College of Electronics Engineering Ninevah University

Systems and Control Engineering Department

Lecture_1

Subject: Basics of Optimal Control

Lecturer: Ass. Prof. Dr. Ibrahim Khalaf Mohammed

Syllabus:

1. Basic of Optimal Controller
2. Calculus of Variations
3. Optimal Control Theory
4. Optimal Control Design in ContinuousTime-Domain
5. Optimal Control Design in Discrete Time-Domain
6. Steady-state Quadratic Optimal Control of Discrete-Time Systems
7. Steady-State Servo Optimal Control System in Discrete-Time Form
8. Linear Quadratic Gaussian (LQG) Control

Motivation of Optimal Control

1. Get the best performance of controller.
2. Learn the system limits.
3. Design some acceptable control.

Optimization

Optimization is the task of making the best choice among a set of given alternatives or it is a collection process of finding the set of conditions required to achieve the best from a given situation.

Why Optimization?

1. To improve the performance of the system.
2. To achieve the best performance in the minimum time with minimum error.

How to Perform the Optimization

Optimization process compares between different choices through an objective function (index function).

Objective Function: It is the performance measure of a system and is chosen so that the important system sepcifications can be meet. Choosing the category of objective function (maximum or minimum) depandes on the natural of the control problem. It should be formulated in mathematical form.

Steps Used To Solve Optimisation Problems

1. Analyse the process in order to make a list of all the variables.
2. Determine the optimisation criterion and specify the objective function.
3. Develop the mathematical model of the process to identify the independent and dependent variables to obtain the number of degrees of freedom.
4. If the problem formulation is too large or complex simplify it if possible.
5. Apply a suitable optimisation technique.
6. Check the result and examine it’s sensitivity to changes in model parameters and assumptions.

Classification of Optimisation Problems

Properties of f(x)

1. Single variable or multivariable
2. Linear or nonlinear
3. Sum of squares
4. Quadratic
5. Smooth or non-smooth
6. Sparsity

Types of Optimizations

1. Static optimisation: variables have numerical values, fixed with respect to time.
2. Dynamic optimisation: variables are functions of time.

Typical Examples of Application

Static Optimisation

1. Plant design (sizing and layout).

2. Operation (best steady-state operating condition).

3. Parameter estimation (model fitting).

4. Allocation of resources.

5. Choice of controller parameters (e.g. gains, time constants) to minimise a given performance index (e.g. overshoot, settling time, integral of error squared).

Dynamic Optimisation

1. Determination of a control signal u(t) to transfer a dynamic system from an initial state to a desired final state to satisfy a given performance index.

2. Optimal plant start-up and/or shut down.

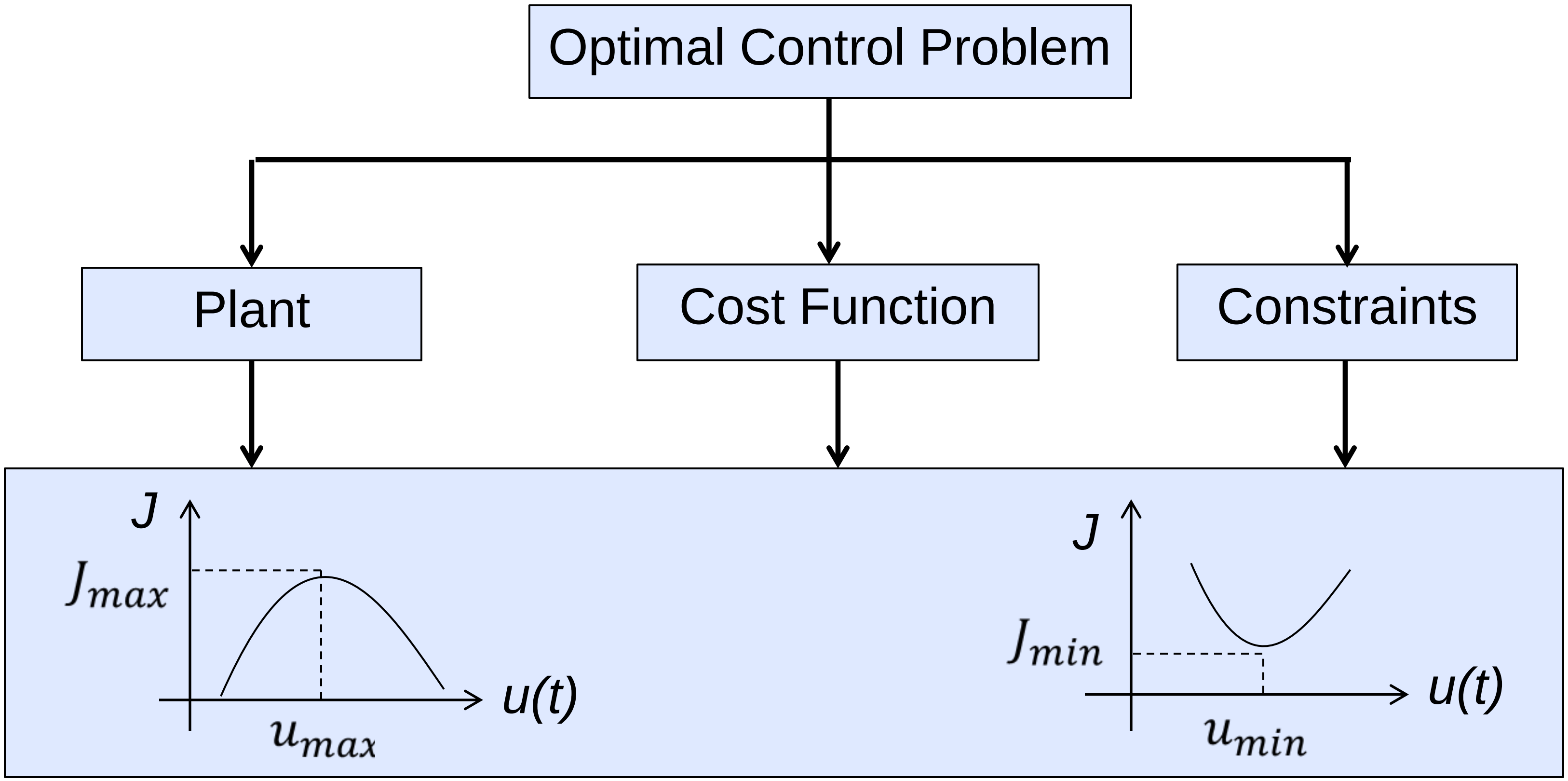
3. Minimum time problems

Design Types of Control System

1. Classical design approach: It is used to control simple dynamic behavior systems like single-input single-output (SISO) systems with zero initial conditions.

2. Optimal design approach: It is used to control multi-input multi-output (MIMO) systems with complex dynamic behavior where the classical design procedure can not achieve the required performance specifications.

The following figure summarizes the optimal control problem



Optimal Control Principles

1. Model the system in state space form with state vector:

$$x(t) = [x_1(t), x_2(t), x_3(t), \dots \dots \dots . x_n(t),]^T$$

And control vector:

$$\mathrm{u}(t) = [u_1(t), u(t), u(t), \dots \dots \dots . u(t),]^T$$

2. Minimize the cost function (J)
3. Initial and final states can be known, unkown or constriant or free.
4. System states and control input must be constriansed.
5. The final time may be known, uknown or free.
6. Derivation the control function u(t).

- For unconstriant (unbounded) control systems, Euler-lagrange and Hamiltonian are used to find u(t).
- For constriant (bounded) control systems, Pontryagin maximum or minimum principle must be used.

Control System Performance

The control system has a good performance if the following requiements are satisfied:

1. Minimum steady state error.
2. Good stability performance.
3. Reasonable system response speed.
4. The effect of disturbances is very small or neglected.

Performance Index

Optimum Control System

It is a quantities measure of the performance of a system and chosen so that the important system specifications can be meet.

Some Typical Performance Criteria:

- maximum profit
- minimum cost
- minimum effort
- minimum error
- minimum waste
- maximum throughput
- best product quality

Conditions of Performance Index

1. Reliability.
2. Easy to apply.

Performance Indices Classification Based on System Error

1. The integral of the error (IE)

$$IE = \int_0^T e(t)dt$$

2. The integral of the square of the error (ISE)

$$ISE = \int_0^T e^2(t)dt$$

3. The integral of the absolute magnetude of the error (IAE)

$$IAE = \int_{0}^{T} |e(t)| dt$$

4. The integral of the time multiplied by absolute value of the error (ITAE)

$$ITAE = \int_0^T t|e(t)|\, dt$$

The general form of the performance index is as follows:

$$J = S(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt$$

Where the first term is called scalar function while the second term is called traiectorv function, $x(t_f)$ is **the** state trajectory at the final time t_f, ***x(t)*** **is state** trajectory at the time t $(0 < t < t_f)$ and $u(t)$ is the control vector input.

Calculus of Variations; It is a field of mathematical analysis that uses variations, which are small changes in functions and functionals, to find maxima and minima of functionals.

The simplest form of the calculus of variation is as follows:

$$J(x) = \int_{t_0}^{t_1} F\big(t, x(t), \dot{x}(t)\big)dt$$

Where $J(x)$ is the performance index could be distance, length, surface etc F is a given function of three variables called Lagrange function, $x(t)$ is an unknown function on the interval $[t_0, t_1]$, t_0, t_1 are given numbers, initial and terminal values of $x(t)$ must be known:

$$x(t_0) = x_0$$
$$x(t_1) = x_1$$

The problem is finding the function $x = f(t)$ (trajectory equation between t_0 and t_1) that maximizes or minimizes $J(x)$.

Theorm: (Main theorm of the Calculus of Variations)

Assume that F is a function defined on R^3(Real valued three variables function). Consider the integral:

$$* \int_{t_0}^{t_1} F\big(t, x(t), \dot{x}(t)\big) dt$$

If x(t) satisfies the following Euler-Lagrange function:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$$

For $t \in [t_0,\ t_1]$, then x^* maximizes or minimizes the integral among all functions $x = f(t)$ on $[t_0,\ t_1]$,.

Example: Solve

$$min\int_0^1 (x^2+\dot{x}^2)dt, \qquad x(0)=0, \qquad x(1)=e^2-1$$

Solution:

$$F\big(t,x(t),\dot{x}(t)\big)=x^2(t)+\dot{x}(t)^2$$

$$\frac{\partial F}{\partial x}=2x(t)\,, \qquad \frac{\partial F}{\partial \dot{x}}=2\dot{x}(t)$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)=\frac{d}{dt}(2\dot{x}(t)\ )=2\ddot{x}(t)$$

Applying the following Euler-Lagrange equation yields:

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)=0$$

$$2x(t)-2\ddot{x}(t)=0$$

The solution of the ordinary differential equation:

$$2x - 2\ddot{x} = 0$$

$$2\ddot{x} - 2x = 0$$

$$\ddot{x} - x = 0$$

$$r^2 - \mathrm{r} = 0$$

$$r = \mp 1$$

Then the general solution is:

$$x(t) = Ae^{t} + Be^{-t}$$

Using the endpoint condition:

$$x(0) = 0$$

$$0 = Ae^{0} + Be^{0} \rightarrow A = -B$$

$$x(1) = e^2 - 1$$

$$e^2 - 1 = Ae^1 + Be^{-1} = A(e^1 - e^{-1}) \rightarrow A = e, \quad B = -e$$

Then:

$$x(t) = e^{t+1} - e^{1-t}$$

$$\dot{x}(t) = e^{t+1} + e^{1-t}$$

The minimum of the performance index is as below:

$$\int_0^1 (x^2 + \dot{x}^2)dt$$

$$\int_0^1 ((e^{t+1} - e^{1-t})^2 + (e^{t+1} + e^{1-t})^2)dt$$

$$\int_0^1 (e^{2t+2} - 2e^2 + e^{2-2t} + e^{2t+2} + 2e^2 + e^{2-2t})dt$$

$$\int_0^1 (e^{2t+2} + e^{2-2t} + e^{2t+2} + e^{2-2t})dt$$

$$\int_0^1 e^2(e^{2t} + e^{-2t} + e^{2t} + e^{-2t})dt = \int_0^1 e^2(2e^{2t} + 2e^{-2t})dt$$

$$J_{min} = 2e^2\left[\frac{e^{2t}}{2} - \frac{e^{-2t}}{2}\right]^1 = e^4 - 1$$

Ready to answer your questions

Thank you

Systems and Control Eng. Department

Lecture_3

Subject: Optimal Control Theory

Lecturer: Asst. Prof. Dr. Ibrahim K. Mohammed

Optimal Control Theory

Optimal Control Theory: It is modern technique used to solve dynamic optimization problems. The new approach is developed to exceed the limitations associated with calculus of variations which are: Differential functions and dealing with interior solutions

What is the object of optimal control theory

1. Determine the control signals that will cause a process to satisfy the physical constriants and minimize or maximize some performance criterion.
2. Find a control law u(t) for a given system such that a certain optimality criterion is achieved.

Optimal Control Theory

Optimal control theory differs from Calculus of Variations in that it uses control variables to optimize the functionals.

In general: the optimal control problem is to find control input $u(t)$ which causes a given system (state equation):

$$\dot{x}(t) = g\big(t, x(t), u(t)\big)$$
$$u(t) = \dot{x}(t)$$

To follow an optimal trajectory $x(t)$ that minimizes the following cost or index function:

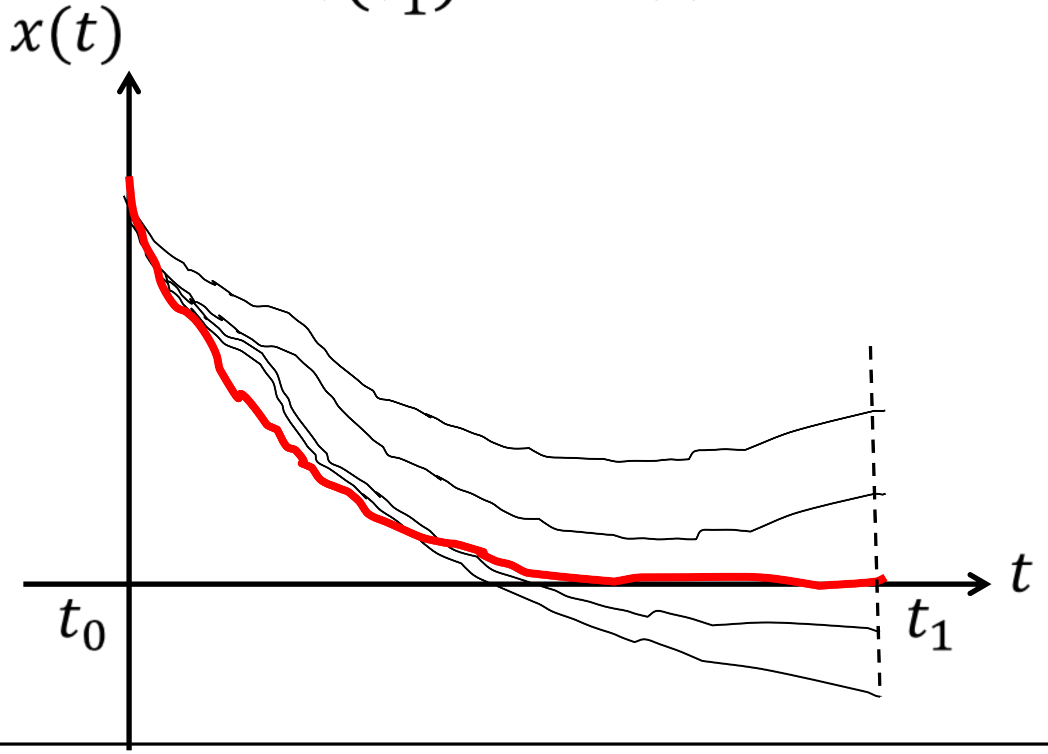
$$J(x) = \int_{t_0}^{t_1} f\big(t, x(t), u(t)\big)dt$$

Optimal Control Theory

Where the inital value of $x(t)$ at $t = t_0$ is given, while its value at $t = t_1$ is free

The endpoint condition is as follows:

$$x(t_0) = A$$
$$x(t_1) = Free$$



Optimal Control Theory

$u(t) \in \mathbb{R}, t \in [t_0, t_1]$ piecewise continuous and called control variable.

Pair of function $(\mathrm{x}(\mathrm{t}), u(t))$ that satisfies the end points conditions and the state equation $\dot{x}(t) = g(t, x(t), u(t))$ are called **admissible pairs**.

Theorem (The maximum Principle I);

Assume that $(x^*(t), u^*(t))$ is an optimal pair for the given problem:

$$\int_{t_0}^{t_1} f(t, x(t), u(t))dt$$

Subject to state equation: $\dot{x}(t) = g(t, x(t), u(t))$

Optimal Control Theory

Then there exist a continuous function **called adjoint function $\boldsymbol{\rho(t)}$**, such that for all $t \in [t_0, t_1]$, the following conditions are satisfied:

a) $u^*(t)$ maximizes the **Hamilton function** $H(t, x^*(t), u(t), \rho(t))$ $u(t) \in \mathbb{R}$ that is:

$$H\big(t, x^*(t), u^*(t), \rho(t)\big) = f\big(t, x(t), u(t)\big) + \rho\dot{x}(t)$$

$$H(t, x^*(t), u(t), \rho(t)) \leq H(t, x^*(t), u^*(t), \rho(t))$$

$$\frac{\partial H}{\partial u} = 0$$

b) The function $\rho(t)$ satisfies the following:

$$\dot{\rho}(t) = -\frac{\partial H}{\partial x}(t, x^*(t), u^*(t), \rho(t))$$

Optimal Control Theory

c) The function $\rho(t)$ satisfies **the transversality condition (T).**

$$\rho(t_1) = 0$$

For the sake of simplicity:

$$\frac{\partial H^*}{\partial x} = \frac{\partial H}{\partial x}(t, x^*(t), u^*(t), \rho(t))$$

$$f^* = f(t, x^*(t), u^*(t))$$

$$g^* = g(t, x^*(t), u^*(t))$$

$$H^* = H(t, x^*(t), u^*(t), \rho(t))$$

$$H\big(t, x(t), u(t), \rho(t)\big) = f\big(t, x(t), u(t)\big) + \rho\dot{x}(\mathrm{t})$$

$$\dot{\rho}(\mathrm{t}) = -\frac{\partial H^*}{\partial x} = -\frac{\partial f^*}{\partial x}$$

Optimal Control Theory

Since $u(t) \in \mathbb{R}$, there are no endpoints to consider then

$$0 = \frac{\partial H^*}{\partial u} = \frac{\partial f^*}{\partial u} + \rho(t)$$

$$\rho(t) = -\frac{\partial f^*}{\partial u} = -\frac{\partial f^*}{\partial \dot{x}}$$

$u(t) = u^*(t)$ will maximize H, then the above equation becomes:

$$\dot{\rho}(t) = -\frac{d}{dt}\left(\frac{\partial f^*}{\partial u}\right)$$

Mangasarian Theorem: Let the notation be as the statement of the maximum principle. If $H(t, x, u, \rho)$ is concave for each $t \in [t_0, t_1]$, then each admissible pair (x^*, u^*) that

Optimal Control Theory

satisfies condition (a), (b) and (c) of the maximum principle, will give a maximum.

Example 1: Solve the following problem

$$max \int_{o}^{T} [1 - tx(t) - u^2(t)]$$

Subject to the state equation: $\dot{x}(t) = u(t)$,

$$\boldsymbol{x(0) = x_0} \quad , x(\tau) = free$$

Where $x(0)$ and T are positive constants.

Solution: $f(t, x, u) = 1 - tx(t) - u^2(t)$

The Hamiltom equation is:

$$\mathrm{H}(t, x, u) = f(t, x, u) + \rho(t)\dot{x}(t)$$

Optimal Control Theory

$$\mathrm{H}(t,x,u) = 1 - tx(t) - u^2(t) + \rho(t)u(t)$$

$u(t) = u^*(t)$ shall maximize H

From condition (a):

$$\frac{\partial H}{\partial u}\big(t, x^*(t), u(t), \rho(t)\big) = 0$$

$$\frac{\partial H}{\partial u} = -2u(t) + \rho(t) = 0$$

$$\mathrm{u}(t) = \tfrac{1}{2}\rho(t)$$

$u(t)$ will maximize H as $\frac{\partial 2_H}{\partial u^2} = -2 < 0$

Hence, $u^*(t) = \frac{\rho(t)}{2} \qquad t \in [0, T]$

Optimal Control Theory

From condition (b):

$$\frac{\partial H^*}{\partial x}\left(t, x^*(t), u^*(t), \rho(t)\right) = -\dot{\rho}(t)$$

$$-t = -\dot{\rho}(t)$$

$$\dot{\rho}(t) = t \;\rightarrow\; \frac{d\rho}{dt} = t \;\rightarrow d\rho = t\,dt \;\rightarrow \int d\rho = \int t\,dt$$

$$\rho(t) = \tfrac{1}{2}t^2 + A \qquad (1)$$

From the tranversality condition (c):

$$\rho(t_1) = 0$$

At $t_1 = T$ $\qquad \rho(T) = 0$

Based on the equation (1):

$$\rho(\mathrm{t}) = \tfrac{1}{2}t^2 + A \qquad \rightarrow\; 0 = \tfrac{1}{2}T^2 + A \quad \rightarrow\; A = -\frac{1}{2}T^2$$

Optimal Control Theory

Using equation (1):

$$\rho(t) = \frac{1}{2}t^2 - \frac{1}{2}T^2 \rightarrow \rho(t) = \frac{1}{2}(t^2 - T^2)$$

But $u^*(\mathrm{t}) = \frac{\rho(t)}{2}$, then $u^*(\mathrm{t}) = \frac{1}{4}(t^2 - T^2)$

Based on the state equation:

$$\dot{x}^*(\mathrm{t}) = u^*(t)$$

$$\dot{x}^*(\mathrm{t}) = \frac{1}{4}(t^2 - T^2) = \frac{1}{4}t^2 - \frac{1}{4}T^2$$

$$\int \dot{x}^* dt = \int \left(\frac{1}{4}t^2 - \frac{1}{4}T^2\right) dt$$

$$x^*(t) = \frac{1}{12}t^3 - \frac{1}{4}T^2 t + B \qquad (2)$$

Optimal Control Theory

At $t = 0, \quad x(0) = x_0$

Using equation (2):

$$x_0 = \frac{1}{12}0^3 - \frac{1}{4}T^2 0 + B$$

$$\mathrm{B} = x_0$$

Then equation (2) becomes:

$$x^*(t) = \frac{1}{12}t^3 - \frac{1}{4}T^2 t + x_0$$

$(x^*(t), u^*(t))$ is the optimal pair of the problem.

Optimal Control Theory

<u>Example 2:</u> Solve the following problem

$$max \int_{o}^{T} [x(t) - u^2(t)]$$

Subject to the state equation: $\dot{x}(t) = x(t) + u(t)$,

$$x(0) = 0, x(T) = free$$

Where $x(0)$ and T are positive constants.

<u>Solution</u>: $f(t, x, u) = x(t) - u^2(t)$

The Hamiltom equation is:

$$\mathrm{H}(t, x, u) = f(t, x, u) + \rho(t)\dot{x}(t)$$

$$\mathrm{H}(t, x, u) = x(t) - u^2(t) + \rho(t)(x(t) + u(t))$$

$$\mathrm{H}(t, x, u) = -u^2(t) + \rho(t)u(t) + \big(1 + \rho(t)\big)x(t)$$

Optimal Control Theory

$u(t) = u^*(t)$, shall maximize H

From (a) condition:

$$\frac{\partial H}{\partial u} = 0$$

$$\frac{\partial H}{\partial u} = -2u(t) + \rho(t) = 0$$

$u(t) = \frac{\rho(t)}{2}$, this yields a maximum, since $\frac{\partial 2_H}{\partial u^2} = -2 < 0$

Hence, $u^*(t) = \frac{\rho(t)}{2} \quad t \in [0, T]$

From (b) condition:

$$\frac{\partial H^*}{\partial x} = -\dot{\rho}(t)$$

Optimal Control Theory

$$1+\rho(t) = -\dot{\rho}(t)$$

$$1+\rho(t) = -\frac{d\rho(t)}{dt}$$

$$\frac{1}{1+\rho(t)}d\rho = -dt \quad \text{[Taking integral]}$$

$$\int \frac{1}{1+\rho(t)}d\rho = \int -dt$$

$$\ln(1+\rho) = -t + C_1 \rightarrow 1+\rho = e^{-t} + e^{C_1}$$

$$1+\rho = Ae^{-t}$$

$$\rho = Ae^{-t} - 1 \qquad (1)$$

From (c) condition (Transverality Condittion):

$$\rho(T) = 0$$

Optimal Control Theory

At $t = T$, using equation (1):

$$\rho(T) = Ae^{-T} - 1$$

$$0 = Ae^{-T} - 1 \rightarrow \quad A = e^{T}$$

$$\rho(t) = e^{T}e^{-t} - 1 = e^{T-t} - 1$$

But $u^{*}(t) = \frac{\rho(t)}{2}$ then,

$$u^{*}(t) = \frac{1}{2}(e^{T-t} - 1)$$

As $\dot{x}(t) = x(t) + u(t)$

To find x^{*},

$$\dot{x}^{*}(t) - x^{*}(t) = u^{*}(t)$$

$$\dot{x}^{*}(t) - x^{*}(t) = \frac{1}{2}(e^{T-t} - 1)$$

Optimal Control Theory

$$x^*(t) = -\frac{1}{4}e^{T-t} + \frac{1}{2} + De^t$$

Using endpoints condition;

$$x(0) = 0$$

$$0 = -\frac{1}{4}e^T + \frac{1}{2} + D, \rightarrow D = \frac{1}{4}e^T - \frac{1}{2}$$

$$\dot{x}^*(t) - x^*(t) = \frac{1}{2}(e^{T-t} - 1)$$

$$x^*(t) = -\frac{1}{4}e^{T-t} + \frac{1}{2} + e^t\left(\frac{1}{4}e^T - \frac{1}{2}\right)$$

$$x^*(t) = \frac{1}{4}(e^{T+t} - e^{T-t}) + \frac{1}{2}(1 - e^t)$$

College of Electronics Eng. Ninevah University

Systems and Control Eng. Department

Lecture_4

Subject: Optimal Control Design in Time-domain

Lecturer: Asst. Prof. Dr. Ibrahim K. Mohammed

Control Problems

There are many types of control problems which are given by:

1. The terminal control problem

This is used to bring the system output state as closed as possible to a given terminal state within a given period of time.

2. The minimum time control problem

It is used to reach terminal state in a shortest possible time periode. For example breaking a car at the terminal point as hard as possible.

3. The minimum energy control problem

It is employed to transfer the system from the initial state to a final state with minimum control energy.

Control Problems

4. The regulator control problem

It is used with the system initially displaced for equilibrium . It will return the system output states to the equilibrium states in such manner so as to minimize a given performance index.

5. The tracking control problem

This is used to cause the system state to track desired state while minimizing the cost function (performance index).

Performance Index

The general form of performance index (J) is given by:

$$J = \int_{t_0}^{t_1} (x^T Qx + u^T Ru)dt$$

Where Q and R are the state and control weighting matrices respectively and they are always square and positive semidefinite matrices, J always a scalar quantity.

In procedure of design of control systems the cost function should be minimize to a smallest value.

Linear Quadratic Regulator in Continuous Time

Linear Quadratic Regulator (LQR) is state feedback controller system, which is classified as an optimal control system.

LQR is a multi variable controller technique used in many industrial applications as it basically seeks a compromise between the best performance and minimum control input.

It worth considering that this technique is highly recommended in the precision movement applications due to its good tracking performance.

Consider a system with the following state space representation :

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) + Du(t) \quad (2)$$

Linear Quadratic Regulator in Continuous Time

Where A is system matrix, B is input matrix, C is output matrix and D is feed forward matrix.

The controller approach involves applying the input matrix:

$$u(t) = -K\,x(t) \tag{3}$$

To track the input commands while minimizing the following performance index:

$$J = \int_{t_0}^{t_1} (x^T Q x + u^T R u)\,dt \tag{4}$$

Linear Quadratic Regulator in Continuous Time

Where K is the optimal state feedback gain matrix of LQR controller that its expression can be derived based optimal control theory approach. As follows:

The Hamilton equation is given by :

$$H = f + \rho\dot{x}$$

Using equations (1) and (4), the above equation becomes:

$$H = x^T Qx + u^T Ru + \rho(Ax + Bu)$$

$$H = x^T Qx + \rho Ax + u^T Ru + \rho Bu$$

Based on (a) condition:

$$\frac{\partial H}{\partial u} = 0$$

$$Ru + B^T\rho = 0$$

$$Ru = -B^T\rho$$

$$u = -R^{-1}B^T\rho$$

Linear Quadratic Regulator in Continuous Time

Assume $\rho(t) = P(t)x(t)$ where $P(t)$ is a positive semidefintion matrix, then the closed loop feedback control effort is as follows:

$$u = -R^{-1}B^T P(t)x(t)$$

Comparing equation (3) with the above equation, the feedback gain matrix is as follows:

$$K = R^{-1}B^T P(t)$$

Where $P(t)$ is the stabilizing solution of the following Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

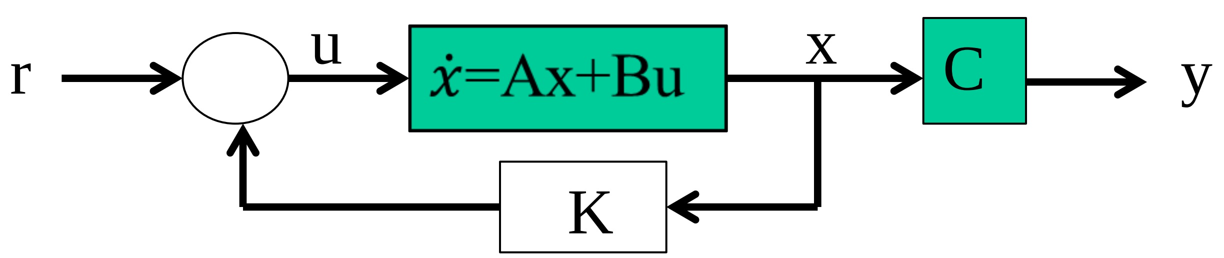
Linear Quadratic Regulator in Continuous Time

The LQR controller weighting matrices Q and R should be set properly using numerical optimization algorithms such as Genetic algorithm, Particle Swarm Optimization (PSO), Artificial Bee Colony (ABC) etc such that the control system can achieve best performance.

System Eigen values

The stability of systems can be studied through calculating their eigen values which corresponding to poles positions of the systems. The system is stable if its poles are located in the left hand side of the s-plan.

The block diagram of regultor system is shown below:



Linear Quadratic Regulator in Continuous Time

$$u = r - Kx$$

For regulation system $r = 0$, then

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$$

The state space representation is as follows:

State equation: $\dot{x} = Ax + Bu$ (1)

Output equation: $y = Cx$ (2)

Where x is state vector of the system with dimension (n x 1). The state and output equation in the s-domain are given by:

$$sX(s) = AX(s) + BU(s) \quad (3)$$

$$Y(s) = CX(s) \quad (4)$$

Based on equation (3):

Linear Quadratic Regulator in Continuous Time

$$sX(s) - AX(s) = BU(s)$$

$$X(s)(s - A) = BU(s) \quad (5)$$

The control law in the s-domain is given by:

$$U(s) = -K\,X(s) \quad (6)$$

Substitute equation (6) in equation (5) yields:

$$X(s)(sI - A) = -\,BKX(s)$$

$$X(s)|sI - A + BK| = 0$$

Based on the above equation the term:

$$|sI - A + BK|$$

is called characterstics equation that can be used to determine the eigen values of the closed- loop control system.

Linear Quadratic Regulator in Continuous Time

<u>Example 1:</u> A regulator control system contains a plant that described by the following state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

And has a performance index:

$$J = \int_0^\infty \left[x^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x + u^2 \right] dt$$

Determine (a). The Riccati matrix P

(b). The state feedback gain matrix K

(c). The closed-loop eigenvalues

Linear Quadratic Regulator in Continuous Time

Solution: $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, B= $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, Q= $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $R = scalar = 1$

Solving (a):

The Riccati equation is:

$$PA + A^T P + Q - PBR^{-1}B^T P = 0$$

$$PA = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -p_{12} & p_{11} - 2p_{12} \\ -p_{22} & p_{21} - 2p_{22} \end{bmatrix}$$

$$A^T P = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}^T \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$A^T P = \begin{bmatrix} -p_{21} & -p_{22} \\ p_{11} - 2p_{21} & p_{12} - 2p_{22} \end{bmatrix}$$

Linear Quadratic Regulator in Continuous Time

$$PBR^{-1}B^TP = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$PBR^{-1}B^TP = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \begin{bmatrix} p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{12}p_{21} & p_{12}p_{22} \\ p_{22}p_{21} & {p_{22}}^2 \end{bmatrix}$$

Applying Riccati equation:

$$\begin{bmatrix} -p_{12} & p_{11} - 2p_{12} \\ -p_{22} & p_{21} - 2p_{22} \end{bmatrix} + \begin{bmatrix} -p_{21} & -p_{22} \\ p_{11} - 2p_{21} & p_{12} - 2p_{22} \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$- \begin{bmatrix} p_{12}p_{21} & p_{12}p_{22} \\ p_{22}p_{21} & {p_{22}}^2 \end{bmatrix} = 0$$

Since P is symmetric, $p_{21} = p_{12}$, the above equation can be expressed as follows:

Linear Quadratic Regulator in Continuous Time

$$-p_{12} - p_{12} + 2 - {p_{12}}^2 = 0 \qquad (1)$$
$$p_{11} - 2p_{12} - p_{22} - p_{12}p_{22} = 0 \qquad (2)$$
$$-p_{22} + p_{11} - 2p_{12} - p_{12}p_{22} = 0 \qquad (3)$$
$$p_{12} - 2p_{22} + p_{12} - 2p_{22} + 1 - {p_{22}}^2 = 0 \qquad (4)$$

Equation (2) and (3) are the same, from equation (1):

$${p_{12}}^2 + 2p_{12} - 2 = 0$$

Solving the above equation yields:

$$p_{12} = p_{21} = 0.732,$$
$$p_{12} = p_{21} = -2.732$$

The positive one is the right solution

Based on equation (4):

$$2p_{12} - 4p_{22} + 1 - \boldsymbol{p_{22}}^2 = 0$$

Linear Quadratic Regulator in Continuous Time

$$p_{22}{}^2 + 4p_{22} - 1 - 2 * 0.732 = 0$$

$$p_{22}{}^2 + 4p_{22} - 2.464 = 0$$

Solving the above equation gives:

$$p_{22} = 0.542 \quad and \quad -4.542$$

The solution is $p_{22} = 0.542$

Using equation (3)

$$p_{11} - 2 * 0.732 - 0.542 - 0.732 * 0.542 = 0$$

$$p_{11} = 2.403$$

Then the Riccati matrix is given by:

$$P = \begin{bmatrix} 2.403 & 0.732 \\ 0.732 & 0.542 \end{bmatrix}$$

Solving (b):

$$K = R^{-1}B^TP = 1\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 2.403 & 0.732 \\ 0.732 & 0.542 \end{bmatrix} = \begin{bmatrix} 0.732 & 0.542 \end{bmatrix}$$

Linear Quadratic Regulator in Continuous Time

Solving (c):

To find the closed-loop eigenvalues of the system the following equation is used:

$$|sI - A + BK| = 0$$

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0.732 & 0.542 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.732 & 0.542 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} s & -1 \\ 1.732 & s+2.542 \end{bmatrix}\right| = 0$$

$$s^2 + 2.542s + 1.732 = \mathbf{0}$$

$s_1 = -1.271 + j0.341,\ s_2 = -1.271 - j0.341$

The system is stable.

Linear Quadratic Regulator in Continuous Time

Example 2: Consider a system with the following state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$Q = eye(2), \mathrm{R} = 1$$

Show that the system can not be stabilized by the state feedback control scheme.

$$u = -Kx$$

Whatever the gain matrix is chosen.

Solution:

The state and input matrices are given below:

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And the gain matrix for second order system is $K = [k_1 \quad k_2]$

Linear Quadratic Regulator in Continuous Time

$$|sI - A + BK| = 0$$

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} s+1 & -1 \\ 0 & s-2 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} s+1+k_1 & k_2-1 \\ 0 & s-2 \end{bmatrix}\right| = 0$$

$$(s+1+k_1)(s-2) = 0$$

$$s_1 = -1 - k_1, \qquad s_2 = 2$$

The system is unstable whatever the gain matrix K is chosen as the second pole $s_2 = 2$ is in the right - hand -side of the s-plane.

Linear Quadratic Regulator in continuous time

Example 3: Consider a system with the following state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

If its block diagram is given in the following figure.

(1). Design LQR controller.

(2). Find state equation of the system.

(3). Find the unit step response of the controlled system.

Linear Quadratic Regulator in Continuous Time

Example 3: Consider a system with the following state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]$$

If its block diagram is given in the following figure.

(1). Design LQR controller.

(2). Find state equation of the system.

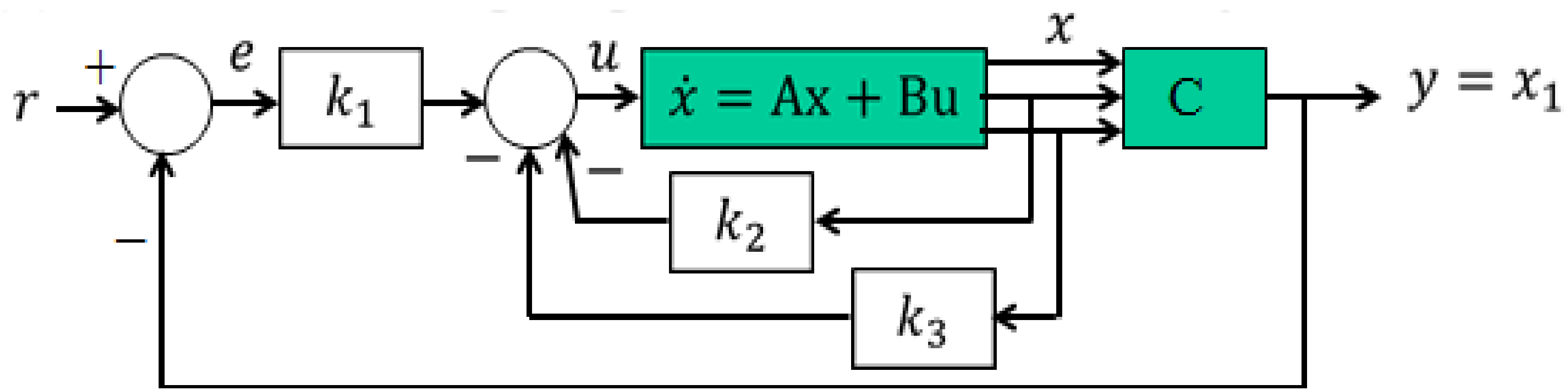
(3). Find the unit step response of the controlled system.

Linear Quadratic Regulator in Continuous Time

4. If the Riccati matrix is given below, find the LQR gain matrix.

$$P = \begin{bmatrix} 55.12 & 14.67 & 1 \\ 14.67 & 7.02 & 0.53 \\ 1 & 0.53 & 0.11 \end{bmatrix}$$

5. If the state vector is $x = \begin{bmatrix} 0.2 \\ 0.1 \\ 0 \end{bmatrix}$, find the the control input value.



Linear Quadratic Regulator in Continuous Time

Solution:

The error is:

$$e = r - x_1$$

The control signal is:

$$u = k_1 e - k_2 x_2 - k_3 x_3$$

$$u = k_1(r - x_1) - k_2 x_2 - k_3 x_3$$

$$u = k_1 r - k_1 x_1 - k_2 x_2 - k_3 x_3$$

$$u = k_1 r - \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$u = k_1 r - Kx$$

For regulator system $r = 0$, then

Linear Quadratic Regulator in Continuous Time

$$u = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The state feedback gain matrix is:

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

The standard state and input weighting matrices are given by:

$$Q = \begin{bmatrix} q_{11} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{bmatrix}, \quad R = [1]$$

To get fast output response q_{11} must be sufficient large compared with q_{22} and q_{33} and R.

Linear Quadratic Regulator in Continuous Time

Let q_{11}=100 , $q_{22} = 1, q_{33}$=1 and $R =$ 0.01, using "*lqr*" Matlab command the feedback gain matrix is calculated as follows:

$$K = lqr(A, B, Q, R)$$

$$K = [100 \quad 53.12 \quad 11.6711]$$

Solving (2):

$$\dot{x} = Ax + Bu$$

But

$$u = k_1 r - Kx$$

$$\dot{x} = Ax + B(k_1 r - Kx)$$

$$\dot{x} = Ax - BKx + B\, k_1 r$$

The state equation of the system is as follows:

$$\dot{x} = (A - BK)x + B\, k_1 r \qquad (1)$$

From equation (1) the state of the system is x while the input is r

Linear Quadratic Regulator in Continuous Time

Solving (3):

Based on equation (1)

$$\dot{x} = A_{cl}x + B_{cl}u$$
$$y = C_{cl}x + D_{cl}u$$

$A_{cl} = \mathrm{A} - \mathrm{BK}$

$B_{cl} = Bk_1$

$C_{cl} = C$

$D_{cl} = D$

To find the unit step response

$[y, x, t] = step(A_{cl}, B_{cl}, C_{cl}, D_{cl})$

LQR controller in continuous time.

4. If the Riccati matrix is given below, find the controller gain matrix.

$$P = \begin{bmatrix} 55.12 & 14.67 & 1 \\ 14.67 & 7.02 & 0.53 \\ 1 & 0.53 & 0.11 \end{bmatrix}$$

$$K = R^{-1}B^{T}P$$

$$K = [0.01]^{-1}[0 \quad 0 \quad 1]^{T}\begin{bmatrix} 55.12 & 14.67 & 1 \\ 14.67 & 7.02 & 0.53 \\ 1 & 0.53 & 0.11 \end{bmatrix}$$

$$K = [100 \quad 53.12 \quad 11.67]$$

LQR controller in continuous time

5. If the state vector is $x = \begin{bmatrix} 0.2 \\ 1.5 \\ 0.75 \end{bmatrix}$, find the the control input value.

$$u = rk_1 - Kx$$

$$u = 1 * 100 - [100 \quad 53.12 \quad 11.67] \begin{bmatrix} 0.2 \\ 1.5 \\ 0.75 \end{bmatrix}$$

$$u = 100 - (20 + 79.685.3 + 8.73)$$

$$u = 100 - 108.4 = -8.4$$

College of Electronics Eng. Ninevah University

Systems and Control Eng. Department

Lecture_5

Subject: Optimal Control Design in Discrete-domain

Lecturer: Asst. Prof. Dr. Ibrahim K. Mohammed

Linear Quadratic Regulator in Discrete-Time

Assume a completely state controllable linear system has a discrete-time form as follows:

$$x(k+1) = Gx(k) + Hu(k)$$

Where $x(k)$ is state vector (n-vector)

$u(k)$ is control vector (input vector) (n-vector)

G is (n x n) nonsingular state matrix.

H is (n x r) input matrix, r is the number of inputs

The objective of LQR controller is to find the optimal control sequence $u(0), u(1), u(2), \ldots . u(N-1)$ that minimizes the following performance index:

$$J = \frac{1}{2}X^T(N)SX(N) + \frac{1}{2}\sum_{k=0}^{N-1}[X^T(k)QX(k) + U^T(k)RU(k)$$

LQR_in Discrete-Time

Where Q is (n x n) real symmetric matrix or Hermitian matrix.

R is (r x r) positive definite Hermitian matrix.

S is (n x n) positive definite or positive semi-definite Hermitian matrix.

$$S = P(N)$$

The LQR gain matrix in discrete-time given by:

$$K(k) = R^{-1}H^{T}(G^{T})^{-1}[P(k) - Q)$$

Where $P(k)$ is calculated by solving the following equation:

$$\mathrm{P}(k) = Q + G^{T}P(k+1)[I + HR^{-1}H^{T}P(k+1)]^{-1}\mathrm{G}$$

LQR_in Discrete-Time

Minimum of performance index:

The minimum value of performance index (J) can be calculated using the following equation:

$$J_{min} = \frac{1}{2} X^T(0) P(0) X(0)$$

Where $P(0)$ is Hermitian matrix

Control Problems

Example 1: Consider a discrete-time control system defined by:

$$x(k+1) = 0.3679x(k) + 0.6321u(k), \qquad x(0) = 1$$

Determine the optimal control law ($u(k)$) required to minimize the following performance index:

$$J = \frac{1}{2}[x(10)]^2 + \frac{1}{2}\sum_{k=0}^{k=9}[X^2(k) + U^2(k)]$$

Where $S = 1, Q = 1$ and $R = 1$.

Then determine the minimum value of the cost function (J).

LQR in Discrete-Time

Solution:

$$P(k) = Q + G^T P(k+1)[I + HR^{-1}H^T P(k+1)]^{-1} G$$

$G = 0.3679, H = 0.6321$

$P(k) = 1 + 0.3679P(k+1)\ [1 + 0.6321(1)(0.6321)$
$P(k+1)]^{-1} 0.3679$

The above equation can be simplified to the following equation

$$P(k) = 1 + 0.1354P(k+1)\ [1 + 0.3996P(k+1)]^{-1} \qquad (1)$$

The boundary condition for $P(k+1)$

$P(N) = S$

$N = 10. S = 1, P(10) = S = 1,$

Computing of $P(k)$ from $k = 9$ to $k = 0$ is given by:

LQR in Discrete-Time

At $k = 9$, using equation (1):

$$P(k) = 1 + 0.1354P(k+1)\,[1 + 0.3996P(k+1)]^{-1} \qquad (1)$$

$$P(9) = 1 + 0.1354P(10)\,[1 + 0.3996P(10)]^{-1}$$

$$P(9) = 1 + 0.1354 * 1\,[1 + 0.3996 * 1]^{-1}$$

$$P(9) = 1.1354\,[1.3996]^{-1} = 1.0967$$

At $k = 8$, using equation (1):

$$P(8) = 1 + 0.1354P(9)\,[1 + 0.3996P(9)]^{-1}$$

$$P(8) = 1 + 0.1354 * 1.0967\,[1 + 0.3996 * 1.0967]^{-1}$$

$$P(8) = 1.103$$

At $k = 7$, using equation (1):

$$P(7) = 1 + 0.1354P(8)\,[1 + 0.3996P(8)]^{-1} = 1.1036$$

LQR in Discrete-Time

At $k = 6$, using equation (1):

$$P(6) = 1 + 0.1354P(7)\,[1 + 0.3996P(7)]^{-1} = 1.1037$$

At $k = 5,4,3,2,1,0$, using equation (1):

$$P(k) = 1.1037$$

The steady state value of P matrix (P_{ss}) can be obtained using equation (1) as follows: $P(k) = P(k+1) = P_{ss}$

$$P_{ss} = 1 + 0.1354P_{ss}[1 + 0.3996P_{ss}]^{-1}$$

$$0.3996{P_{ss}}^2 + 0.465P_{ss} - 1 = 0$$

$$P_{ss} = 1.1037 \quad \text{or} \quad P_{ss} = -2.2674$$

P_{ss} must be positive, then

$$P_{ss} = 1.1037$$

LQR in Discrete-Time

The feedback gain (K) is calculated as follows:

$$K(k) = R^{-1}H^{T}(G^{T})^{-1}[P(k) - Q]$$

$$K(k) = 1(0.6321)\,(0.3679)^{-1}[P(k) - 1]$$

$$K(k) = 1.7181[P(k) - 1] \qquad (2)$$

At k=10, using equation (2):

$$K(10) = 1.7181[P(10) - 1]$$

$$K(10) = 1.7181[1 - 1] = 0$$

At k=9, using equation (2):

$$K(9) = 1.7181[P(9) - 1]$$

$$K(9) = 1.7181[1.0967 - 1] = 0.1662$$

By using the same manner:

$K(8) = 0.1773, K(7) = 0.1781,$

LQR in Discrete-Time

$$K(6) = K(5) = K(4) \dots \dots .. K(0) = 0.1781$$

To find states values:

$$x(k+1) = Gx(k) + Hu(k)$$

$$x(k+1) = 0.3679x(k) + 0.6321u(k)$$

But: $u(k) = -K(k)x(k)$

$$x(k+1) = \big(0.3679 - 0.6321K(k)\big)x(k) \qquad (3)$$

At $k = 0$, using equation (3):

$$x(1) = \big(0.3679 - 0.6321K(0)\big)x(0)$$

$$x(1) = (0.3679 - 0.6321 * 0.1781)1 = 0.2553$$

At $k = 1$, using equation (3):

$$x(2) = \big(0.3679 - 0.6321K(1)\big)x(1) = 0.0652$$

LQR in Discrete-Time

At $k = 2$, using equation (3):

$$x(3) = \big(0.3679 - 0.6321K(2)\big)x(2) = 0.0166$$

At $k = 3$, using equation (3):

$$x(4) = \big(0.3679 - 0.6321K(3)\big)x(3) = 0.00424$$

The values of $x(k)$ for $k = 5,6,7,8,9,10$ approache to zero rapidly.

The optimal control sequence $u(k)$ is now obtained as follows:

$$u(k) = -K(k)x(k)$$

For $k = 0$

$$u(0) = -K(0)x(0) = -0.1781 * 1 = -0.1718$$

LQR in Discrete-Time

For $k = 1$

$$u(1) = -K(1)x(1) = -0.1781 * 0.2553 = -0.0455$$

For $k = 2$

$$u(2) = -K(2)x(2) = -0.1781 * 0.0652 = -0.0116$$

For $k = 3$

$$u(3) = -K(3)x(3) = -0.1781 * 0.0166 = -0.00296$$

For $k = 4$

$$u(4) = -K(4)x(4) = -0.1781 * 0.00424 = -0.000756$$

$$u(k) = 0 \quad \text{for} \quad k = 5,6,7,8,9,10$$

The minimum value of the performance index (J)

$$J_{min} = \frac{1}{2} x^T(0) P(0) \mathrm{x}(0) = \frac{1}{2}(1 * 1.1037 * 1) = 0.5518$$

College of Electronics Eng. Ninevah University

Systems and Control Eng. Department

Lecture_6

Subject: Steady-state Quadratic Optimal Control of Discrete-Time Systems

Lecturer: Dr. Ibrahim Khalaf Mohammed

Steady-State of Discrete-Time Optimal Control

To find steatdy-state gain matrix (K) of optimal control systems, we need steady-state solution of the following Riccati equation:

$$P = Q + G^T P[I + HR^{-1}H^T P]^{-1}G$$

Where the steady-state matrix $P(k)$ is defined as P. The solution begins the with $P = 0$ and iterate the equation until a stationary solution is obtained.

Based on the steady-state solution of the above Riccati equation, the steady state gain matrix can be calculated using the following equation:

$$K = R^{-1}H^T(G^T)^{-1}(P - Q)$$

Steady-State of Discrete-Time Optimal Control

Example: consider the folowing discrete time system:

$$x(k+1) = Gx(k) + Hu(k)$$

Where $G = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$, $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the performance index (J) is given by:

$$J = \frac{1}{2} \sum_{k=0}^{\infty} \left(x^T(k) Q x(k) + u^T(k) R u(k) \right)$$

Where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$, $R = 1$

The control law that minimizes J is given by:

$$u(k) = -Kx(k)$$

Derermine the steady state gain matrix.

LQR in Discrete-Time

Solution: Start the steady state solution of the Riccati equation with $P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

First iteration: $P = Q + G^T P[I + HR^{-1}H^T P]^{-1}G$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 [1 \quad 1] \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}]^{-1} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Second iteration: $P = Q + G^T P[I + HR^{-1}H^T P]^{-1}G$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 [1 \quad 1] \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}]^{-1} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

LQR in Discrete-Time

$$P = \begin{bmatrix} 1.024 & -0.016 \\ -0.016 & 0.564 \end{bmatrix},$$

Third iteration: $P = Q + G^T P[I + HR^{-1}H^T P]^{-1} G$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1.024 & -0.016 \\ -0.016 & 0.564 \end{bmatrix} [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 [1 \quad 1] \begin{bmatrix} 1.024 & -0.016 \\ -0.016 & 0.564 \end{bmatrix}]^{-1} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.0251 & -0.0186 \\ -0.0186 & 0.5723 \end{bmatrix}$$

Forth iteration: $P = Q + G^T P[I + HR^{-1}H^T P]^{-1} G$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1.0251 & -0.0186 \\ -0.0186 & 0.5723 \end{bmatrix} [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 [1 \quad 1] \begin{bmatrix} 1.0251 & -0.0186 \\ -0.0186 & 0.5723 \end{bmatrix}]^{-1} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

LQR in Discrete-Time

$$P = \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5723 \end{bmatrix}$$

Fifth iteration: $P = Q + G^T P [I + HR^{-1}H^T P]^{-1} G$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5723 \end{bmatrix} [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 [1 \quad 1] \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5723 \end{bmatrix}]^{-1} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5724 \end{bmatrix}$$

Sixth iteration: $P = Q + G^T P [I + HR^{-1}H^T P]^{-1} G$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5724 \end{bmatrix} [\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1 [1 \quad 1] \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5724 \end{bmatrix}]^{-1} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

LQR in Discrete-Time

$$P = \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5724 \end{bmatrix}$$

When P matrix stays constant, steady state is reached.

$$P_{ss} = \begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5724 \end{bmatrix}$$

Steady state gain matrix

$$K = R^{-1}H^T(G^T)^{-1}(P - Q)$$

$$K = 1\begin{bmatrix} 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 1.0252 & -0.0189 \\ -0.0189 & 0.5724 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\right) = \begin{bmatrix} 0.0786 & 0.0865 \end{bmatrix}$$

College of Electronics Eng. Ninevah University

Systems and Control Eng. Department

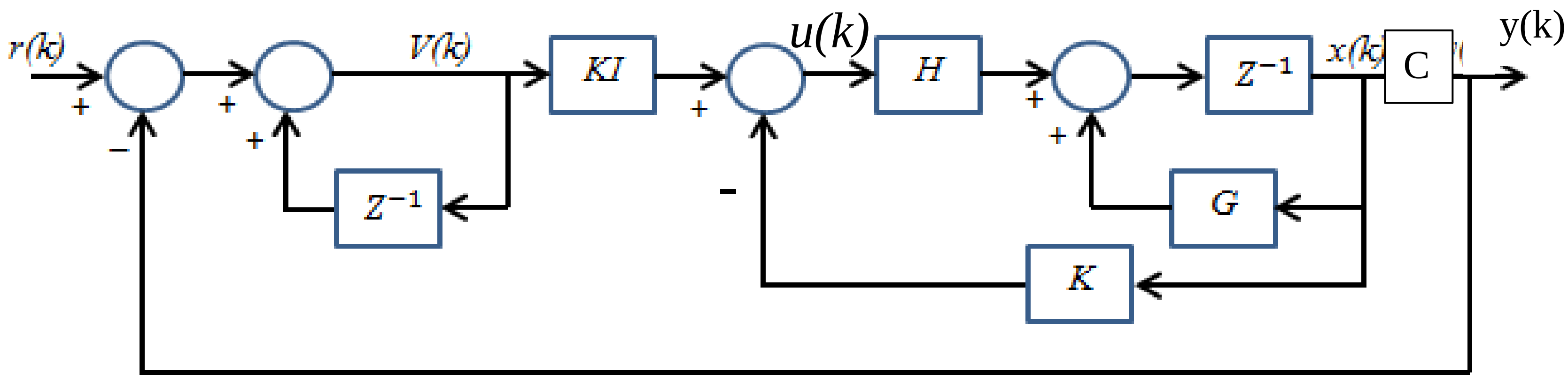
Lecture_7

Subject: Steady-State Servo Optimal Control System in Discrete-Time Form

Lecturer: Dr. Ibrahim Khalaf Mohammed

Steady-State of Discrete-Time Optimal Control

Consider a servo control system shown in the following figure.



With the following state and output equations:

$$x(k+1) = Gx(k) + Hu(x) \quad (1)$$

$$y(k) = Cx(k) \quad (2)$$

Steady-State Servo Control System

The plant of the system does not involve an integrator. This system uses state feedback and integral control.

The equation for the integrator is:

$$V(k) = V(k-1) + r(k) - y(k) \tag{3}$$

$$u(k) = -Kx(k) + KI\,V(k) \tag{4}$$

Equation (3) can be rewritten it as follows:

$$V(k+1) = V(k) + r(k+1) - y(k+1)$$

$$V(k+1) = V(k) + r(k+1) - Cx(k+1)$$

Based on equation (1):

$$V(k+1) = V(k) + r(k+1) - C[Gx(k) + Hu(k)]$$

Based on equation(4), the above equation becomes:

$$V(k+1) = V(k) + r(k+1) - CGx(k) - CH(-Kx(k) + KIV(k)]$$

$$V(k+1) = (1 - CHKI)V(k) + (-CG + CHK)x(k) + r(k+1) \quad (5)$$

Steady-State Servo Control System

Based on equation (1) and equation (4):

$$x(k+1) = Gx(k) + H(-Kx(k) + KIV(k))$$

$$x(k+1) = (G - HK)x(k) + HKI\,V(k) \qquad (6)$$

Based on equation (5) and equation (6), the state and output equations are given by:

$$\begin{bmatrix} x(k+1) \\ V(k+1) \end{bmatrix} = \begin{bmatrix} G - HK & HKI \\ -CG + CHK & 1 - CHKI \end{bmatrix} \begin{bmatrix} x(k) \\ V(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{r}(k+1)$$

$$y(k) = [C\ 0] \begin{bmatrix} x(k) \\ V(k) \end{bmatrix}$$

For a step input $r(k) = r$

Steady-State Servo Control System

$$\begin{bmatrix} x(k+1) \\ V(k+1) \end{bmatrix} = \begin{bmatrix} G - HK & HKI \\ -CG + CHK & 1 - CHKI \end{bmatrix} \begin{bmatrix} x(k) \\ V(k) \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix} \qquad (7)$$

$$y(k) = [C \; 0] \begin{bmatrix} x(k) \\ V(k) \end{bmatrix}$$

For a step input, as k approaches infinity

$$x(k) \rightarrow x(\infty)$$

$$u(k) \rightarrow u(\infty)$$

$$V(k) \rightarrow V(\infty)$$

Based on equation (3)

$$V(\infty) = V(\infty) + r(\infty) - y(\infty)$$

$$y(\infty) = r(\infty) = r$$

There is no steady state error. By substituting $k = \infty$ into equation (7), it can obtain:

Steady-State Servo Control System

$$\begin{bmatrix} x(\infty) \\ V(\infty) \end{bmatrix} = \begin{bmatrix} G - HK & HKI \\ -CG + CHK & 1 - CHKI \end{bmatrix} \begin{bmatrix} x(\infty) \\ V(\infty) \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix} \qquad (8)$$

Define:

$$x_e(k) = x(k) - x(\infty)$$
$$V_e(k) = V(k) - V(\infty)$$

$$x_e(k+1) = x(k+1) - x(\infty)$$
$$V_e(k+1) = V(k+1) - V(\infty)$$

Subtract equation (8) from equation(7):

$$\begin{bmatrix} x(k+1) - x(\infty) \\ V(k+1) - V(\infty) \end{bmatrix} = \begin{bmatrix} G - HK & HKI \\ -CG + CHK & 1 - CHKI \end{bmatrix} \begin{bmatrix} x(k) \\ V(k) \end{bmatrix} + \begin{bmatrix} 0 \\ r \end{bmatrix} - \begin{bmatrix} G - HK & HKI \\ -CG + CHK & 1 - CHKI \end{bmatrix} \begin{bmatrix} x(\infty) \\ V(\infty) \end{bmatrix} - \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Steady-State Servo Control System

$$\begin{bmatrix} x_e(k+1) \\ V_e(k+1) \end{bmatrix} = \begin{bmatrix} G - HK & HKI \\ -CG + CHK & 1 - CHKI \end{bmatrix} \begin{bmatrix} x_e(k) \\ V_e(k) \end{bmatrix}$$

$$\begin{bmatrix} x_e(k+1) \\ V_e(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x_e(k) \\ V_e(k) \end{bmatrix} + \begin{bmatrix} H \\ -CH \end{bmatrix} \begin{bmatrix} -K & KI \end{bmatrix} \begin{bmatrix} x_e(k) \\ V_e(k) \end{bmatrix}$$

Let $W(k) = \begin{bmatrix} -K & KI \end{bmatrix} \begin{bmatrix} x_e(k) \\ V_e(k) \end{bmatrix}$

The above equation becomes:

$$\begin{bmatrix} x_e(k+1) \\ V_e(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x_e(k) \\ V_e(k) \end{bmatrix} + \begin{bmatrix} H \\ -CH \end{bmatrix} W(k) \qquad (10)$$

$$W(k) = -Kx_e(k) + KIV_e(k)$$

Using equation (4):

$$u(k) = -Kx(k) + KIV(k)$$

$$u_e(k) = -Kx_e(k) + KIV_e(k) \qquad (11)$$

Steady-State Servo Control System

$$W(k) = u_e(k) \tag{12}$$

Based on equation (11), equation (10) becomes:

$$\begin{bmatrix} x_e(k+1) \\ V_e(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x_e(k) \\ V_e(k) \end{bmatrix} + \begin{bmatrix} H \\ -CH \end{bmatrix} u_e(k) \tag{13}$$

Let

$$x_1(k) = x_e(k)$$
$$x_2(k) = V_e(k)$$

Based on the above assumption, equation (11) becomes:

$$u_e(k) = -Kx_1(k) + KIx_2(k)$$

$$u_e(k) = W_k = [-K \quad KI] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$
$$u_e(k) = W_k = -\mathbb{K}x(k)$$

Steady-State Servo Control System

$$\mathbb{K} = [K \quad -KI]$$

And equation (13) becomes:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} H \\ -CH \end{bmatrix} W(k) \qquad (13)$$

$$x(k+1) = \mathbb{G}x(k) + \mathbb{H}W(k)$$

$$W_k = -\mathbb{K}x(k)$$

$$\mathbb{G} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix}, \mathbb{H} = \begin{bmatrix} H \\ -CH \end{bmatrix}$$

The performance index of the servo system is as follows:

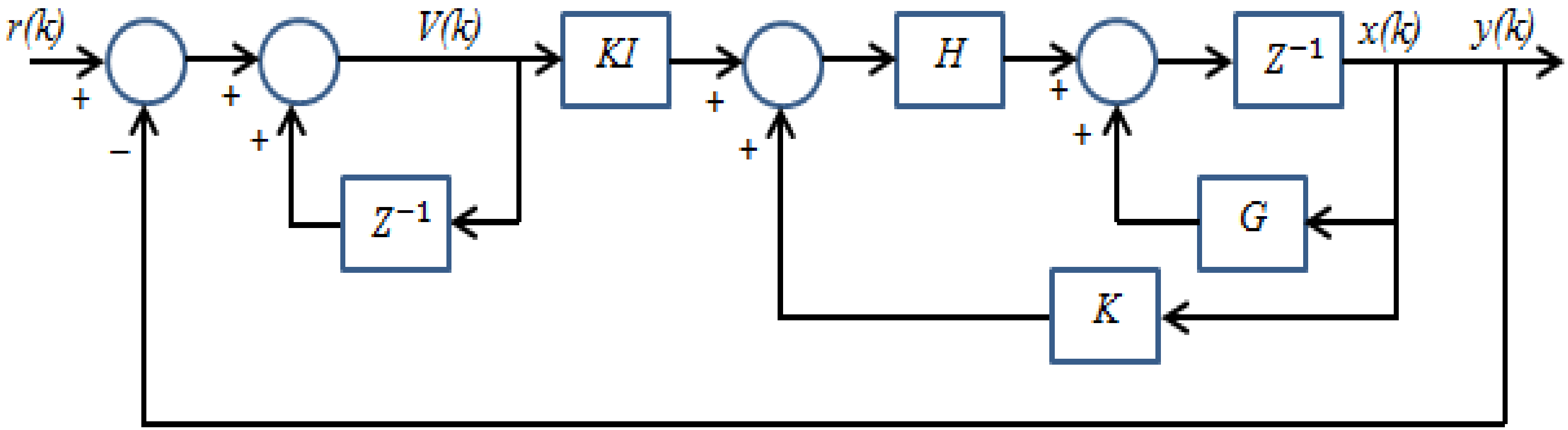
$$J = \frac{1}{2}\sum_{k=0}^{\infty}[X^T(k)QX(k) + W^T(k)RW(k)]$$

Steady-State Servo Control System

Example: Consider a servo system with the following state and output equations.

$$x(k+1) = 0.5\,x(k) + 2\,u(k)$$
$$y(k) = x(k)$$

An integral **controller** is included to the system as shown in the following figure:



Steady-State Servo Control System

If the weighting matrices of the LQR controller are given by:

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1;$$

Determine integral gain (KI) and feedback gain (K) so that the system is stable and will exhibit an acceptable transient response to the unit step input if the sampling time T = 0.1 sec.

Solution:

$$x(k+1) = 0.5\, x(k) + 2\, u(k)$$
$$y(k) = x(k)$$

$G = 0.5, \quad H = 2, C = 1$

The state equation of the system with integral action is given by:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} H \\ -CH \end{bmatrix} W(k)$$

$$x(k+1) = \hat{G}x(k) + \hat{H}W(k)$$

Steady-State Servo Control System

$$\text{Ǥ} = \begin{bmatrix} G & 0 \\ -CG & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1 \end{bmatrix}, \qquad \text{Ƕ} = \begin{bmatrix} H \\ -CH \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = Q + G^T P [I + HR^{-1} H^T P]^{-1} G$$

Begin the solution with matrix $P = 0$ and iterate the above Riccati equation until a stationary solution (steady state P matrix) is obtained.

After many iteration, the steady state P matrix is given by:

$$P = \begin{bmatrix} 100 & -0.0119 \\ -0.0119 & 10.5168 \end{bmatrix}$$

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Steady-State Servo Control System

Using the above steady state P matrix, the feedback control is calculated as follows:

$$\mathbb{K} = (R + H^T P H)^{-1} H^T P G$$

$$\mathbb{K} = [K \quad -KI]$$

$$\mathbb{K} = [0.2494 \quad -0.0475]$$

The feedback gain matrix $K = 0.2494$

and the integral gain $KI = 0.0475.$