$$

$$y_m = K_1 G(s) u_c \tag{3}$$

Based on the above equation:

$$u_c = \frac{y_m}{K_1 G(s)} \tag{4}$$

$$e = G(s)\theta u_c - K_1 G(s) u_c$$

The sensitivity derivative $\frac{de}{d\theta}$ can be computed from the above equation:

$$\frac{de}{d\theta} = G(s)u_c$$

Model Reference Adaptive Controller (MRAC)

Based on (4) thre above equation becomes:

$$\frac{de}{d\theta} = G(s)\frac{y_m}{K_1 G(s)} = \frac{y_m}{K_1}$$

Using MIT rule:

$$\frac{d\theta}{dt} = -\gamma\, e(t)\,\frac{de}{d\theta} = -\gamma\, e(t)\,\frac{y_m}{K_1}$$

Let

$$\gamma_n = \frac{\gamma}{K_1}$$

Then

$$\frac{d\theta}{dt} = -\gamma_n\, y_m e$$

It is clear from the above equation that the rate of change of the parameter θ should be made proprotional to the product of the error and the model output.

By taking laplace transform for the above equation:

$$s\,\theta = -\gamma_n\, y_m e$$

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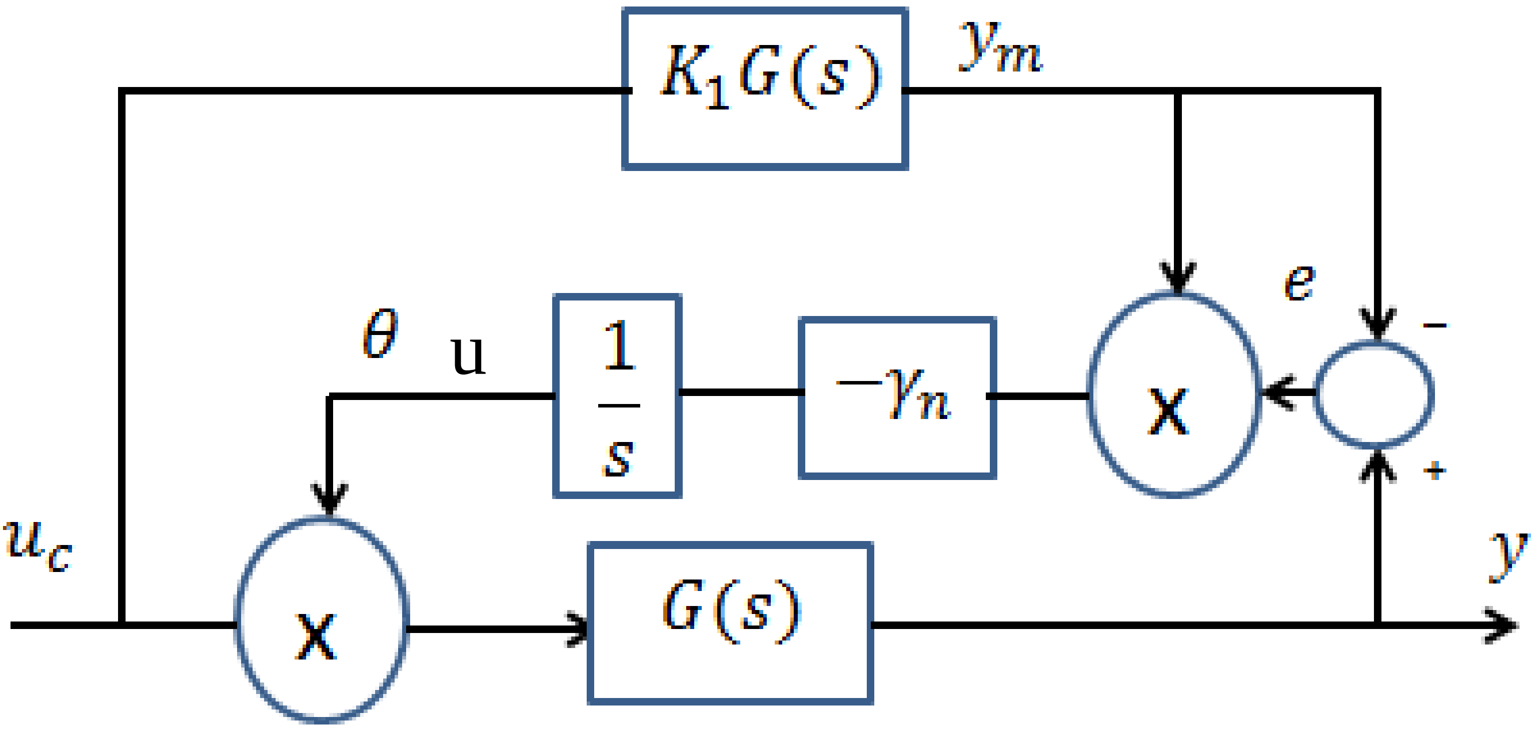
The final form for the controller gain parameter is as follows:

$$\theta = \frac{-1}{s}\gamma_n\, y_m e$$

MRAC Design

The block diagram of the MRAC system is below:



For design model reference block

The standard form of the second order model reference is as follows:

$$G_m(s) = \frac{{W_n}^2}{s^2 + 2\xi w_n s + {W_n}^2}$$

Where ξ is damping ratio, and w_n is un-damped natural frequency, which is the frequency of oscillation without damping.

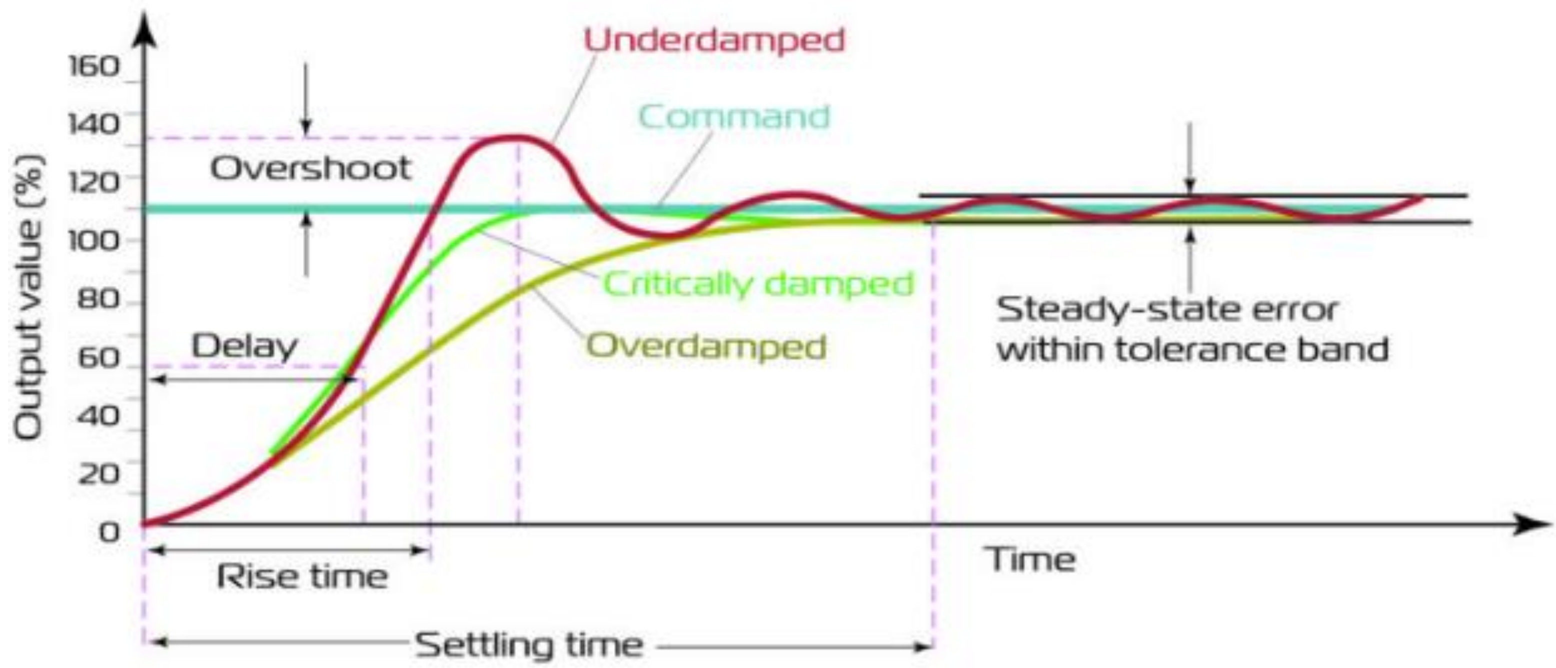
According to damping ratio value, a second order system can be set into the following categories:

1- Over damped response ($\xi > 1$)

2- Critical damped response (ξ =1)

3- Under damped response (0< ξ < 1)

4- No damped response (ξ =0)



Rise time :

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{w_n\sqrt{1-\xi^2}}$$

Settling time:

- ❖ Setting time (2%)

$$t_s = \frac{4}{\xi w_n}$$

- ❖ Settling time (5%)

$$t_s = \frac{3}{\xi w_n}$$

Adaptive Control

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Systems and Control Engineering Department

Lecture 7

Asst. Prof. Dr. Ibrahim Khalaf Mohammed

Model Reference Adaptive Controller (MRAC)

Example: Consider a system with an output model

$$\frac{dy}{dt} = -ay + bu$$

where u is the control input. Assume it is desirable to obtain a closed-loop system described by:

$$\frac{dy_m}{dt} = -a_m y + b_m u_c$$

with an adaptation error formula:

$$e = y - y_m$$

The controller formula is described by:

$$\mathrm{u} = k_1 u_c - k_2 y$$

Where $k_1 = \frac{b_m}{b}$ and $k_2 = \frac{a_m - a}{b}$ where $a_m > a$

Design a MRAC system using MIT rule

Model Reference Adaptive Controller (MRAC)

Solution:

The transfer function of the system plant is as follows:

$$\frac{dy}{dt} = -ay + bu$$

$$sY(s) = -aY(s) + bU(s)$$

$$(s + a)Y(s) = bU(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s + a}$$

For the model reference

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

The transfer function of the model is:

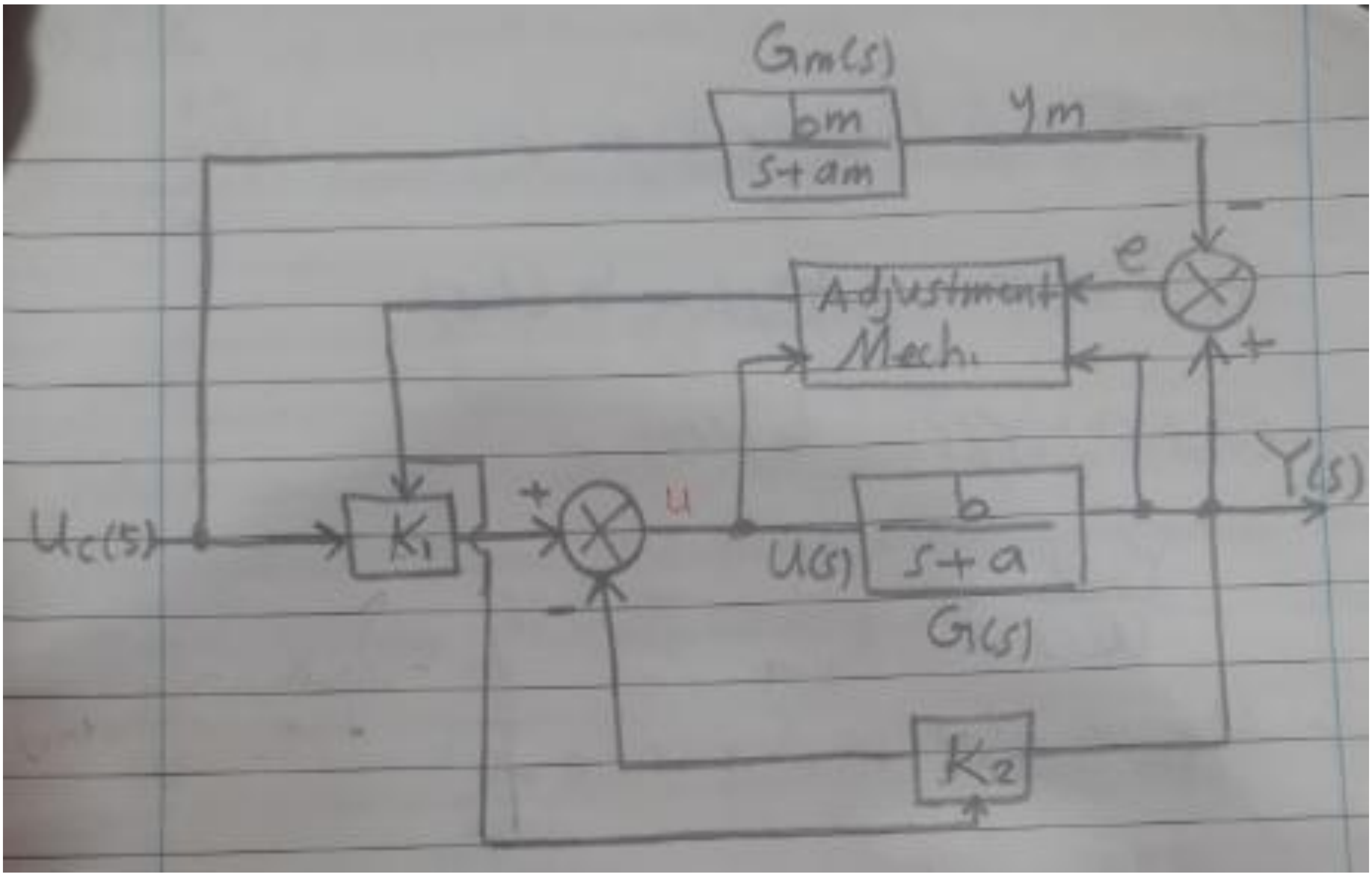
$$sY_m(s) = -a_m Y_m(s) + b_m U_c(s)$$

$$(s + a_m)Y_m(s) = b_m U_c(s)$$

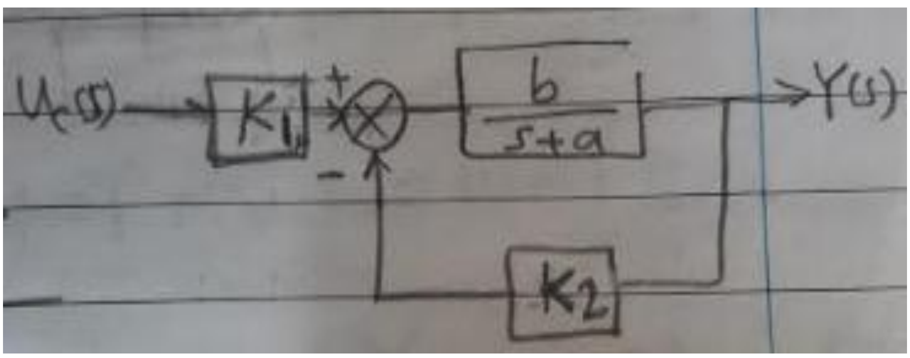
Model Reference Adaptive Controller (MRAC)

$$G_m(s) = \frac{Y_m(s)}{U_c(s)} = \frac{b_m}{s + a_m}$$

Based on the above equations the block diagram of th
system is given by:



For the below closed-loop subsystem, the transfer function
Is given by:



$$\frac{Y(s)}{k_1 U_c(s)} = \frac{\frac{b}{s+a}}{1+\frac{b}{s+a}k_2}$$

$$\frac{Y(s)}{k_1 U_c(s)} = \frac{\frac{b}{s+a}}{\frac{s+a+bk_2}{s+a}} = \frac{b}{s+a+bk_2}$$

$$Y(s) = \frac{bk_1}{s+a+bk_2} U_c(s) \qquad (1)$$

The error equation is

$$e = y - y_m$$

$$e = Y(s) - Y_{m(s)}$$

$$e = \frac{bk_1}{s+a+bk_2} U_c(s) - \frac{b_m U_c(s)}{s+a_m}$$

The parameters of the controller are k1 and k2:

The sensitivity derivative is :

$$\frac{de}{dk_1} = \frac{bU_c(s)}{s+a+bk_2} \qquad (2)$$

and

$$\frac{de}{dk_2} = \frac{-b^2 k_1 U_c(s)}{(s+a+bk_2)^2}$$

$$\frac{de}{dk_2} = -\frac{b}{s+a+bk_2}\,\frac{bk_1 U_c(s)}{s+a+bk_2}$$

Based on equation (1) the above equation becomes:

$$\frac{de}{dk_2} = -\frac{b}{s+a+bk_2}\;Y(s) \qquad (3)$$

The formula $\frac{\mathrm{de}}{\mathrm{dk}_1}$ and $\frac{\mathrm{de}}{\mathrm{dk}_2}$ can not be used because the process parameters a and b are not known. Approximation is required in order to obtain realization parameter adjustment.

$$k_2 = \frac{a_m - a}{b}$$

$$s + a + bk_2 = s + a + b\left(\frac{a_m - a}{b}\right) = s + a_m$$

Applying MIT rule for k_1 parametrer gives:

$$\frac{dk_1}{dt} = -\gamma e \frac{de}{dk_1}$$

Based on equation (2) te last equation becomes:

$$\frac{dk_1}{dt} = -\gamma e \frac{de}{dk_1} = -\gamma e \frac{bU_c(s)}{s + a + bk_2}$$

$$\frac{dk_1}{dt} = -\gamma e \frac{bU_c(s)}{s + a_m}$$

Let $\gamma_n = \gamma b$

$$\boldsymbol{\frac{dk_1}{dt} = -\gamma_n e \frac{U_c(s)}{s + a_m}} \qquad \textbf{(4)}$$

23 For k_2 parameter

$$\frac{dk_2}{dt} = -\gamma e \frac{de}{dk_2}$$

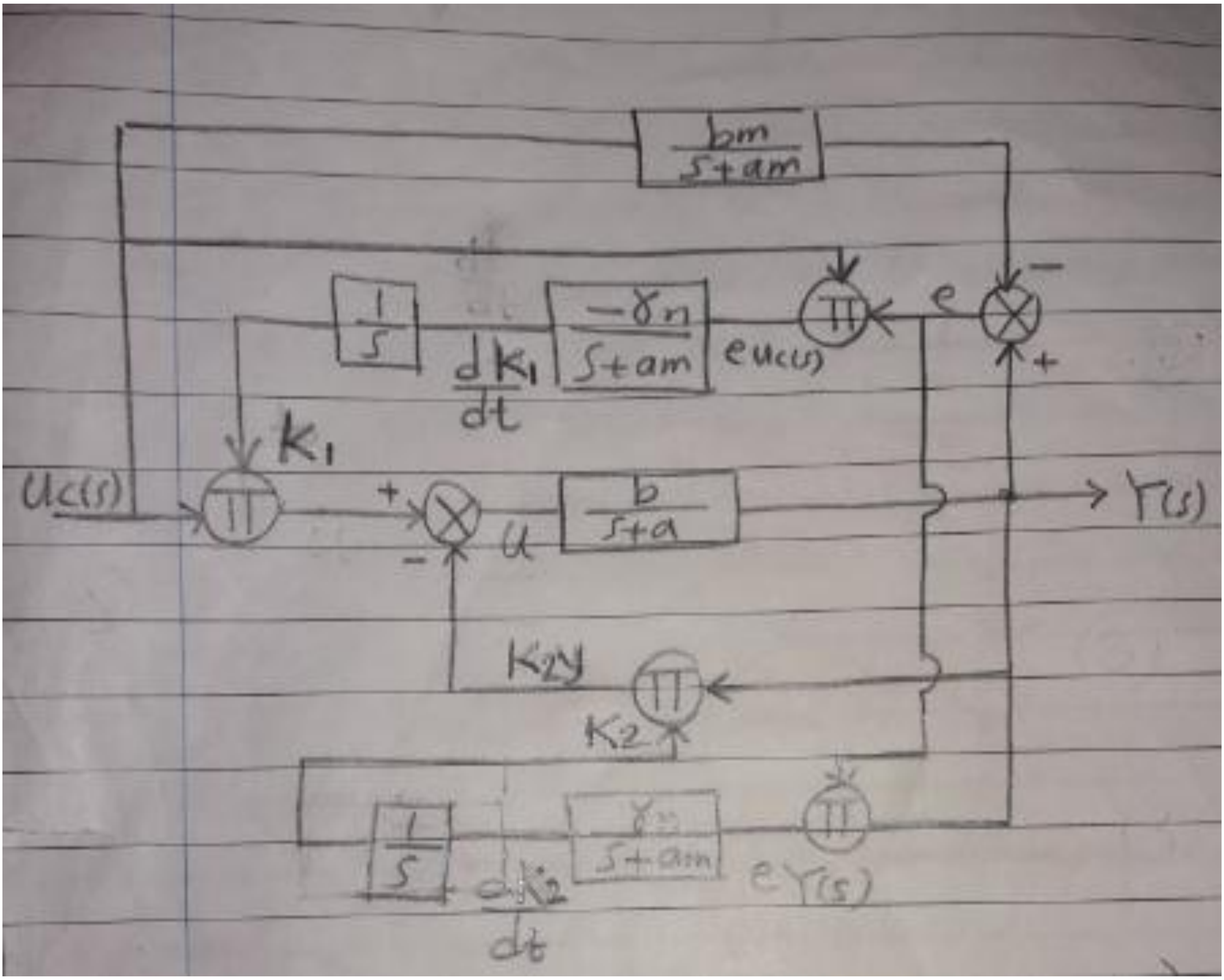
Based on equation (3), the above equation becomes:

$$\frac{dk_2}{dt} = -\gamma e * -\frac{bY(s)}{s + a + bk_2}$$

$$\frac{dk_2}{dt} = \gamma e \frac{bY(s)}{s + a_m}$$

$$\boldsymbol{\frac{dk_2}{dt} = \gamma_n e \frac{Y(s)}{s + a_m}} \qquad \textbf{(5)}$$

Based on equation (4) and (5), the block diagram of the MRAC system is given below:



Adaptive Control

Ninevah University **College of Electronics Eng.**

Systems and Control Engineering Department

Lecture 8

Asst. Prof. Dr. Ibrahim Khalaf Mohammed

Model Reference Adaptive Controller (MRAC)

Example: Consider a system with an output model

$$\frac{dy}{dt} = -y + 0.5u$$

where u is the control input. Assume it is desirable to obtain a closed-loop system described by:

$$\frac{dy_m}{dt} = -2y_m + 2u_c$$

with an adaptation error formula:

$$e = y - y_m$$

The controller formula is described by:

$$\text{u} = k_1 u_c - k_2 y$$

Where

Design a MRAC system using MIT rule for $\gamma_n = 1$

Solution:

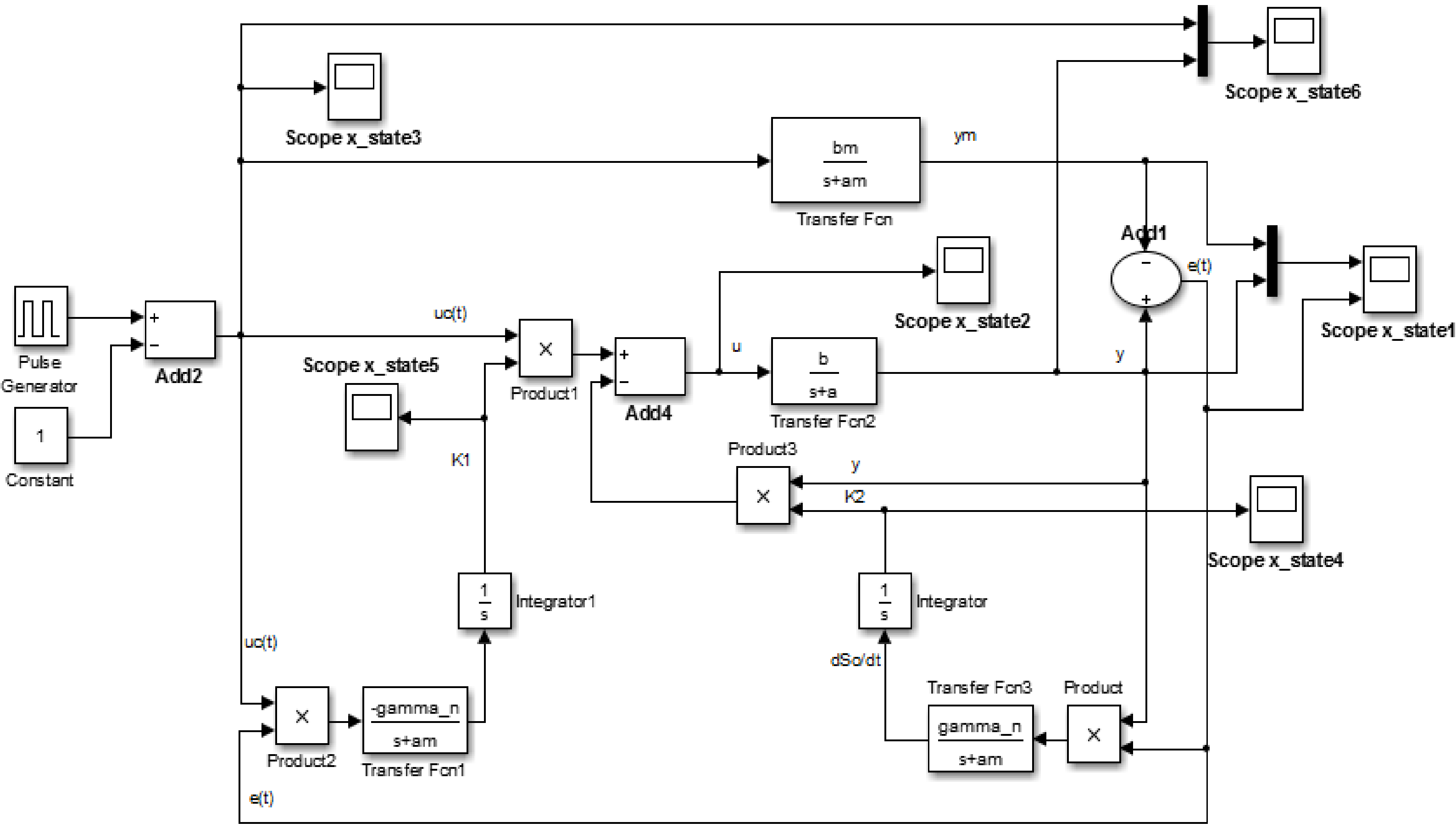
Using the same design procedure mentioned in the previous example (example 2), the gain parameters expressions are given below:

$$\frac{dk_1}{dt} = -\gamma_n e \frac{U_c(s)}{s+2} \tag{1}$$

$$\frac{dk_2}{dt} = \gamma_n e \frac{Y(s)}{s+2} \tag{2}$$

MRAC_MIT Technique

The simulink model of the system is shown below:



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Adaptive Control

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Systems and Control Engineering Department

Lecture 9

Asst. Prof. Dr. Ibrahim Khalaf Mohammed

MRAC Based Lyapunov Theorm

Example: Consider a system with an output model

$$\frac{dy}{dt} = -y + bu$$

where u is the control input, b is unknown, $b > 0$. Assume it is desirable to obtain a closed-loop system described by:

$$\frac{dy_m}{dt} = -y_m + u_c$$

Where u_c is set point.
with an adaptation error formula:

$$e = y - y_m$$

The controller formula is described by:

$$\text{u} = \theta u_c$$

Design a MRAC system using Lyapunov thorem
Solution:

$\dot{e} = \dot{y} - \dot{y_m}$

$$\dot{e} = -y + bu + y_m - u_c$$

$$\dot{e} = -(y - y_m) + b\theta u_c - u_c$$

$$\dot{e} = -e + b(\theta - \tfrac{1}{b})u_c \qquad (1)$$

Candidate Lyapunov function:

$$V(e,\theta) = \frac{1}{2}e^2 + \frac{b}{2\gamma}(\theta - \frac{1}{b})^2$$

$$\dot{V}(e,\theta) = e\dot{e} + \tfrac{b}{\gamma}\left(\theta - \tfrac{1}{b}\right)\dot{\theta} \qquad (2)$$

Substitute equation(1) in equation (2):

$$\dot{V}(e,\theta) = e[-e + b\left(\theta - \frac{1}{b}\right)u_c] + \frac{b}{\gamma}\left(\theta - \frac{1}{b}\right)\dot{\theta}$$

$$\dot{V}(e,\theta) = -e^2 + b\left(\theta - \frac{1}{b}\right)[eu_c + \frac{\dot{\theta}}{\gamma}]$$

$$eu_c + \frac{\dot{\theta}}{\gamma} = 0$$

$$\dot{\theta} = -\gamma e u_c \qquad (3)$$

$$\ddot{V}(e,\theta) = -2e\dot{e}$$

Using equation (1), the above equation becomes:

$$\ddot{V}(e,\theta) = -2e[-e + b(\theta - \frac{1}{b})u_c]$$

$$\ddot{V}(e,\theta) = 2e^2 - 2(b\theta - 1)eu_c$$

Example: Design MRAC using Lyapunov method for the first order system with:
Model reference

$$\frac{dy_m}{dt} = -a_m y_m + b_m u_c$$

Plant:

$$\frac{dy}{dt} = -ay + bu$$

a and b are unknown, $b > 0$,
Control law:

$$u = \theta_1 u_c - \theta_2 y$$

Solution:

$$\dot{y} = -ay + bu$$

$$\dot{y} = -ay + b(\theta_1 u_c - \theta_2 y)$$

$$\dot{y} = (-a - b\theta_2)y + b\theta_1 u_c$$

$$e = y - y_m$$

$$\dot{e} = \dot{y} - \dot{y_m}$$

$$\dot{e} = (-a - b\theta_2)y + b\theta_1 u_c + a_m y_m - b_m u_c$$

$$\dot{e} = -a_m(y - y_m) + (a_m - a - b\theta_2)y \ + (b\theta_1 - b_m)u_c$$

$$\dot{e} = -a_m e + (a_m - a - b\theta_2)y \ + (b\theta_1 - b_m)u_c \qquad (1)$$

Candidate Lyapunov function:

$$V(e, \theta_1, \theta_2) = \frac{1}{2}e^2 + \frac{1}{2\gamma b}(a_m - a - b\theta_2)^2 + \frac{1}{2\gamma b}(b\theta_1 - b_m)^2$$

$$\dot{V}(e, \theta_1, \theta_2) = e\dot{e} - \frac{1}{\gamma b}(a_m - a - b\theta_2)b\dot{\theta}_2 + \frac{1}{\gamma b}(b\theta_1 - b_m)\, b\dot{\theta}_1$$

$$\dot{V}(e, \theta_1, \theta_2) = e[-a_m e + (a_m - a - b\theta_2)y + (b\theta_1 - b_m)u_c]$$

$$-\frac{1}{\gamma}(a_m - a - b\theta_2)\dot{\theta}_2 + \frac{1}{\gamma}(b\theta_1 - b_m)\,\dot{\theta}_1 \qquad (2)$$

$$ey - \frac{1}{\gamma}\dot{\theta}_2 = 0$$

$$\dot{\theta}_2 = \gamma e y \qquad (3)$$

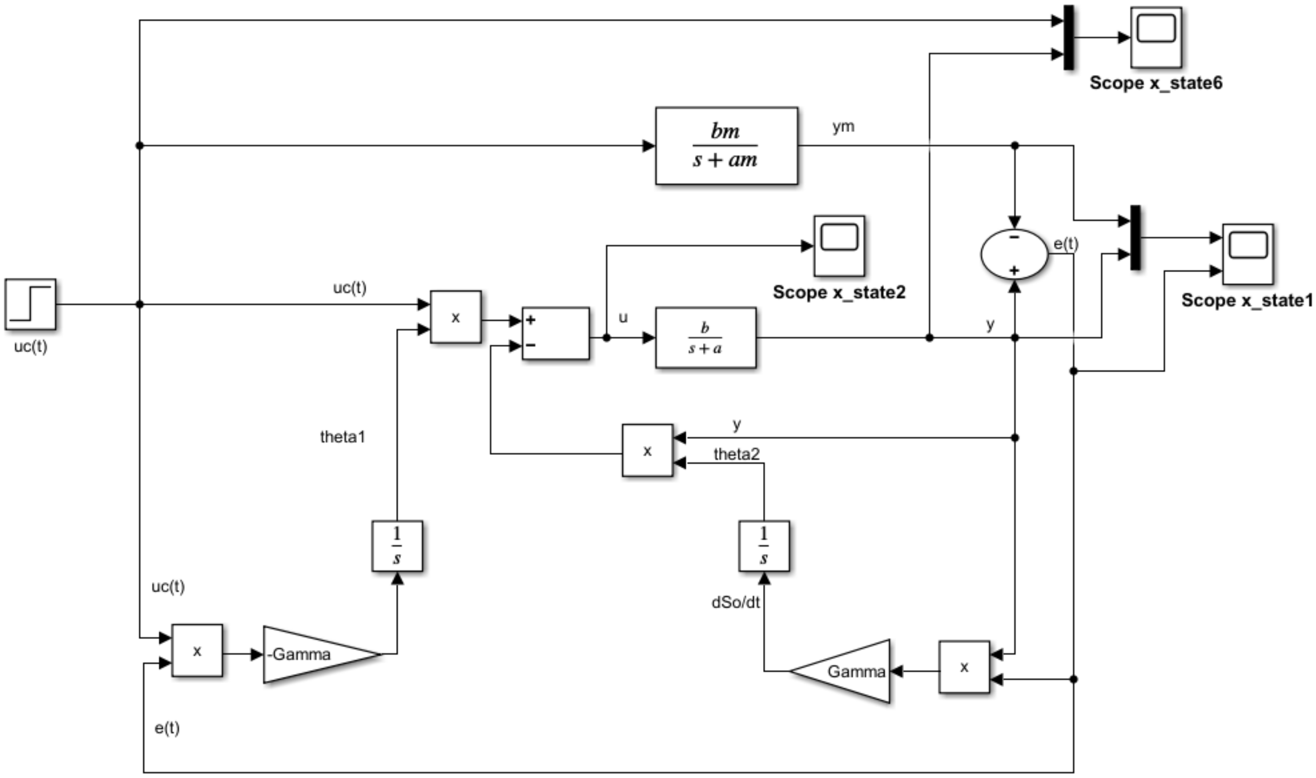
$$\dot{\theta}_1 = -\gamma e u_c \qquad (4)$$

For simulink implementation of the MRAC system:
Laplace transformation of equation (3) and (4) yields:

$$\theta_1 = -\frac{1}{s}\gamma e u_c$$

$$\theta_2 = \frac{1}{s}\gamma e y$$

- Simulink model of the system based on Lyapunnov function method.



Adaptive Control

Ninevah University **College of Electronics Eng.**

Systems and Control Engineering Department

Lecture 10

Asst. Prof. Dr. Ibrahim Khalaf Mohammed

Model Reference Adaptive Controller (MRAC)

<u>Example:</u> Consider a system with an output model

$$\frac{dy}{dt} = -y + 0.5u$$

where u is the control input. Assume it is desirable to obtain a closed-loop system described by:

$$\frac{dy_m}{dt} = -2y_m + 2u_c$$

with an adaptation error formula:

$$e = y - y_m$$

The controller formula is described by:

$$\text{u} = k_1 u_c - k_2 y$$

Where

1. Design a MRAC system using Lyapunov function based on $\gamma_n = 1$.
2. Find the gain and settling time of the model reference mathematically.

Solution:

Using the same design procedure mentioned in the previous example (example 2), the gain parameters expressions are given below:

$$\frac{d\theta_1}{dt} = -\gamma_n e u_c \qquad (1)$$

$$\frac{d\theta_2}{dt} = \gamma_n e y \qquad (2)$$

$$G_m(s) = \frac{2}{s+2}$$

The standard form of the model reference:

$$G_m(s) = \frac{K}{\tau s + 1}$$

Compare to the standard form

$$G_m(s) = \frac{1}{0.5s + 1}$$

Gain K=1, time constant $\tau = 0.5\ s$,
settling time $t_s = 5\tau = 2.5$ s.

MRAC_Lyapunov functionTechnique

The simulink model of the system based on Lyapunov function method is shown below: