Digital Control Systems

LECTURE 1
INTRODUCTION

Prepared by: Mr. Abdullah I. Abdullah

Course Description

This course introduces fundamental concepts in the **theory, analysis and design of discrete control systems**.

يقدم هذا المقرر المفاهيم الأساسية في نظرية وتحليل وتصميم أنظمة التحكم المنفصلة.

Course Objectives:

Knowledge and understanding (المعرفة والفهم)

- **Model** and analyze discrete control systems نمذجة وتحليل أنظمة التحكم المنفصلة
- Evaluate the performance of discrete control systems تقييم أداء أنظمة التحكم المنفصلة

Professional and practical skills (المهارات المهنية والعملية)

- **Design and simulate** industrial and practical systems تصميم ومحاكاة الأنظمة الصناعية والعملية
- Improve performances of discrete control systems تحسين أداء أنظمة التحكم المنفصلة

General and transferable skills (المهارات العامة والقابلة للتحويل)

- Understand the requirements and operations of discrete control systems (فهم متطلبات وعمليات أنظمة التحكم المنفصلة)
- Design and tuning techniques for performance improvement تقنيات التصميم والضبط لتحسين الأداء

Syllabus:

UNIT – I:

Introduction

Introduction to analog and digital control systems – Advantages of digital systems – Typical examples – why digital control- A/D converter and D/A converter–Sampling theorem.

UNIT–II:

Z–transformations

Z–Transforms – Theorems – Finding inverse z–transforms – Formulation of difference equations and solving.

UNIT–III:

Block diagram representation – Pulse transfer functions and finding open loop and closed loop responses -Zero Order Hold transfer function –Time response .

UNIT – IV

System Response Characteristics: Time Domain Specifications; Mapping s-domain to z-domain -Primary strips and Complementary Strips

UNIT – V:

Stability analysis: Factorization Method -Modified Routh's stability criterion and jury's stability test- steady state error.

UNIT – VI:

Design of sampled data control systems

Root locus technique in the z–plane- Controller design using root locus-Root locus based controller design using MATLAB .

Text Book:

M. Sami **Fadali** and A. Visioli , *Digital Control Engineering Analysis and Design* (2*nd* ed.), Elsevier, 2012.

References:

1-G F **Franklin**, J D Powell & M Workman , *Digital Control of Dynamic Systems* (3*rd* ed.),Addison Wesley, 1998.

2-R C **Dorf** & R H Bishop, *Modern Control Systems*, Pearson Prentice Hall, 2008.

3-Anastasia Veloni Nikolaos I. Miridakis, *Digital Control Systems Theoretical Problems and Simulation Tools* by Taylor & Francis Group, LLC, 2018.

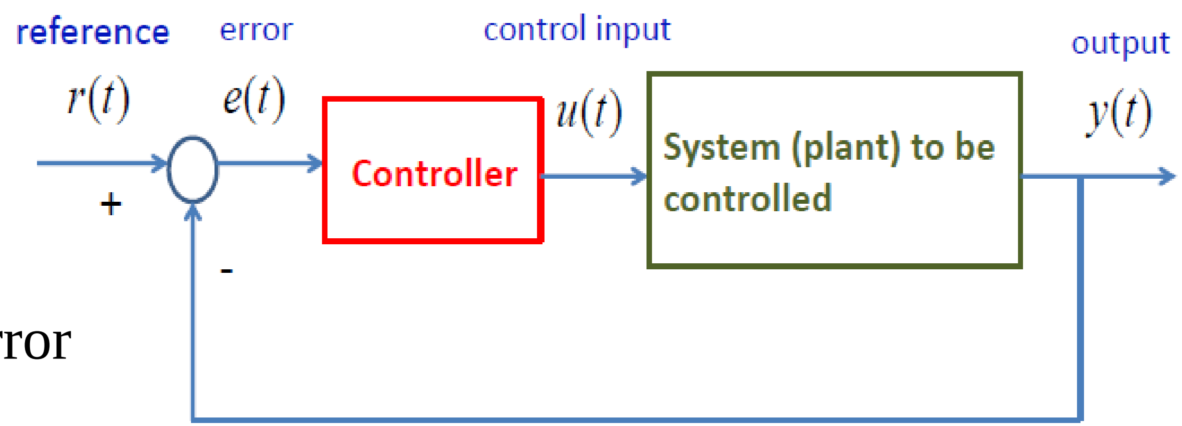
4- J.D'**Azzo** and C.H.**Houpis** , *Linear Control Systems Analysis and Design .*

5- C. **Kuo**, *Automatic Control Systems* , by Benjamin Seventh edition

Classical Control Systems

mechanical, optical, or electronic device, or set of devices, that manages, commands, directs or regulates the behavior of other devices or systems **to maintain a desired output**.

In Feedback Control Systems, we learned how to make an analog controller D(s) to control a linear-time-invariant (LTI) plant G(s).



Objective:

1-Small steady-state error
2- Closed-loop stable
3- Good transient response
4- Disturbance rejection

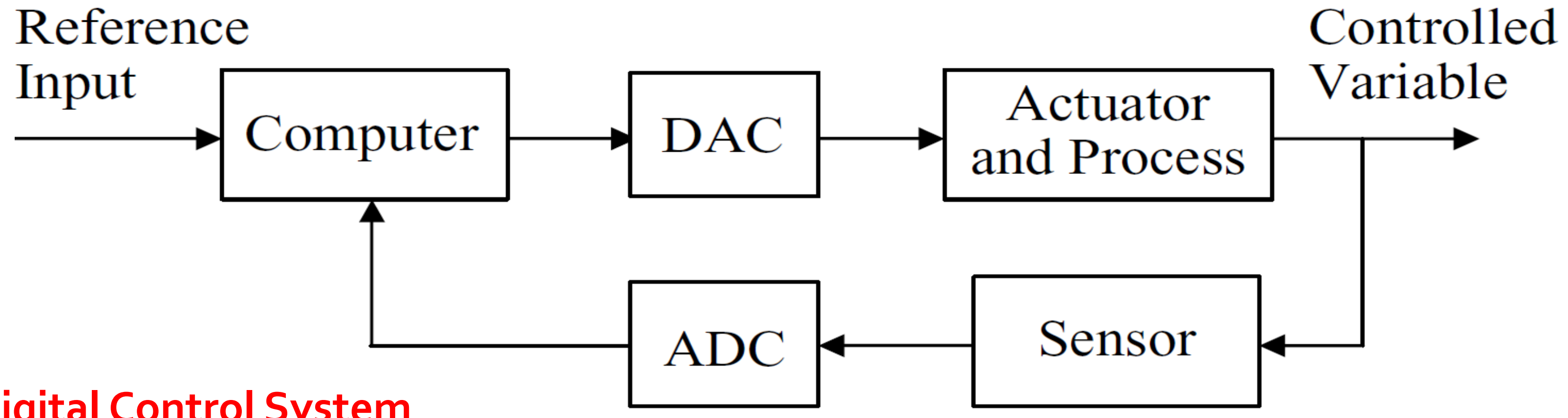
- Analog controllers difficult to modify or redesign once implemented in hardware.

Digital control

A digital control system model can be viewed from different perspectives including control algorithm, computer program, conversion between analog and digital domains, system performance etc. One of the most important aspects is the sampling process level.

يمكن عرض نموذج نظام التحكم الرقمي من وجهات نظر مختلفة بما في ذلك خوارزمية التحكم ، وبرنامج الكمبيوتر ، والتحويل بين المجالات التناظرية والرقمية ، وأداء النظام ، إلخ. أحد أهم الجوانب هو مستوى عملية أخذ العينات.

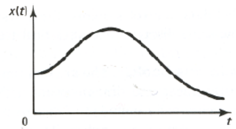
A digital control system consists of an A/D conversion for converting analog input to digital format for the machine, D/A conversion for converting digital output to a form that can be the input for a plant, and a digital controller in the form of a computer, **microcontroller** or a **microprocessor**. Such devices are light, fast and economical.



Digital Control System

- The difference between the continuous and digital systems is that the digital system operates on samples of the sensed plant rather than the continuous signal and that the control provided by the digital controller D(s) must be generated by algebraic equations.
- In this regard, we will consider the action of the analog-to-digital (A/D) converter on the signal. This device samples a physical signal, mostly voltage, and convert it to binary number that usually consists of 10 to 16 bits.
- Conversion from the analog signal $y(t)$ to the samples $y(kt)$, occurs repeatedly at instants of time T seconds apart.
- A system having both discrete and continuous signals is called sampled data system.
- The sample rate required depends on the closed-loop bandwidth of the system. Generally, sample rates should be about 20 times the bandwidth or faster in order to assure that the digital controller will match the performance of the continuous controller

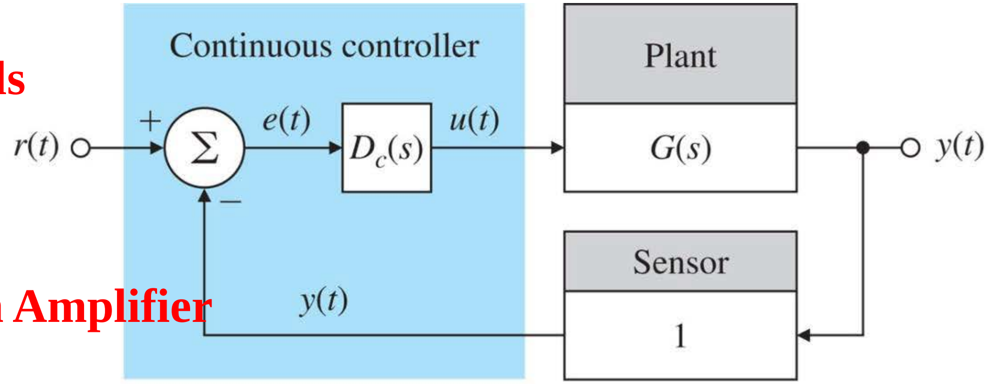
Continuous vs Discrete Control



1-Continuous-time signals

2-Analog signals

3- Controller :Operation Amplifier



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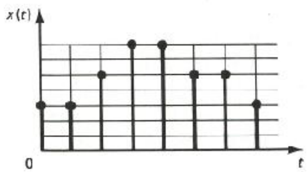
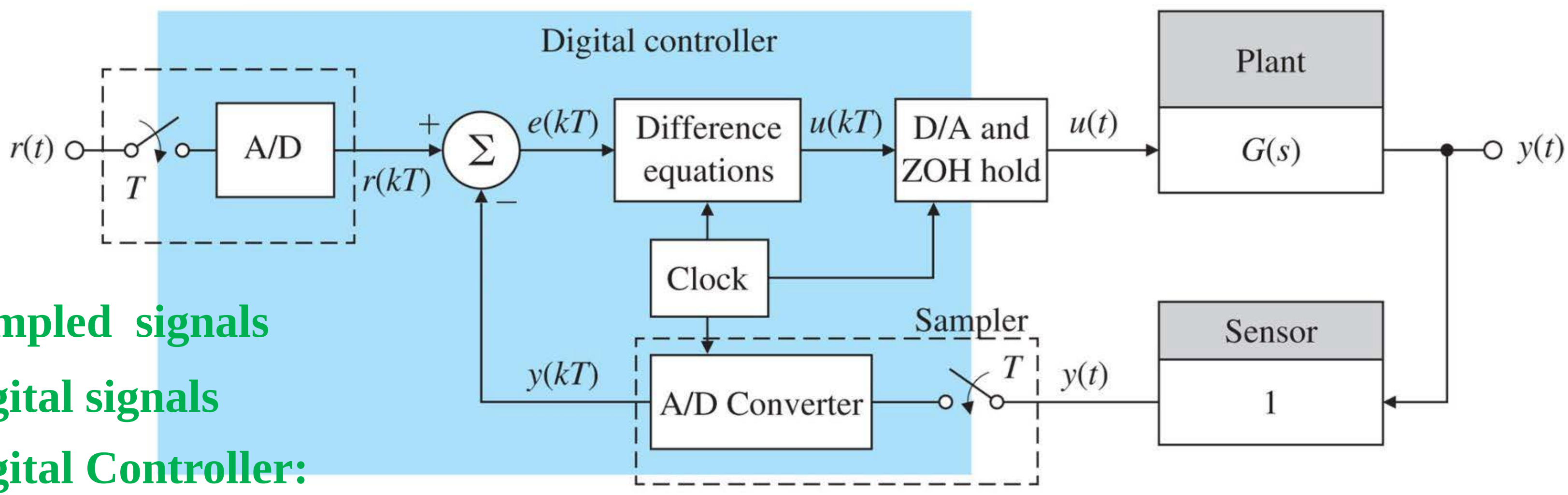




1-Sampled signals

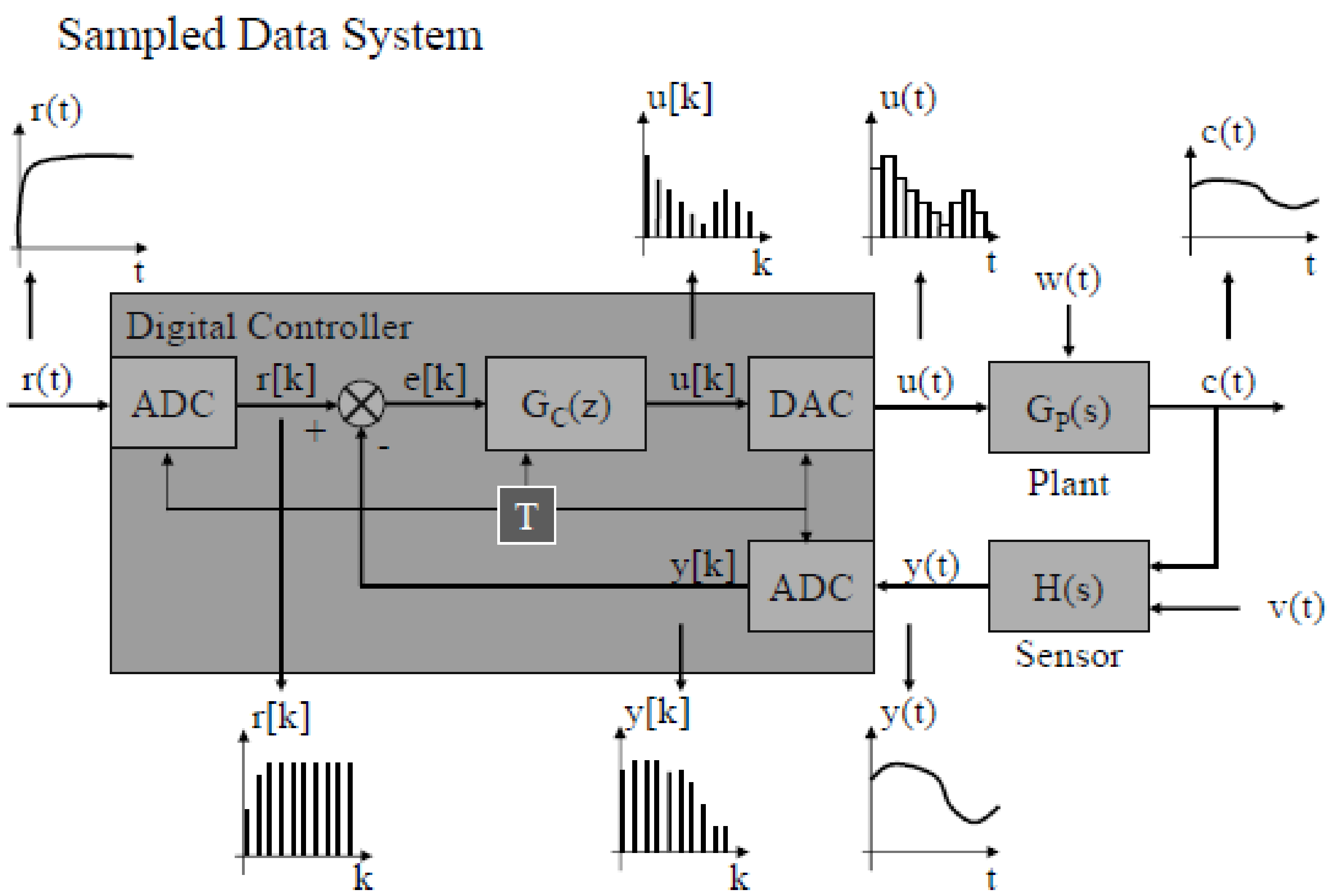
2-Digital signals

3-Digital Controller:

microcontroller, or microprocessors

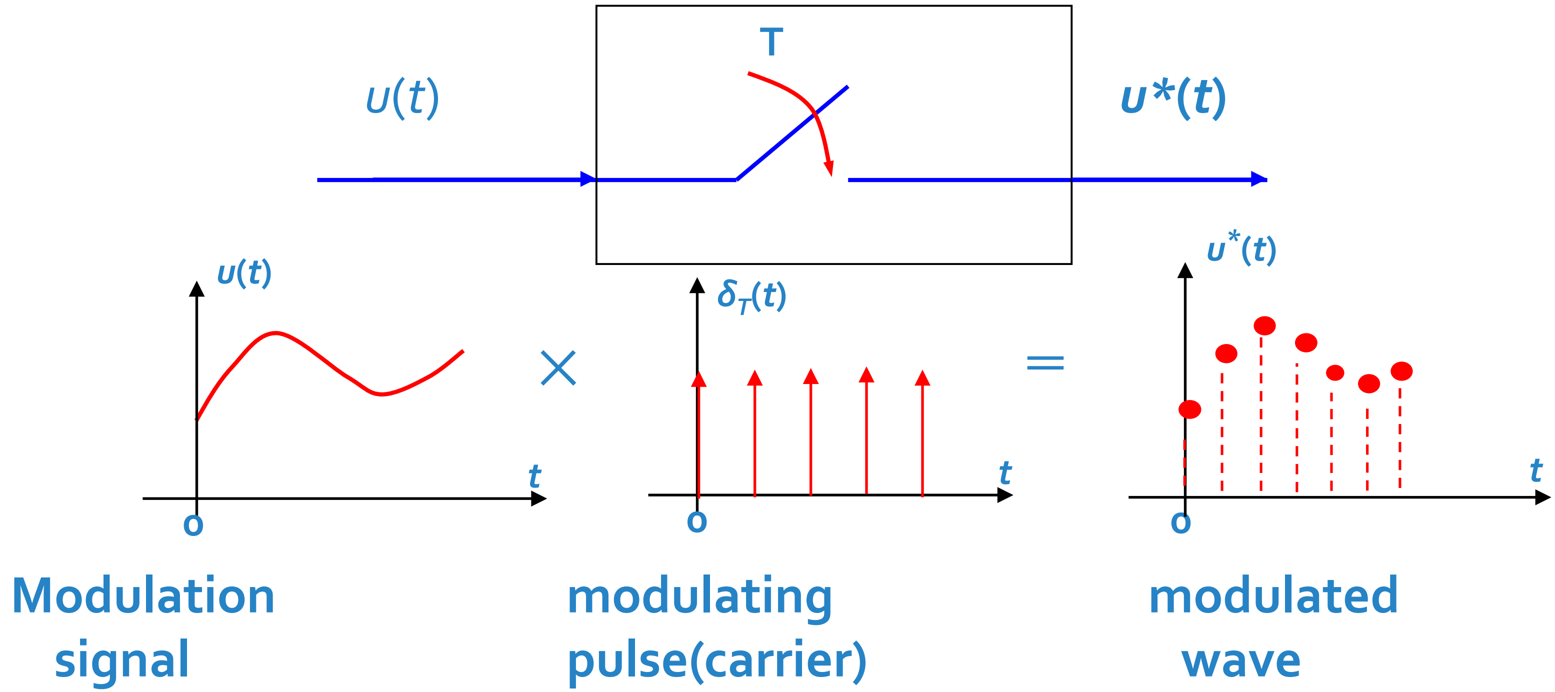
(b)

Sampled Data System



(ADC Model)

Samples analog signal (typically a voltage) and then converts these samples into an integer number (**quantization**) suitable for processing by digital computer

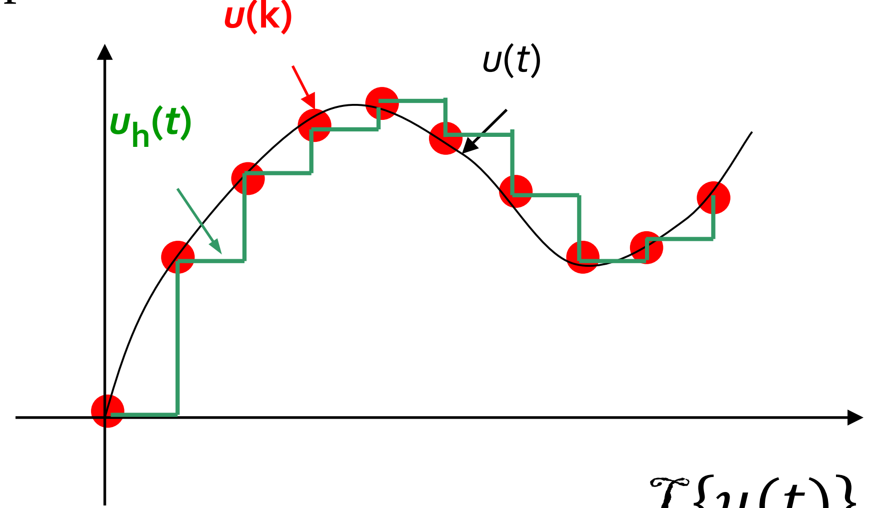


$$u^*(t) = \sum_{k=0}^{\infty} u(t)\delta(t-kT)$$

It converts a voltage level into a corresponding (binary) number representation at a particular instant of time.

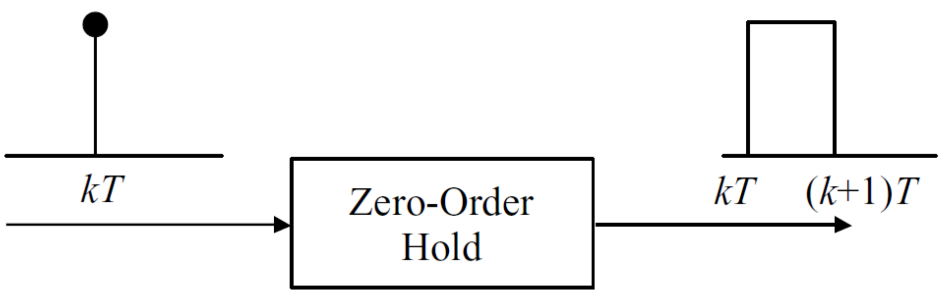
DAC Model

Converts the digital (integer) number calculated by the computer into a voltage so as to drive the output of the plant as desired.

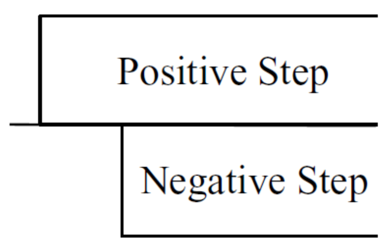


$$u(k) \xrightarrow{ZOH} u_h(t), \quad kT \le t \le (k+1)T$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$



$$\mathcal{L}\{-u(t-T)\} = -\frac{e^{-Ts}}{s}$$



$$G_{ZOH}(s) = \frac{1 - e^{-Ts}}{s}$$

ADVANTAGES

- Digital control offers distinct advantages over analog control that explain its popularity.
- **Accuracy:** Digital signals are more accurate than their analogue counterparts.
- **Implementation Errors:** Implementation errors are negligible.
- **Flexibility:** Modification of a digital controller is possible without complete replacement.
- **Speed:** Digital computers may yield superior performance at very fast speeds
- **Cost:** Digital controllers are more economical than analogue controllers.

DISADVANTAGES

- Sampling and quantization process will degrade system performance
ستؤدي عملية أخذ العينات والكمية إلى تدهور أداء النظام

- Software errors أخطاء البرامج

- Lose information during conversions due to technical problems.
تفقد المعلومات أثناء التحويلات بسبب مشاكل فنية.

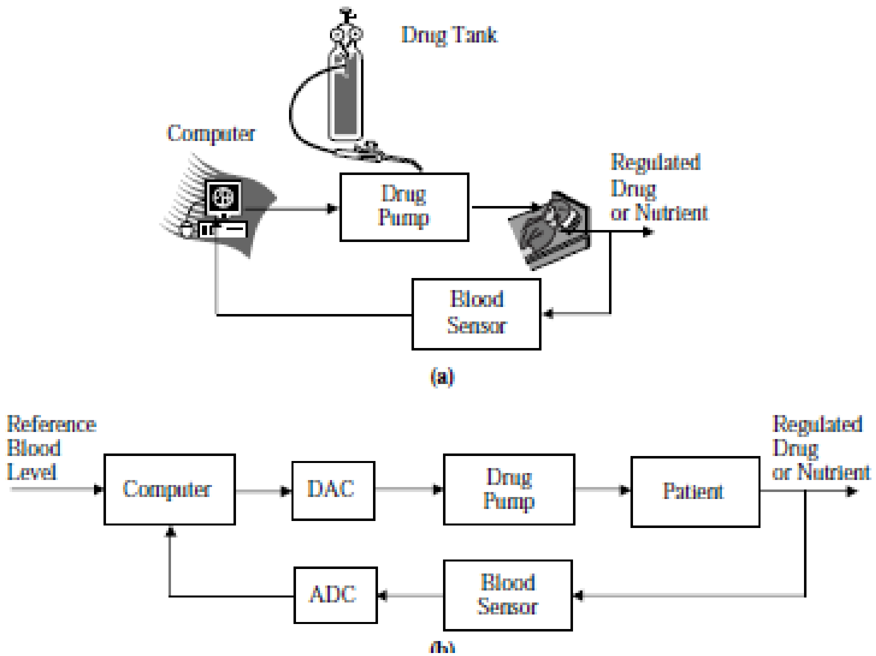
- From the tracking performance side, the analog control system exhibits good performances than digital control system.

- Digital control system will introduce a delay in the loop.

APPLICATIONS

1-Closed-loop drug delivery system

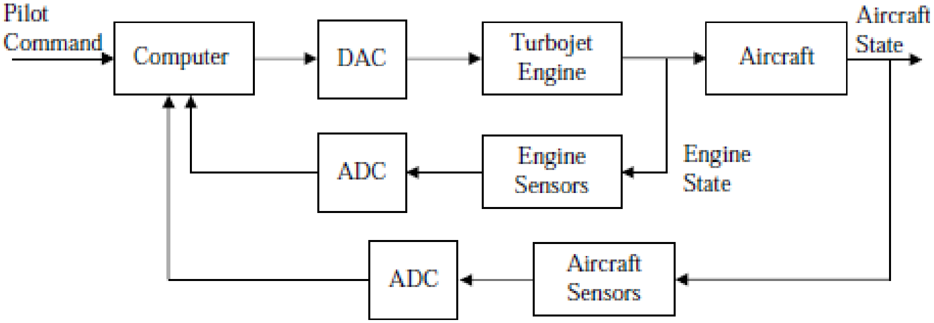
Several chronic diseases require the regulation of the patient's blood levels of a specific drug or hormone. For example, some diseases involve the failure of the body's natural closed-loop control of blood levels of nutrients. Most prominent among these is the disease diabetes, where the production of the hormone insulin that controls blood glucose levels is impaired



2-Computer control of an aircraft turbojet engine

To achieve the high performance required for today's aircraft, turbojet engines employ sophisticated computer control strategies

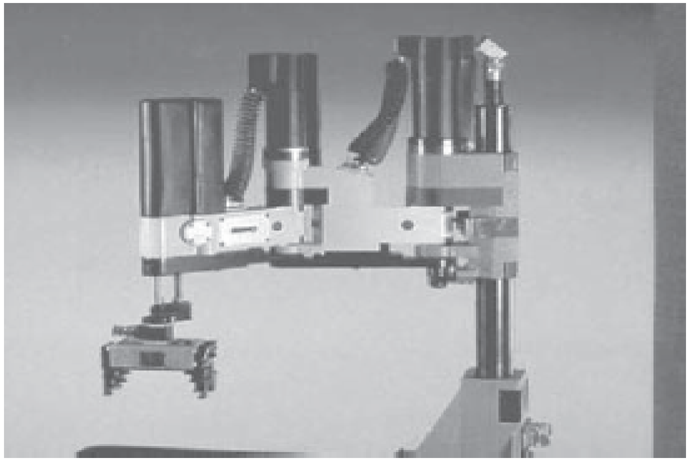
(a)



(b)

3-Control of a robotic manipulator

Robotic manipulators are capable of performing repetitive tasks at speeds and accuracies that far exceed those of human operators. They are now widely used in manufacturing processes such as spot welding and painting.



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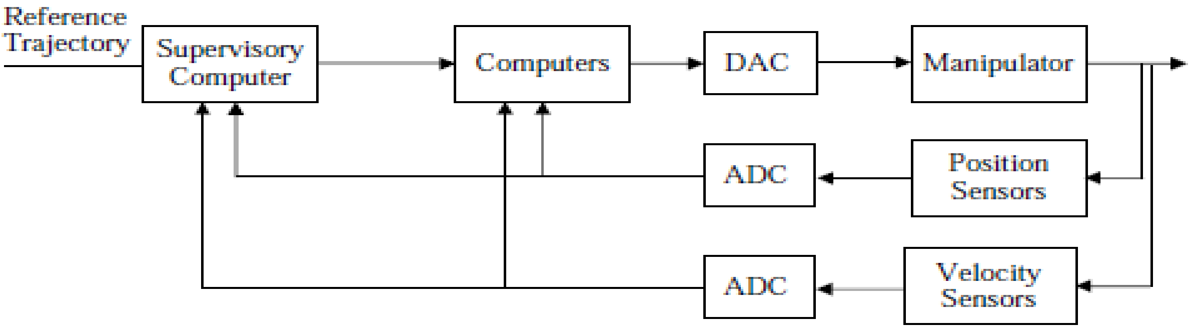

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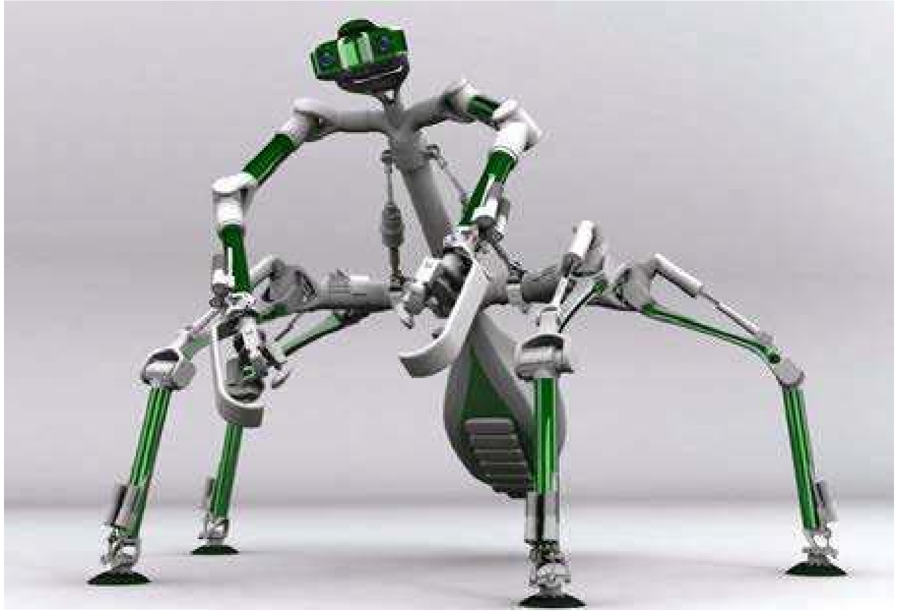

Digital Control Systems

LECTURE 2

The z Transform

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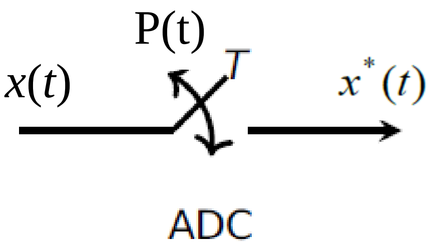
Sampled-Data Systems

Ideal sampling

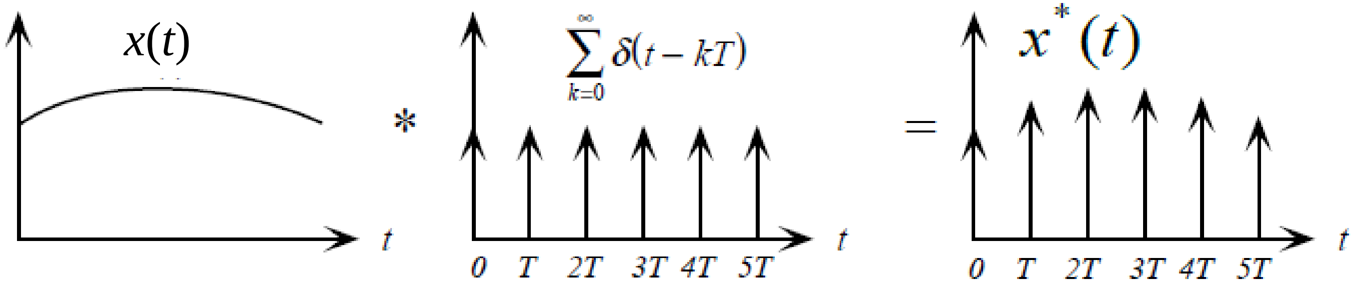
- Ideal sampling of a continuous signal can be considered as a multiplication of the signal, $x(t)$, with an impulse train $P(t)$

- an impulse train P(t) is defined as: $P(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$



Thus, the sampled signal $x^*(t)$ is:

$$x^*(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT) \quad \ldots\ldots(1)$$



Where is a unit impulse at $\delta(t)$ at t=0 and $\delta(t - nT)$ is a unit impulse at t= nT.

The Laplace transform of the sampled signal is

$$X^*(s) = \mathcal{L}\left[x^*(t)\right] = x^*(0) + x^*(T)e^{-Ts} + x^*(2T)e^{-2Ts} + \ldots$$

$$= \sum_{n=0}^{\infty} x(nT)e^{-nTs} \qquad \ldots..(2)$$

The Z Transform

The simple substitution $z = e^{Ts}$

Convert the Laplace transform to the z transform. Making this substitution in Eq(2) gives

$$z[x^*(t)] = X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n} \qquad \ldots..(3)$$

(power series representation of discrete – time sequence)

Where X(z) designates the z transform of x*(t).Because only values of the signal at the sampling instant are considered, the z transform of x(t) is the same as that of x*(t).

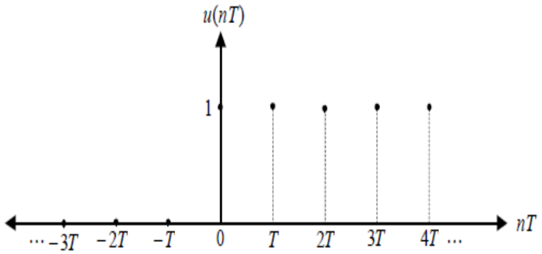
1. Z Transform by Definition

Ex. 1: Determine the z transform for a unit step function $u(nT) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

Sol: for this function u(nT) = 1 for n= 1,2,3,…., thus application of Eq (3) gives

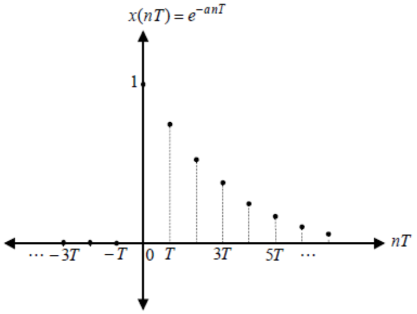
$$X(z) = \sum_{n=0}^{\infty} x(nT) z^{-n}$$

$$Z[u^*] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots = \frac{z}{z-1}$$



Ex2: Determine the z transform of the exponential e^{-at}, thus

Sol: for this function $x(nT) = e^{-anT}$



$$Z(e^{-at}) = 1 + \frac{e^{-aT}}{z} + \frac{e^{-2aT}}{z^2} + \frac{e^{-3aT}}{z^3} + \cdots = \frac{z}{z - e^{-at}}$$

Ex.3: obtain the z transform of x(t) where $X(t) = \sin \omega t$ for $t > 0$

since $Z[e^{-at}] = \dfrac{z}{z-e^{-aT}}$

We obtain $Z[\sin \omega t] = Z\left[\dfrac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]$

$$= \frac{1}{2j}\left[\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}}\right]$$

$$= \frac{1}{2j}\,\frac{z\left(e^{j\omega T} - e^{-j\omega T}\right)}{z^2 - z\left(e^{j\omega T} + e^{-j\omega T}\right) + 1}$$

$$= \frac{z\ \sin \omega T}{z^2 - 2z\cos \omega T + 1}$$

Ex .4: Find z transform of $x(t) = a^t$, where a is a constant.

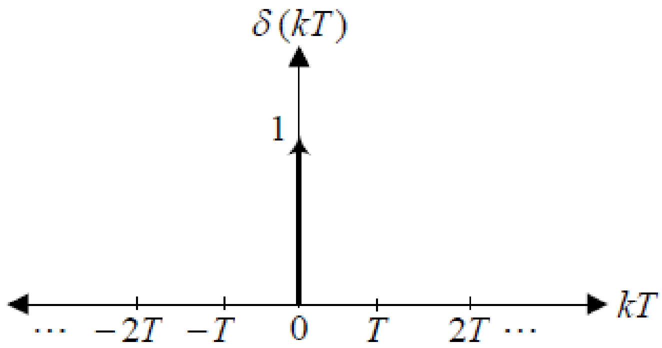
$$X(z) = Z[a^{nT}] = \sum_{n=0}^{\infty} x(nT) z^{-n} = \sum_{n=0}^{\infty} a^{nT} z^{-n}$$

$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ \dots = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Ex.5: Find z transform of unit impulse function is defined as

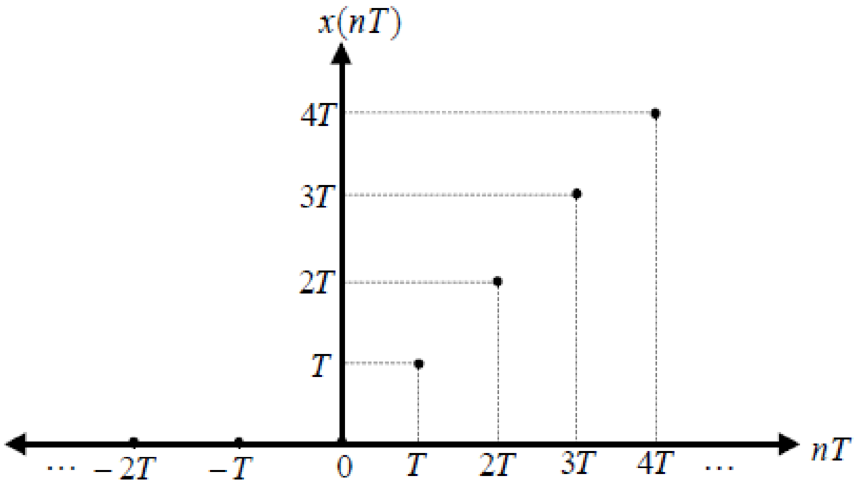
$$Z\{\delta(nT)\} = \sum_{n=0}^{\infty} u_1(nT)\, z^{-n} = z^0 = 1$$

$$\delta(nT) = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases}$$



Ex.6: Find z transform of unit ramp function is defined as $x(nT)=\begin{cases} nT & n\ge 0 \\ 0 & n<0 \end{cases}$

$$X(z)=\sum_{n=0}^{\infty} nT\,z^{-n}=T\sum_{n=0}^{\infty} n\,z^{-n}$$



since $n\,z^{-n}=-z\,\dfrac{d}{dz}\left(z^{-n}\right)$, then

$$X(z)=-Tz\,\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^{-n}\right)$$

since $\displaystyle\sum_{n=0}^{\infty} z^{-n}=\frac{z}{(z-1)}$, then $\displaystyle X(z)=\frac{zT}{(z-1)^2}$

Table (1) z transforms

Time function	Laplace Transform	Discrete Time function	Z transform
$\delta(t)$	1	$\delta(nT)$	1
$u(t)$	$\frac{1}{s}$	$u(nT)$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	nT	$\frac{zT}{(z-1)^2}$
$\frac{t^2}{2}$	$\frac{1}{s^3}$	$\frac{(nT)^2}{2}$	$\frac{z(z+1)T^2}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	e^{-anT}	$\frac{z}{z-e^{-aT}}$
$t\,e^{-at}$	$\frac{1}{(s+a)^2}$	$nT\,e^{-anT}$	$\frac{zT\,e^{-aT}}{(z-e^{-aT})^2}$
$a^{t/T}$	$\frac{1}{s-(1/T)\ln(a)}$	a^n	$\frac{z}{z-a} \quad (a>0)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega nT)$	$\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\cos(\omega nT)$	$\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$

2-Z Transform Using Partial Fraction

When the Laplace transform of a function is known, the corresponding z transform may be obtained by the partial fraction

Ex.1 : Determine the z transform for the function whose Laplace transform is

$$F(s) = \frac{1}{s\ (s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

From Table (1), the z transform corresponding to $1/s$ *is* $z/z-1$

and that corresponding to $1/s+1$ is $z/z-e^{-T}$.

$$F(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{z\ (1-e^{-T})}{(z-1)(z-e^{-T})}$$

Ex.2: Determine the z transform of $\cos(\omega t)$.

$$\mathcal{L}\{\cos(\omega t)\}\ \frac{s}{s^2+\omega^2} = \frac{A}{(s+jw)} + \frac{B}{(s-jw)} = \frac{1/2}{(s+jw)} + \frac{1/2}{(s-jw)}$$

$$Z(\cos wt) = F(z) = \frac{1}{2}\frac{z}{z-e^{-jwT}} + \frac{1}{2}\frac{z}{z-e^{jwT}}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{-jwT}} + \frac{z}{z-e^{jwT}}\right]$$

$$= \frac{z^2 - z\cos(wT)}{z^2 - 2z\cos(wT) + 1}$$

3.Z Transform Using Residue Method

This is a powerful technique for obtaining z transforms. The z transform of $f^*(t)$ may be expressed in the form

$$F(z) = Z\,[f^*(t)] = \sum residues\ of\ F(s)\frac{z}{z-e^{sT}} \qquad at\ poles\ of\ F(s)$$

When the denominator of $F(s)$ contains a linear factor of the form s-r such that $F(s)$ has a first-order pole at $s = r$, the corresponding residue R is

$$R = \lim_{s\to r}\,(s-r)\left[F(s)\,\frac{z}{z-e^{sT}}\right]$$

When $F(s)$ contains a repeated pole of order q , the residue R is

$$R = \frac{1}{(q-1)!}\lim_{s\to r}\frac{d^{\,q-1}}{ds^{\,q-1}}\left[(s-r)^q\;F(s)\;\frac{z}{z-e^{sT}}\right]$$

Ex 1: Determine the z transform of a unit step function u(t).

For F(s)=1/s , there is but one pole at s=0. The corresponding residue is

$$R = \underbrace{\lim}_{s \to r} (s-r)\left[F(s)\frac{z}{z-e^{sT}}\right]$$

Simple Pole at s=0

$$R = \underbrace{\lim}_{s \to 0} s\left[\frac{1}{s}\frac{z}{z-e^{sT}}\right] = \frac{z}{z-1}$$

Ex 2: Determine the z transform of e^{-at}.

For this function $F(s) = 1/(s+a)$, which has but one pole at s=-a. Thus,

$$R = \underbrace{\lim}_{s \to -a} (s+a)\left[\frac{1}{(s+a)}\frac{z}{z-e^{sT}}\right] = \frac{z}{z-e^{-aT}}$$

Ex.3 : Determine the z transform of for the function whose Laplace transform is $F(s) = \frac{1}{s\ (s+1)}$

The poles of $F(s)$ occur at s=0 and s=-1.

The residue due to the pole at s=0 is

$$R_1 = \lim_{s \to 0} s \left[\frac{1}{s\ (s+1)} \frac{z}{z - e^{sT}} \right] = \frac{z}{z-1}$$

The residue due to the pole at s=-1 is

$$R_2 = \lim_{s \to -1} (s+1) \left[\frac{1}{s(s+1)} \frac{z}{z - e^{sT}} \right] = -\frac{z}{z - e^{-T}}$$

Adding these two residues results in

$$R = \sum_{i=1}^{2} R_i = R_1 + R_2 = \frac{z}{z-1} - \frac{z}{z - e^{-T}} = \frac{z\ (1 - e^{-T})}{(z-1)(z - e^{-T})}$$

Ex.4: Determine the z transform of cos(*w t*) .

The Laplace transform is

$$F(s)=\frac{s}{s^2+\omega^2}=\frac{s}{(s-j\omega)(s+j\omega)}$$

The poles are at $s = jw$ and $s = -jw$. Thus,

$$R_1=\lim_{s\to jw}(s-jw)\left[\frac{s}{(s-jw)(s+jw)}\frac{z}{z-e^{sT}}\right]=\frac{1}{2}\frac{z}{z-e^{jwT}}$$

$$R_2=\lim_{s\to -jw}(s+jw)\left[\frac{s}{(s-jw)(s+jw)}\frac{z}{z-e^{sT}}\right]=\frac{1}{2}\frac{z}{z-e^{-jwT}}$$

Adding these verify the previous result

$$R=\sum_{i=1}^{2}R_i=R_1+R_2=\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$$

Ex.5: Determine the z transform corresponding to the function $f(t) = t$.

The Laplace transform is

$$F(s) = \frac{1}{s^2}$$

This has a second-order pole at s=0. Thus, the residue becomes

$$R = \frac{1}{(q-1)!} \lim_{s \to r} \frac{d^{q-1}}{ds^{q-1}} \left[(s-r)^q \; F(s) \; \frac{z}{(z-e^{sT})} \right]$$

$$R = \frac{1}{(2-1)!} \lim_{s \to 0} \frac{d^{2-1}}{ds^{2-1}} \left[(s-r)^2 \; F(s) \; \frac{z}{z-e^{sT}} \right]$$

$$R = \frac{1}{1!} \lim_{s \to 0} \frac{d}{ds} \left[s^2 \; \frac{1}{s^2} \; \frac{z}{(z-e^{sT})} \right] = \frac{zT}{(z-1)^2}$$

Digital Control Systems

LECTURE 3

Properties of z Transform

Prepared by: Mr. Abdullah I. Abdullah

Properties of z-Transform

1- Linearity of z-transform

$$\mathcal{Z}\{\alpha f_1(n) + \beta f_2(n)\} = \alpha F_1(z) + \beta F_2(z)$$

Example: Find the z-transform for the signal $x(nT) = 3*2^n - 4*3^n$

$$Zx(nT) = 3Z2^n - 4Z3^n = \frac{3z}{z-2} - \frac{4z}{z-3}$$

2- Time Delay:

$$\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$$

Example: If $f(k) = \begin{cases} 2^{k-1} & k > 0, \\ 0 & k \leq 0 \end{cases}$ **Find F(z)**

$$F(z) = z^{-1}\frac{z}{z-2}$$

3-Time Advance:

Shifting n sampling period

$$\mathcal{Z}\{f(k+n)\} = z^n F(z) - z^n f(0) - z^{n-1} f(1) - \cdots - z f(n-1)$$

$$Z\left[f(kT+T)\right] = z\,F(z) - zf(0)$$

Shifting one sampling period

$$Z\left[f(kT+2T)\right] = z^2\,F(z) - z^2\,f(0) - zf(1)$$

Shifting two sampling period

4.Multiplication by k

$$\mathscr{Z}\{k^m f(k)\} = \left(-z\frac{d}{dz}\right)^m F(z)$$

Example : Find G(z) for $g(k) = n\,2^n$

$$f(n) = 2^n \Leftrightarrow F(z) = \frac{z}{z-2}$$

$$G(z) = -z\frac{d}{dz}F(z) = \frac{2z}{(z-2)^2}.$$

5-Final value of the time response:

$$f(\infty) = \lim_{n\to\infty} f(n) = \lim_{z\to 1}(1 - z^{-1})\,F(z)$$

Example

Find the final value of g(n), if $G(z) = \dfrac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}$,

$$g_\infty = \lim_{n\to\infty} g(n) = \lim_{z\to 1}(1 - z^{-1})G(z)$$

$$= \lim_{z\to 1}(1 - z^{-1})\frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)},$$

$$= \lim_{z\to 1}\frac{0.792}{(z^2 - 0.416z + 0.208)} = 1.$$

6-Initial value theorem

Suppose $f(nT)$ has z transform $F(z)$ and $\lim_{z\to\infty} F(z)$ exist, then the initial value $f(0)$ of $f(nT)$ is given by

$$f(0)=\lim_{z\to\infty} F(z)$$

Example 1: *If* $F(z)=\dfrac{z}{z-1}$ *Find* $f(0)$ ⇒ $f(0)=\lim_{z\to\infty} F(z)=\lim_{z\to\infty}\dfrac{z}{z-1}=\lim_{z\to\infty}\dfrac{z}{z(1-1/z)}=1$

Example 2: For a discrete data system with transfer function

$$H(z)=\frac{Y(z)}{U(z)}=\frac{z+1}{z^2-1.4z+0.48}$$

and a unit step input for which the z transform is

find the final value of the response sequence $y(nT)$.

$$Y(z)=\frac{z\,(z+1)}{(z^2-1.4z+0.48)(z-1)} \Rightarrow y(\infty)=\lim_{z\to 1}\left[\frac{z-1}{z}\,Y(z)\right]=25$$

Properties of the z transforms

$f(nT)$	$Z[f(nT)]$
$a\ f(nT)$	$a\ F(z)$
$f_1(nT)+f_2(nT)$	$F_1(z)+F_2(z)$
$f(kT-nT)$	$z^{-n}\ F(z)$
$f(kT+T)$	$z\,F(z)-z\,f(0)$
$f(kT+2T)$	$z^2\ F(z)-z^2\ f(0)-z\ f(T)$
$f(kT+nT)$	$z^n\ F(z)-z^n\ f(0)-z^{n-1}\ f(T)-\cdots-z\ f[(n-1)T]$
$nT\ f(nT)$	$-zT\dfrac{d}{dz}(F(z))$
$e^{-a\,(nT)}\ f(nT)$	$F\left(z\,e^{aT}\right)$
$a^{(nT)}\ f(nT)$	$F(z/a)$
$\dfrac{\partial}{\partial a}\ f(nT,a)$	$\dfrac{\partial}{\partial z}\ F(z/a)$

Digital Control Systems

LECTURE 4

Inverse z-Transform

Prepared by: Mr. Abdullah I. Abdullah

Inverse z-Transform

Given the z-transform, Y (z), of a function, it is required to find the time-domain function y(n).
There are three methods: power series (long division) ,partial fractions and Residue Method .

1-power series: long division.

▲ This method involves dividing the denominator of Y (z) into the numerator to obtain a power series of the form:

$$Y(z) = y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3} + \cdots$$

▲ values of y(n) are, directly, the coefficients in the power series.

2-partial fractions:

◮ a partial fraction expansion of Y (z) is found, and then tables of z-transform can be used to determine the inverse z-transform.

3-Residue Method: The third method of finding the inverse z transform is to use the inversion integral.

Method 1: Power Series (long division)

Example 1:

use power series method to find the inverse z-transform for:

$$Y(z) = \frac{z^2 + z}{z^2 - 3z + 4}$$

$$1 + 4z^{-1} + 8z^{-2} + 8z^{-3}$$
$$z^2 - 3z + 4 \overline{\big) z^2 + z}$$
$$z^2 - 3z + 4$$
$$4z - 4$$
$$4z - 12 + 16z^{-1}$$
$$8 - 16z^{-1}$$
$$8 - 24z^{-1} + 32z^{-2}$$
$$8z^{-1} - 32z^{-2}$$
$$8z^{-1} \quad 24z^{-2} + 32z^{-3}$$

Dividing the denominator into the numerator gives:

from coefficients of power serie $y_k = \{1, 4, 8, 8, \cdots\}$

The required sequence:

$$y(t) = \delta(t) + 4\,\delta(t - T) + 8\,\delta(t - 2T) + 8\,\delta(t - 3T) + \cdots$$

Example-2: Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.2z + 0.1}$$

Long Division

$$\begin{array}{r|l} & z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \ldots\ldots \\ \hline z^2 + 0.2z + 0.1 & z + 1 \\ & z + 0.2 + 0.1z^{-1} \\ \cline{2-2} & 0.8 - 0.10z^{-1} \\ & 0.8 + 0.16z^{-1} + 0.08z^{-2} \\ \cline{2-2} & -0.26z^{-1} - \ldots \end{array}$$

Thus

$$F(z) = 0 + z^{-1} + 0.8z^{-2} - 0.26z^{-3} + \cdots$$

Inverse z-transform

$$f(n) = \{0, \quad 1, \quad 0.8, \quad -0.26, \quad \ldots\}$$

in MATLAB, you can use the following commands:

```
%%%%%%%%%%%%% Long Divition
method
clc
Delta =[1 zeros(1,4)] ;
num = [0    1    1];
den = [1    0.2    0.1];
yk = filter (num , den , Delta )


yk =


        0    1.0000    0.8000   -0.2600   -0.0280
```

disadvantage of power series method: it does not give a closed form of the resulting sequence.

Method 2: Partial Fractions

Looking at z-transform table,

- there is usually a z term in numerator.
- It is therefore more convenient to find the partial fractions of Y(z)/z
- then multiply the partial fractions by z to obtain a z term in the numerator.

Example 1 : Find the inverse z-transform of $Y(z)=\dfrac{z^2+3z-2}{(z+5)(z-0.8)(z-2)^2}$

$$\frac{Y(z)}{z}=\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}$$

$$=\frac{A}{z}+\frac{B}{z+5}+\frac{C}{z-0.8}+\frac{D}{(z-2)}+\frac{E}{(z-2)^2}$$

$$A=z\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\Bigg|_{z=0}=0.125,$$

$$B = (z+5)\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=-5} = 0.0056,$$

$$C = (z-0.8)\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0.8} = 0.16,$$

$$E = (z-2)^2\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=2} = 0.48,$$

$$D = \left[\frac{d}{dz}\frac{z^2+3z-2}{z(z+5)(z-0.8)}\right]\bigg|_{z=2}$$

$$= \frac{(2z+3)z(z+5)(z-0.8)-(z^2+3z-2)(3z^2+8.4z-4)}{[z(z+5)(z-0.8)]^2}\bigg|_{z=2} = -0.29$$

$$Y(z) = 0.125 + \frac{0.0056z}{z+5} + \frac{0.016z}{z-0.8} - \frac{0.29z}{(z-2)} + \frac{0.48z}{(z-2)^2}$$

$$y(n) = 0.125\,\delta(n) + 0.0056\,(-5)^n + 0.016\,(0.8)^n - 0.29\,(2)^n + 0.24\,n\,(2)^n$$

Note: for last term, we used the multiplication by k property which is equivalent to a z-differentiation.

Example-2: Obtain the inverse z-transform of the function

$$F(z) = \frac{z+1}{z^2 + 0.3z + 0.02}$$

Solution

$$\frac{F(z)}{z} = \frac{z+1}{z(z^2 + 0.3z + 0.02)}$$

$$\frac{F(z)}{z} = \frac{z+1}{z(z^2 + 0.1z + 0.2z + 0.02)}$$

$$\frac{F(z)}{z} = \frac{z+1}{z(z+0.1)(z+0.2)}$$

$$\frac{F(z)}{z} = \frac{A}{z} + \frac{B}{z+0.1} + \frac{C}{z+0.2}$$

$$A = z\frac{F(z)}{z}\bigg|_{z=0} = F(0) = \frac{1}{0.1\times 0.2} = \frac{1}{0.02} = 50$$

$$B = (z+0.1)\frac{F(z)}{z}\bigg|_{z=-0.1} = (z+0.1)\frac{1}{z}\frac{z+1}{(z+0.1)(z+0.2)}\bigg|_{z=-0.1} = \frac{-0.1+1}{(-0.1)(-0.1+0.2)} = -90$$

$$C = (z+0.2)\frac{F(z)}{z}\bigg|_{z=-0.2} = (z+0.2)\frac{1}{z}\frac{z+1}{(z+0.1)(z+0.2)}\bigg|_{z=-0.2} = \frac{-0.2+1}{(-0.2)(-0.2+0.1)} = 40$$

$$\frac{F(z)}{z} = \frac{50}{z} - \frac{90}{z+0.1} + \frac{40}{z+0.2}$$

$$F(z) = 50 - \frac{90z}{z+0.1} + \frac{40z}{z+0.2}$$

- Taking inverse z-transform (using z-transform table)

$$f(n) = 50\delta(n) - 90(-0.1)^n + 40(-0.2)^n$$

In MATLAB, you can find the partial fraction expansion of a ratio of two polynomials F(z) with:

Example : Find the inverse z-transform of $F(z) = \dfrac{2z^3 + z^2}{z^3 + z + 1}$

```
num = [2    1    0      0];
den  = [1    0    1      1];
[r,p,k] = residue (num ,den)
```

residue returns the complex roots and poles, and a constant term in k,

representing the partial fraction expansion

r =

0.5354 + 1.0390 i

0.5354 - 1.0390 i

-0.0708 + 0.0000 i

p =

0.3412 + 1.1615 i

0.3412 - 1.1615 i

-0.6823 + 0.0000 i

k =

2

$$F(z) = \frac{0.5354 + 1.0390i}{z - (0.3412 + 1.1615j)} + \frac{0.5354 - 1.0390i}{z - (0.3412 - 1.1615j)}$$

$$+ \frac{-0.0708}{z + 0.6823} + 2$$

Method 3: Residue Method:

The third method of finding the inverse z transform is to use the inversion integral. Note that

$$x(nT) = \frac{1}{2\pi j} \oint X(z)\ z^{k-1}\ dz$$

$$x(nT) = \sum [\mathit{residues\ of}\ X(z) z^{n-1}\ \mathit{at\ poles\ of}\ X(z)]$$

In particular, the residue due to a first order pole at $z = r$ is

$$R = \lim_{z \to r} (z - r)\ [F(z)\, z^{n-1}]$$

Similarly, the residue due to a repeated pole of order q is

$$R = \frac{1}{(q-1)!} \lim_{z \to r} \frac{d^{q-1}}{dz^{q-1}} [(z-r)^q\ F(z)\, z^{n-1}]$$

Example1 : Using residue method, find $f\,(nT)$ if $F(z)$ is given by

$$F(z)=\frac{(1-e^{-T})\,z}{(z-1)(z-e^{-T})}$$

$$R_1=\lim_{z\to -1}(z-1)\left[\frac{(1-e^{-T})\,z}{(z-1)(z-e^{-T})}z^{n-1}\right]=1$$

$$R_2=\lim_{z\to e^{-T}}(z-e^{-T})\left[\frac{(1-e^{-T})\,z}{(z-1)(z-e^{-T})}z^{n-1}\right]=-e^{-nT}$$

$$f(nT)=1-e^{-nT}\qquad\qquad n=0,1,2,3,\ \ldots$$

Example 2: Determine the inverse z transform for the function $F(z)=\dfrac{Tz}{(z-1)^2}$

$$R=\frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\ \frac{Tz}{(z-1)^2}z^{n-1}\right]=nT$$

For $f\,(nT)=nT$, the corresponding time function is $f\,(t)=t$.

z-Transform solution of difference equations

Linear difference equations may be solved by constructing the Z-Transform of both sides of the equation. The method will be illustrated with linear difference equations of the first and second orders (with constant coefficients).

Example 1: Solve the linear difference equation $u_{n+1} - 2u_n = (3)^{-n}$, given that $u_0 = 2/5$.

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

$$Z[f(kT+T)] = zF(z) - zf(0)$$

second shifting theorem

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z-\frac{1}{3}},$$

$$Z\{u_n\} = \frac{2}{5}.\frac{z}{z-2} + \frac{z}{\left(z-\frac{1}{3}\right)(z-2)} \equiv \frac{2}{5}.\frac{z}{z-2} + z.\left[\frac{-\frac{3}{5}}{z-\frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$\equiv \frac{z}{z-2} - \frac{3}{5}.\frac{z}{z-\frac{1}{3}}.$$

$$\{u_n\} \equiv \left\{(2)^n - \frac{3}{5}(3)^{-n}\right\}.$$

Example 2 : Solve the linear difference equation $u_{n+2} - 7u_{n+1} + 10u_n = 16n,$

given that $u_0 = 6$ and $u_1 = 2.$

second shifting theorem

$$Z\left[f(kT+T)\right] = z\,F(z) - zf(0)$$

$$Z\left[f(kT+2T)\right] = z^2\,F(z) - z^2 f(0) - zf(1)$$

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7\left[z.Z\{u_n\} - 6z\right] + 10Z\{u_n\} = \frac{16z}{(z-1)^2},$$

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2};$$

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

$$Z\{u_n\} = z.\left[\frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1}\right].$$

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

Home Work

- Find the inverse transforms of the following functions

a) $F(z) = 1 + 3z^{-1} + 4z^{-2}$

b) $F(z) = 5z^{-1} + 4z^{-5}$

c) $F(z) = \frac{z}{z^2+0.3z+0.02}$

d) $F(z) = \frac{z-0.1}{z^2+0.04z+0.25}$

e) $F(z) = \frac{z}{(z+0.1)(z+0.2)(z+0.3)}$

Digital Control Systems

LECTURE 5

Modeling of Digital Control Systems

Prepared by: Mr. Abdullah I. Abdullah

Discrete-time Block Diagrams

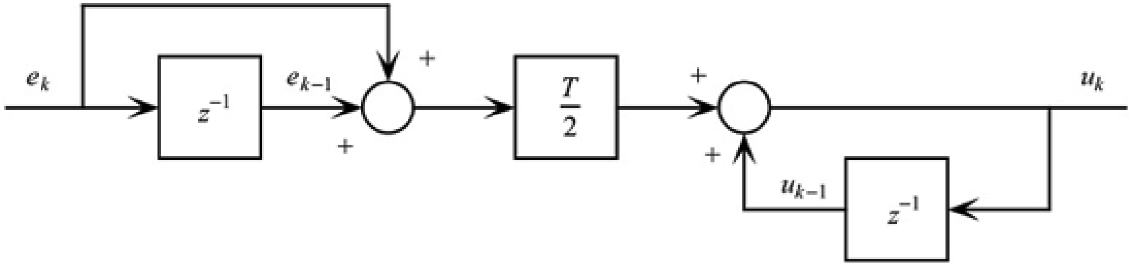
- Derive the pulse transfer function of a continuous-time system driven by discrete input.
- Manipulate block diagrams of open and closed-loop discrete-time systems.
- All linear difference equations are composed of delays, multiplies, and adds, and we can represent these operations in block diagrams.
- block diagrams are often helpful in system visualization.

Example

Consider the difference equation for trapezoidal integration:

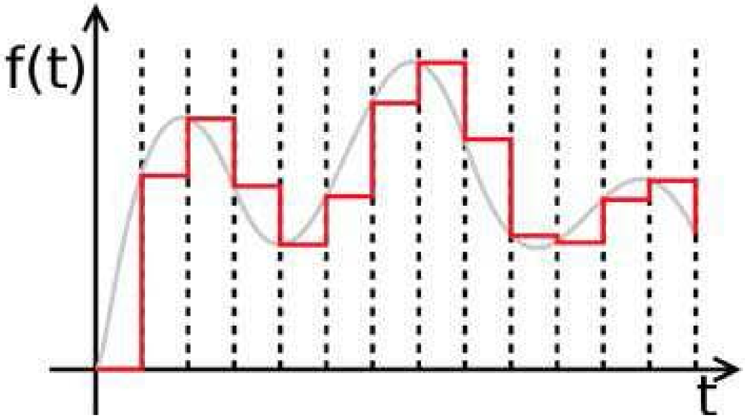
$$U_K = U_{K-1} + \frac{T}{2}(e_K + e_{K-1})$$

This difference equation is represented by the block diagram shown.



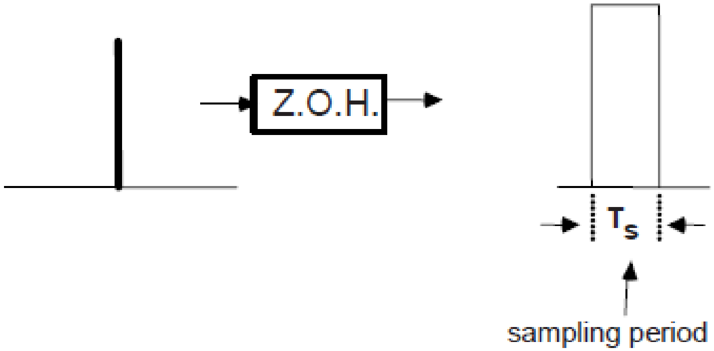
The ZOH transfer function

- A zero-order hold as a way to reconstruct continuous signal from discrete samples
- The ZOH remembers the last information until a new sample is obtained, i.e. it takes the value $r(kT)$ and holds it constant for $kT \le t < (k+1)T$.



$$g_{ZOH}(t) = u(t) - u(t - T_s)$$

$$G_{ZOH}(s) = \frac{1}{s} - \frac{e^{-sT_s}}{s} = \frac{1 - e^{-sT_s}}{s}$$



$$\therefore G_{ZOH}(z) = Z\left[\frac{1 - e^{-sT_s}}{s} G(s)\right]$$

$$= (1 - z^{-1}) Z\left[\frac{G(s)}{s}\right]$$

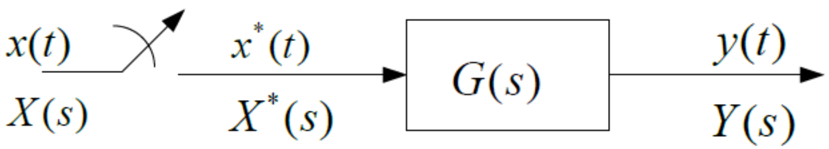
This is exactly the behavior of a DAC in converting a sampled signal $r^*(t)$ into continuous $r(t)$.

Pulse Transfer Function

The transfer function for the continuous-time system relate the Laplace transform of the continuous-time output to that of the continuous time input, while the pulse transfer function relate the z transform of the output at the sampling instant to that of the sampled input.

Consider the two different cases below:

Case 1:

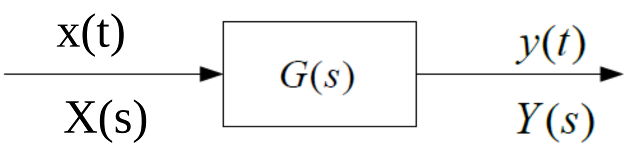


$$Y(s) = G(s)X^*(s)$$

$$Y^*(s) = [G(s)X^*(s)]^* = G^*(s)X^*(s)$$

$$Y(z) = G(z)X(z)$$

Case 2:



The Laplace transform of the output $y(t)$ is

$$Y(s) = G(s)X(s)$$

$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^*$$

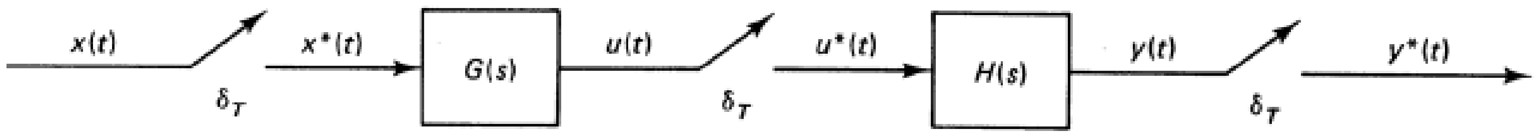
Or in term z transform

$$Y(z) = \mathcal{Z}[G(s)X(s)] = \mathcal{Z}[GX(s)]$$

$$Y(z) = GX(z)$$

$$GX(z) \neq G(z)X(z)$$

Pulse Transfer Function of Cascade Element

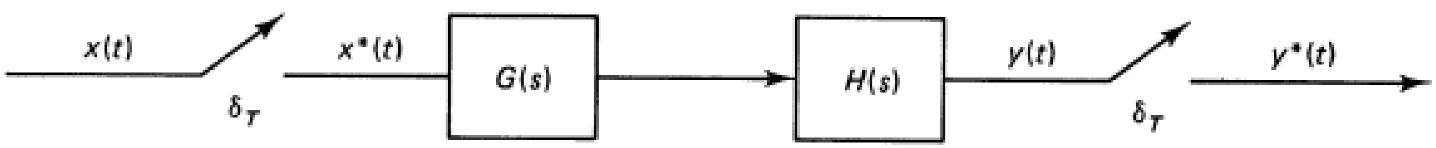


$$U(s) = G(s)X^*(s), \qquad Y(s) = H(s)U^*(s)$$

$$U^*(s) = G^*(s)X^*(s), \qquad Y^*(s) = H^*(s)U^*(s)$$

$$\Rightarrow \quad Y^*(s) = H^*(s)U^*(s) = H^*(s)G^*(s)X^*(s)$$

$$\Rightarrow \quad Y(z) = G(z)H(z)X(z) \Rightarrow \frac{Y(z)}{X(z)} = G(z)H(z)$$



$$Y(s) = G(s)H(s)X^*(s) = GH(s)X^*(s)$$

$$Y^*(s) = [GH(s)]^* X^*(s)$$

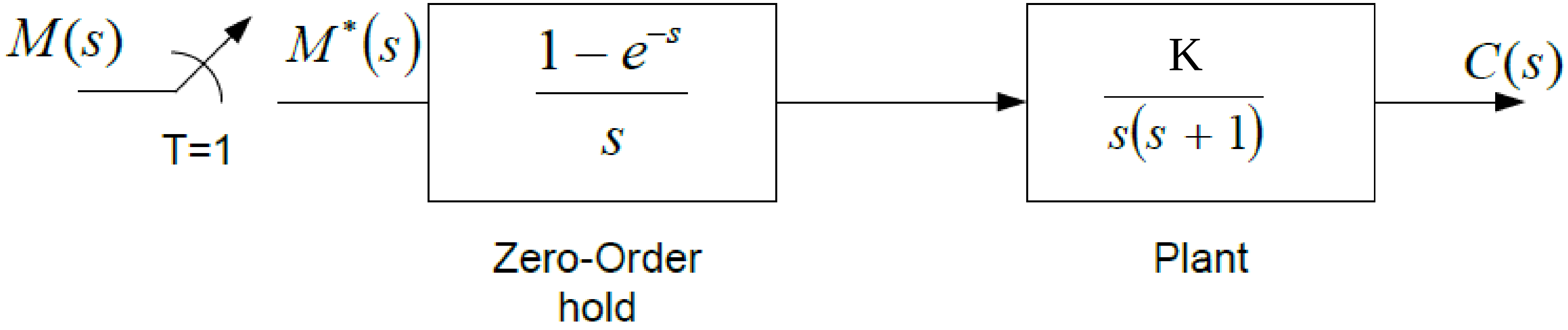
$$Y(z) = GH(z)X(z)$$

$$\frac{Y(z)}{X(z)} = GH(z) = \mathrm{Z}[GH(s)]$$

Note that

$$G(z)H(z) \neq GH(z)$$

Example: obtain the pulse transfer function of the system shown in figure below:



$$G(z) = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z}Z\left\{\frac{K}{s^2(s+1)}\right\} \quad \text{for } T_s = 1$$

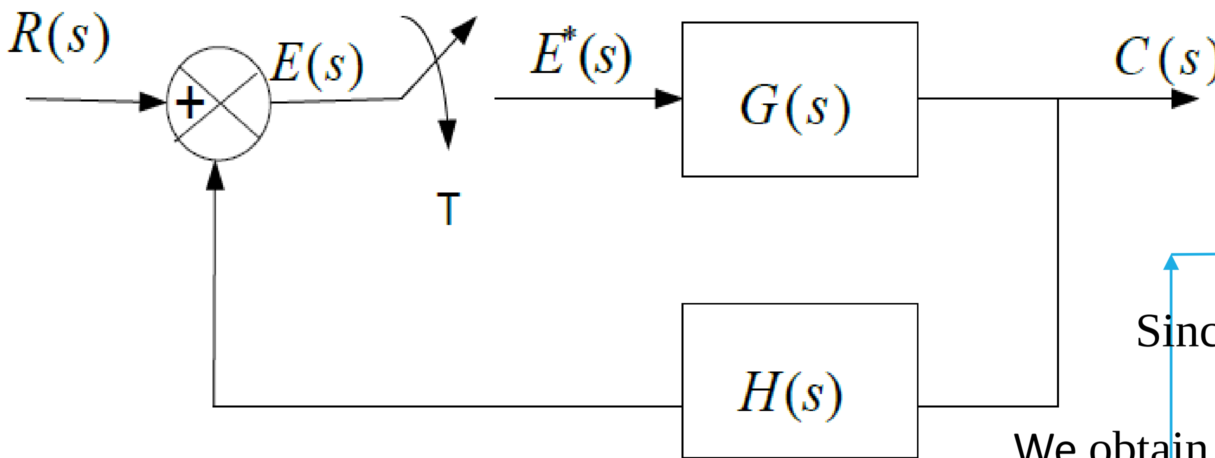
$$G(z) = \frac{z-1}{z}Z\left\{\frac{K}{s^2(s+1)}\right\} = \frac{z-1}{z}Z\left\{\frac{K}{s^2} - \frac{K}{s} + \frac{K}{(s+1)}\right\}$$

$$G(z) = \frac{z-1}{z}\left\{\frac{T_s Kz}{(z-1)^2} - \frac{Kz}{(z-1)} + \frac{Kz}{z - e^{-T_s}}\right\}$$

$$G(z) = \frac{K(0.368z + 0.264)}{z^2 - 1.368z + 0.368}$$

Pulse Transfer Function of Closed-Loop System

Consider the closed loop system shown below. In this system, the actuating error is sampled.



$$E(s) = R(s) - H(s)C(s) \quad \text{----- 1}$$

$$C(s) = G(s)E^*(s) \quad \text{----- 2}$$

Sub. :2 in 1

$$E(s) = R(s) - H(s)G(s)E^*(s) \quad \text{----- 3}$$

$$E^*(s) = R^*(s) - GH^*(s)E^*(s)$$

Or

$$E^*(s) = \frac{R^*(s)}{1 + GH^*(s)} \quad \text{--- 4}$$

Since $C^*(s) = G^*(s)E^*(s)$

We obtain $C^*(s) = \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$

In term of z-transform C(z) is given by

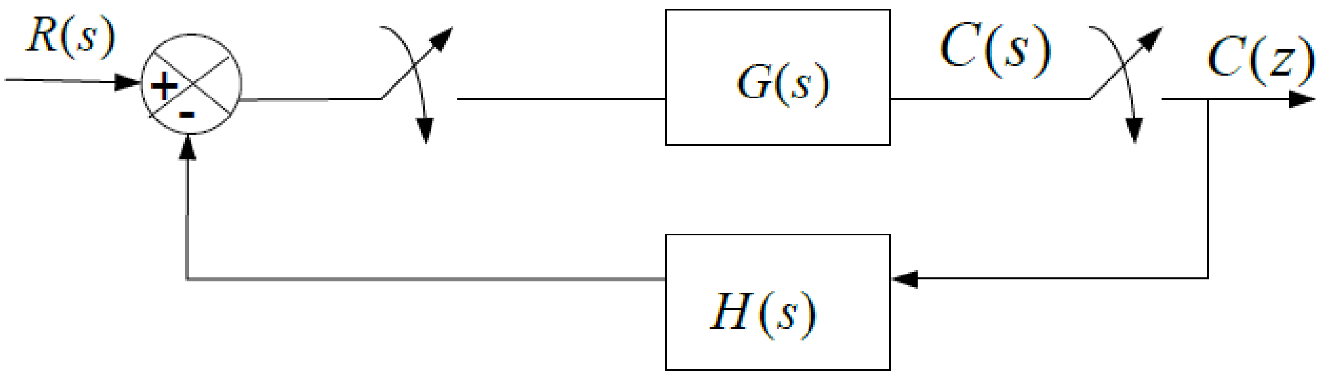
$$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

The pulse transfer function is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} \quad \text{--- 5}$$

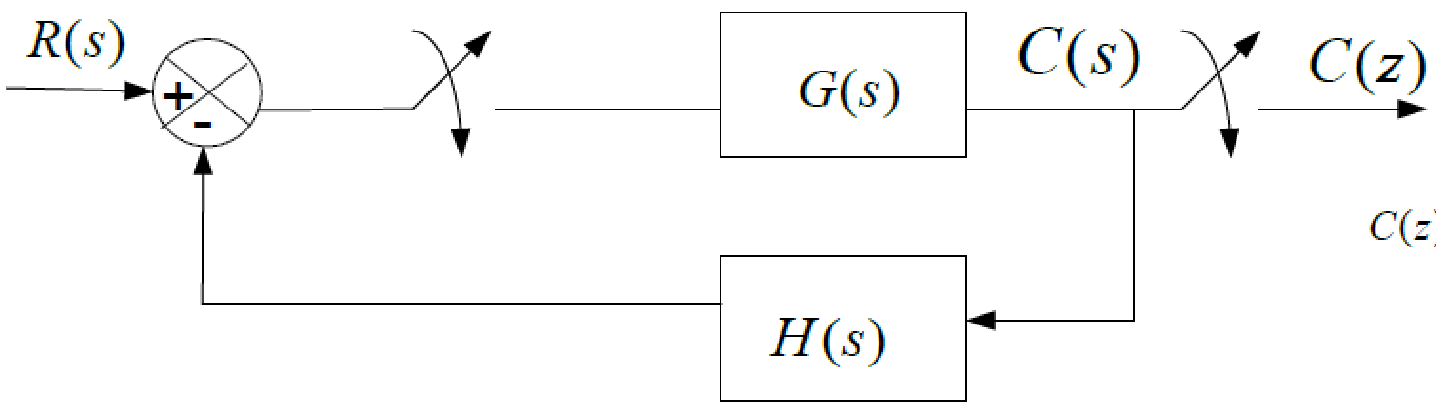
Typical configuration of closed loop discrete-time systems and the corresponding outputs $C(z)$ are listed below:

Case 1



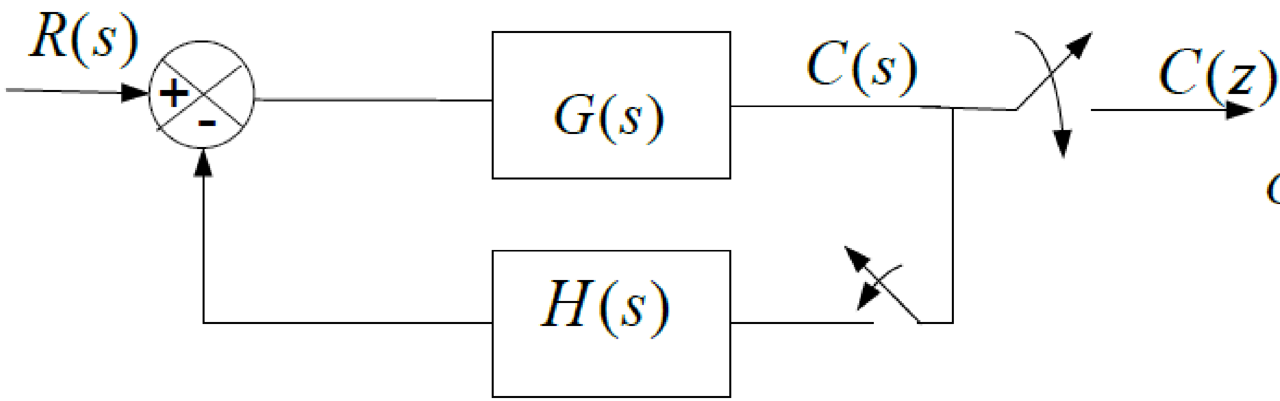
$$C(z) = \frac{G(z)R(z)}{1 + G(z)H(z)}$$

Case 2



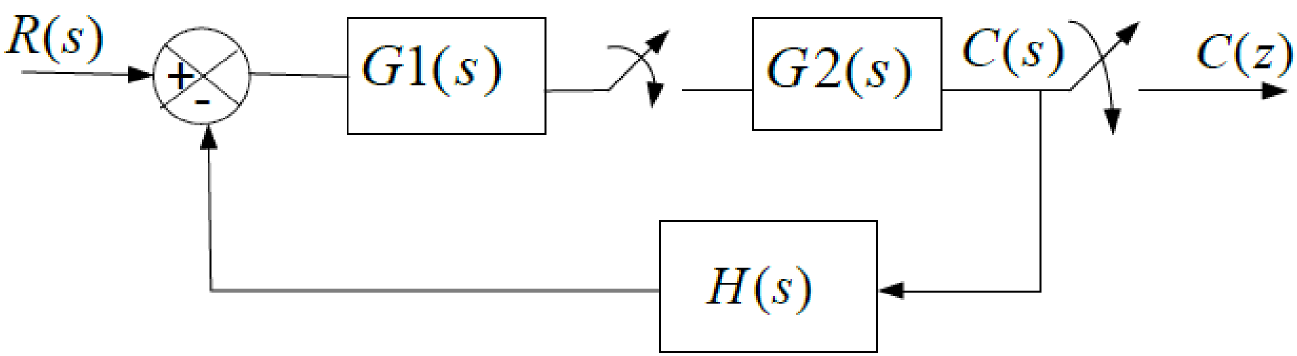
$$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$

Case 3



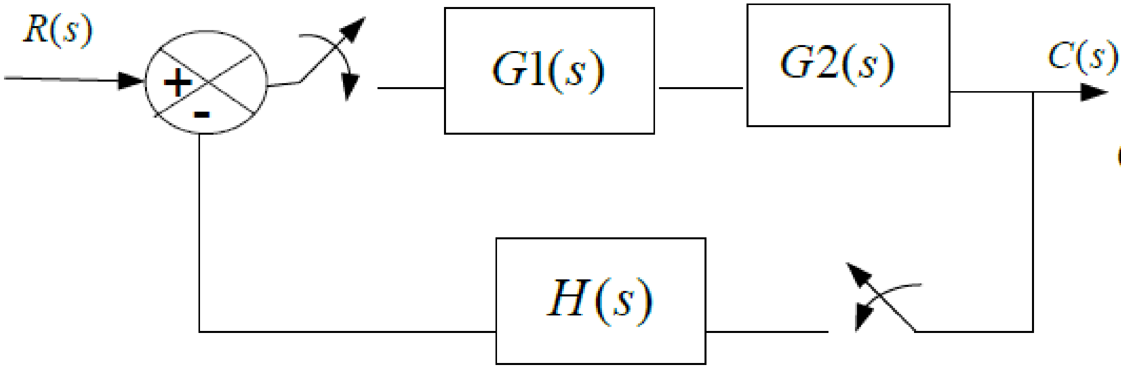
$$C(z) = \frac{RG(z)}{1 + HG(z)}$$

Case 4



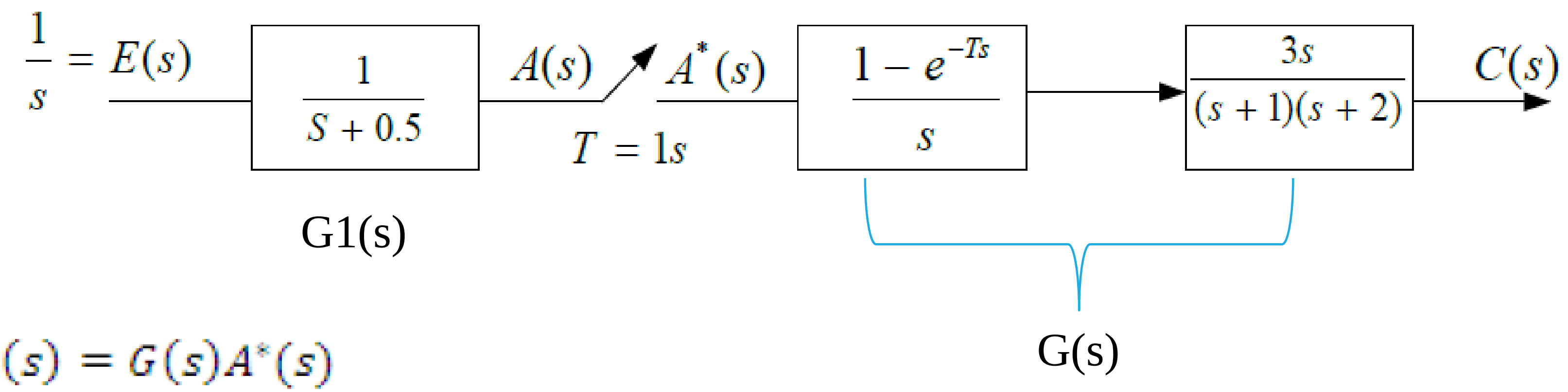
$$C(z) = \frac{G2(z)RG1(z)}{1 + G1G2H(z)}$$

Case 5



$$C(z) = \frac{G1G2(z)R(z)}{1 + H(z)G1G2(z)}$$

Example 1: find the discrete system response C(z) to a unit step input for the system shown below:



$$C(s) = G(s)A^*(s)$$

$$C(z) = G(z)A(z)$$

$$A(s) = G1(s)E(s)$$

$$G1(s) = \frac{1}{s+0.5}$$

$$G(s) = \frac{3\,(1-e^{-Ts})}{(s+1)(s+2)}$$

$$A(z) = Z\,\{G1(s)E(s)$$

$$= Z\left\{\frac{1}{s+0.5}\;\frac{1}{s}\right\} = \frac{1}{0.5}\,Z\left\{\frac{0.5}{s(s+0.5)}\right\}$$

$$= 2\;\frac{z\,(1-e^{-0.5T})}{(z-1)(z-e^{-0.5T})}$$

$$G(s) = \frac{3}{(s+1)(s+2)}\ (1 - e^{-Ts}) = 3\ \frac{1}{(s+1)(s+2)}\ (1 - e^{-Ts})$$

$$G(z) = 3(1 - z^{-1})\frac{(e^{-T}-e^{-2T})z}{(z-e^{-T})(z-e^{-2T})} = 3\,\frac{z-1}{z}\,\frac{(e^{-1}-e^{-2})z}{(z-e^{-1})(z-e^{-2})}$$

$$= \frac{3(z-1)(e^{-1}-e^{-2})}{(z-e^{-1})(z-e^{-2})}$$

$$C(z) = G(z)A(z)$$

$$\therefore C(z) = \frac{3(z-1)(e^{-1}-e^{-2})}{(z-e^{-1})(z-e^{-2})}\ \frac{2z}{z-1}\ \frac{(1-e^{-0.5})}{z-e^{-0.5}}$$

Example 2

Find the pulse transfer function of the system given by

$$y(kT)+3y(kT-T)+4y(kT-2T)+5y(kT-3T)=r(kT)-3r(kT-T)+2r(kT-2T)$$

with zero initial condition.

Solution :

z transform of the system is given by

$$Y(z)+3Y(z)z^{-1}+4Y(z)z^{-2}+5Y(z)z^{-3}=R(z)-3R(z)z^{-1}+2R(z)z^{-2}$$

The pulse transfer function of the system can be obtained as

$$G(z)=\frac{Y(z)}{R(z)}=\frac{1-3z^{-1}+2z^{-2}}{1+3z^{-1}+4z^{-2}+5z^{-3}}=\frac{z^3-3z^2+2z}{z^3+3z^2+4z+5}$$

Example 3

Find the difference equation of the system whose transfer function is

$$G(z)=\frac{z^4+3z^3+2z^2+z+1}{z^4+4z^3+5z^2+3z+2}$$

Solution : $G(z)=\frac{Y(z)}{R(z)}=\frac{1+3z^{-1}+2z^{-2}+z^{-3}+z^{-4}}{1+4z^{-1}+5z^{-2}+3z^{-3}+2z^{-4}}$

so we have

$$Y(z)(1+4z^{-1}+5z^{-2}+3z^{-3}+2z^{-4})=R(z)(1+3z^{-1}+2z^{-2}+z^{-3}+z^{-4})$$

The difference equation of the system can be obtained by the inverse z transform

$$y(kT)+4y(kT\text{-}T)+5y(kT\text{-}2T)+3y(kT\text{-}3T)+2y(kT\text{-}4T)=$$
$$r(kT)+3r(kT\text{-}T)+2r(kT\text{-}2T)+r(kT\text{-}3T)+r(kT\text{-}4T)$$

Digital Control Systems

LECTURE 6

Time Response

Prepared by: Mr. Abdullah I. Abdullah

Time Response

In this chapter the time response of the sampled data system of Fig.(1) to unit step input will be determined. Three methods will be explained: long division , difference equations and partial fraction expansion.

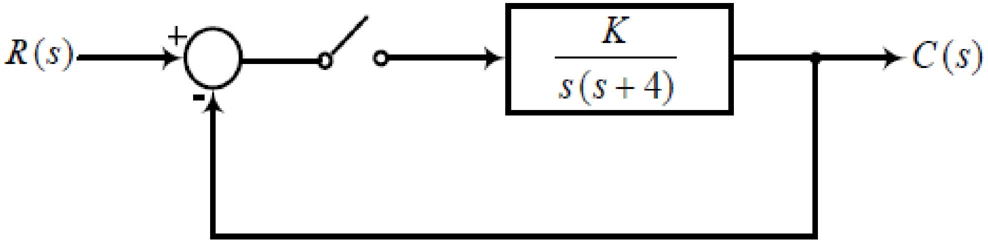


Figure (1) *Sampled data system*

$$G(s)=\frac{k}{s(s+4)}=\frac{A}{s}+\frac{B}{(s+4)}=\frac{k/4}{s}-\frac{k/4}{(s+4)}$$

$$G(z)=\frac{k}{4}\frac{z}{(z-1)}-\frac{k}{4}\frac{z}{(z-e^{-4T})}$$

$$G(z)=\frac{k}{4}\frac{z(z-e^{-4T})-z(z-1)}{(z-1)(z-e^{-4T})}$$

$$G(z)=\frac{z(K/4)\,(1-e^{-4T})}{(z-1)\,(z-e^{-4T})}$$

$$G(z) = \frac{z(K/4)\,(1-e^{-4T})}{(z-1)(z-e^{-4T})}$$

Letting $K = 1$ and $T = 0.25$ sec, then

$$G(z) = \frac{0.158z}{(z-1)(z-0.368)}$$

The pulse transfer function $\frac{C(z)}{R(z)}$ is

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)}$$

$$C(z) = \frac{0.158z}{[(z-1)(z-0.368)+0.158z]}R(z)$$

$$= \frac{0.158z}{(z-0.61)^2}R(z)$$

1- Long division method:

For unit step input, $R(z)=\dfrac{z}{z-1}$. Then

$$C(z) = \frac{0.158z}{(z-0.61)^2}R(z)$$

$$C(z)=\frac{0.158z^2}{(z-0.61)^2(z-1)}$$

Using the long-division method to determine the inverse gives

$$0.158z^{-1}+0.349z^{-2}+0.522z^{-3}+\cdots$$

$$z^3-2.21z^2+1.58z-0.368\ \overline{)\,0.158z^2}$$

$$0.158z^2-0.349z+0.249642-0.058144z^{-1}$$ بالطرح

$$0.349z-0.249642+0.058144z^{-1}$$

$$C(z)=\sum_{n=0}^{\infty} c(nT)\,z^{-n} = c(0)+c(T)\,z^{-1}+c(2T)\,z^{-2}+-------$$

$$c(z)=0.185\,z^{-1}+0.349\,z^{-2}+0.522\,z^{-3}+-----$$

then c(0)=0, c(T)=0.158, c(2T)=0.349, and c(3T)=0.522

A plot of the response $c(nT)$ at the sampling instants is shown in Fig.(2).The long division method becomes quite cumbersome for computing $c(nT)$ for larger values of n. A more convenient procedure results from expressing the solution in the form of a difference equation.

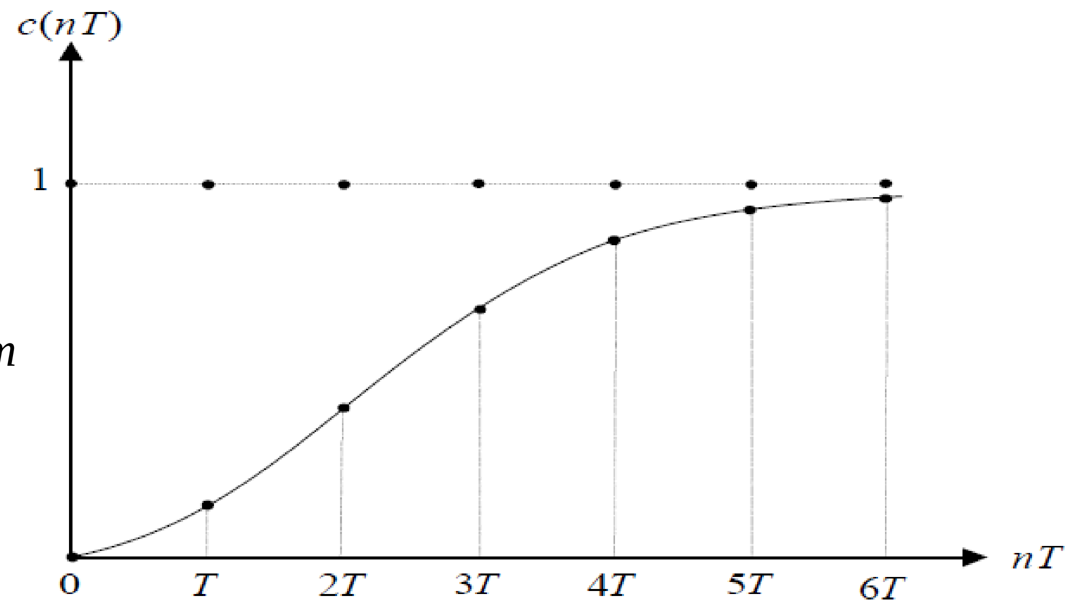


Figure (2) *Sampled data system*

2-Difference Equations:

To determine the inverse z transform by this method, one can write the equation for $C(z)$ in the form

$$G(z) = \frac{0.158z}{(z-1)(z-0.368)}$$

The pulse transfer function $\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)}$

From previous example

$$C(z) = \frac{0.158z}{[z^2 - 1.21z + 0.368]} R(z)$$

Thus $C(z) - 1.21z^{-1}C(z) + 0.368z^{-2}C(z) = 0.158z^{-1}R(z)$

Application of right shifting property $Z[f(nT-kT) = z^{-n}F(z)$

Then the preceding expression yields directly the difference equation

$$c(nT) = 1.21\, c(nT-T) - 0.368\, c(nT-2T) + 0.158\, r(nT-T)$$

This difference equation gives the value $c(nT)$ at the *nth* sampling instants in terms of values at the preceding sampling instants. Application of this result to obtain the values at the sampling instants gives

$$c(nT) = 1.21\, c(nT - T) - 0.368\, c(nT - 2T) + 0.158\, r(nT - T)$$

$$c(0) = 0,$$
$$c(T) = 0.158\, r(0) = 0.158$$
$$c(2T) = 1.21 c(T) + 0.158\, r(T) = 0.349$$
$$c(3T) = 1.21\, c(2T) - 0.368\, c(T) + 0.158\, r(2T) = 0.522$$

Such recurrence relationships lend themselves very well to solution by a digital computer.

3-Partial-fraction expansion:

The response $c(nT)$ at the sampling instants may be also be obtained by performing a partial fraction expansion and then inverting. Thus

$$C(z) = \frac{0.158z^2}{(z-0.61)^2(z-1)}$$

From previous example

$$C(z) = z\left[\frac{0.158z}{(z-1)(z-0.61)^2}\right] = z\left[\frac{A}{(z-1)} + \frac{B_1}{(z-0.61)^2} + \frac{B_2}{(z-0.61)}\right]$$

The partial-fraction expansion constants are $A = 1$, $B1 = -0.24$, and $B2 = -1.0$. Thus, $C(z)$ becomes

$$C(z) = \frac{z}{(z-1)} - 0.39\frac{0.61z}{(z-0.61)^2} - \frac{z}{(z-0.61)}$$

By noting that

$$Z^{-1}\left[\frac{z}{z-1}\right]=1,\qquad Z^{-1}\left[\frac{z}{z-a}\right]=a^{nT},\quad \text{and}\quad Z^{-1}\left[\frac{az}{(z-a)^2}\right]=nT\,a^{nT}$$

$$C(z)=\frac{z}{(z-1)}-0.39\frac{0.61z}{(z-0.61)^2}-\frac{z}{(z-0.61)}$$

Note B=0.24
0.24=0.61*0.39

The inverse is found to be

$$c(nT)=1-0.39\,nT(0.61)^{nT}-(0.61)^{nT}\qquad n=0,1,2,3,-------$$

With this method, the value $c(nT)$ at any sampling instants may be calculated directly without the need to compute the value at all the preceding instants

Figure 3. should just be a reminder of how we can characterize a transient response. It shows five measurements, **the delay time, the rise time, time to the first peak, the peak value, and the settling time**

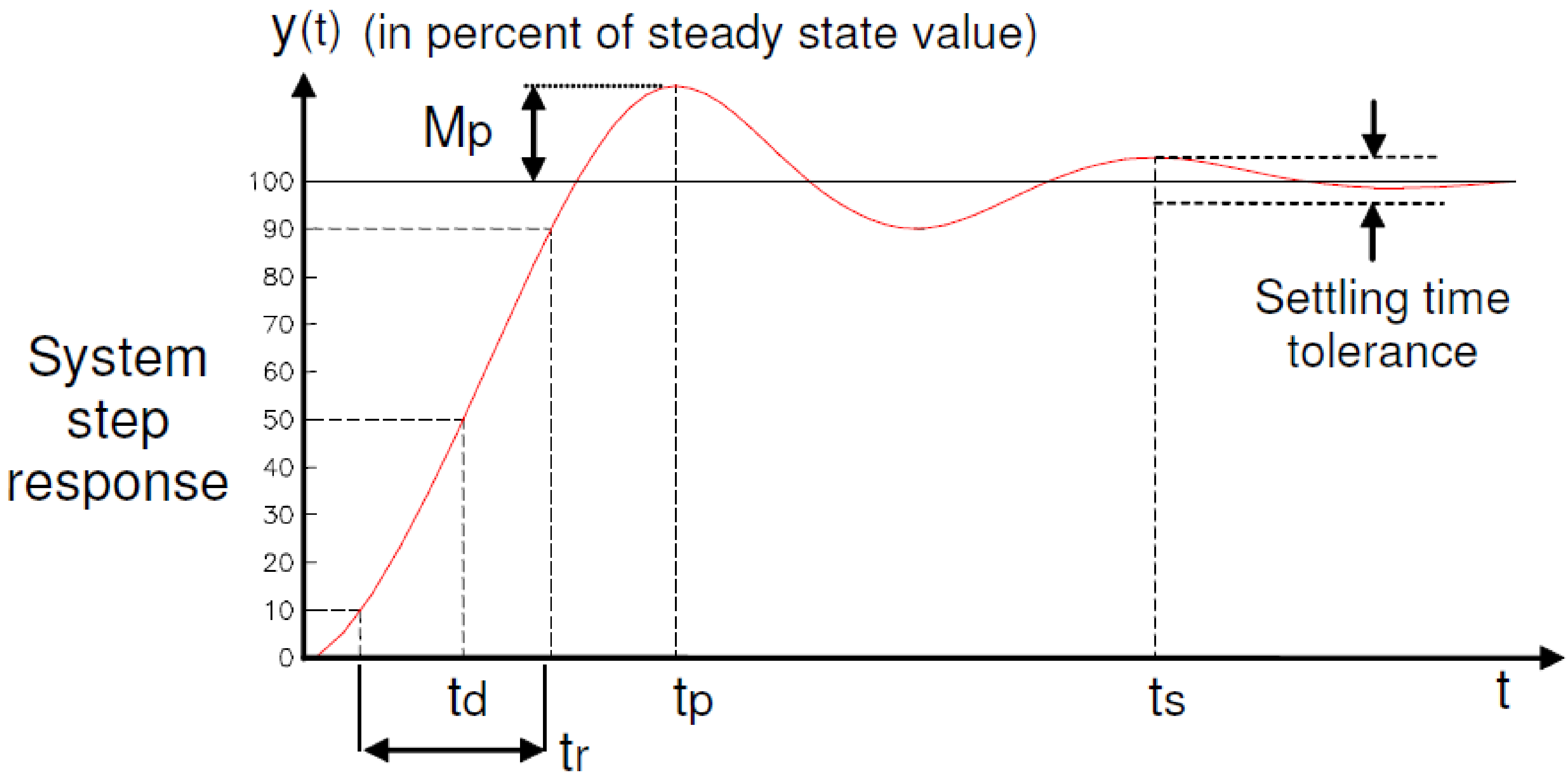


Figure 3.13: Transient response characteristics

t_d	Delay time: time to reach 50% of SSV
t_r	Rise time: time to go from 10% to 90% of SSV,or 0-100%, depending on situation.
t_p	Time at which first peak occurs - if any (peak time)
M_p	Maximum overshoot as % of SSV
t_s	Settling time: time for output to stabilise within a given tolerance band (usually 2%)

SSV = Steady State Value

$$t_r = \frac{\pi - \cos^{-1}(\varsigma)}{\omega_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$(0 \to 100\%)$$

$$M_p = 100e^{-\left(\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}\right)}$$

$$t_s \approx \frac{4}{\varsigma\omega_n} \ (to\ 2\%)$$

Digital Control Systems

LECTURE 7

Mapping of s-plane to z-plane

Prepared by: Mr. Abdullah I. Abdullah

Mapping of s-plane to z-plane

It is possible to map from the s plane to the z plane using the relationship $z = e^{sT}$

Where $s = \sigma + j\omega$.

$$z = e^{(\sigma + j\omega)T}$$

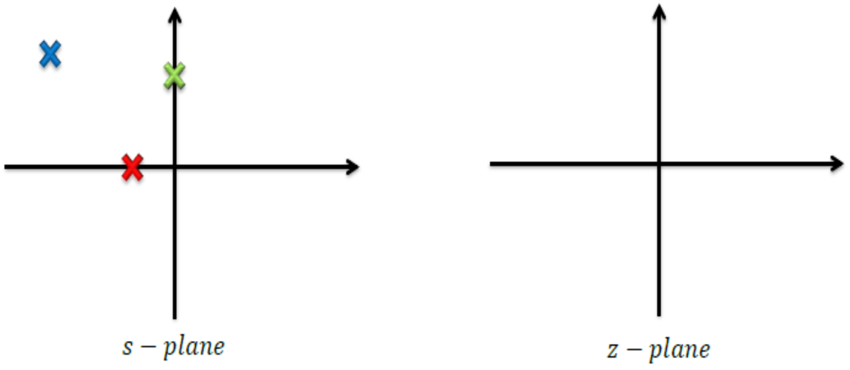
Then z in polar coordinates is given by

$$z = e^{\sigma T} e^{j\omega T}$$

$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

We will discuss following cases to map given points on s-plane to z-plane.

- **Case-1:** Real pole in s-plane ($s = \sigma$)
- **Case-2:** Imaginary Pole in s-plane ($s = j\omega$)
- **Case-3:** Complex Poles ($s = \sigma + j\omega$)



Case-1: Real pole in s-plane ($s = \sigma$)

$$|z| = e^{\sigma T} \quad , \quad \angle z = 0$$

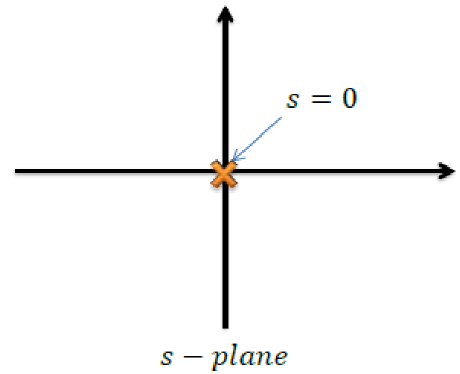
We know

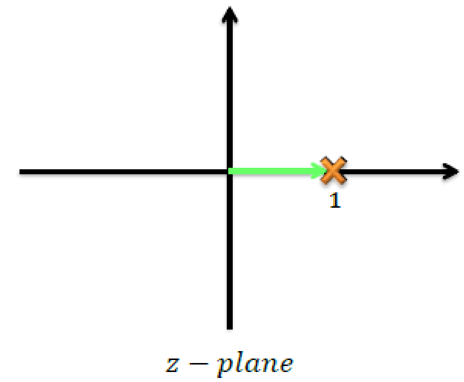
$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

When $s = 0$

$$|z| = e^{0T} = 1$$

$$\angle z = 0T = 0$$

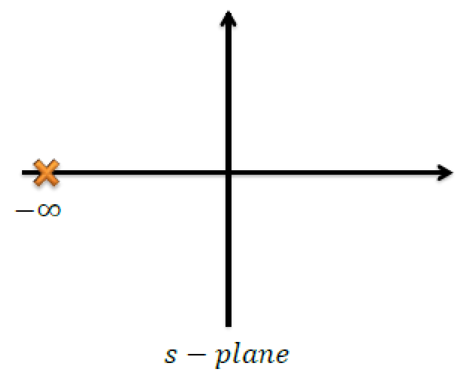


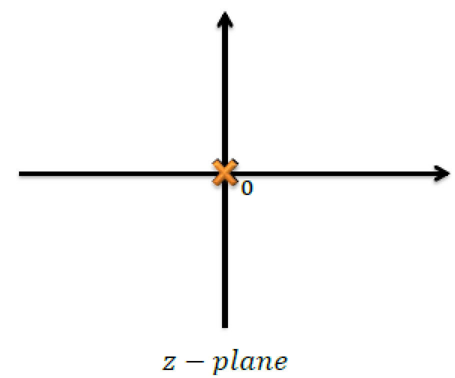


When $s = -\infty$

$$|z| = e^{-\infty T} = 0$$

$$\angle z = 0$$

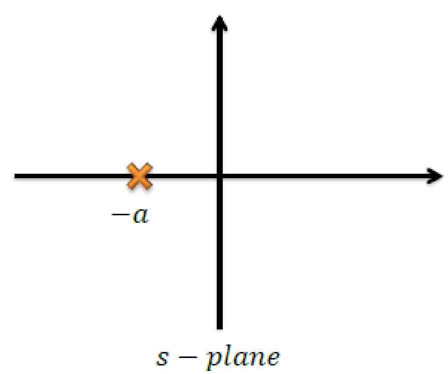


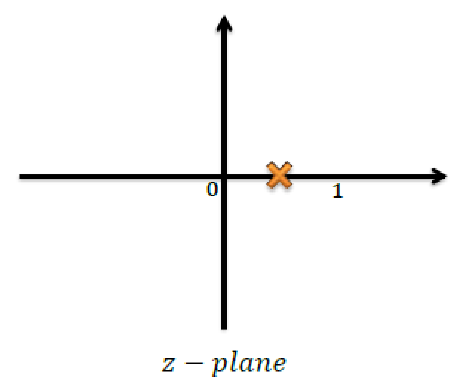


Consider $s = -a$

$$|z| = e^{-aT}$$

$$\angle z = 0$$





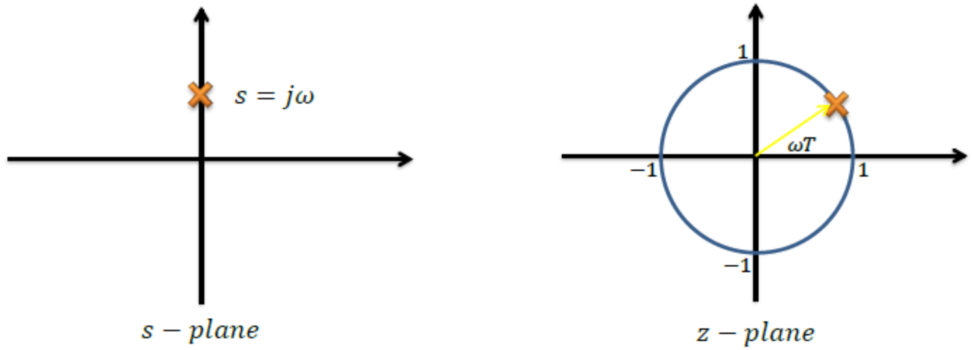
Case-2: Imaginary pole in s-plane ($s = \pm j\omega$)

$$|z| = e^{\sigma T} \qquad \angle z = \omega T$$

Consider $s = j\omega$

$$|z| = e^{0T} = 1$$

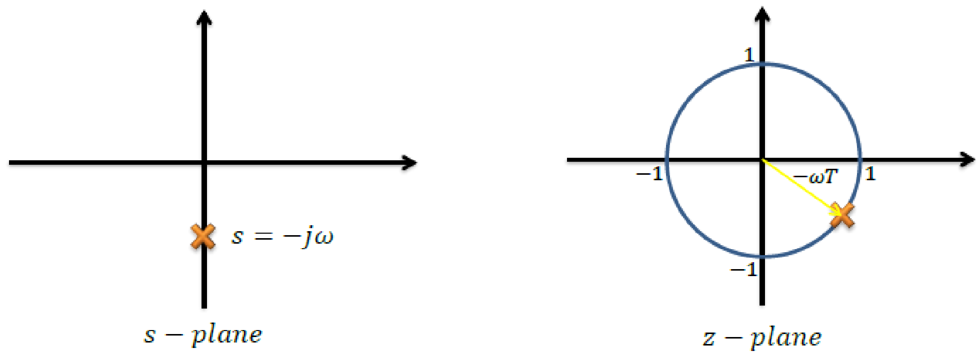
$$\angle z = \omega T$$



When $s = -j\omega$

$$|z| = e^{0T} = 1$$

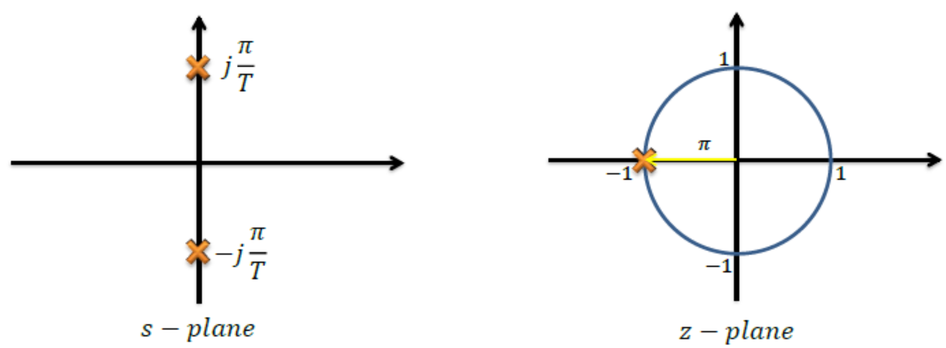
$$\angle z = -\omega T$$



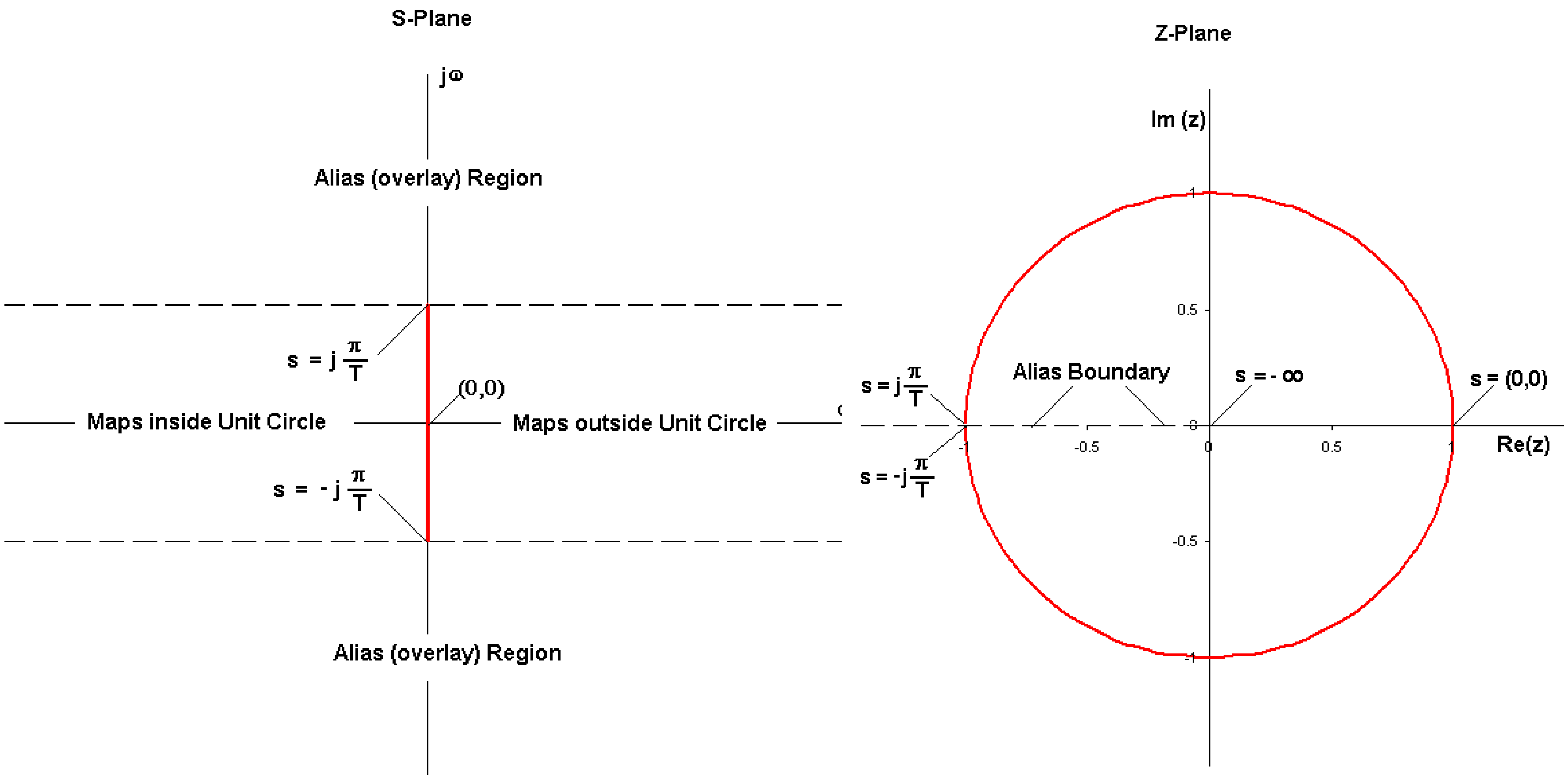
When $s = \pm j\frac{\pi}{T}$

$$|z| = e^{0T} = 1$$

$$\angle z = \pm \frac{\pi}{T} T = \pm \pi$$

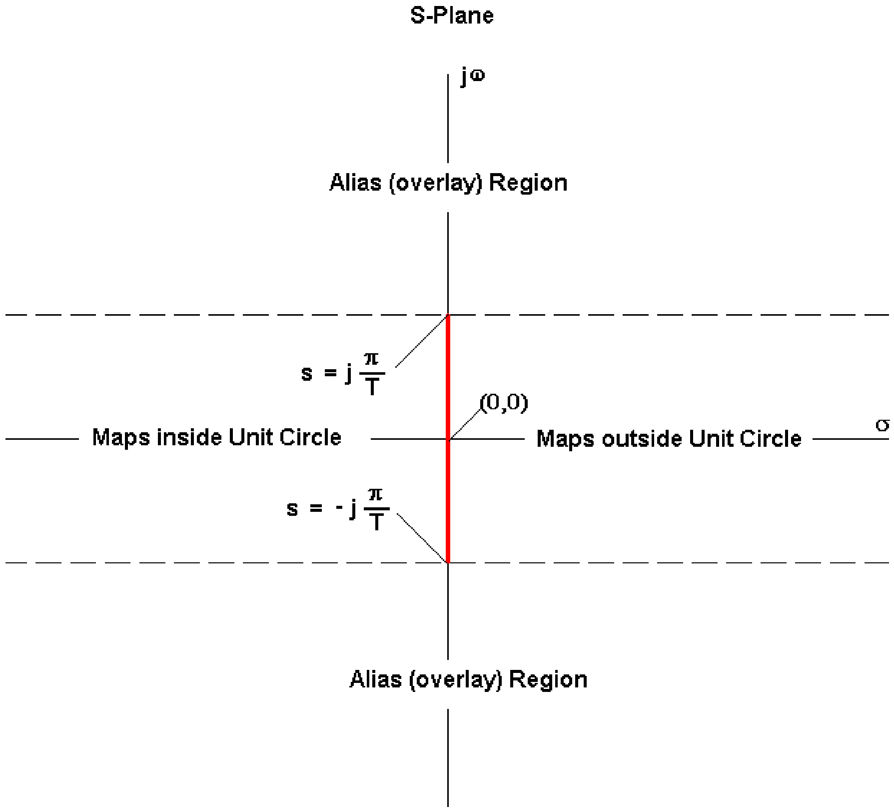


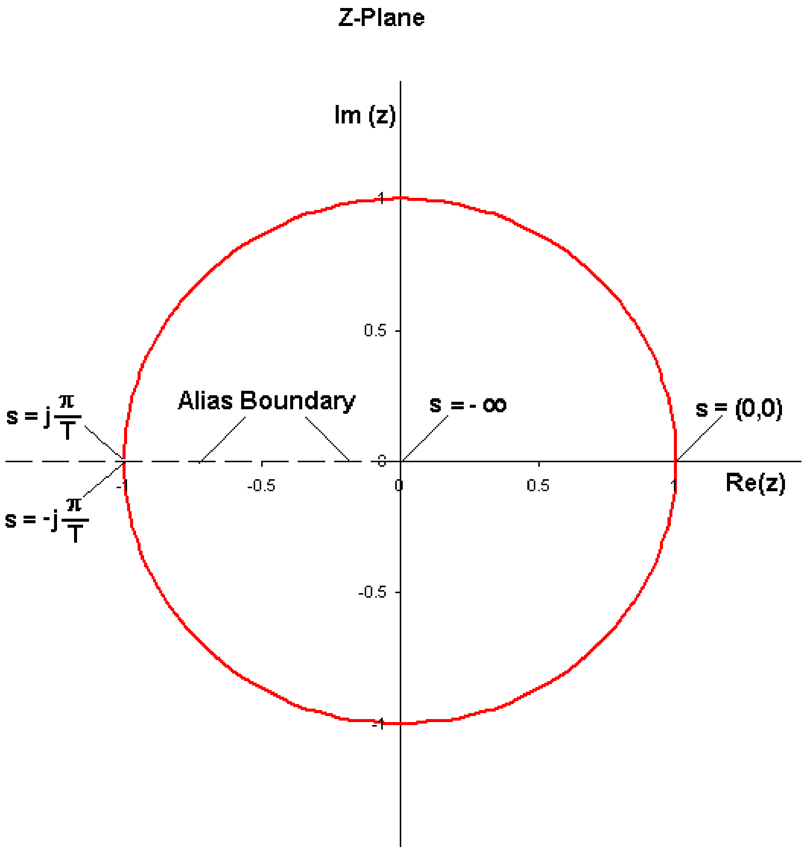
- Anything in the Alias/Overlay region in the S-Plane will be overlaid on the Z-Plane along with the contents of the strip between $\pm j\frac{\pi}{T}$.



- In order to avoid aliasing, there must be nothing in this region, i.e. there must be no signals present with radian frequencies higher than w = p/T, or cyclic frequencies higher than f = 1/2T

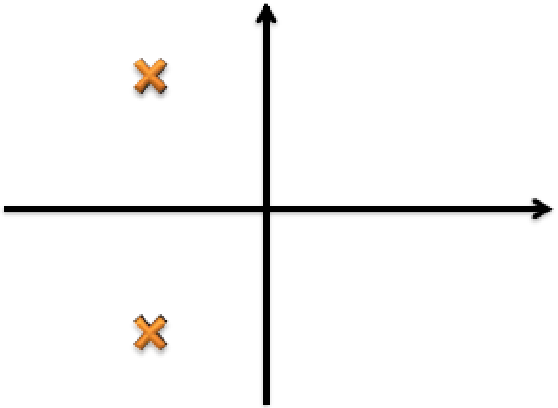
- Stated another way, the sampling frequency must be at least twice the highest frequency present (Nyquist rate).

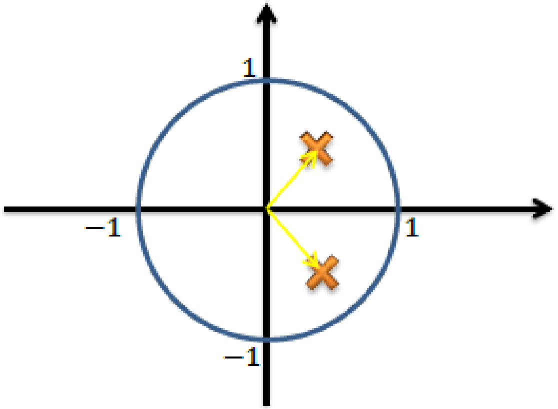




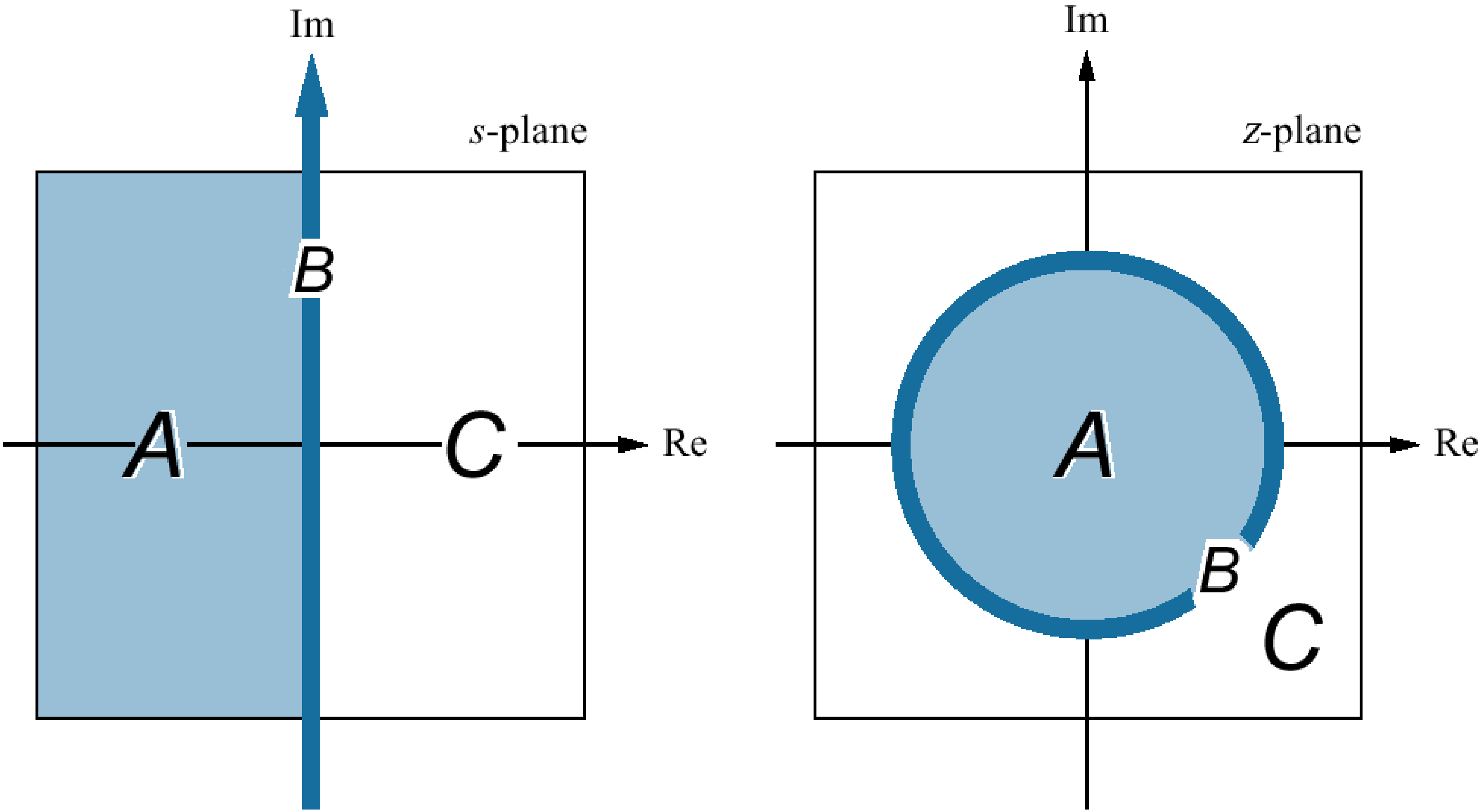
Case-3: Complex pole in s-plane ($s = \sigma \pm j\omega$)

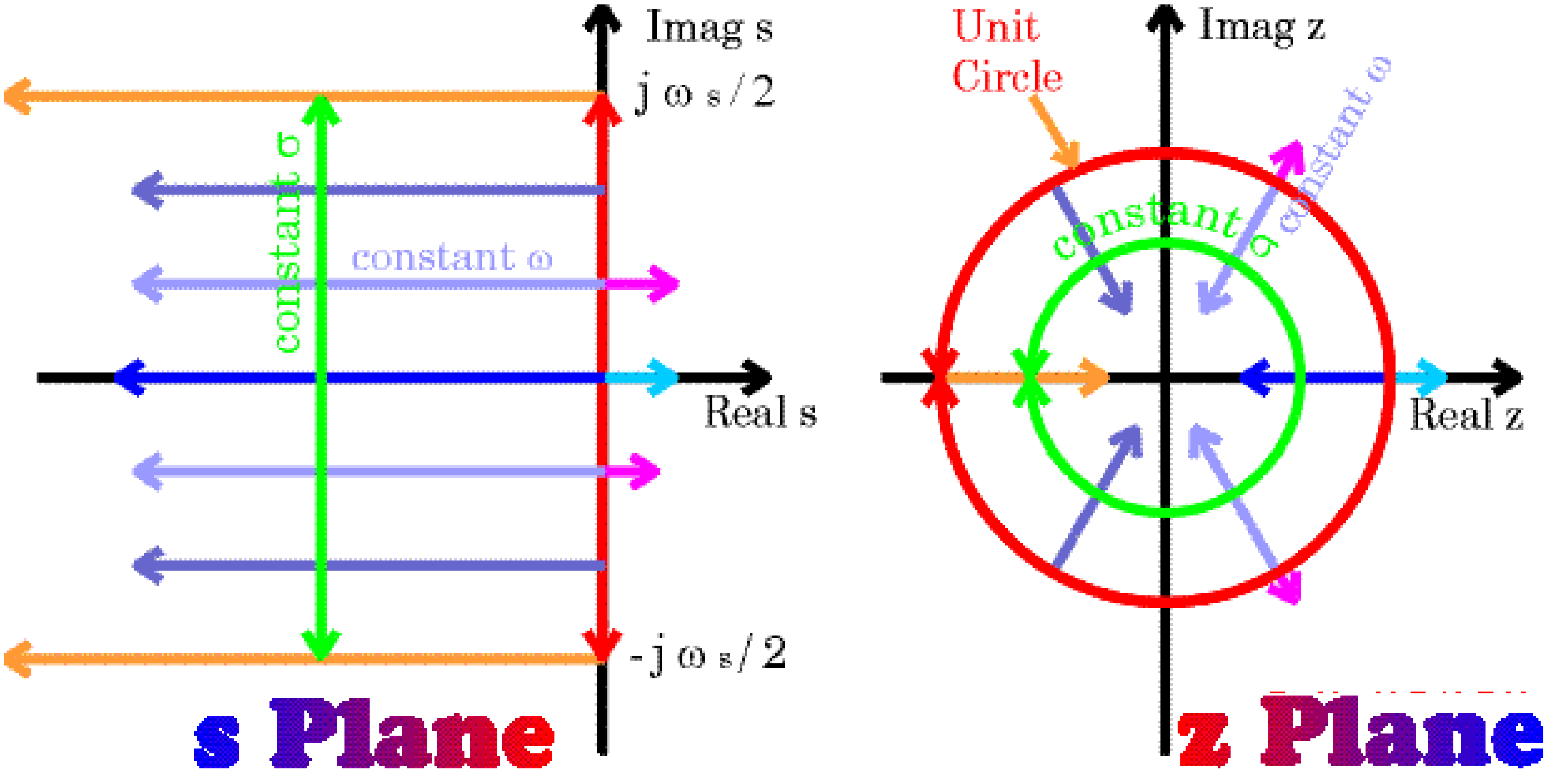
$|z| = e^{\sigma T}$ $\angle z = \pm\omega T$





Mapping regions of the *s*-plane onto the *z*-plane



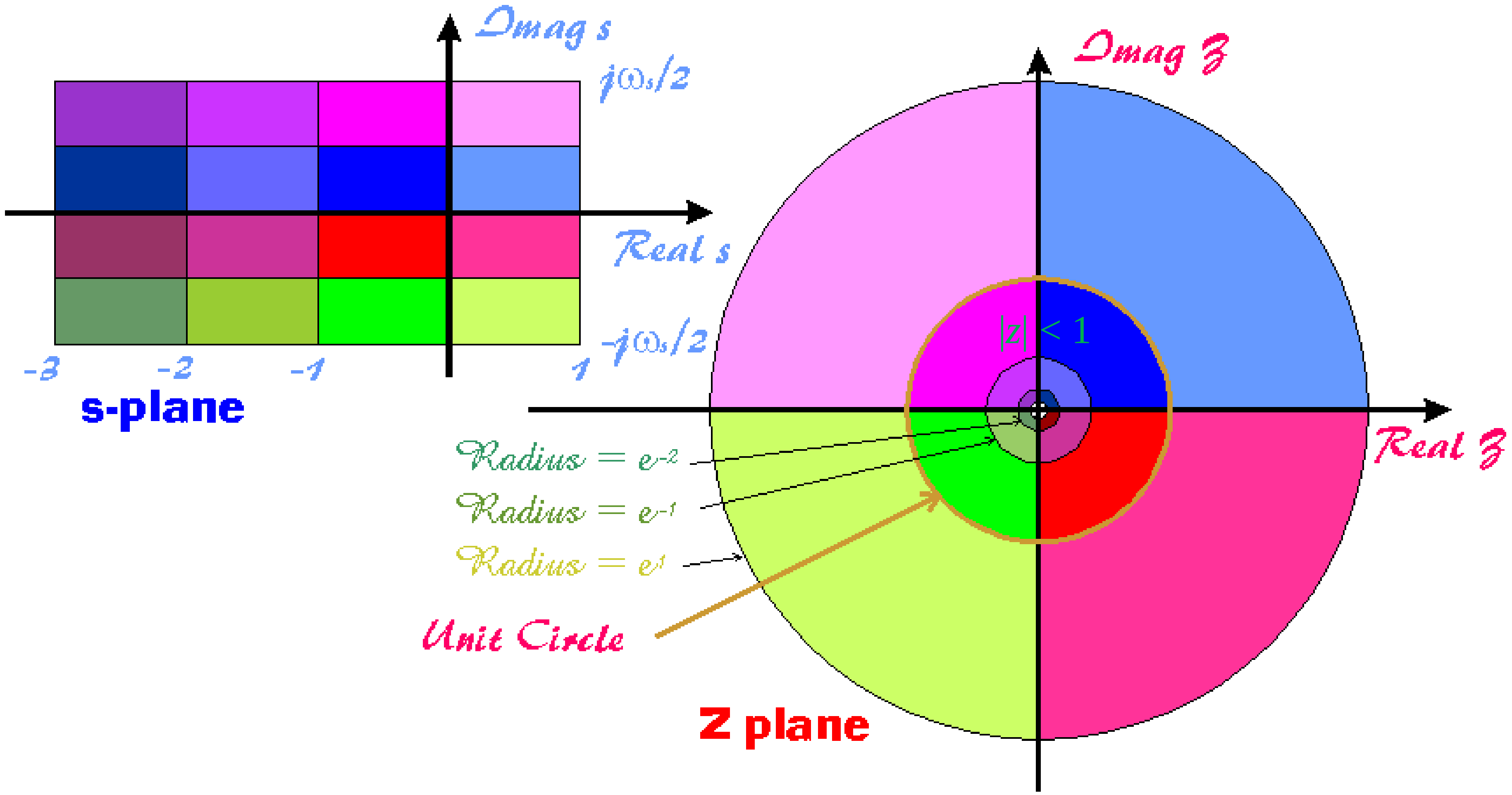


$$Z = e^{sT} \qquad , s = j\frac{w}{2}$$

$$Z = e^{j\frac{w}{2}T} \quad , Where \;\; w = \frac{2\pi}{T}$$

$$Z = e^{j\pi} = 1$$

Mapping regions of the *s*-plane onto the z-plane



Stability of Discrete Systems

There are several methods to check the stability of a discrete-time system such as:

1-Factorizing $D(z)$ and finding its roots.

2-Jury Test.

3-Routh–Hurwitz criterion .

1-Factorization

Suppose that we have the following transfer function of a closed-loop discrete-time system:

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} = \frac{N(z)}{D(z)}$$

The system is **stable** if **all** poles* lie inside the unit circle in z-plane.

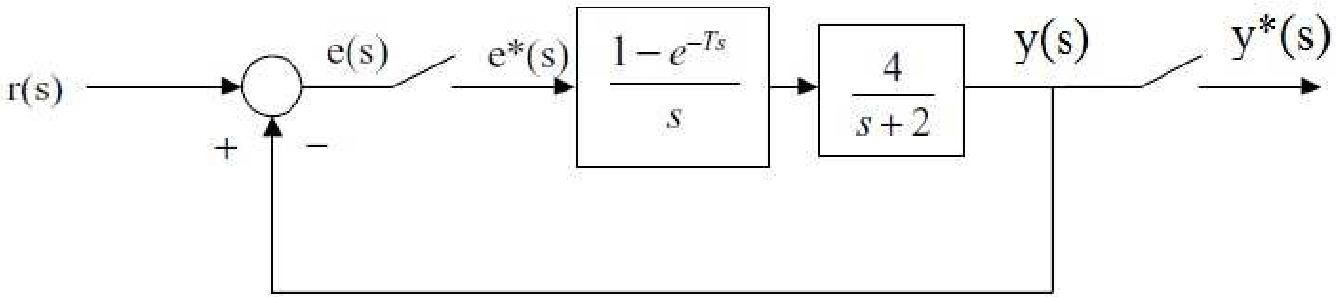
The direct method to check system stability is to factorize the characteristic equation, $1 + GH(z) = 0$

◮ determine its roots, and check if their **magnitudes** are all less than 1.

$$|z| < 1$$

Example 1

Check the stability of the following closed-loop discrete system. Assume that $T = 1$ s



The transfer function of the closed-loop system is: $\dfrac{Y(z)}{R(z)} = \dfrac{G(z)}{1+G(z)}$

Where $$G(z) = \mathscr{Z}\left\{\frac{1-e^{-Ts}}{s}\frac{4}{s+2}\right\}$$

$$= (1-z^{-1})\frac{2z(1-e^{-2T})}{(z-1)(z-e^{-2T})} = \frac{2(1-e^{-2T})}{(z-e^{-2T})}\Bigg|_{T=1\text{ sec}}$$

$$= \frac{1.729}{z-0.135}$$

The characteristic equation is thus:

$$G(z) = \frac{1.729}{z - 0.135}$$

$$1 + G(z) = 0$$
$$z + 1.594 = 0$$
$$z = -1.594$$
$$|z| > 1 \quad \Rightarrow \text{system is } \textbf{unstable}$$

Example 2

In the previous example, find the value of *T* for which the system is stable.

From the previous example, we found:

$$G(z) = \frac{2(1 - e^{-2T})}{(z - e^{-2T})}$$

The characteristic equation is:

$$1 + G(z) = 0$$
$$z - 3e^{-2T} + 2 = 0$$
$$z = 3e^{-2T} - 2$$

For stability, the condition $|z| < 1$ must be satisfied; $z = 3e^{-2T} - 2$

$$|z| = |3e^{-2T} - 2| < 1$$

$$-1 < 3e^{-2T} - 2 < 1$$

$$\ln\left(\frac{1}{3}\right) < -2T < 0$$

$$-0.5\ln\left(\frac{1}{3}\right) > T > 0$$

$$0 < T < 0.549$$

Thus the system is stable as long as $T < 0.549$.

Digital Control Systems

LECTURE 8

Stability of Digital Control Systems

Jury Test

Prepared by: Mr. Abdullah I. Abdullah

Stability of Digital Control Systems

- The difference between the stability of the continuous system and digital system is the effect of sampling rate on the transient response.
- Changes in sampling rate not only change the nature of the response from overdamped to underdamped, but also can turn the system to an unstable.

Jury Test

- Stability test method presented by Eliahu Ibraham Jury.
- Jury stability test is similar to Routh–Hurwitz stability criterion used for continuous systems.
- In Jury test, the characteristic equation of a discrete system of order *n* is expressed as:

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0,$$

where $a_n > 0$

Jury table

- Row of the Jury table is a listing of $F(z)$ coefficients in order of **increasing** power of z.

- The table has $2n-3$ rows (always odd)

	z^0	z^1	z^2	$\cdots$	z^{n-1}	z^n
1	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_1	a_0
3	b_0	b_1	b_2	$\cdots$	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_0	
5	c_0	c_1	c_2	$\cdots$		
6	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$		
.	.	.	.	$\cdots$		
$2n-3$	r_0	r_1	r_2			

The elements of this array are defined as follows:

- Elements of **even**-numbered row are the elements of the preceding row, in reverse order.
- Elements of the **odd**–numbered rows are defined as given by *bk* , *ck* , …

$$b_k = \begin{vmatrix} a_o & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad \boldsymbol{k = 0, 1, \ldots, n-1}$$

$$c_k = \begin{vmatrix} b_o & b_{n-k} \\ b_n & b_k \end{vmatrix}, \qquad \boldsymbol{k = 0, 1, \ldots, n-2}$$

$$r_o = \begin{vmatrix} s_o & s_3 \\ s_3 & s_o \end{vmatrix}, \qquad r_1 = \begin{vmatrix} s_o & s_2 \\ s_3 & s_1 \end{vmatrix}, r_2 = \begin{vmatrix} s_o & s_1 \\ s_3 & s_2 \end{vmatrix}$$

The *necessary and sufficient* conditions for the characteristic equation to have all roots inside the unit circle are given as:

(I) *Necessary conditions*

$$F(1) > 0,$$
$$(-1)^n F(-1) > 0,$$
$$|a_0| < a_n,$$

(II) *Sufficient conditions*

$$|b_0| > |b_{n-1}|$$
$$|c_0| > |c_{n-2}|$$
$$|d_0| > |d_{n-3}|$$
$$\vdots$$

Jury Test is applied as follows:

- Check the three conditions (I) and stop if any of them is not satisfied.
- Construct Jury array and check the conditions (II) . Stop if any condition is not satisfied.

Example-1

- Test the stability of the polynomial.

$$F(z) = z^5 + 2.6z^4 - 0.56z^3 - 2.05z^2 + 0.0775z + 0.35$$

Solution

- Develop Jury's Table [(2n-3) rows].

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	-2.05	-0.56	2.6	1
2	1	2.6	-0.56	-2.05	0.0775	0.35
3	b_o	b_1	b_2	b_3	b_4	
4	b_4	b_3	b_2	b_1	b_o	
5	c_o	c_1	c_2	c_3		
6	c_3	c_2	c_1	c_o		
7	d_o	d_1	d_2			

- 3rd row is calculated using

$$b_k = \begin{vmatrix} a_o & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

k=0,1,2,3,….n-1

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	-2.05	-0.56	2.6	1
2	1	2.6	-0.56	-2.05	0.0775	0.35
3	-0.8775	-2.5728	-0.1575	1.854	0.8352	
4	b_4	b_3	b_2	b_1	b_o	
5	c_o	c_1	c_2	c_3		
6	c_3	c_2	c_1	c_o		
7	d_o	d_1	d_2			

$$b_o = \begin{vmatrix} a_o & a_5 \\ a_5 & a_o \end{vmatrix} = \begin{vmatrix} 0.35 & 1 \\ 1 & 0.35 \end{vmatrix} = -0.8775$$

$$b_1 = \begin{vmatrix} a_o & a_4 \\ a_5 & a_1 \end{vmatrix} = \begin{vmatrix} 0.35 & 2.6 \\ 1 & 0.0775 \end{vmatrix} = -2.5728$$

$$b_2 = \begin{vmatrix} a_o & a_3 \\ a_5 & a_2 \end{vmatrix} = \begin{vmatrix} 0.35 & -0.56 \\ 1 & -2.05 \end{vmatrix} = -0.1575$$

$$b_3 = \begin{vmatrix} a_o & a_2 \\ a_5 & a_3 \end{vmatrix} = \begin{vmatrix} 0.35 & -2.05 \\ 1 & -0.56 \end{vmatrix} = 1.854$$

$$b_4 = \begin{vmatrix} a_o & a_1 \\ a_5 & a_4 \end{vmatrix} = \begin{vmatrix} 0.35 & 0.0775 \\ 1 & 2.6 \end{vmatrix} = 0.8352$$

- 4rth row is same as 3rd row in reverse order

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	-2.05	-0.56	2.6	1
2	1	2.6	-0.56	-2.05	0.0775	0.35
3	-0.8775	-2.5728	-0.1575	1.854	0.8352	
4	0.8352	1.854	-0.1575	-2.5728	-0.8775	
5	c_o	c_1	c_2	c_3		
6	c_3	c_2	c_1	c_o		
7	d_o	d_1	d_2			

- 5^{th} row is calculated using

$$c_k = \begin{vmatrix} b_o & b_{n-k} \\ b_n & b_k \end{vmatrix}$$

k=0,1,2,….n-2

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	-2.05	-0.56	2.6	1
2	1	2.6	-0.56	-2.05	0.0775	0.35
3	-0.8775	-2.5728	-0.1575	1.854	0.8352	
4	0.8352	1.854	-0.1575	-2.5728	-0.8775	
5	0.077	0.7143	0.2693	0.5151		
6	c_3	c_2	c_1	c_o		
7	d_o	d_1	d_2			

- 6^{th} row is same as 5^{th} row in reverse order

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	-2.05	-0.56	2.6	1
2	1	2.6	-0.56	-2.05	0.0775	0.35
3	-0.8775	-2.5728	-0.1575	1.854	0.8352	
4	0.8352	1.854	-0.1575	-2.5728	-0.8775	
5	0.077	0.7143	0.2693	0.5151		
6	0.5151	0.2693	0.7143	0.077		
7	d_o	d_1	d_2			

- 7^{th} row is calculated using

k=0,1,2,….n-3

$$d_k = \begin{vmatrix} d_o & d_{n-k} \\ d_n & d_k \end{vmatrix}$$

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	-2.05	-0.56	2.6	1
2	1	2.6	-0.56	-2.05	0.0775	0.35
3	-0.8775	-2.5728	-0.1575	1.854	0.8352	
4	0.8352	1.854	-0.1575	-2.5728	-0.8775	
5	0.077	0.7143	0.2693	0.5151		
6	0.5151	0.2693	0.7143	0.077		
7	-0.2593	-0.0837	-0.3472			

- Now we need to evaluate following conditions

5th order System

(1). $F(1) > 0$

(2). $(-1)^5 F(-1) > 0$

(3). $|a_o| < a_5$

(4). $|b_0| > |b_4|$

(5). $|c_0| > |c_3|$

(6). $|d_0| > |d_2|$

- The first two conditions require the evaluation of *F*(*z*) at *z* = ±1.

$$F(z) = z^5 + 2.6z^4 - 0.56z^3 - 2.05z^2 + 0.0775z + 0.35$$

$$F(1) = 1 + 2.6 - 0.56 - 2.05 + 0.0775 + 0.351 = 1.4175$$

$$F(-1) = -1 + 2.6 + 0.56 - 2.05 - 0.0775 + 0.35 = 0.3825$$

(1). $F(1) > 0$ ✓ Satisfied

(2). $(-1)^5 F(-1) > 0$ ✗ Not Satisfied

- Next four conditions require Jury's table

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	-2.05	-0.56	2.6	1
2	1	2.6	-0.56	-2.05	0.0775	0.35
3	-0.8775	-2.5728	-0.1575	1.854	0.8352	
4	0.8352	1.854	-0.1575	-2.5728	-0.8775	
5	0.077	0.7143	0.2693	0.5151		
6	0.5151	0.2693	0.7143	0.077		
7	-0.2593	-0.0837	-0.3472			

(3). $|a_o| < a_5$ Satisfied

(5). $|c_0| > |c_3|$ Not Satisfied

(4). $|b_0| > |b_4|$ Satisfied

(6). $|d_0| > |d_2|$ Not Satisfied

- The system is unstable ,because the roots on or outside the unit circle. .

The polynomial can be factored as $F(z) = (z - 0.7)(z - 0.5)(z + 0.5)(z + 0.8)(z + 2.5) = 0$

and has a root at -2.5 outside the unit circle.

Example-2

- Test the stability of the polynomial.

$$F(z) = z^2 - 0.25$$

Solution

- Develop Jury's Table [(2n-3) rows].

Row	z^0	z^1	z^2
1	-0.25	0	1

$$F(1) = 1 - 0.25 = 0.75$$

$$F(-1) = 1 - 0.25 = 0.75$$

(1). $F(1) > 0$ Satisfied

(2). $(-1)^2 F(-1) > 0$ Satisfied

(3). $|a_o| < a_2$ Satisfied

- Since all the conditions are satisfied, the system is stable.

Example 3

The closed-loop transfer function of a system is given by $\dfrac{G(z)}{1+G(z)}$, where

$G(z) = \dfrac{0.2z+0.5}{z^2-1.2z+0.2}$ Determine the stability of this system using Jury Test.

The characteristic equation is $1 + G(z) = 0$

$$1 + \frac{0.2z+0.5}{z^2-1.2z+0.2} = 0$$

$$z^2 - z + 0.7 = 0$$

Applying Jury Test:

$$F(1) = 0.7 > 0, \quad F(-1) = 2.7 > 0,$$
$$|a_0| = 0.7 < 1 = a_2$$

All conditions are satisfied, so the system is **stable**.

Example 4

Determine the stability of a system having the following characteristic equation:

$$F(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0$$

Applying Jury test:

$$a_3 = 1, a_2 = -2, a_1 = 1.4, a_0 = -0.1$$

$$F(1) = 0.3 > 0, \quad F(-1) = -4.5 < 0, \quad |a_0| = 0.1 < 1 = a_3$$

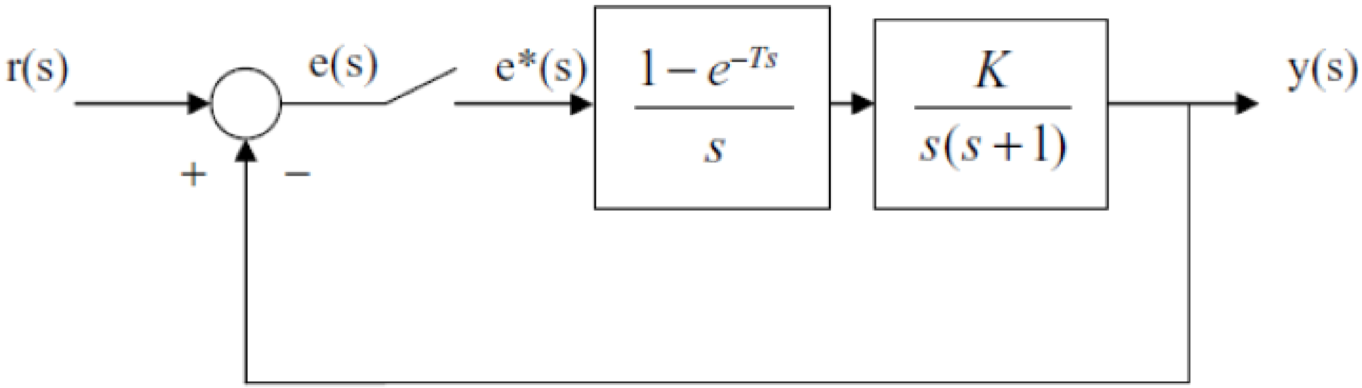
The first conditions are satisfied. Applying the other condition:

$$\begin{vmatrix} -0.1 & 1 \\ 1 & -0.1 \end{vmatrix} = -0.99 \quad \text{and} \quad \begin{vmatrix} -0.1 & 1.4 \\ 1 & -2 \end{vmatrix} = -1.2$$

since $|-0.99| < |-1.2|$, the system is **stable**

Example 5

The block diagram of a sampled data system is shown below. Use Jury Test to determine the value of K for which the system is stable. Assume that $K > 0$ and $T = 1$ s.



The characteristic equation is:

$$1 + G(z) = 0$$

$$G(z) = \mathscr{Z}\left\{\frac{1-e^{-Ts}}{s}\frac{K}{s(s+1)}\right\} = (1 - z^{-1})\mathscr{Z}\left\{\frac{k}{s^2(s+1)}\right\}$$

$$= \frac{K(0.368z + 0.264)}{(z-1)(z-0.368)}$$

$$z^2 - z(1.368 - 0.368K) + 0.368 + 0.264K = 0$$

Apply Jury test for 2nd order equation:

$$F(z) = a_2 z^2 + a_1 z + a_0 = 0, \quad where \quad a_2 > 0$$

$$z^2 - z(1.368 - 0.368K) + 0.368 + 0.264K = 0$$

$$F(1) > 0, \quad F(-1) > 0, \quad |a_0| < a_2$$

$$F(1) = 0.632K > 0 \quad \Rightarrow K > 0$$

$$F(-1) = 2.736 - 0.104K > 0 \quad \Rightarrow K < 26.3$$

The third condition is:

$$|a_0| < a_2$$

$$|0.368 + 0.264K| < 1$$

$$-1 < 0.368 + 0.264K < 1$$

$$-5.18 < K < 2.4$$

Combining all inequalities together, the system is stable for 0 < *K* < 2.4

Example 6

Determine the stability of the system having the following characteristic equation:

$$F(z) = z^4 + z^3 + 2z^2 + 2z + 0.5 = 0$$

z^0	z^1	z^2	z^3	z^4
0.5	2	2	1	1
1	1	2	2	0.5
−0.75	0	−1	**−1.5**	
−1.5	−1	0	−0.75	
−1.6875	−1.5	**0.75**		

$$F(1) = 6.5 > 0, \checkmark$$

$$(-1)^4 F(-1) = 1 - 1 + 2 - 2 + 0.5 > 0, \checkmark$$

$$|a_0| = 0.5 < 1 = a_4 \checkmark$$

$$|b0| = 0.75 > |b3| = 1.5 \times$$

$$|c0| = 1.6875 > |c2| = 0.75 \checkmark$$

System is **unstable**

Digital Control Systems

LECTURE 9

Stability of Digital Control Systems

Routh–Hurwitz Criterion

Prepared by: Mr. Abdullah I. Abdullah

Routh–Hurwitz Criterion

- The stability of a sampled data system can be analyzed by transforming the system characteristic equation into the s-plane and then applying the well-known Routh–Hurwitz criterion.
- A bilinear transformation is usually used to transform the interior of the unit circle in the z-plane into the left-hand s-plane (w-plane).

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \qquad z = \left(\frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}\right)$$

Some engineers replace s by "w" and call the resulting operation as *"w" transform"*

For this transformation, z is replaced by: $z = \frac{1+w}{1-w} \Leftrightarrow w = \frac{1+z}{1-z}$

$$F(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_o$$

$$F(w) = b_n \left(\frac{1+w}{1-w}\right)^n + b_{n-1}\left(\frac{1+w}{1-w}\right)^{n-1} + \cdots + b_o$$

Routh–Hurwitz criterion

Number of roots of the characteristic equation in the right hand s-plane is equal to the number of sign changes of the coefficients in the first column of the array.

Routh-Hurwitz array is formed as:

First two rows are obtained from the equation directly and the other rows are calculated as:

$$\begin{array}{l|llll}
\omega^{n} & b_{n} & b_{n-2} & b_{n-4} & \cdots \\
\omega^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
\omega^{n-1} & c_{1} & c_{2} & c_{3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots \\
\omega^{1} & j_{1} & & & \\
\omega^{0} & k_{1} & & &
\end{array}$$

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}}, \quad c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}}, \quad \cdots$$

Thus, for a stable system **all coefficients** in **first column** must have the **same sign**.

Example 1

The characteristic equation of a sampled data system is given by

$$2z^3 + z^2 + z + 1 = 0$$

Determine the stability of the system using the Routh–Hurwitz criterion.

$$2\left(\frac{1+\omega}{1-\omega}\right)^3 + \left(\frac{1+\omega}{1-\omega}\right)^2 + \left(\frac{1+\omega}{1-\omega}\right) + 1 = 0$$

$$2(1+\omega)^3 + (1-\omega)(1+\omega)^2 + (1-\omega)^2(1+\omega) + (1-\omega)^3 = 0$$

$$\omega^3 + 7\omega^2 + 3\omega + 5 = 0$$

Now, we form Routh array:

$$C_1 = \frac{7*3-5*1}{7} = \frac{16}{7} \qquad D_1 = \frac{16/7*5-0*5}{16/7} = 5$$

ω^3	1	3
ω^2	7	5
ω^1	16/7	0
ω^0	5	

No sign change in the first column, so the system is **stable**.

Roots of the characteristic equation: $2z^3 + z^2 + z + 1 = 0$

can be found using MATLAB

roots([2 1 1 1])

abs(roots([2 1 1 1]))

$0.1195 + j\,0.8138$
$0.1195 - j\,0.8138$
-0.7390

0.8226
0.8226
0.7390

all roots are less than one, i.e. the roots lie inside unit circle. Hence, we can conclude that the system is **stable**.

Example 2

- By using Routh-Hurwitz stability criterion, determine the stability of the following digital systems whose characteristic are given as.

Solution $$z^2 - 0.25 = 0$$

r1,2 = 0.500 , -0.5000

- Transforming the characteristic equation $z^2 - 0.25 = 0$ into $w - domain$ by using the bilinear transformation $z = \frac{1+w}{1-w}$ gives:

$$0.75w^2 + 2.5w + 0.75 = 0$$

Since there are no sign changes in the first column of the Routh array therefore the system is stable.

w^2	0.75	0.75
w^1	2.5	0
w^0	0.75	

Example-3

- By using Routh-Hurwitz stability criterion, determine the stability of the following digital systems whose characteristic are given as.

$$z^3 - 1.2z^2 - 1.375z - 0.25 = 0$$

Solution

- Transforming the characteristic equation into $w - domain$ by using the bilinear transformation $z = \frac{1+w}{1-w}$ gives:

$$-1.875w^3 + 3.875w^2 + 4.875w + 1.125 = 0$$

w^3	-1.875	4.875
w^2	3.875	1.125
w^1	5.419	0
w^0	1125	

- From the table above, since there is one sign change in the first column above equation has one root in the right-half of the w-plane.
- This, in turn, implies that there will be one root of the characteristic equation outside of the unit circle in the z-plane.

```
clc
clear
P=[1 -1.2 -1.375 -0.25];
r=roots(P)
d=abs(r)
```

```
r =
    1.9646
   -0.5199
   -0.2448


d =
    1.9646
    0.5199
    0.2448
```

Example-4

By using Routh-Hurwitz stability criterion, determine K for the stable digital systems whose feed forward G(z) are given as.

$$G(z) = \frac{0.632\,K\,z}{z^2 - 1.368\,z + 0.368}$$

The characteristic equation $1 + G(z) = 1 + \frac{0.632\,K\,z}{z^2 - 1.368\,z + 0.368} = 0$

$$z = \frac{1+w}{1-w}$$

$$1 + \frac{0.632\,K\,z}{z^2 - 1.368\,z + 0.368} = 0 \Rightarrow \quad 0.632K\,w^2 + 1.264\,w + (2.736 - 0.362K) = 0$$

In terms of Routh criterion

$0.632K$	$2.736 - 0.362K$
1.624	0
$2.736 - 0.362K$	

We have $0 < K < 7.558$

Example 5

Determine the stability of the following system (with T = 1 sec) using Routh's criterion:

$$\frac{Y(z)}{R(z)} = F(z) = 0.0484 \frac{z + 0.9672}{(z-1)(z-0.9048)}$$

Solution:

Tustin's (bilinear) transformation leads to

$$z = \left(\frac{1 + s\frac{T}{2}}{1 - s\frac{T}{2}}\right) = \left(\frac{1 + s\frac{1}{2}}{1 - s\frac{1}{2}}\right)$$

$$F(z)\Big|_{z=\frac{1+\frac{s}{2}}{1-\frac{s}{2}}} \cong F'(s) = 0.0484 \frac{\frac{1+\frac{s}{2}}{1-\frac{s}{2}} + 0.9672}{\left(\frac{1+\frac{s}{2}}{1-\frac{s}{2}} - 1\right)\left(\frac{1+\frac{s}{2}}{1-\frac{s}{2}} - 0.9048\right)}$$

Thus,

$$F'(s) = -\frac{\left(\frac{s}{120}+1\right)\left(\frac{s}{2}-1\right)}{s\left(\frac{s}{0.0999}+1\right)}$$

The characteristic polynomial becomes

$$A'(s) = 10.01s^2 + s + 0$$

Routh Array:

s^2:	10.01	0
s^1:	1	0
s^0:	$0 \approx \varepsilon$	

All first-column coefficients are bigger than zero ⇒ system is *(marginally) stable*.

Digital Control Systems

LECTURE 10

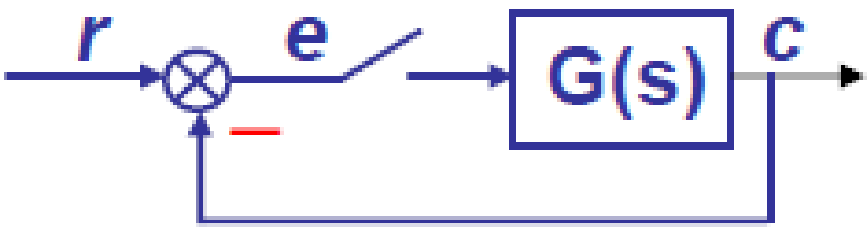
Steady State Error

Prepared by: Mr. Abdullah I. Abdullah

Steady State Error

- An important characteristic of a control system is its ability to follow, or track, certain inputs with a minimum of error. The control system designer attempts to minimize the system error to certain anticipated inputs.

- Consider the unity feedback block diagram shown in following figure.



- The error ratio can be calculated as

$$E(z) = R(z) - C(z) = R(z) - \frac{R(z)G(z)}{1 + G(z)} = \frac{R(z)}{1 + G(z)}$$

- Applying the final value theorem yields the steady-state error.

$$e_{ss} = \lim_{z \to 1} \frac{z-1}{z} E(z) = \lim_{z \to 1} \frac{z-1}{z} \frac{R(z)}{1+G(z)}$$

$$e_{ss} = \lim_{z\to 1} \frac{z-1}{z} \frac{R(z)}{1+G(z)}$$

$$e_{ss} = \begin{cases} \dfrac{1}{1+K_p^*} & \text{for unit step input} \quad r(t) = 1(t) \Leftrightarrow R(z) = \dfrac{z}{z-1}; \quad K_p^* = \lim_{z\to 1} G(z) \\ \dfrac{T}{K_v^*} & \text{for unit ramp input} \quad r(t) = t \Leftrightarrow R(z) = \dfrac{Tz}{(z-1)^2}; \quad K_v^* = \lim_{z\to 1} (z-1)G(z) \\ \dfrac{T^2}{K_a^*} & \text{for unit Parabolic input} \quad r(t) = t^2 \Leftrightarrow R(z) = \dfrac{T^2 z(z+1)}{(z-1)^3}; \quad K_a^* = \lim_{z\to 1} (z-1)^2 G(z) \end{cases}$$

where

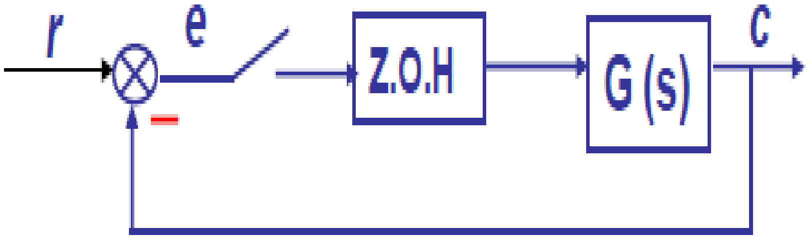
K_p^* , *Position Error Constant*

K_v^* , *Velocity Error Constant*

K_a^* , *Parabolic Error Constant*

Example

For the unity feedback control system with the transfer functions $G(s) = \dfrac{K}{s(s+5)}$ *and* $T = 1\sec$



1) Determine K for the stable system.

2) If r(t) = 1+t , determine e_{ss} .

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+5)}\right]$$

$$= (1-e^{-Ts})Z\left[\frac{K}{s^2(s+5)}\right]$$

$$= (1-e^{-Ts})Z\left[\frac{K/5}{s^2} + \frac{-K/5}{s} + \frac{K/25}{s+5}\right]$$

$$= (1-z^{-1})\left(\frac{KTz/5}{(z-1)^2} - \frac{Kz/5}{z-1} + \frac{Kz/25}{z-e^{-5T}}\right)\Bigg|_{T=1}$$

$$\approx -\frac{K}{5}\cdot\frac{z^2 - 2.2067\,z + 0.2135}{(z-1)(z-0.0067)}$$

The characteristics equation is $1+G(z)=1-\frac{K}{5}\cdot\frac{z^2-2.2067z+0.2135}{(z-1)(z-0.0067)}=0$

$$(5\text{-}K)z^2+(2.2067K-5.0335)z+(0.0335-0.2135K)=0 \qquad z=\frac{1+w}{1-w}$$

$$0.9932w^2+(9.993-1.573K)w+(10.067-2.4202K)=0$$

$$0<K<4.16$$

2- r(t) = 1+ t I/P

$$K_p^*=\lim_{z\to1}G(z)=\lim_{z\to1}\frac{K}{5}\cdot\frac{z^2-2.2067z+0.2135}{(z-1)(z-0.0067)}=\infty$$

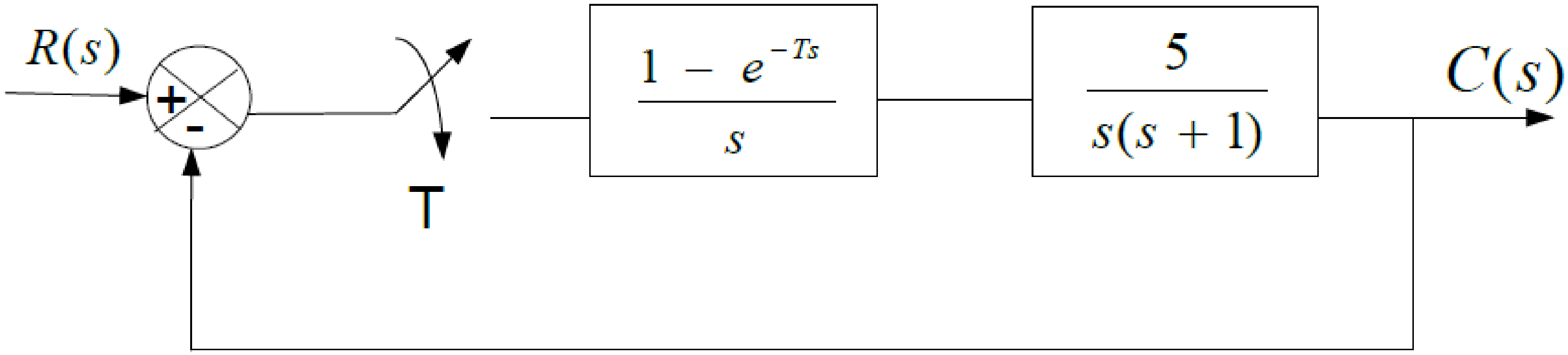
$$K_v^*=\lim_{z\to1}(z-1)G(z)=\lim_{z\to1}-\frac{K}{5}\cdot\frac{z^2-2.2067z+0.2135}{(z-0.0067)}\approx0.2K$$

$$e_{ss}=\frac{1}{1+K_p^*}+\frac{T}{K_v^*}=0+\left.\frac{T}{0.2K}\right|_{T=1}=\frac{5}{K}$$

Steady State Error and System Type

System	Steady-state errors in response to		
	Step input $r(t) = 1$	Ramp input $r(t) = t$	Acceleration input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K_p}$	∞	∞
Type 1 system	0	$\frac{1}{K_v}$	∞
Type 2 system	0	0	$\frac{1}{K_a}$

Example:2 for the unit feedback system find the steady state error.



The open-loop transfer function is

$$G(s) = \frac{1-e^{-Ts}}{s} \cdot \frac{5}{s(s+1)}$$

$$= \frac{5(1-e^{-Ts})}{s^2(s+1)} = 5(1-e^{-Ts})\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$$G(z) = 5(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

For step input: $$e^*(\infty) = \frac{1}{1+K_p}$$

$$K_p = \lim_{z\to 1} G(z) = \lim_{z\to 1} 5(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= \lim_{z\to 1} 5(z-1)\left[\frac{T}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{z-e^{-T}}\right]$$

$$= \lim_{z\to 1} 5\left[\frac{T}{(z-1)} - \frac{1}{1} + \frac{z-1}{z-e^{-T}}\right] = 5\left[\frac{T}{0} - 1 + 0\right] = \infty$$

Then

$$e^*(\infty) = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

For a unit ramp input: $e^*(\infty) = \dfrac{1}{K_v}$

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z)$$

$$= \frac{1}{T}\lim_{z\to 1}(z-1)5(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= \frac{5}{T}\lim_{z\to 1}(z-1)(z-1)\left[\frac{T}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{z-e^{-T}}\right]$$

$$= \frac{5}{T}\lim_{z\to 1}\left[T - (z-1) + \frac{(z-1)^2}{z-e^{-T}}\right]$$

$$= \frac{5}{T} \times T = 5$$

then $e^*(\infty) = \dfrac{1}{K_v} = \dfrac{1}{5} = 0.2$

For parabolic input

$$e^*(\infty) = \frac{1}{K_a} \qquad \textit{where} \quad \boldsymbol{K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z)}$$

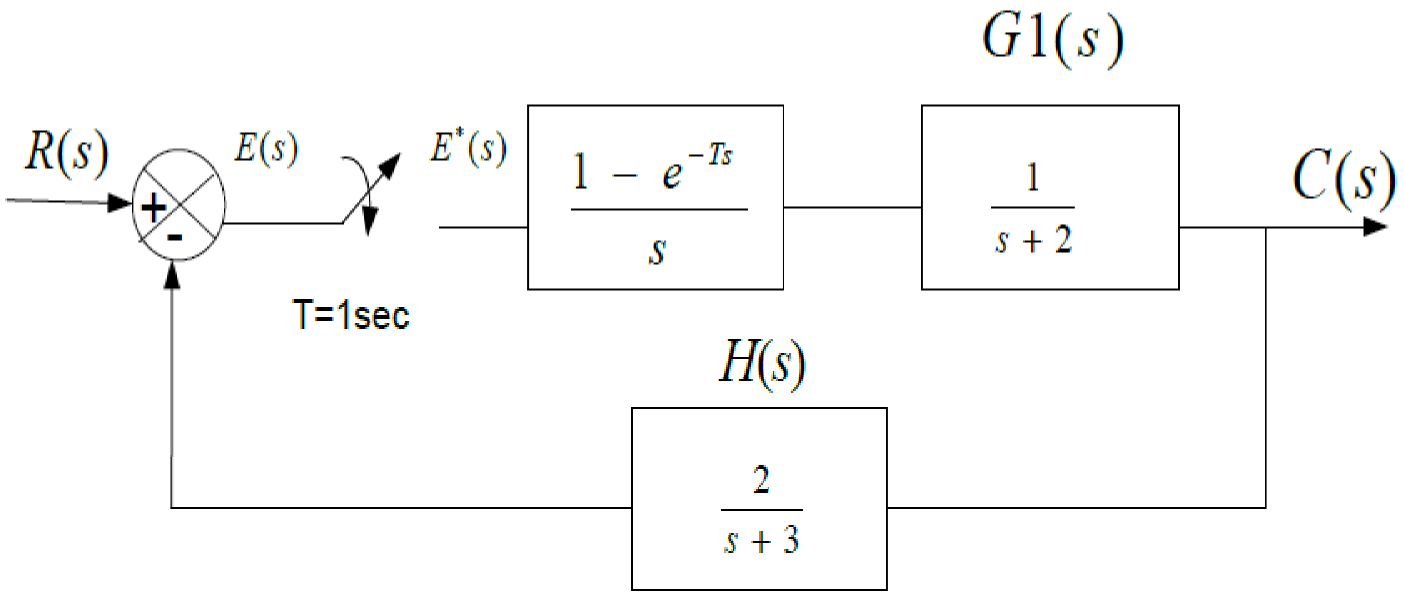
$$K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 5(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= \frac{5}{T^2}\lim_{z\to 1}\left[T(z-1) - (z-1)^2 + \frac{(z-1)^3}{z-e^{-T}}\right] = 0$$

$$\textit{then} \qquad e^*(\infty) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

H.W.:

For the discrete control system shown below, Find the steady state error of unit step, ramp and parabolic input



Digital Control Systems

LECTURE 11

Root Locus in the z-plane

Prepared by: Mr. Abdullah I. Abdullah

Root-Locus

Definition:

The root-locus method is a plot of the roots of the characteristic equation of the closed-loop system as a function of the open-loop gain constant K which is varied from 0 to infinity $0 \le K < \infty$.

In many LTI discrete time control systems, the characteristics equation may have either of the following two forms.

$$1 + KG(z)H(z) = 0$$

or

$$1 + KGH(z) = 0$$

Where , $G(z)H(z)$ or $GH(z)$ as known open loop pulse transfer function is equal to L(z)

$$L(z) = G(z)H(z) = -\frac{1}{K}$$

$$or \quad L(z) = GH(z) = -\frac{1}{K}$$

The characteristics equation should be rearranged in the following form

$$1 + KL(z) = \frac{K(z - z_1) \ldots (z - z_i) \ldots (z - z_m)}{(z - p_1) \ldots (z - p_c) \ldots (z - p_n)}$$

The closed-loop system will remain stable providing the loci remain within the unit circle

Since L(z) is a complex quantity it can be split into two equations by equating angles and magnitudes of two sides.
This gives us the angle and magnitude criteria as

Magnitude condition:

$$|L(z)| = 1$$

Angle condition:

$\angle L(z) = \mp 180^{\circ}(2n+1)$ Where n =0,1,2,3,…..

Rules for Drawing Root Locus

1. The root locus is symmetric about real axis. (Number of root locus branches equals the number of open loop poles).

2-The root locus starts at the open-loop poles and terminates at the open-loop zeros or at infinity.

3-The angles of the asymptotes of the root locus that end at infinity are determined by

$$\gamma = \frac{(1+2n)180^0}{[no.of\ poles\ (n)]-[no.of\ zeros\ (m)]} \qquad n=0,1,....(n-m-1)$$

4. The real-axis intercept of the asymptotes is

$$\delta = \frac{\sum_{c=1}^{n} Re(P_c) - \sum_{i=1}^{m} Re(z_i)}{n-m}$$

5-Breakaway (Break in) points or the points of multiple roots are the solution of the following equation $\frac{dK}{dz} = 0$, where K is expressed as a function of z from the characteristic equation.

6. The unit circle crossing of the root locus can be determined by setting up the Jury's array from closed-loop characteristic equation. Determine the range of values that K must have to satisfy the necessary and sufficient conditions for a stable system.

7-The angle of departure from a complex open loop pole is given by

$$\emptyset_p = 180^{\circ} + \emptyset$$

Where $\emptyset$ is the net angle contribution of all other open loop poles and zeros to that pole.

$$\emptyset = \sum_{i} \varphi_i - \sum_{j \neq p} \gamma_j$$

φ_i are the angles contributed by zeros and γ_j are the angles contributed by the poles.

8-The angle of arrival at a complex zero is given by $\emptyset_z = 180^{\circ} - \emptyset$

where is same as in the above rule.

9-The gain at any point z_o on the root locus is given by $K = \frac{\prod_{j=1}^{n} |z_o + p_j|}{\prod_{i=1}^{m} |z_o + z_i|}$

1. Root Locus without Zero Order Hold

Example 1 : Sketch the root locus for the diagram shown in Fig.(1)

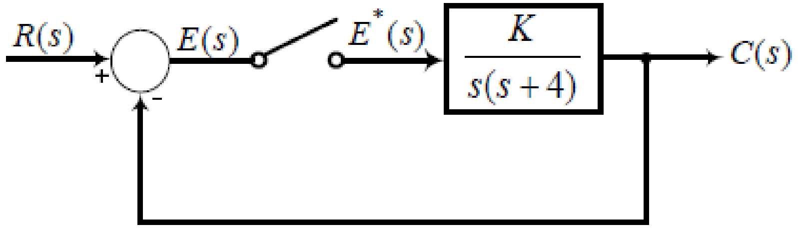


Figure (1) *Sample-data system*

The z-transform for the output *C*(*z*) is $$C(z)=\frac{G(z)}{1+G(z)}R(z)$$

The z-transformed characteristic equation is $1+G(z)=0$

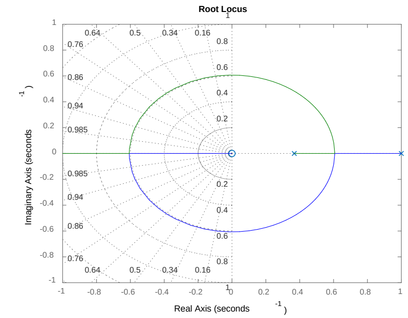
$$G(s)=\frac{K}{s(s+4)}=\frac{A}{S}+\frac{B}{(s+4)}=\left(\frac{K/4}{s}-\frac{K/4}{(s+4)}\right) \quad , \quad G(s)=\frac{K}{4}\left(\frac{1}{s}-\frac{1}{s+4}\right)$$

The corresponding z transform is

$$G(z)=\frac{K}{4}\left(\frac{z}{z-1}-\frac{z}{z-e^{-4T}}\right)=\left(\frac{K}{4}\right)\frac{z(1-e^{-4T})}{(z-1)(z-e^{-4T})}=K\frac{0.158\,z}{(z-1)(z-0.368)}$$, For T=0.25 sec.

Open-loop poles and zeros:
Poles: z =1 and z = 0.368
Zeros: z = 0

$$G(z) = K\frac{0.158\,z}{(z-1)(z-0.368)}$$



- **Number of branches:** Number of branches equals No. of poles=2.
- **Root locus locations on the real axis:** The root locus on the real axis lies between poles (z =1 and z = 0.368) and to the left of zero (z=0).
- **Break away and in points:**

The characteristic equation is
$$1+G(z) = 1+K\frac{0.158\,z}{(z-1)(z-0.368)} = 0$$

$$= \frac{z^2 - 1.368\,z + 0.368 + 0.158\,k\,z}{z^2 - 1.368\,z + 0.368} = 0 \;\Rightarrow\; z^2 - 1.368\,z + 0.368 + 0.158\,K\,z = 0$$

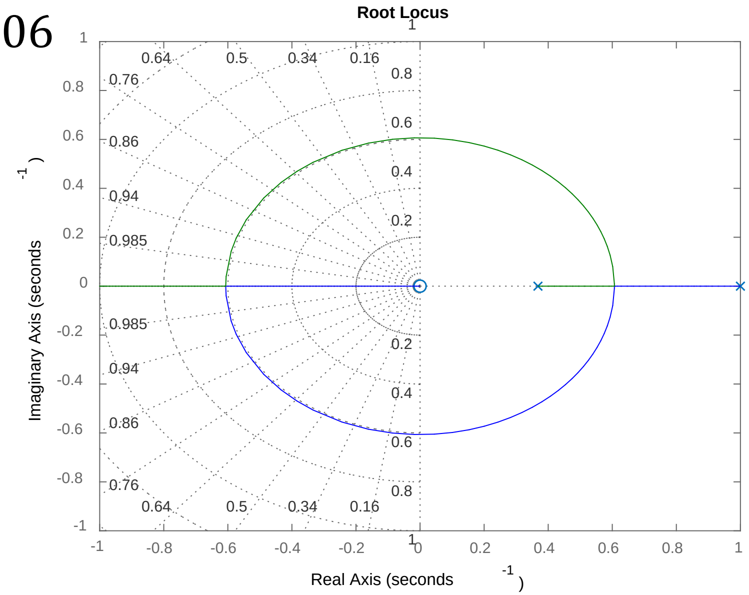
$$k = -\frac{1}{0.158}\,\frac{z^2 - 1.368z + 0.368}{z}$$

$$\frac{dk}{dz} = -\left(\frac{1}{0.158}\right)\frac{[z(2z-1.368)-(z^2-1.368\,z+0.368)]}{z^2} = 0 \;\Rightarrow -[z(2z-1.368)-(z^2-1.368\,z+0.368)] = 0$$

$$-z^2 + 0.368 = 0 \;\Rightarrow\; z^2 - 0.368 = 0 \;\Rightarrow\; (z-0.606)(z+0.606) = 0 \;\Rightarrow\; \begin{aligned} z_1 &= 0.606 \\ z_2 &= -0.606 \end{aligned}$$

To find the value of K at break away and in points, we use the magnitude condition:

The gain K at breakaway point: $z = 0.606$



$$1+\frac{0.158\,k\,z}{(z-1)(z-0.368)}=0$$

$$\frac{k\;0.158\,z}{(z-1)(z-0.368)}=-1$$

$$K=\left|\frac{(z-1)(z-0.368)}{0.158z}\right|_{z=0.606}=\left[\frac{|(z-1)||(z-0.368)|}{|0.158z|}\right]_{z=0.606}=0.979$$

The gain K at break in point: $z=-0.606$

$$K=\left|\frac{(z-1)(z-0.368)}{0.158z}\right|_{z=-0.606}=16.337$$

Crossing points of z-plane imaginary axis:

In general $z = a + jb$, and when the root locus crosses the imaginary axis of the z-plane, then the real part becomes zero, or $z = jb$. Substitute this value in the characteristic equation one can obtain:

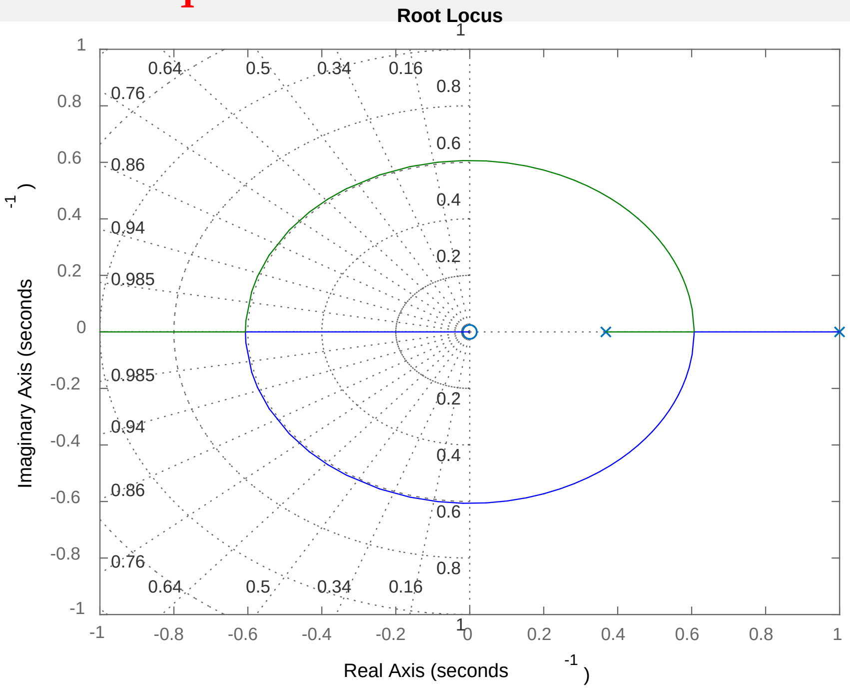
- The characteristic equation is

$$z^2 - 1.368z + 0.368 + 0.158Kz = 0$$

$$(jb)^2 - 1.368(jb) + 0.368 + 0.158K(jb) = 0$$

$$-b^2 - j1.368b + 0.368 + j0.158Kb = 0$$

$$\underbrace{(-b^2 + 0.368)}_{Real} + \underbrace{j(-1.368b + 0.158Kb)}_{Imaginary} = 0$$



Two equations will be obtained: $-b^2 + 0.368 = 0$ and $-1.368b + 0.158Kb = 0$

From the first equation one can obtain the point of interception of root locus with the imaginary axis $-b^2 + 0.368 = 0 \Rightarrow b = \pm 0.606 \rightarrow z = \pm j0.606$

Substitute the value of b at the second equation, the value of gain K at the imaginary axis becomes

$$-1.368b + 0.158Kb = 0 \Rightarrow -1.368 \times 0.606 + 0.158K\,0.606 = 0$$

$$\boxed{K = 8.658}$$

K for marginal stability: Using Routh-Hurwitz criterion (or Jury test), the value of K as the root locus crosses the unit circle into the unstable region is

$$z^2 - 1.368z + 0.368 + 0.158Kz = 0$$ Is the characteristic equation

$$z^2 - (1.368 - 0.158k)z + 0.368 = 0$$

$$F(1) = 1^2 - (1.368 - 0.158k)*1 + 0.368 = 0 \Rightarrow k > 0$$

$$(-1)^n F(-1) = -1^2(-1^2 - (1.368 - 0.158k)(-1) + 0.368 = 0 \Rightarrow 2.736 - 0.158k = 0$$

$$\Rightarrow k = 17.316$$

Unit circle crossover: Inserting K =17.316 into the characteristic equation

$$1+G(z)=1+K\frac{0.158\,z}{(z-1)(z-0.368)}=0 \quad \Rightarrow 1+17.316\times\frac{0.158\,z}{(z-1)(z-0.368)}=0$$

$$\Rightarrow\ z^2+1.367z+0.368\ \Rightarrow \text{ The roots are } z=\pm 1$$

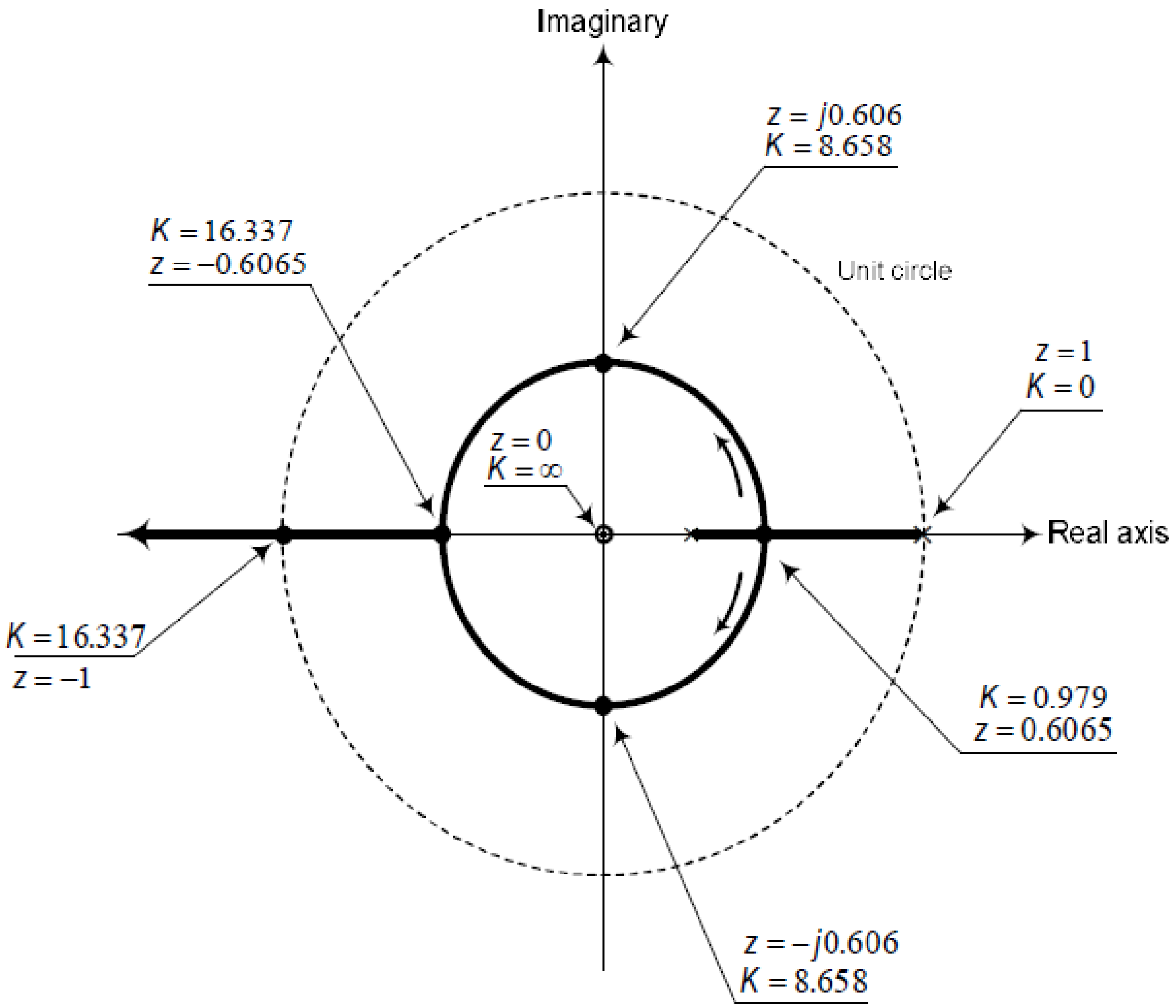
Angle of asymptotes $\lambda=\dfrac{(2n+1)180}{p-z}$ n=0,1,2,3

where p=number of poles and z is the number of zeros. Thus λ becomes λ =180

The real axis interception of the asymptotes is $\sigma_x=\dfrac{\sum_0^p z_p-\sum_0^z z_z}{p-z}$

$$=\frac{1+0.368-0}{2-1}=1.368$$

The complete root-locus plot may now be constructed as shown in the following figure



Example 2: Draw the root locus for the characteristic equation

$$1+\frac{0.368K(z+0.717)}{(z-1)(z-0.368)}=0$$

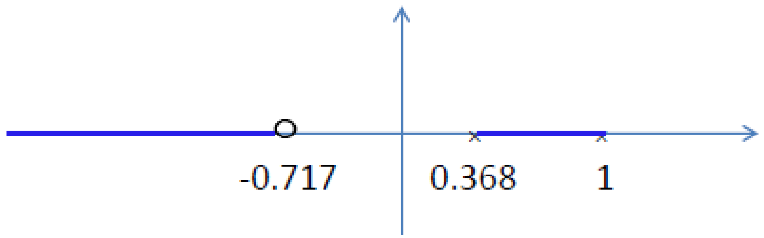
1) Starting point (K=0) $\longrightarrow$ $z=1$ and $z=0.368$

   Ending point ($K=\infty$) $\longrightarrow$ $z=-0.717$ and $z=\infty$

2) The number of asymptote = 2-1=1

   Angle of asymptote = $r180^\circ/(2-1)=180^\circ,\ r=\pm1,\pm3,\ldots$

3) Root loci on real axis (Right-hand-side rule...)



4) Break-in and break-away points

$$\frac{d}{dz}\left(\frac{z+0.717}{(z-1)(z-0.368)}\right)=0 \longrightarrow z^2+1.434z-0.717=0$$

$$z=-2.08\ (k=15) \qquad z=0.65\ (k=0.196)$$

Break-in Break-away

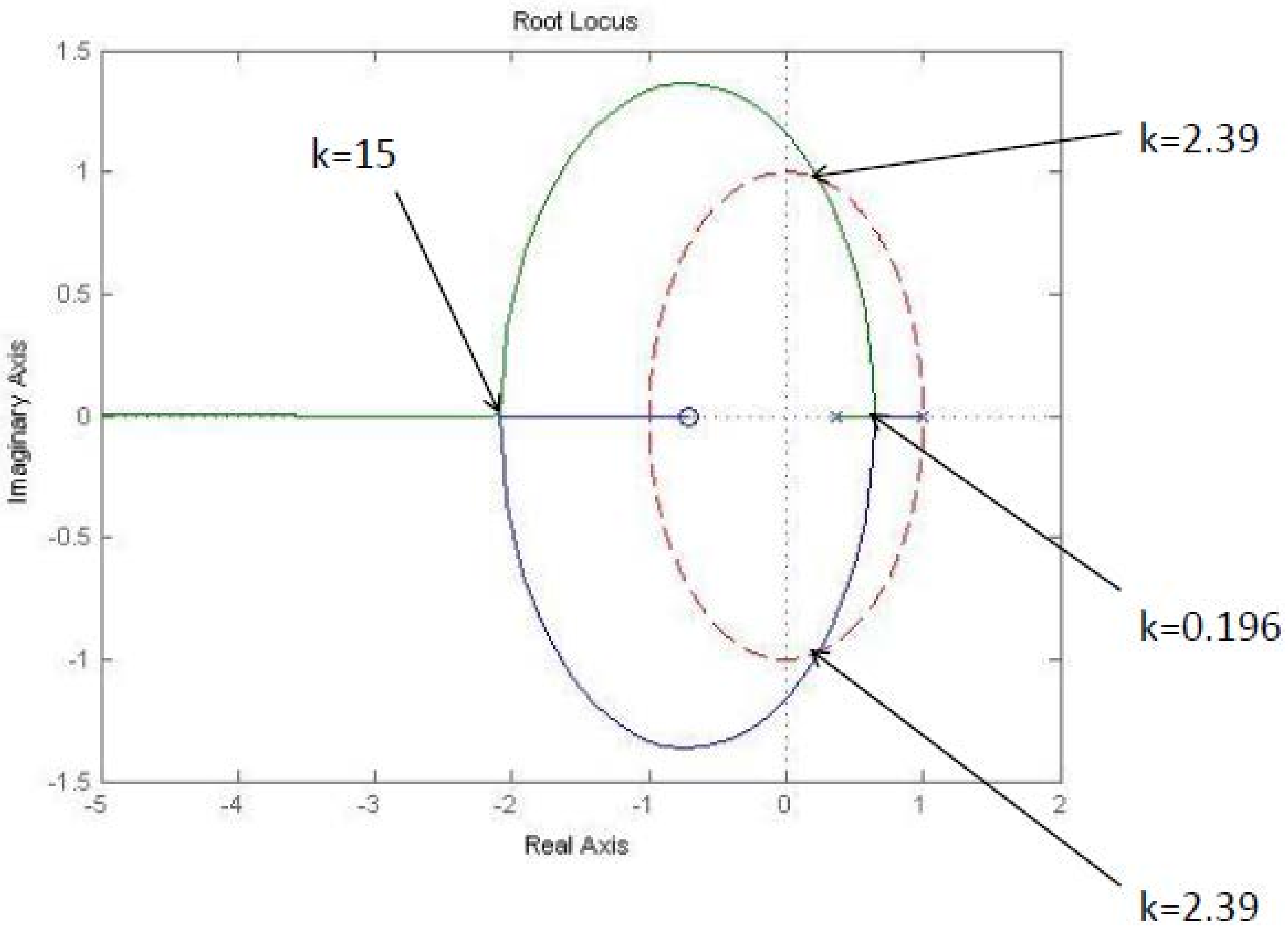
5) Find the crossing point on the unit circle …

Jury's test ⟶ stable when $0<k<2.39$

If $k=2.39 \longrightarrow z^2-0.488z+1=0 \longrightarrow z=0.244\pm j0.970$

6) Sketch the root locus, check with Matlab

```
n=[0 0.368 2.464];
d=[1 -1.368  0.368];
rlocus(n,d)
```


Root Locus
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-5
-4
-3
-2
-1
0
1
2
k=15
k=2.39
k=0.196
k=2.39

Digital Control Systems

LECTURE 12

Discrete PID Controller Design

Prepared by: Mr. Abdullah I. Abdullah

PID controller

The proportional–integral–derivative (PID), also called three-term, is the most widely used controller in process industry.

The output u(t) of the PID controller shown in figure 1 is the sum of three terms:

$$u(t) = K_p\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right] \text{ --- (1)}$$

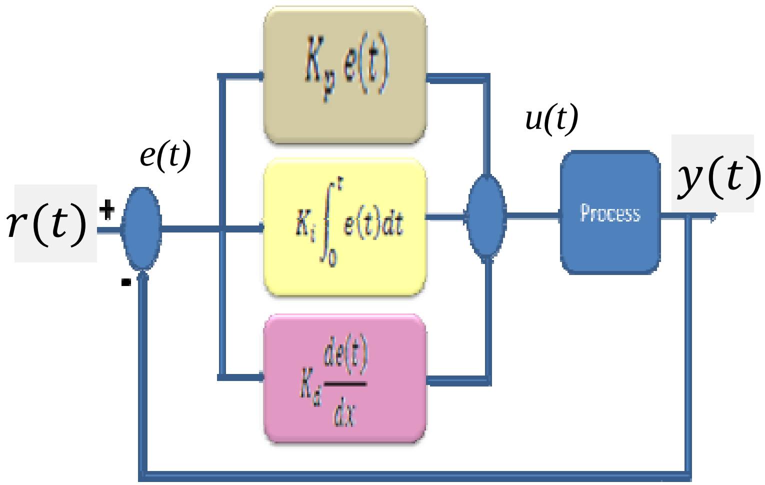


Figure 1 :PID controller

where

- ◭ e(t) = r (t) − y(t), is the error (controller input)
- ◭ r (t) is the reference input
- ◭ y(t) is the plant output.
- ◭ Ti is known as the integral time.
- ◭ Td is known as the derivative time.

PID controller actions

- Proportional: the error is multiplied by a gain. The higher is the gain, the faster is the response. However, very high gain may cause instability.

- Integral: is used to remove steady-state error. However, integral action increases the overshoot and reduces system stability.

- Derivative: is used to improve the transient response by reducing overshoot.

By taking Laplace transform of equation (1) :

$$u(t) = K_p \left[e(t) + \frac{1}{T_i} \int_0^t e(t)dt + T_d \frac{de(t)}{dt} \right]$$

$$\frac{U(s)}{E(s)} = K_p \left(1 + \frac{1}{T_i s} + T_d s \right) = K_p + \frac{K_i}{s} + K_d s \quad \text{--- (2)}$$

With $K_i = \frac{K_p}{T_i}$

$K_d = K_p T_d$

Discrete PID Controller

To implement PID control using a digital computer we convert the following continuous-time equation into a discrete form:

$$u(t) = K_p \left(e(t) + \frac{1}{T_i} \int_0^t e(\tau) d\tau + T_d \frac{de(t)}{dt} \right)$$

To do this, a simple method is to approximate integral and derivative using trapezoidal approximation for the integral and the backward difference approximation for the derivative :

$$\int_0^t e(\tau) d\tau \approx \sum_{k=1}^{n} T\, e(kT), \qquad \text{--- (3)}$$

$$\frac{de(t)}{dt} \approx \frac{e(nT) - e(nT - T)}{T} \qquad \text{--- (4)}$$

Using finite difference approximations, we can write:

$$u(nT) = K_p \left[e(nT) + \frac{1}{T_i} \sum_{k=1}^{n} Te(kT) + T_d \frac{e(nT) - e(nT - T)}{T} \right] \qquad \text{--- (5)}$$

Using subscripts instead of arguments, then eq. (5) become

$$u_n = K_p \left[e_n + \frac{1}{T_i} \sum_{k=1}^{n} T e_k + T_d \frac{e_n - e_{n-1}}{T} \right] \quad \text{--- (6)}$$

where $u_n = u(nT)$, $e_n = e(nT)$ and $e_{n-1} = e(nT - T)$.

This is called the **position form** of discrete PID controller. The drawback of this form is that: to calculate the controller output u_n we need error values $e_k,\ k = 1 \rightarrow n$.

From the position form equation 6 we can write :

$$u_{n-1} = K_p \left[e_{n-1} + \frac{1}{T_i} \sum_{k=1}^{n-1} T e_k + T_d \frac{e_{n-1} - e_{n-2}}{T} \right] \quad \text{--- (7)}$$

Subtracting these two equations (eq. (7) from eq.(6) , we obtain:

$$u_n - u_{n-1}$$

$$u_n = u_{n-1} + K_p[e_n - e_{n-1}] + \frac{K_p T}{T_i} e_n + \frac{K_p T_d}{T}[e_n - 2e_{n-1} + e_{n-2}]$$ --- (8)

where $u_n = u(nT)$, $u_{n-1} = u(nT - T)$, $e_n = e(nT)$,$e_{n-1} = e(nT - T)$ and $e_{n-2} = e(nT - 2T)$.

Here the current control signal u_n is an update of the previous value u_{n-1}. This is called the velocity form.

The velocity form of discrete PID controller is:

$$u_n = u_{n-1} + K_p[e_n - e_{n-1}] + \frac{K_p T}{T_i} e_n + \frac{K_p T_d}{T}[e_n - 2e_{n-1} + e_{n-2}]$$ --- (9)

$$u_n - u_{n-1} = \underbrace{K_p\left(1 + \frac{T}{T_i} + \frac{T_d}{T}\right)}_{K_0} e_n + \underbrace{K_p\left(-1 - 2\frac{T_d}{T}\right)}_{K_1} e_{n-1} + \underbrace{K_p\left(\frac{T_d}{T}\right)}_{K_2} e_{n-2}.$$ --- (10)

Taking z-transform of both sides of eq.(10), we get the transfer function of discrete PID controller:

$$\frac{U(z)}{E(z)} = \frac{K_0 + K_1 z^{-1} + K_2 z^{-2}}{1 - z^{-1}}$$ --- (11)

Transfer function of discrete PID controller is:

$$\frac{U(z)}{E(z)} = \frac{K_0 + K_1 z^{-1} + K_2 z^{-2}}{1 - z^{-1}}$$

where:

$$k_0 = K_p \left(1 + \frac{T}{T_i} + \frac{T_d}{T}\right)$$

$$k_1 = -K_p \left(1 + 2\frac{T_d}{T}\right)$$

$$k_2 = K_p \left(\frac{T_d}{T}\right)$$

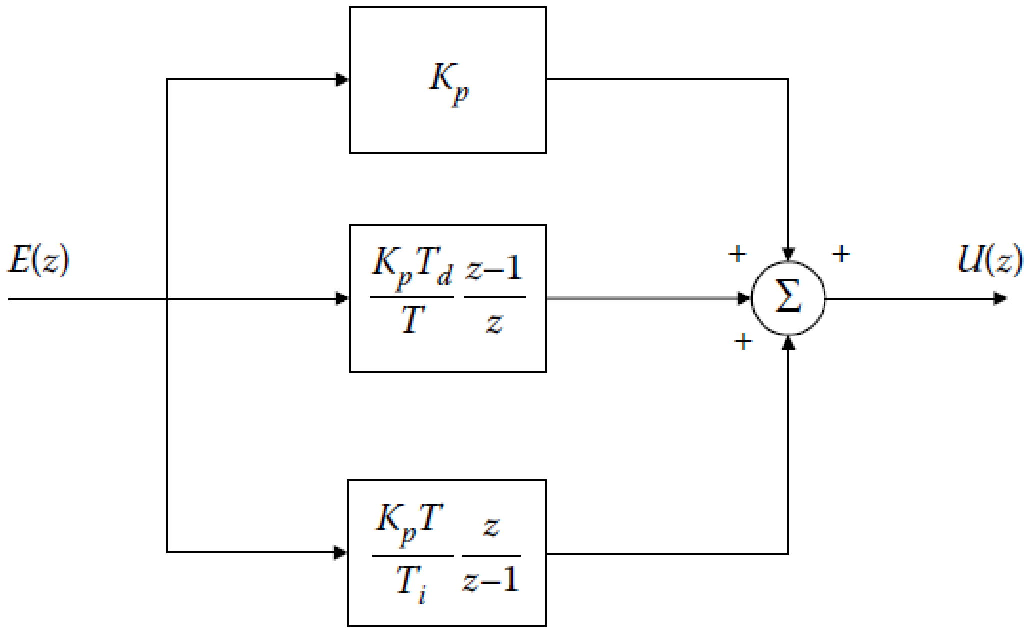
Note :

PID controllers are implemented in discrete time but tuned using a continuous formulation.

PID tuning involves the selection of the best values of K_p , K_i and K_d (or T_p, T_i and T_d). It depends on the process.

The Transfer function of discrete PID controller can be written :

$$\frac{U(z)}{E(z)} = K_p\left(1 + \frac{T}{T_i(1-z^{-1})} + \frac{T_d}{T}(1-z^{-1})\right)$$



Block diagram of a digital PID controller.

Digital Control Systems

LECTURE 13

Discrete PID Controller Tuning

Prepared by: Mr. Abdullah I. Abdullah

PID Tuning

- Tuning the controller involves adjusting the parameters K_p ,T_d and T_i in order to obtain a satisfactory response.
- There are many techniques for tuning a controller, ranging from the first techniques described by J.G. Ziegler and N.B. Nichols (known as the Ziegler–Nichols tuning algorithm), to recent auto-tuning controllers.
- In this section we shall look at the tuning of PID controllers using the Ziegler–Nichols tuning algorithm.
- Ziegler and Nichols suggested values for the PID parameters of a plant based on open-loop or closed-loop tests of the plant.
- According to Ziegler and Nichols, the open-loop transfer function of a system can be approximated with a time delay and a single-order system, i.e.

$$G(s) = \frac{K e^{-sT_D}}{T_1 s + 1},$$

- where T_D is the system time delay (i.e. transportation delay), and T_1 is the time constant of the system.

- For open-loop tuning, we first find the plant parameters by applying a step input to the open loop system.
- The plant parameters K ,T_D and T_I are then found from the result of the step test as shown in figure 2.

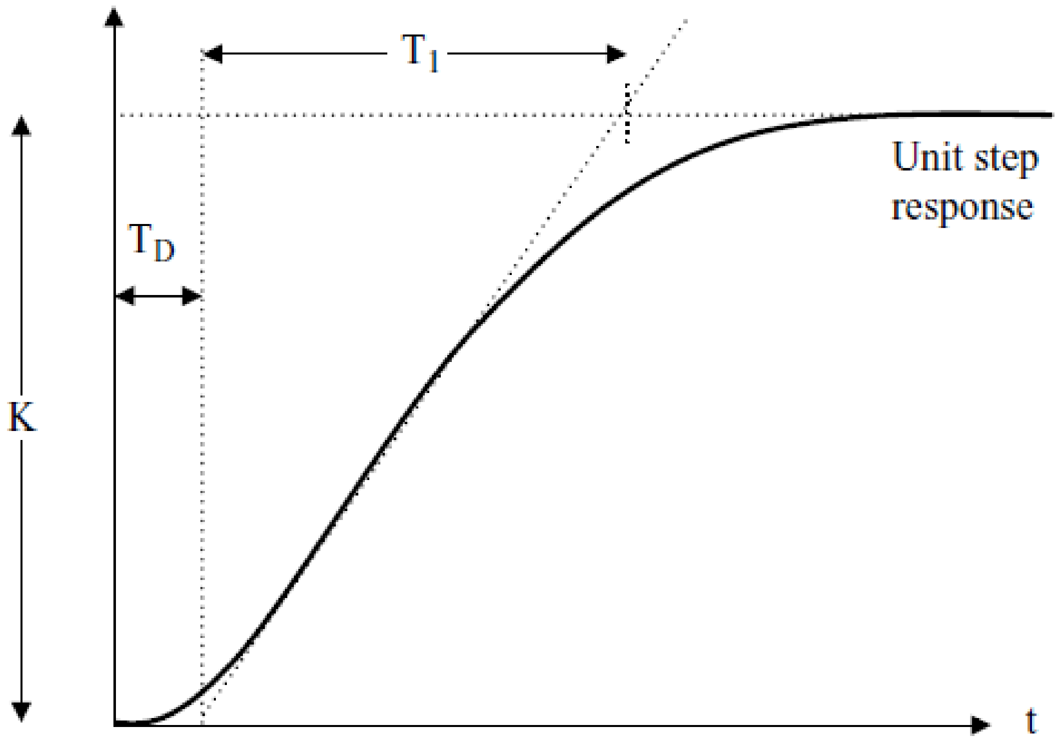


Figure 2: Finding plant parameters K, T_D and T_1

- Ziegler and Nichols then suggest using the PID controller settings given in the Table 1 when the loop is closed.
- These parameters are based on the concept of minimizing the integral of the absolute error after applying a step change to the set-point.

Table 1 Open-loop Ziegler–Nichols settings

Controller	K_p	T_i	T_d
Proportional	$\frac{T_1}{KT_D}$		
PI	$\frac{0.9T_1}{KT_D}$	$3.3T_D$	
PID	$\frac{1.2T_1}{KT_D}$	$2T_D$	$0.5T_D$

Example

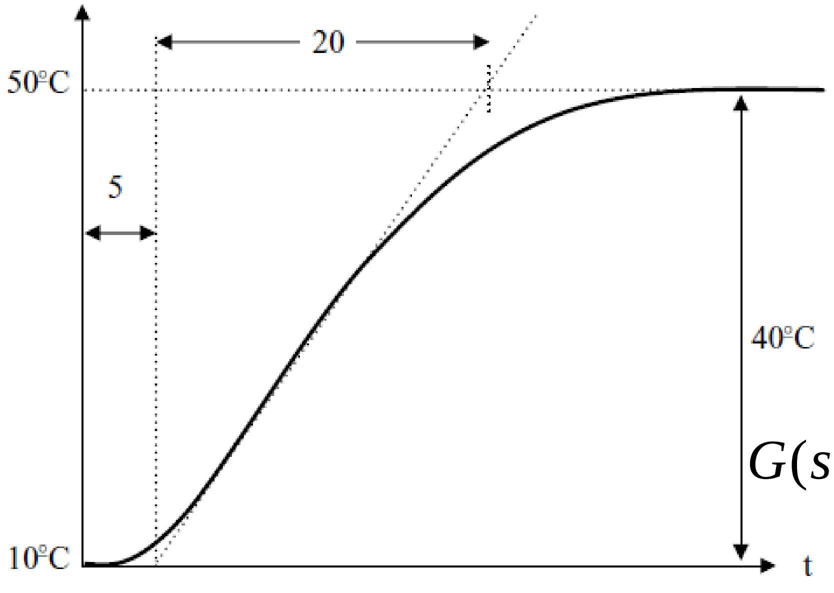
The open-loop unit step response of a thermal system is shown. Obtain the transfer function of this system and use the Ziegler–Nichols tuning algorithm to design:

(a) a proportional controller,

(b) a proportional plus integral (PI) controller, and

(c) a PID controller.

Draw the block diagram of the system in each case.



Solution : From Figure 2, the system parameters are obtained as $K = 40^\circ C$, $T_D = 5$ s and $T_1 = 20$ s, and, hence, the transfer function of the plant is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Ke^{-sT_D}}{T_1 s + 1} \quad , G(s) = \frac{40e^{-5s}}{1 + 20s}$$

(a) *Proportional controller*

- According to the Table of ZN settings for a proportional controller are:

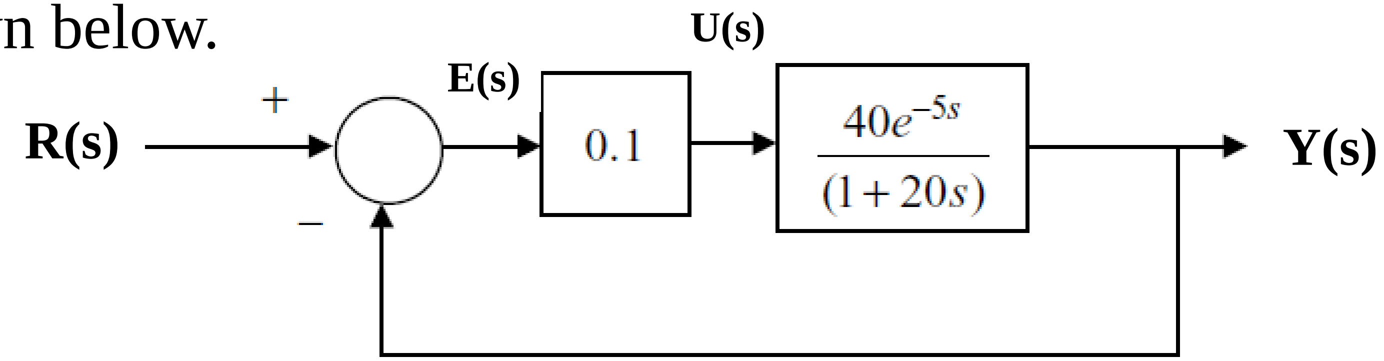
$$K_p = \frac{T_1}{K T_D}.$$

- Thus,

$$K_p = \frac{20}{40 \times 5} = 0.1,$$

The transfer function of the controller is then $\frac{U(s)}{E(s)} = 0.1,$

and the block diagram of the closed-loop system with the controller is shown below.



(b) *PI controller*

PI controller. According to Table 1, the Ziegler–Nichols settings for a PI controller are

$$K_p = \frac{0.9T_1}{KT_D} \quad \text{and} \quad T_i = 3.3T_D.$$

Thus,

$$K_p = \frac{0.9 \times 20}{40 \times 5} = 0.09 \quad \text{and} \quad T_i = 3.3 \times 5 = 16.5.$$

The transfer function of the controller is then

$$\frac{U(s)}{E(s)} = 0.09\left[1 + \frac{1}{16.5s}\right] = \frac{0.09(16.5s + 1)}{16.5s}$$

and the block diagram of the closed-loop system with the controller is shown in Figure 4 .

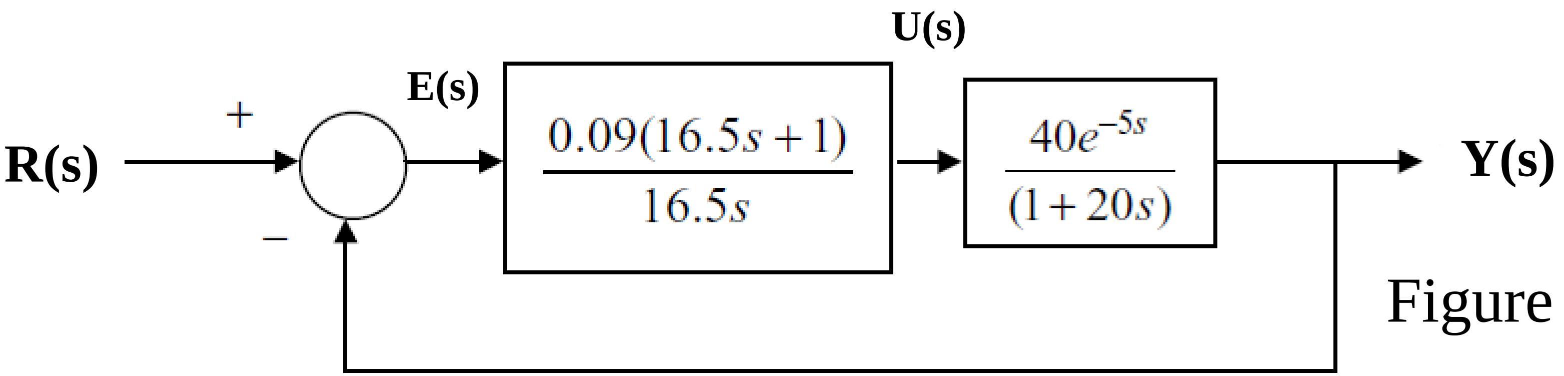


Figure 4

(c) PID controller

PID controller. According to Table 1, the Ziegler–Nichols settings for a PID controller are

$$K_p = \frac{1.2T_1}{KT_D}, \quad T_i = 2T_D, \quad T_d = 0.5T_D.$$

Thus,

$$K_p = \frac{1.2 \times 20}{40 \times 5} = 0.12, \quad T_i = 2 \times 5 = 10, \quad T_d = 0.5 \times 5 = 2.5.$$

The transfer function of the required PID controller is

$$\frac{U(s)}{E(s)} = K_p\left[1 + \frac{1}{T_i s} + T d s\right] = 0.12\left[1 + \frac{1}{10s} + 2.5s\right]$$

or

$$\frac{U(s)}{E(s)} = \frac{3s^2 + 1.2s + 0.12}{10s}.$$

The block diagram of the system, together with the controller, is shown in Figure 5

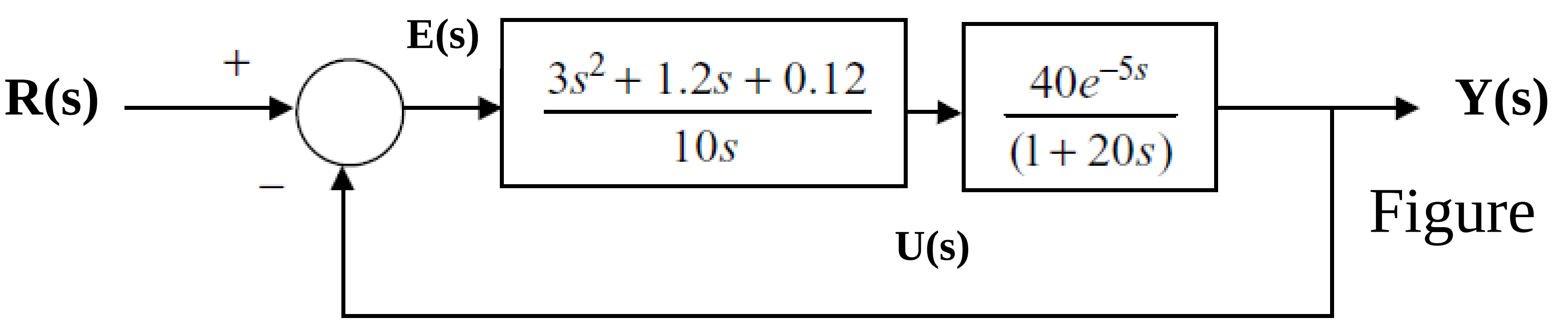
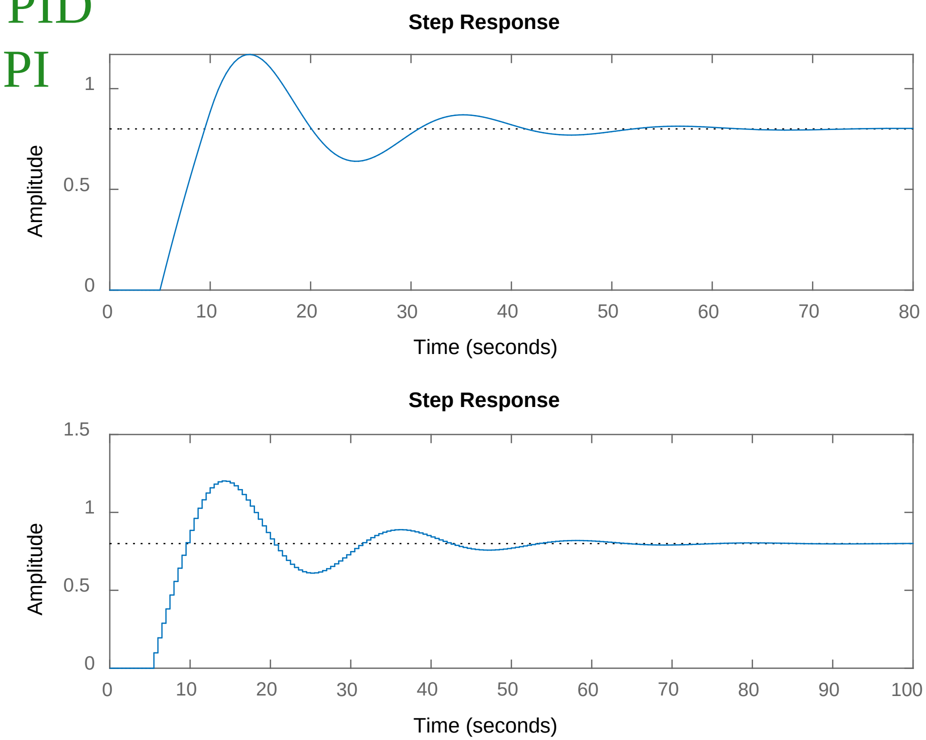
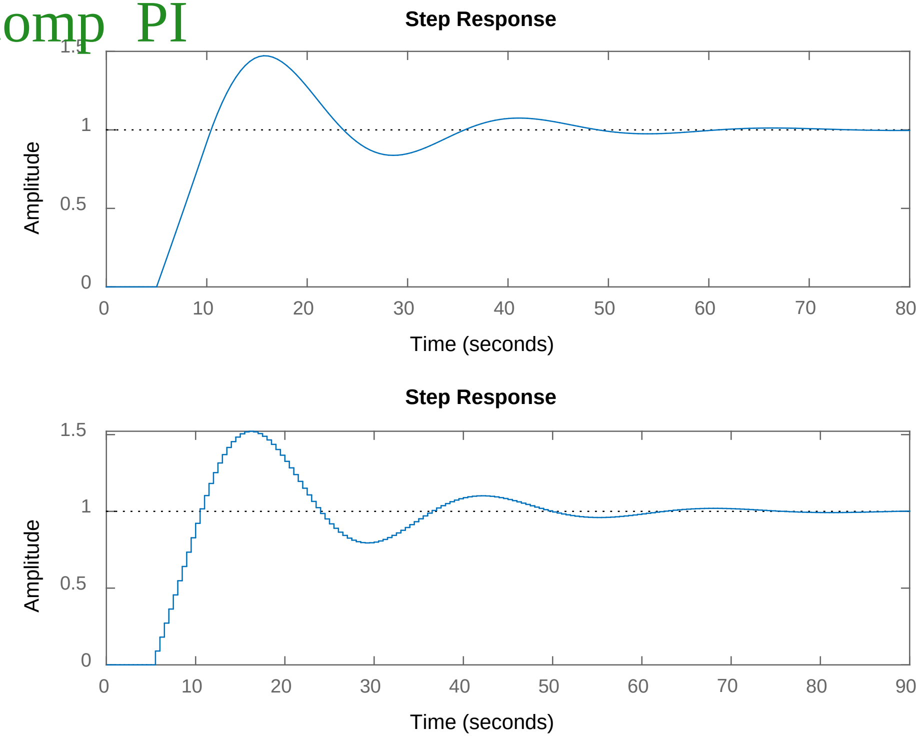


Figure 5

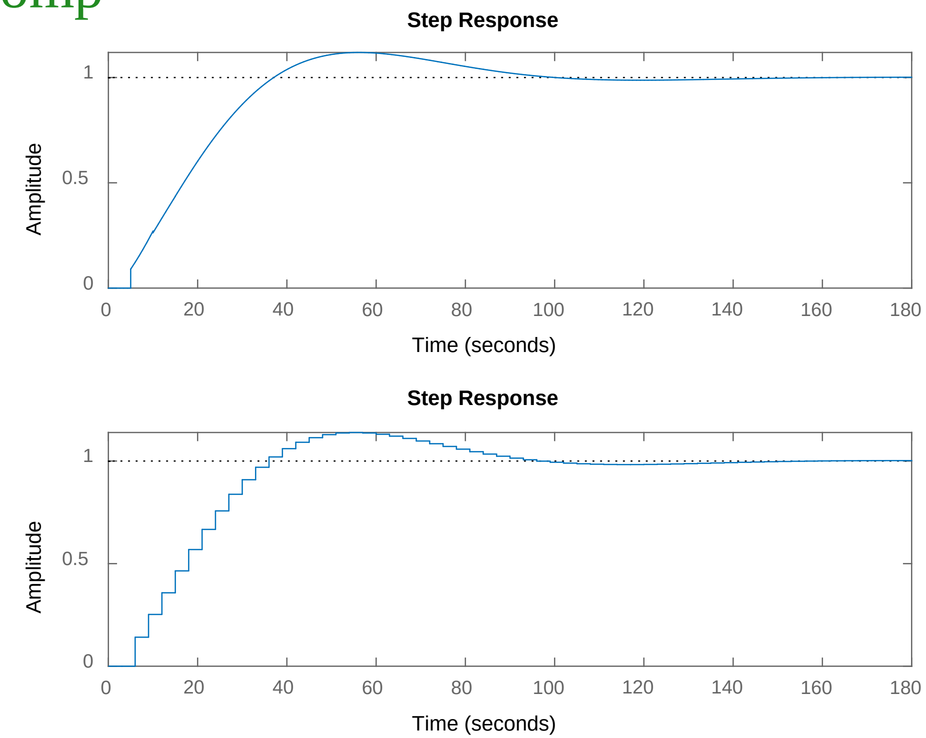
```
%%Comparison between analog and digital P controller
clc
clear
T=0.5;
s=tf('s');
Gp=(40*exp(-5*s))/(1+20*s)
ncomp=[0  0.1];%%%%%   compensator TF num P
dcomp=[0   1];    %%%%   den-comp  P
% ncomp=[3   1.2  0.12];   %%%%%   compensator TF num PID
% dcomp=[10  0];    %%%%   den-comp  PID
% ncomp=[1.485  0.09];% comp.TF num PI
% dcomp=[16.5   0];    %  den-comp  PI
Gc=tf(ncomp,dcomp)
OLc=Gc*Gp
CLc=feedback(OLc,1)
 plant=Gp;
 plantd=c2d(plant,T,'zoh');
 OLd=c2d(OLc,T,'zoh')
 CLd=feedback(OLd,1)
 subplot(211),step(CLc);
 subplot(212),step(CLd);
```



```
%%Comparison between analog and digital PI controller
clc
clear
T=0.5; ];
s=tf('s');
Gp=(40*exp(-5*s))/(1+20*s)
ncomp=[1.485  0.09];%%%%%   compensator TF num PI
dcomp=[16.5   0];    %%%%   den-comp  PI
Gc=tf(ncomp,dcomp)
OLc=Gc*Gp
CLc=feedback(OLc,1)
 plant=Gp;
 plantd=c2d(plant,T,'zoh');
 OLd=c2d(OLc,T,'zoh')
 CLd=feedback(OLd,1)
 subplot(211),step(CLc);
 subplot(212),step(CLd);
```



```
%%Comparison between analog and digital PID controller
clc
clear
T=0.5;
s=tf('s');
Gp=(40*exp(-5*s))/(1+20*s)
ncomp=[3   1.2  0.12];   %%%%%   compensator TF num
dcomp=[10  0];    %%%%   den-comp
Gc=tf(ncomp,dcomp)
OLc=Gc*Gp
CLc=feedback(OLc,1)
 plant=Gp;
plantd=c2d(plant,T,'zoh');
 OLd=c2d(OLc,T,'zoh')
 CLd=feedback(OLd,1)
subplot(211),step(CLc);
 subplot(212),step(CLd);
```



```
%%%%%%%%% discrete PID controller for Z.N second method
clc
Ts=0.2;
ncomp=[0 6.3223 17.999 12.8089];  %%%% comp. TF
dcomp= [0 0 1 0];
num= [0 0 0 1];                   %%%% plant TF
den= [1 6 5 0];
[nol, dol]=series (ncomp, dcomp, num, den);%%%%contin.OL
[ncl,dcl]=cloop(nol,dol);   %%%%%%%%% Closed loop contin.
printsys(nol,dol,'s')   %%%%   OL.C
[numd,dend]=c2dm(num,den,Ts,'tustin');  %%%% discrete Plant
[ncomd,dcomd]=c2dm(ncomp,dcomp,Ts,'tustin'); %%%% Discrete Comp.
printsys(numd,dend,'z') %%%% T.F. discrete Plant
printsys(ncomd,dcomd,'z') %%%% T.F. discrete comp
%%%%%%%%%%%%%%% discrete
[nold,dold]=series(ncomd,dcomd,numd,dend); %%%%% OL. discrete
[ncld,dcld]=cloop(nold,dold); %%%%%%%%% CL. discrete
subplot(211),step(ncl,dcl);
subplot(212),dstep(ncld,dcld);
G=tf(ncl,dcl);
figure(3)
bode(G)
w=1.5; %%%%from  bodeplot
Bw=(w)/(2*pi)
samplingtime=1/(20*(Bw))
```

```
%%%%%%%% discrete second Method ZN optimized PID controller
clc
Ts=0.206;
ncomp=[0 30.332 39.4316 12.8153];  %%%% comp. TF
dcomp= [0 0 1 0];
num= [0 0 0 1];                   %%%% plant TF
den= [1 6 5 0];
[nol, dol]=series (ncomp, dcomp, num, den);%%%%contin.OL
[ncl,dcl]=cloop(nol,dol);   %%%%%%%%% Closed loop contin.
printsys(nol,dol,'s')   %%%%   OL.C
[numd,dend]=c2dm(num,den,Ts,'tustin');  %%%% discrete Plant
[ncomd,dcomd]=c2dm(ncomp,dcomp,Ts,'tustin'); %%%% Discrete Comp.
printsys(numd,dend,'z') %%%% T.F. discrete Plant
printsys(ncomd,dcomd,'z') %%%% T.F. discrete comp
%%%%%%%%%%%%%%% discrete
[nold,dold]=series(ncomd,dcomd,numd,dend); %%%%% OL. discrete
[ncld,dcld]=cloop(nold,dold); %%%%%%%%% CL. discrete
subplot(211),step(ncl,dcl);
subplot(212),dstep(ncld,dcld);
G=tf(ncl,dcl);
figure(3)
bode(G)
w=6.1; %%%%from  bodeplot
Bw=(w)/(2*pi)
samplingtime=1/(5*(Bw))
```

Digital Control Systems

LECTURE 13

Deadbeat Controller

Prepared by: Mr. Abdullah I. Abdullah

Design of Digital Control Systems with the Deadbeat Response

The design objectives of control systems can be classified as follows:

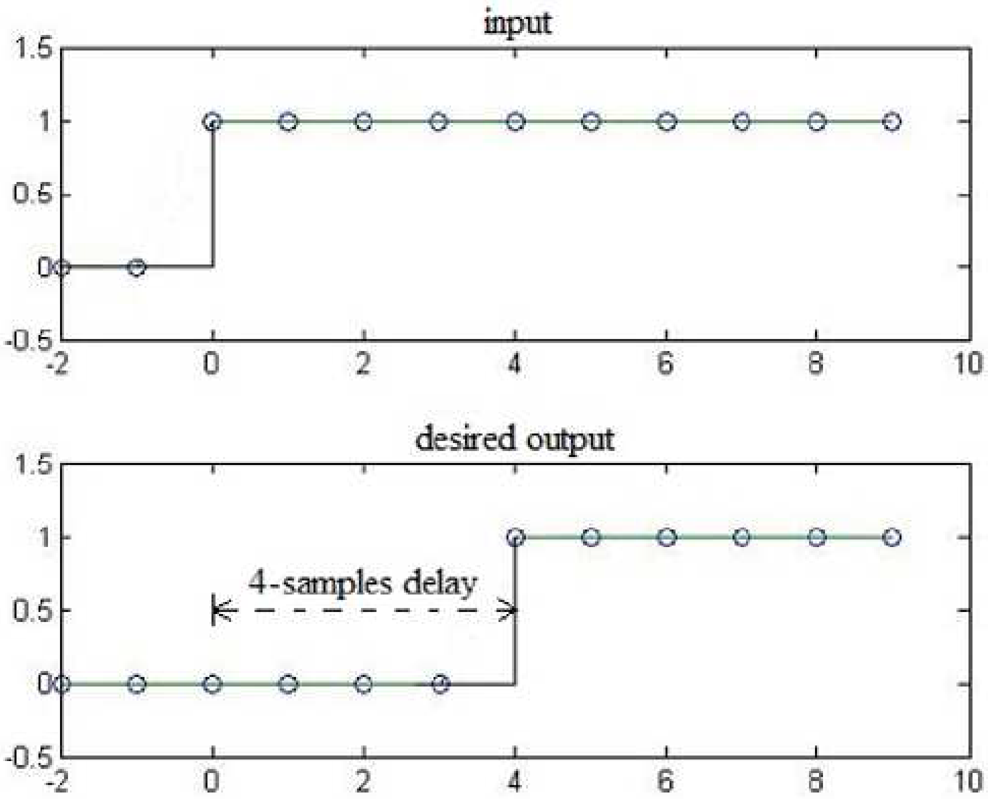
- A large number of control systems are designed with the objective that the responses of the systems should reach respective desired values as quickly as possible. This class of control systems is called minimum-time control systems, or time-optimal control systems.
- With reference to the previous design methods, one of the design objectives is to have a small maximum overshoot and a fast rise time in the step response.

In digital control system we may design the digital compensator Gc(z) to obtain a response (output) with a finite settling time. The output response c(kT) which reaches the desired steady-state value in a finite number of sampling intervals is called a **deadbeat response.**

Deadbeat Controller

Its aim is to bring the output to steady state in smallest number of time steps

◭ assuming, for simplicity, that the set point is a step input.



Therefore, the desired closed-loop transfer function is $T(z) = z^{-k}, \qquad k \geq 1$

and the controller achieving this response is given by:

$$D(z) = \frac{1}{GH(z)} \frac{T(z)}{1 - T(z)} = \frac{1}{GH(z)} \left(\frac{z^{-k}}{1 - z^{-k}} \right) = \frac{1}{GH(z)} \left(\frac{1}{z^k - 1} \right)$$

It is interesting to note that deadbeat control is equivalent to placing all closed-loop poles at z = 0.

These poles correspond to the fastest response possible.

Usually such requirement will come at the expense of large control signal.

Example 1:

The open-loop transfer function of a plant is given by: $G(s) = \frac{e^{-2s}}{10s + 1}$

Design a dead-beat digital controller for the system. Assume that T = 1 s.

The transfer function of the system with a ZOH is given by

$$GH(z) = \mathscr{Z}\left\{ \frac{1 - e^{-Ts}}{s} G(s) \right\} = (1 - z^{-1}) \mathscr{Z}\left\{ \frac{e^{-2s}}{s(10s + 1)} \right\} = (1 - z^{-1}) z^{-2} \mathscr{Z}\left\{ \frac{1}{s(10s + 1)} \right\}$$

From the z-transform tables $\mathscr{Z}\left\{\frac{a}{s(s+a)}\right\} = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

$$GH(z) = (1-z^{-1})z^{-2}Z\left\{\frac{0.1}{s(s+0.1)}\right\} = (1-z^{-1})z^{-2}\frac{z(1-e^{-0.1})}{(z-1)(z-e^{-0.1})} = \frac{0.095}{z^3-0.904z^2}$$

Hence, the dead-beat controller is given by:

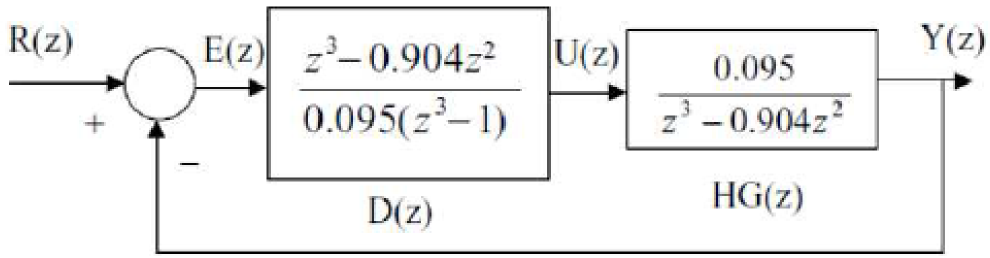
$$D(z) = \frac{1}{GH(z)}\frac{T(z)}{1-T(z)} = \frac{z^3-0.904z^2}{0.095}\left(\frac{1}{z^k-1}\right)$$

For realizability, we must choose $k \geq 3.$

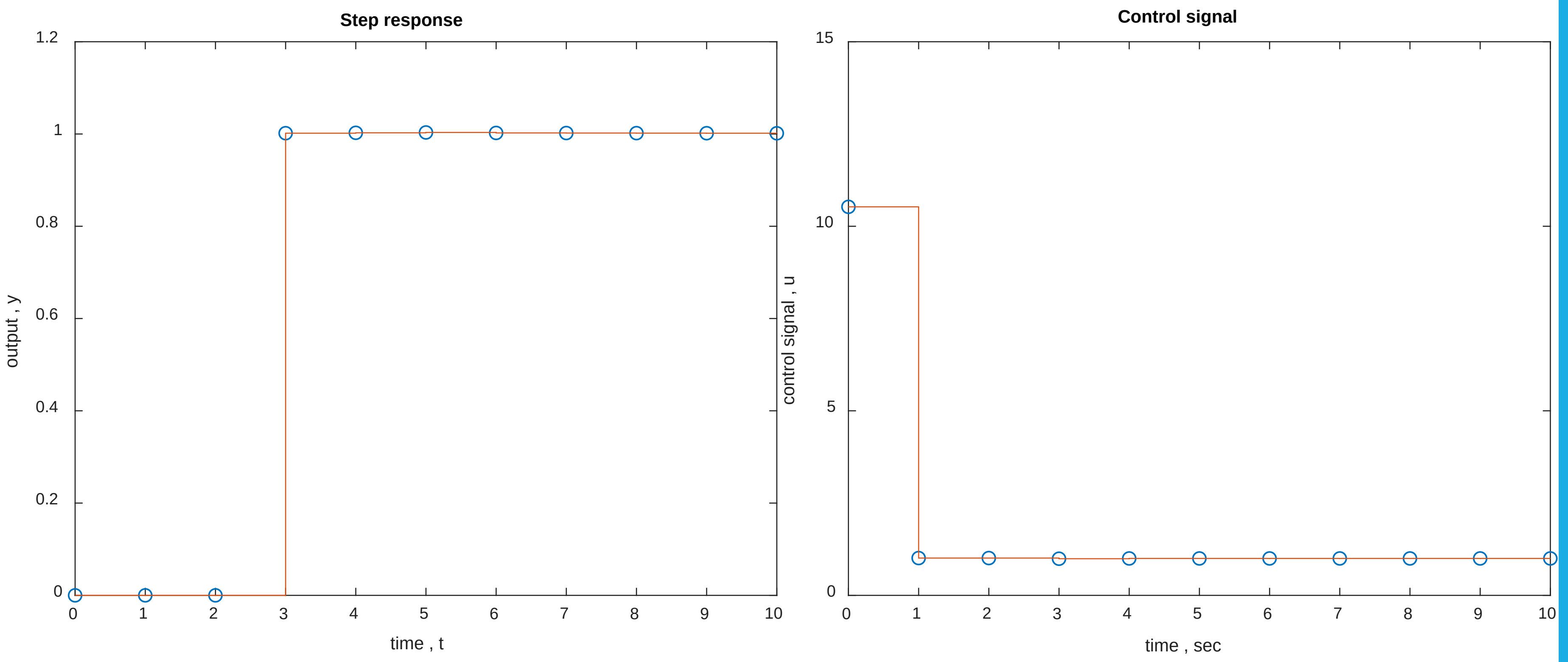
Choosing k = 3, we obtain the controller $D(z) = \frac{z^3-0.904z^2}{0.095}\frac{1}{z^3-1} = \frac{z^3-0.904z^2}{0.095(z^3-1)}$

With this controller, the block diagram of the closed-loop is:

To analyze the designed system performance, we simulate the closed-loop step response and the control signal.



```
% Deadbeat control : D(z) = (z^3 - 0.904 z ^2) / (0.095 (z^3 - 1))
 clear
 clc
 Gp = tf (1 ,[10 1],'iodelay',2);
 Gpd = c2d (Gp ,1);
 Gc = tf ([1 -0.904 0 0] ,[0.095 0 0 -0.095] ,1) ;
 Gcl =Gc* Gpd /(1+ Gc* Gpd );
 t =0:1:10;
 y= step (Gcl ,t)
 figure(1) ; plot (t,y,'o'); hold on;
 stairs (t,y); hold off
 xlabel ('time , t'),
 ylabel ('output , y'),
 axis ([0 10 0 1.2]) ,
 title ('Step response ')
 Gru =Gc /(1+ Gc* Gpd );
 u= step (Gru ,t)
 figure(2) ; plot (t,u,'o'); hold on;
 stairs (t,u); hold off
 xlabel ('time , sec '),
 ylabel ('control signal , u'),
 axis ([0 10 0 15]) ,
 title (' Control signal ')
```



As desired, the step response is unity after 3 seconds.

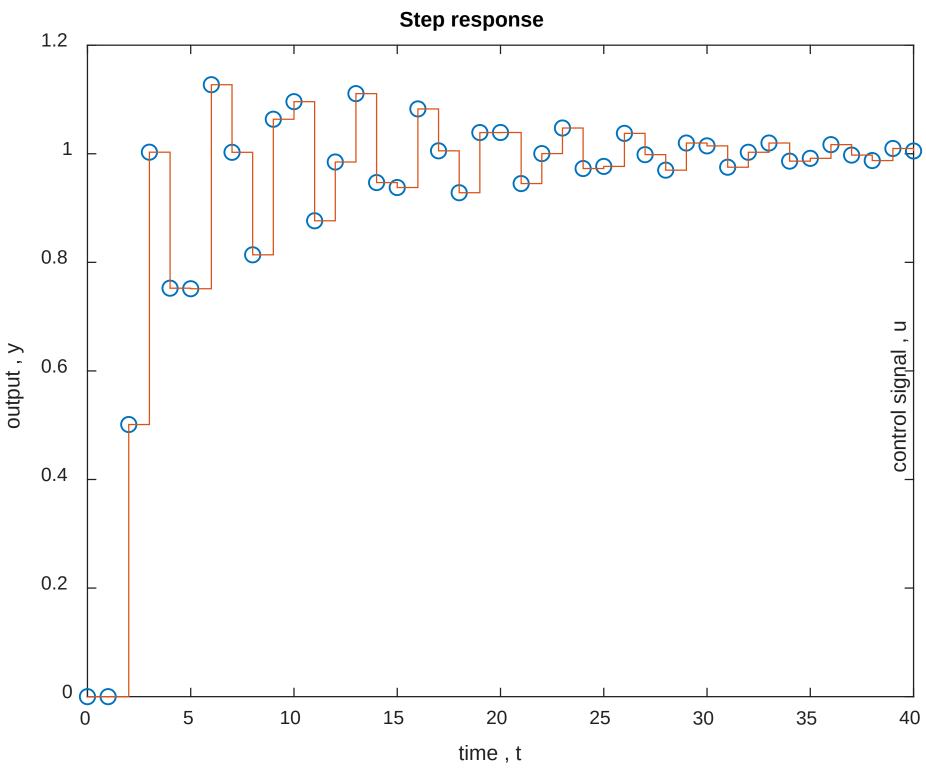
It is, however, important to realize that the response is correct only at the sampling instants and the response can have an oscillatory behavior between samples.

We realize that the magnitude of the control signal is very large at the beginning (11).

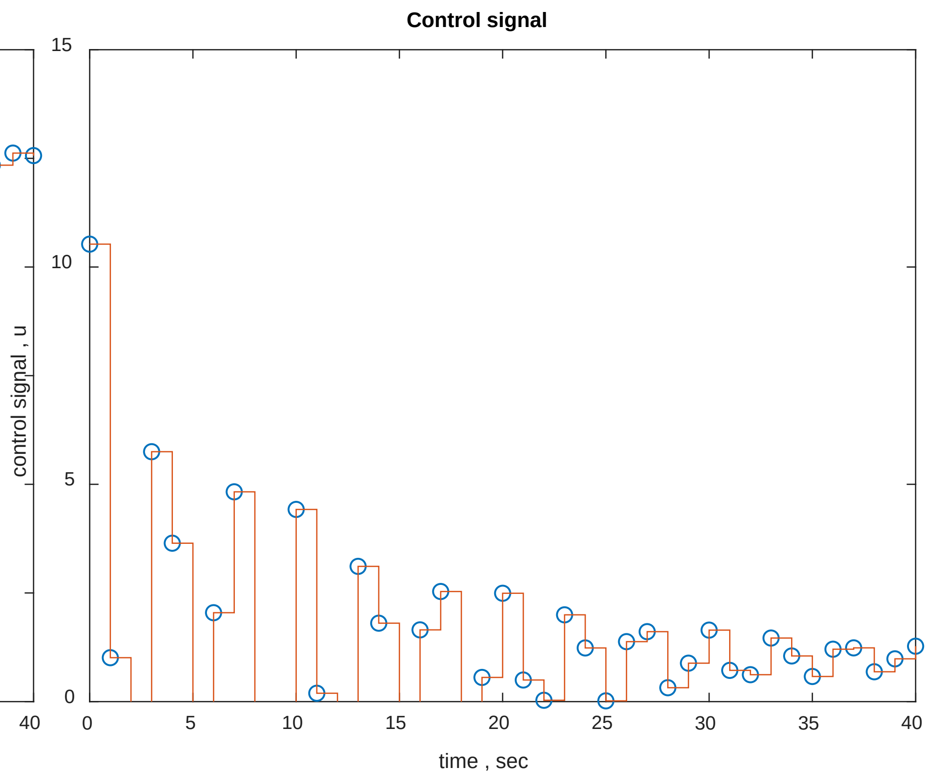
نحن ندرك أن حجم إشارة التحكم كبير جدًا في البداية

The main drawback of dead-beat control is that it requires excessive (large) control efforts which may not be acceptable in practice.

العيب الرئيسي للسيطرة على الضربات الميتة هو أنها تتطلب جهود تحكم مفرطة (كبيرة) والتي قد لا تكون مقبولة في الممارسة.



Step response
output , y
time , t
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
40

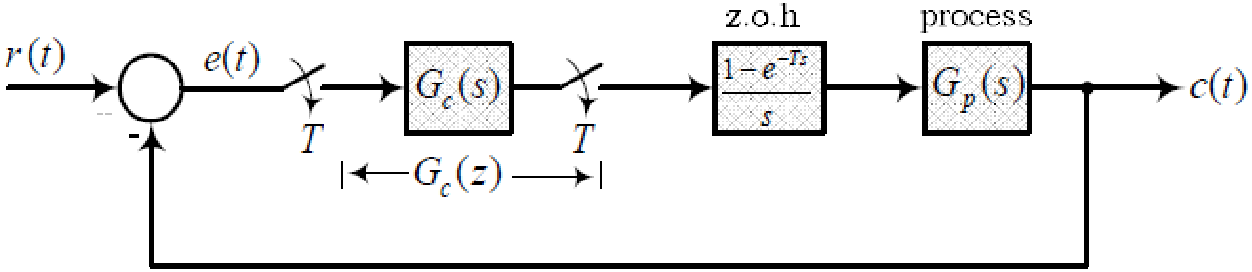


Control signal
control signal , u
time , sec
15
10
5
0
0
5
10
15
20
25
30
35
40

Example 2: The block diagram of a digital control system, shown in Fig.(1), is revisited. Again, the controlled process is represented by the transfer function

$$G_p(s) = \frac{10}{(s+1)(s+2)}$$

Try to find a controller with the objective to cancel all poles and the zeros of the process and then add a pole at z =1.



$$G_{zoh}G_p(z) = (1-z^{-1})\ \mathcal{Z}\left[\frac{10}{s(s+1)(s+2)}\right] = \frac{0.0453(z+0.904)}{(z-0.905)(z-0.819)}$$

The pulse transfer function of the suggested digital controller be

$$G_c = \frac{(z-0.905)(z-0.819)}{0.0453(z-1)(z+0.904)}$$

The open-loop transfer function of the compensated system now simply becomes

$$G_c(z)G_{zoh}G_p(z) = \frac{1}{z-1}$$

The corresponding closed-loop transfer function is $\frac{C(z)}{R(z)} = \frac{1}{z}$

Thus, for a unit step input, the output transform is $C(z) = \frac{1}{z-1} = z^{-1} + z^{-2} + z^{-3} + \cdots$

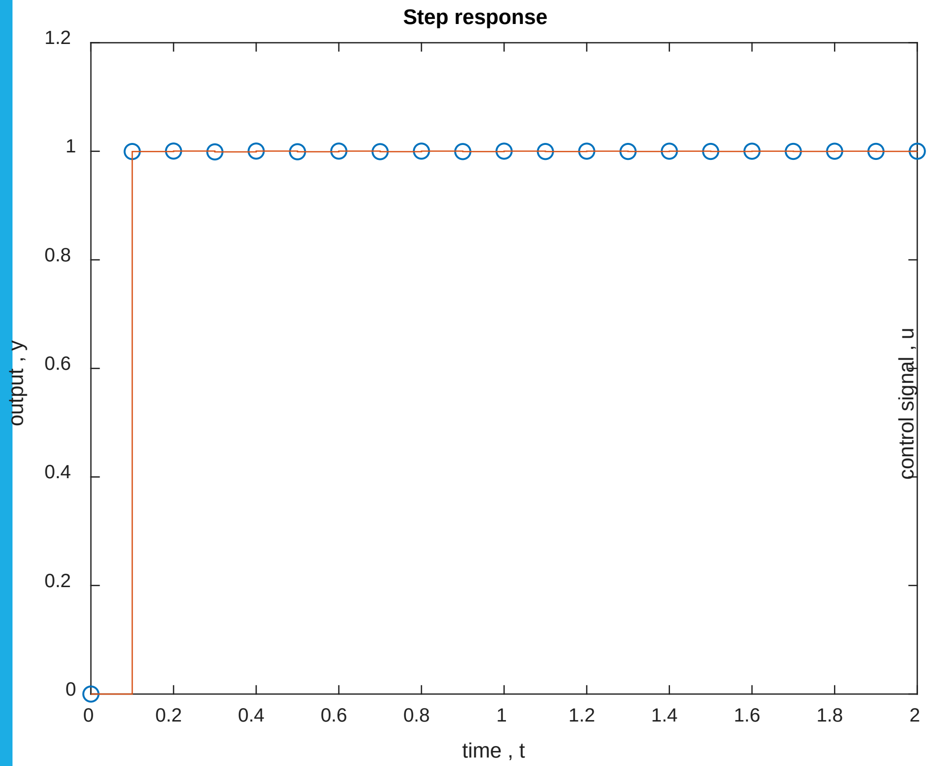
The output response c(kT) reaches the desired steady-state value in one sampling period and stays at that value thereafter.

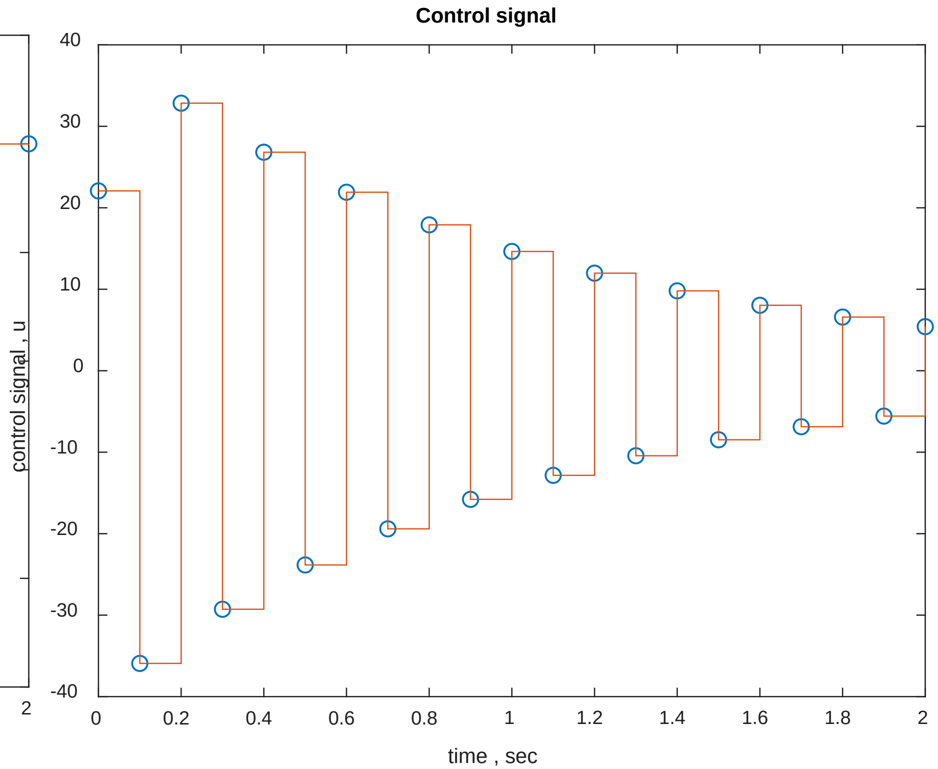
In reality, however, it must be kept in mind that the true judgement on the performance should be based on the behavior of c(t) . In general, although c(kT) may exhibit little or no overshoot, the actual response c(t) may have oscillations between the sampling instants.

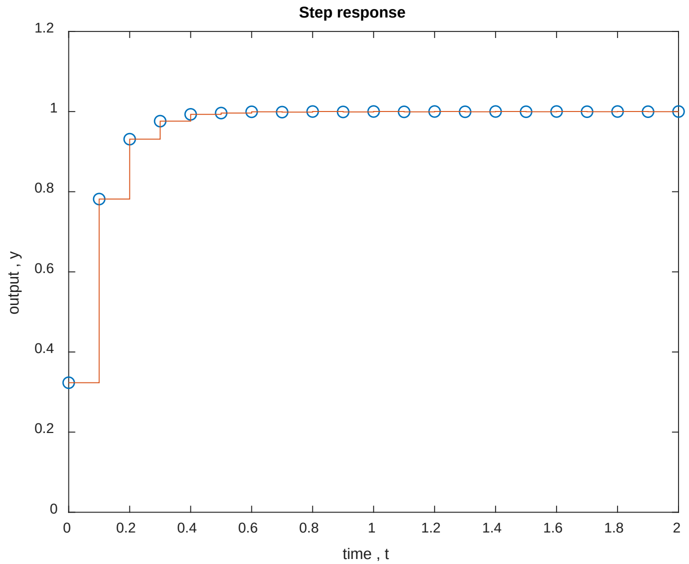
For the present system, since the sampling period T = 0.1sec is much smaller than the time constants of the controlled process, it is expected that c(kT) gives a fairly accurate description c(t) .

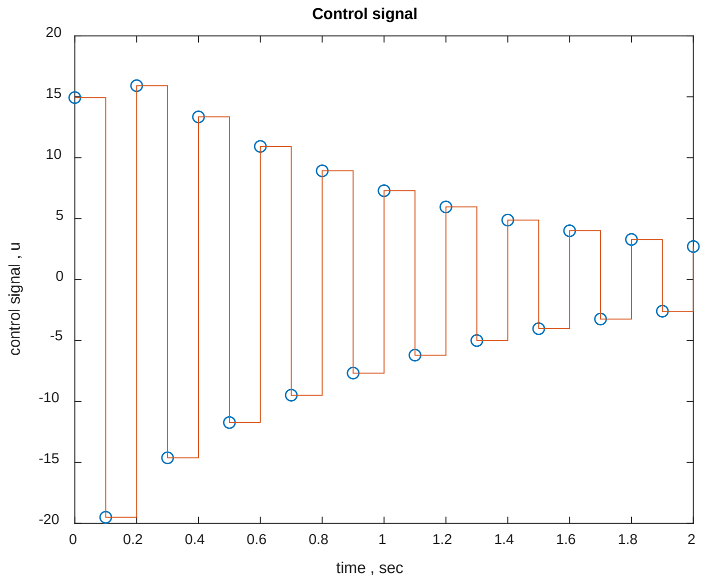
Thus, it is expected that the digital controller will produce a unit-step response that reaches its steady-state value of 0.1 sec, and there should be little or no ripple in between the sampling instants.

This type of response is referred to **deadbeat response.**


Step response
output , y
time , t
1.2
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2


Control signal
control signal , u
time , sec
40
30
20
10
0
-10
-20
-30
-40
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2

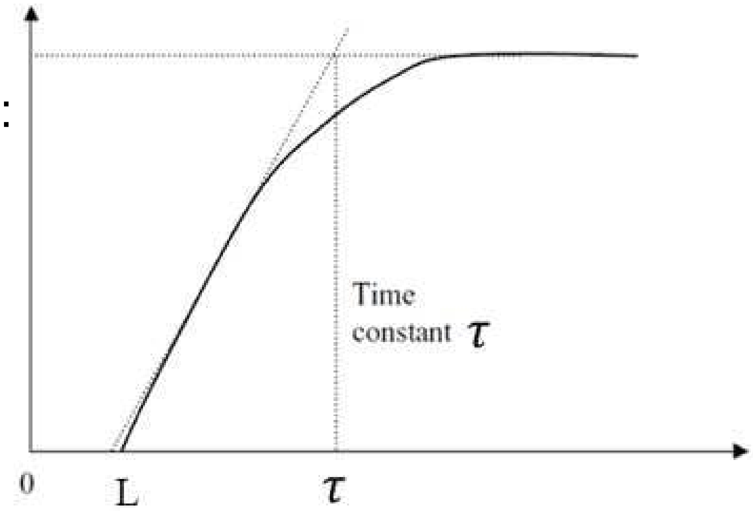

Step response
output , y
time , t
1.2
1
0.8
0.6
0.4
0.2
0
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2


Control signal
control signal , u
time , sec
20
15
10
5
0
-5
-10
-15
-20
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2

Dahlin Controller

Dahlin controller is a modification of the deadbeat controller which produces an exponential response that is smoother than deadbeat response.

The desired closed-loop response for step input looks like:



Hence, the desired closed-loop transfer function is:

$$G_{cl}(s) = \frac{e^{-Ls}}{\tau s + 1}$$

As step input is assumed (which is constant between samples), the desired closed-loop transfer function in the z-domain will be:

$$T(z) = \mathscr{Z}\{G_{zoh}(s)\ G_{cl}(s)\} = \mathscr{Z}\left\{\frac{1 - e^{-Ts}}{s}\frac{e^{-Ls}}{\tau s + 1}\right\}$$

Example

The open-loop transfer function of a plant is given by:

$$G(s) = \frac{e^{-2s}}{10s+1}$$

Design a Dahlin digital controller for the system to achieve a closed-loop time constant of 5 s. Assume that T = 1 s.

From the previous example, this is found to be $GH(z) = \frac{0.095}{z^3 - 0.904z^2}$

The desired closed-loop transfer function, T(z).

$$T(s) = \frac{e^{-Ls}}{5s+1}$$

As the desired closed-loop time constant, τ, is 5 sec,

Therefore, $T(z) = \mathscr{Z}\left\{\frac{1-e^{-sT}}{s}\frac{e^{-Ls}}{5s+1}\right\} = (1-z^{-1})z^{-L/T}\mathscr{Z}\left\{\frac{1}{s(5s+1)}\right\}$

$$= (1-z^{-1})z^{-k}\mathscr{Z}\left\{\frac{0.2}{s(s+0.2)}\right\}$$

$$= (1-z^{-1})z^{-k}\frac{z(1-e^{-0.2T})}{(z-1)(z-e^{-0.2T})} = z^{-k}\frac{(0.181)}{(z-0.819)}$$

The Dahlin controller is thus given by:

$$D(z) = \frac{1}{G(z)}\frac{T(z)}{1 - T(z)} = \frac{z^3 - 0.904z^2}{0.095}\frac{z^{-k}\frac{(0.181)}{(z-0.819)}}{\left(1 - z^{-k}\frac{(0.181)}{(z-0.819)}\right)}$$

$$= \frac{z^3 - 0.904z^2}{0.095}\frac{0.181z^{-k}}{z - 0.819 - 0.181z^{-k}}$$

$$= \frac{0.181z^{3-k} - 0.164z^{2-k}}{0.095z - 0.078 - 0.017z^{-k}}.$$

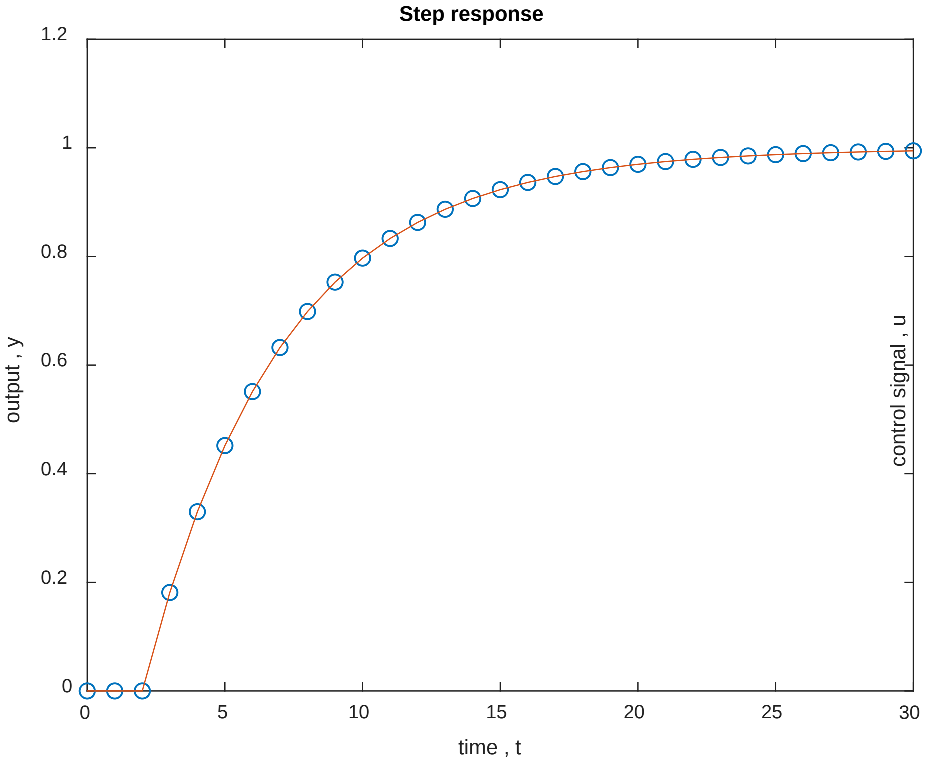
For the controller to be realizable: degree of numerator must be degree of denominator

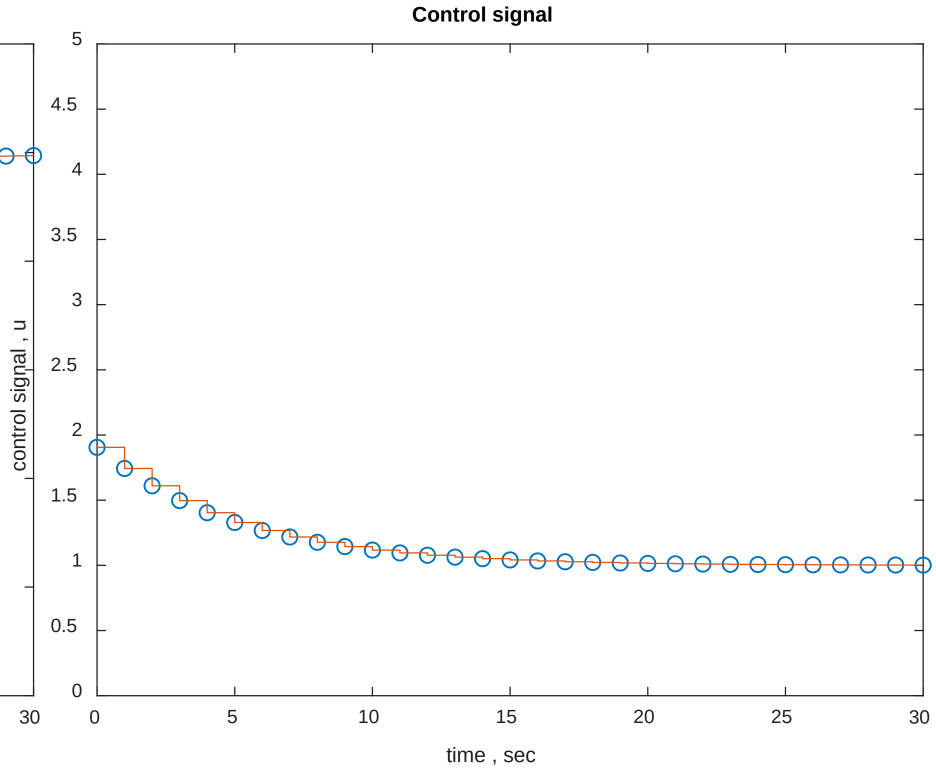
$$3 - k \leq 1 \quad \Rightarrow \quad k \geq 2$$

Choosing k = 2, the controller is, then, given by:

$$D(z) = \frac{0.181z - 0.164}{0.095z - 0.078 - 0.017z^{-2}} = \frac{0.181z^3 - 0.164z^2}{0.095z^3 - 0.078z^2 - 0.017}$$

Using the designed controller, the closed-loop step response and control signal are simulated next.

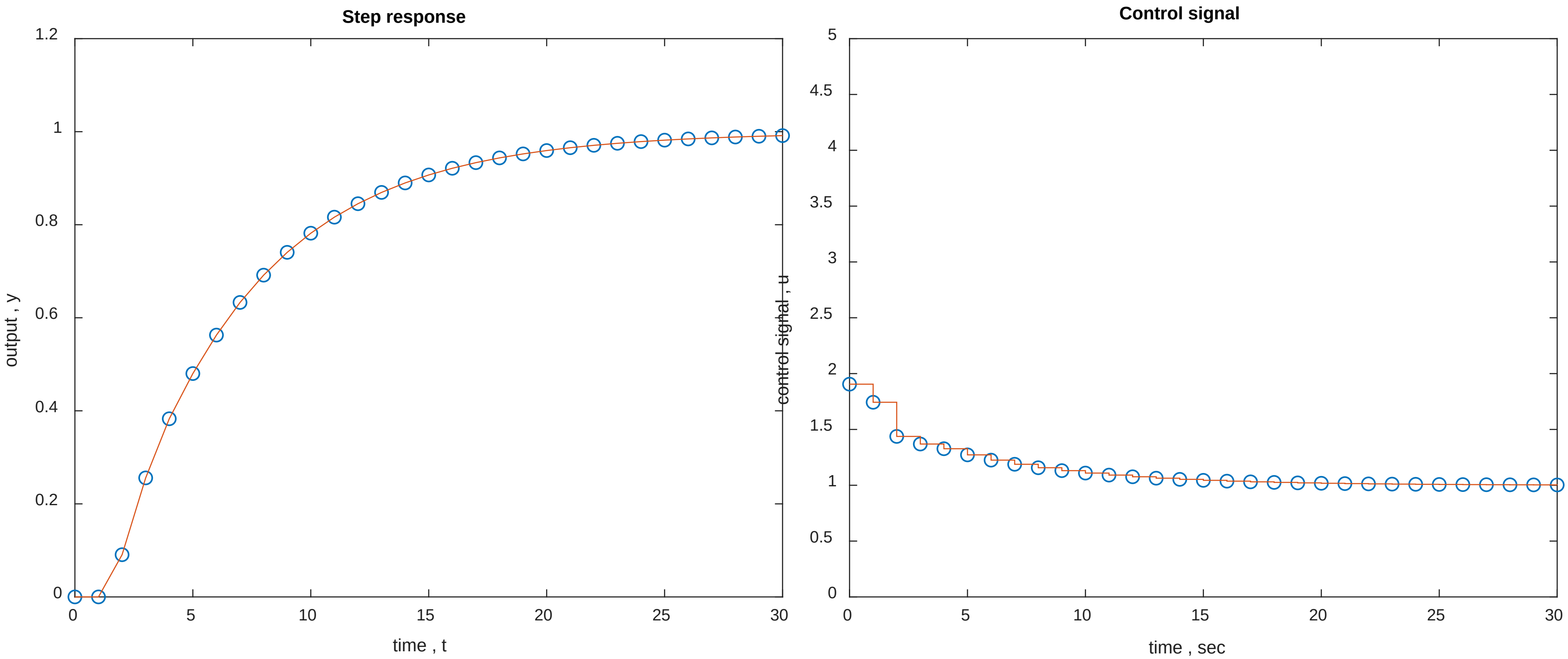




the response is exponential as designed but slower than deadbeat control.

What is the time delay? time constant?

the maximum control signal magnitude (1.9) is much smaller than the control signal obtained using a deadbeat controller (11). This is more acceptable in practice.



Step response
output , y
time , t
1.2
1
0.8
0.6
0.4
0.2
0
5
10
15
20
25
30
Control signal
control signal , u
time , sec
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
5
10
15
20
25
30

Digital Control Systems

LECTURE 14

Root Locus Based Controller Design Using MATLAB

Prepared by: Mr. Abdullah I. Abdullah

Root Locus Based Controller Design Using MATLAB

In this lecture we will show how the MATLAB platform can be utilized to design a controller using root locus technique.

Consider the closed loop discrete control system as shown in Figure 1. Design a digital controller such that the closed loop system has zero steady state error to step input with a reasonable dynamic performance. Velocity error constant kv of the system should at least be 5.

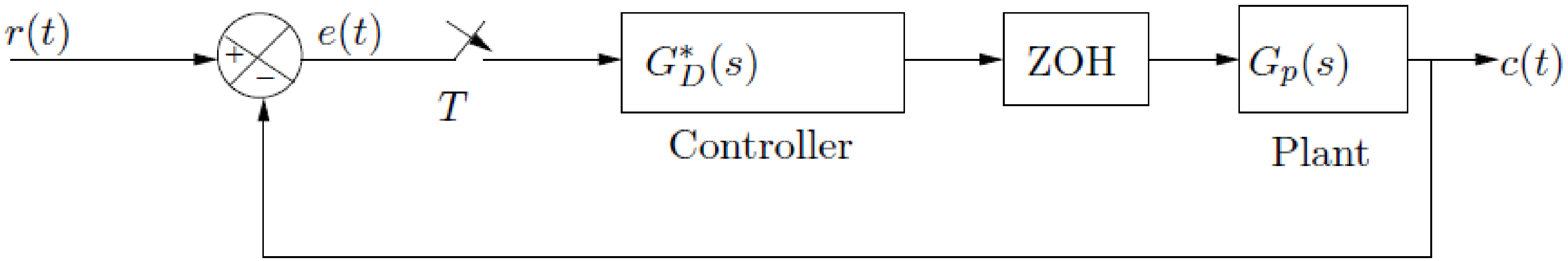


Figure 1: A discrete time control system

$$G_p(s) = \frac{10}{(s+1)(s+2)}, \qquad T = 0.1 \text{ sec}$$

$$G_{h0}G_p(z) = Z\left[\frac{1-e^{-0.1s}}{s}\frac{10}{(s+1)(s+2)}\right] = (1-z^{-1})Z\left[\frac{10}{s(s+1)(s+2)}\right]$$

$$G_{h0}G_p(z) \cong \frac{0.04528(z+0.9048)}{(z-0.9048)(z-0.8187)}$$

The MATLAB script to find out Gh0Gp(z) is as follows.

```
>> s=tf('s');
>> Gp=10/((s+1)*(s+2));
>> GhGp=c2d(Gp,0.1,'zoh');
```

The root locus of the uncompensated system (without controller) is shown in Figure 2 for which the MATLAB command is

```
>> rlocus(GhGp)
```

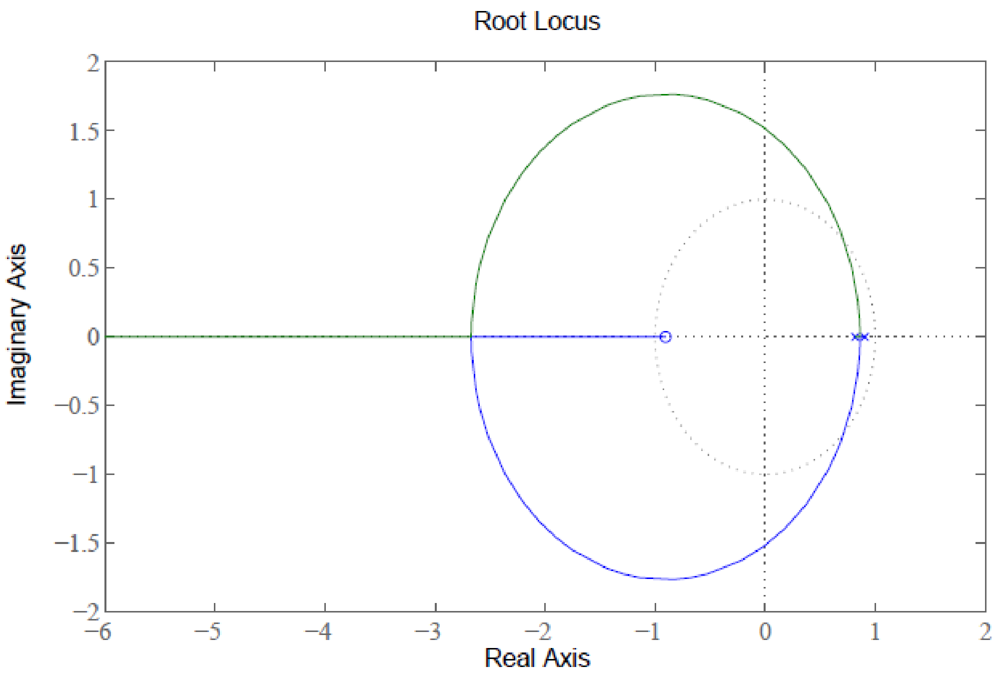


Figure 2: Root locus of the uncompensated system

Pole zero map of the uncompensated system is shown in Figure 3 which can be generated using the MATLAB command

>> pzplot(GhGp)

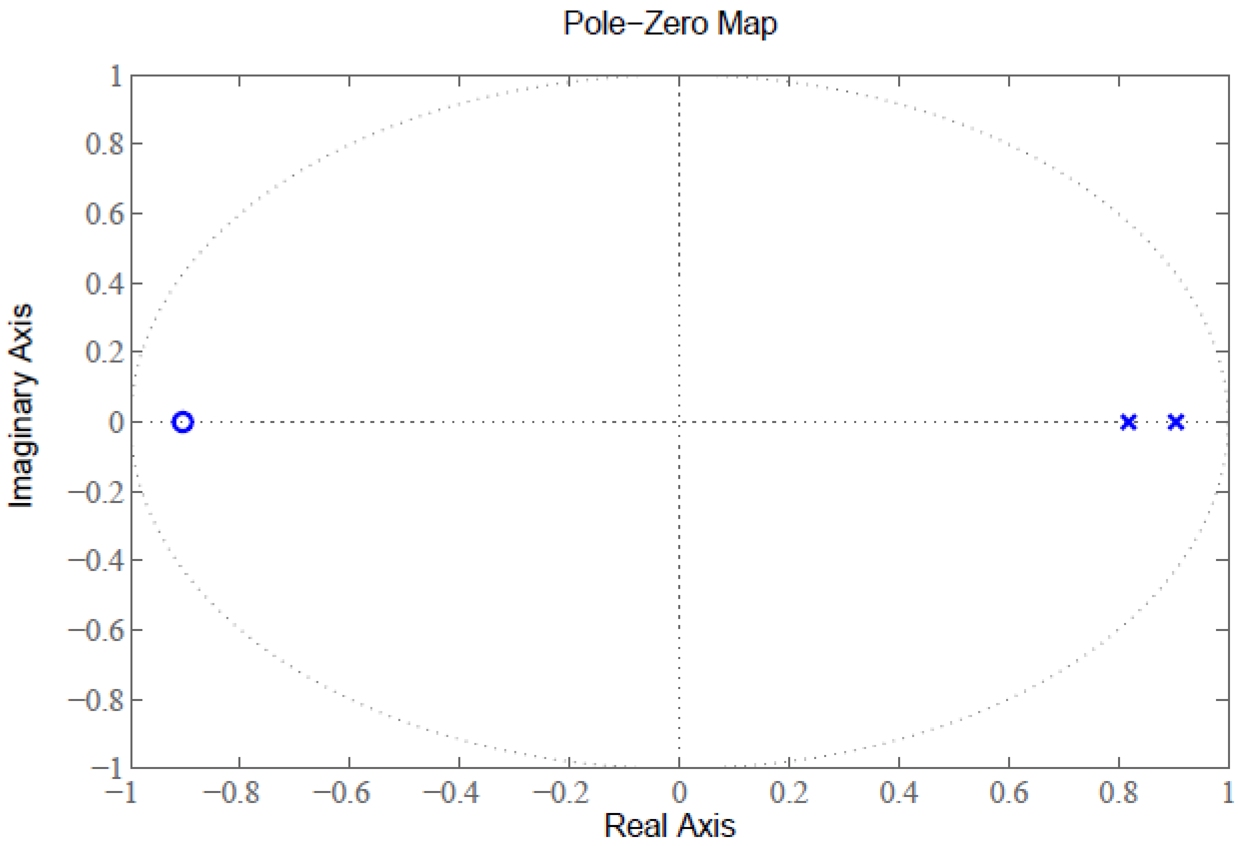


Figure 3: Pole zero map of the uncompensated system

The PI controller transfer function in z-domain when backward rectangular integration is used.

$$G_D(z) = K_p + \frac{K_i T}{z-1} = \frac{K_p z - (K_p - K_i T)}{z-1}$$

The parameter K_i can be designed using the velocity error constant requirement.

$$k_v = \frac{1}{T} \lim_{z \to 1} (z-1) G_D(z) G_{h0} G_p(z) \; = 5K_i \geq 5$$

Above condition will be satisfied if $K_i \geq 1.$

Let us take $K_i = 1$. With $K_i = 1$, the characteristic equation becomes

$$(z-1)(z-0.9048)(z-0.8187) + 0.004528(z+0.9048) + 0.04528K_p(z-1)(z+0.9048) = 0$$

or,

$$1 + \frac{0.04528K_p(z-1)(z+0.9048)}{z^3 - 2.724z^2 + 2.469z - 0.7367} = 0$$

Now, we can plot the root locus of the compensated system with K_p as the variable parameter.

The MATLAB script to plot the root locus is as follows.

```
>> z=tf('z',0.1);
>> Gcomp=0.04528*(z-1)*(z+0.9048)/(z^3 - 2.724*z^2 + 2.469*z - 0.7367);
>> zero(Gcomp);
>> pole(Gcomp);
>> rlocus(Gcomp)
```

The zeros of the system are 1 and −0.9048 and the poles of the system are 1.0114±0.1663i and 0.7013 respectively. The root locus plot is shown in Figure 4. It is clear from the figure that the system is stable for a very small range of Kp.

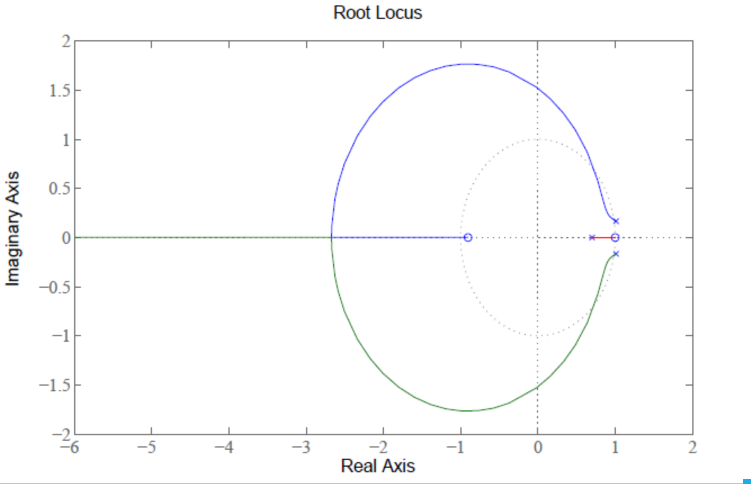


Figure 4: Root locus of the system with PI controller

The stable portion of the root locus is zoomed in Figure 5. The figure shows that the stable range of Kp is 0.239 < Kp < 6.31. The best achievable overshoot is 45.5%, for Kp = 1, which is very high for any practical system.

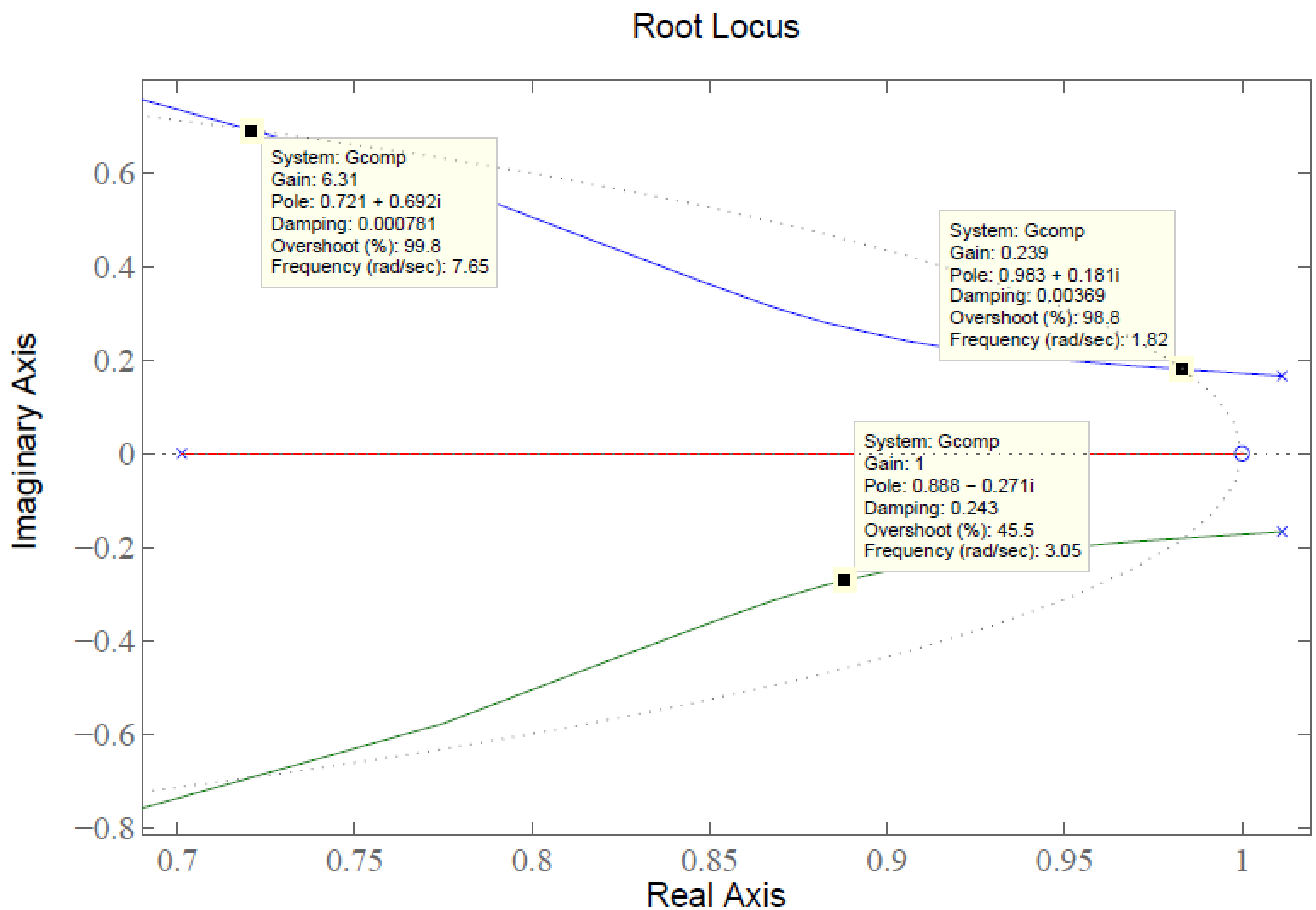


Figure 5: Root locus of the system with PI controller

To improve the relative stability, we need to introduce D action.
Let us modify the controller to a PID controller for which the transfer function in z-domain is given as below.

$$G_D(z) = \frac{(K_pT + K_d)z^2 + (K_iT^2 - K_pT - 2K_d)z + Kd}{Tz(z-1)}$$

To satisfy velocity error constant, $K_i \geq 1$.

If we assume 15% overshoot (corresponding to $\xi \cong 0.5$)

and 2 sec settling time (corresponding to $\omega_n \cong 4$),

the desired dominant poles can be calculated as,

$$\begin{aligned} s_{1,2} &= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \\ &= -2 \pm j3.46 \end{aligned}$$

Thus the closed loop poles in z-plane

$$\begin{aligned} z_{1,2} &= exp(T(-2 \pm j3.46)) \\ &\cong 0.77 \pm j0.28 \end{aligned}$$

The pole zero map including the poles of the PID controller is shown in Figure 6 where the red cross denotes the desired poles.

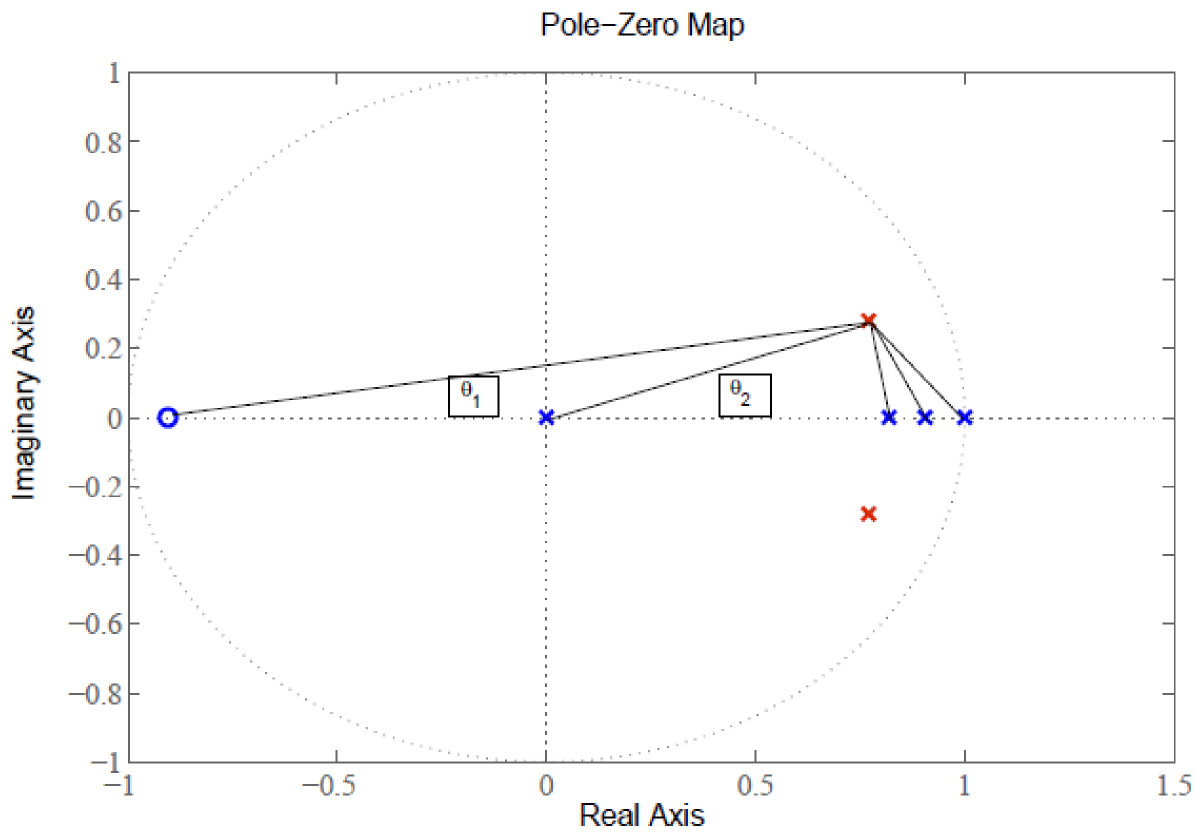


Figure 6: Pole zero map including poles of the PID controller

Let us denote the angle contribution starting from the zero to the right most pole as $\theta_1,\ \theta_2,\ \theta_3,\ \theta_4$ and θ_5 respectively. The angles can be calculated as

$\theta_1 = 9.5^o,\ \theta_2 = 20^o,\ \theta_3 = 99.9^o,\ \ \theta_4 = 115.7^o$ and $\theta_5 = 129.4^o$.

Net angle contribution is $A = 9.5^o - 20^o - 99.9^o - 115.7^o - 129.4^o = -355.5^o$.

Angle deficiency is $-355.5^o + 180^o = -175.5^o$

Thus the two zeros of PID controller must provide an angle of 175.5o.
Let us place the two zeros at the same location, z_{pid}.

Since the required angle by individual zero is 87.75o, we can easily say that the zeros must lie on the left of the desired closed loop pole.

$$
\begin{aligned}
\tan^{-1}\frac{0.28}{0.77 - z_{pid}} &= 87.75^o \\
\text{or,}\ \frac{0.28}{0.77 - z_{pid}} &= \tan(87.75^o) = 25.45 \\
\text{or,}\ 0.77 - z_{pid} &= \frac{0.28}{25.45} = 0.011 \\
\text{or,}\ z_{pid} &= 0.77 - 0.011 = 0.759
\end{aligned}
$$

The controller is then written as $G_D(z) = K\dfrac{(z-0.759)^2}{z(z-1)}$.

The root locus of the compensated system (with PID controller) is shown in Figure 7.

This figure shows that the desired closed loop pole corresponds to K = 4.33

Thus the required controller is $G_D(z) = 4.33\dfrac{z^2 - 1.518z + 0.5761}{z(z-1)}$.

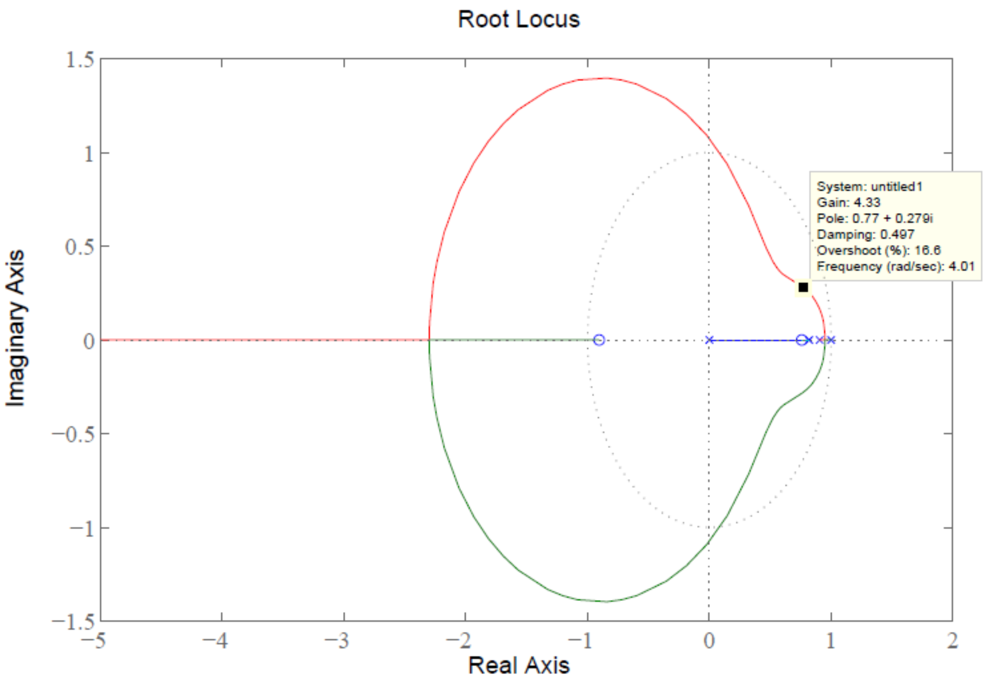


Figure 7: Root locus of compensated system

If we compare the above transfer function with the general PID controller, Kp and Kd can be computed as follows.

$$K_d/T = 0.5761 * 4.33 \quad \Rightarrow \quad K_d = 0.2495$$
$$K_p + K_d/T = 4.33 \quad \Rightarrow \quad K_p = 1.835$$
$$K_iT - K_p - 2K_d/T = -1.518 * 4.33 \quad \Rightarrow \quad K_i = 2.521$$

Note that the above Ki satisfies the constraint $K_i \geq 1$. hould keep in mind that the design is based on second order dominant pole pair approximation. But, in practice, there will be other poles and zeros of the closed loop system which might not be insignificant compared to the desired poles. Thus the actual overshoot of the system may differ from the designed one.

```
% The MATLAB script to find out Gh0Gp(z) is as follows.
s=tf('s');
 Gp=10/((s+1)*(s+2));
 GhGp=c2d(Gp,0.1,'zoh')
 % The root locus of the uncompensated system (without controller) is in fig(1)
 figure(1)
 rlocus(GhGp)
 title('Uncompensated system ');
 %%Pole zero map of the uncompensated system is shown in Figure 2
 figure(2)
 pzplot(GhGp)
 %%%% The MATLAB script to plot the root locus is as follows.
 z=tf('z',0.1);
 Gcomp=0.04528*(z-1)*(z+0.9048)/(z^3 - 2.724*z^2 + 2.469*z - 0.7367);
 zero(Gcomp);
 pole(Gcomp);
 figure(3)
 rlocus(Gcomp)
 figure(4)
 rlocus(Gcomp)
 axis([0.71   1.01   -0.8  0.78]);
 title('Root locus of the system with PI controller')
 figure(5)
 pzplot(Gcomp)
```

Digital Control Simulation Laboratory

lecture 15

Discrete Lead Compensator Design based on Root locus

Prepared by: Mr. Abdullah I. Abdullah

Discrete Lead Compensator Design based on Root locus

The lead compensator has the same purpose as the PD compensator: to improve the transient response of the closed-loop system by reshaping the root locus. The lead compensator consists of a zero and a pole with the zero closer to the origin of the *s* plane than the pole. The zero reshapes a portion of the root locus to achieve the desired transient response. The pole is placed far enough to the left that it does not have much influence of the portion influenced by the zero. Generally Lead compensators are represented by following transfer function

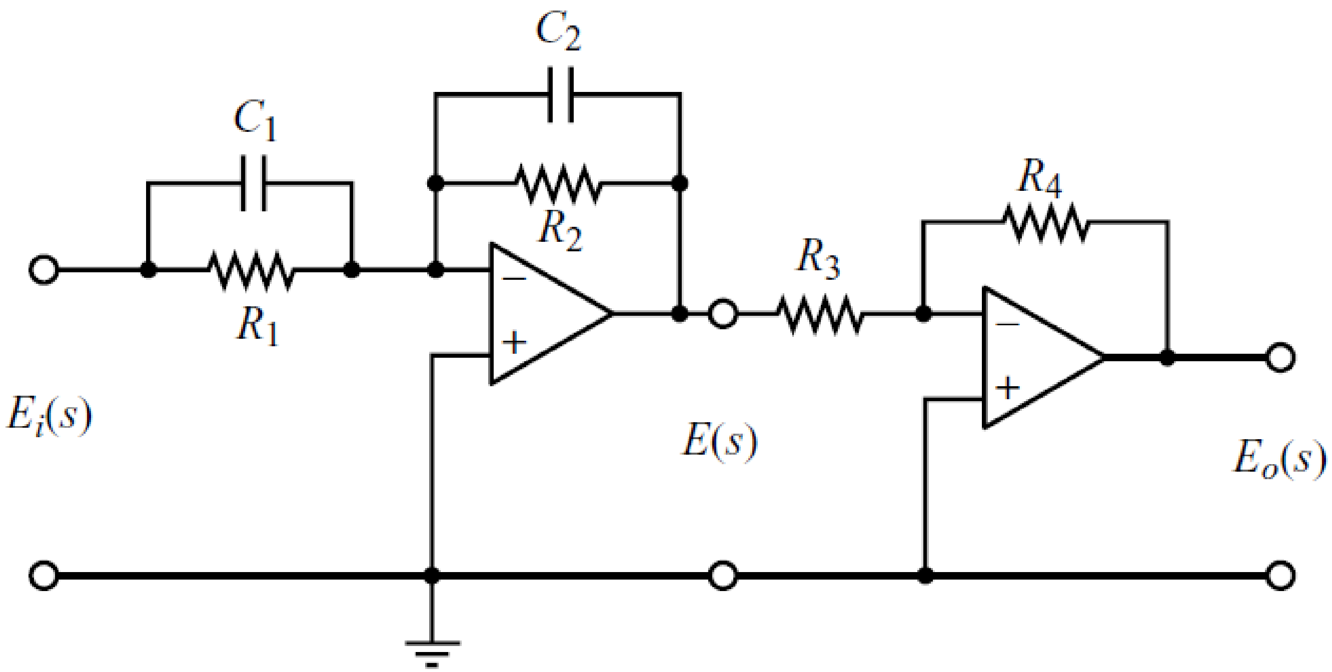
$$G_c(s) = K_c \alpha \frac{Ts+1}{\alpha Ts+1}, \quad (0 < \alpha < 1)$$

or

$$G_c(s) = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}, \quad (0 < \alpha < 1)$$

Electronic Lead Compensator

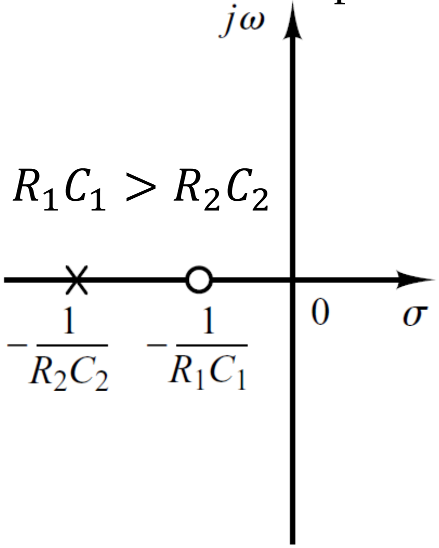
- Following figure shows an electronic lead compensator using operational amplifiers.



$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 R_4}{R_1 R_3} \frac{R_1 C_1 s + 1}{R_2 C_2 s + 1}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4}{R_1R_3}\,\frac{R_1C_1s+1}{R_2C_2s+1}$$

Pole-zero Configuration of Lead Compensator



- This can be represented as

$$\frac{E_o(s)}{E_i(s)} = \frac{R_4C_1}{R_3C_2}\,\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}}$$

- Where,

$T = R_1C_1$ $\quad aT = R_2C_2$ $\quad K_c = \frac{R_4C_1}{R_3C_2}$

- Then,

$$G_c(s) = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}},\qquad (0<\alpha<1)$$

- Notice that

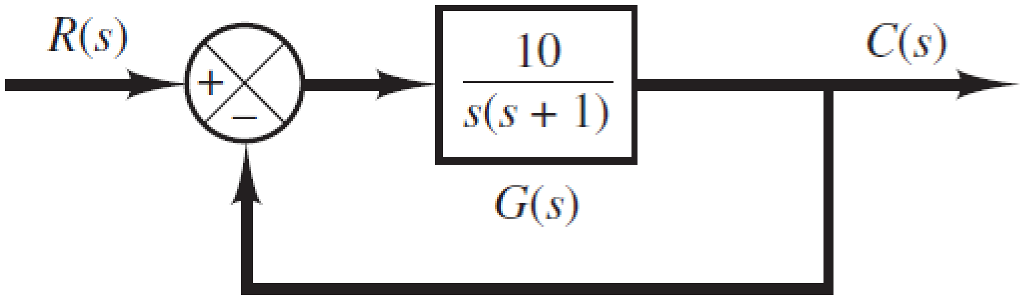
$$R_1C_1 > R_2C_2$$

Lead Compensation Techniques Based on the Root-Locus Approach

- From the performance specifications, determine the desired location for the dominant closed-loop poles.
- By drawing the root-locus plot of the uncompensated system ascertain whether or not the gain adjustment alone can yield the desired closed-loop poles. If not calculate the angle deficiency. This angle must be contributed by the lead compensator.
- If the compensator is required, place the zero of the phase lead network directly below the desired root location.
- Determine the pole location so that the total angle at the desired root location is 180^o and therefore is in the compensated root locus.
- Assume the transfer function of the lead compensator.
- Determine the open-loop gain of the compensated system from the magnitude conditions.

Example

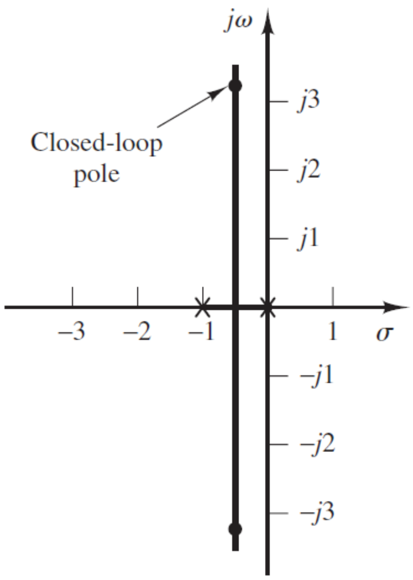
- Consider the position control system shown in following figure.



- It is desired to design an Electronic lead compensator $G_c(s)$ so that the dominant closed poles have the damping ratio 0.5 and undamped natural frequency 3 rad/sec.

Step-1

Draw the root Locus plot of the given system. $G(s)H(s)=\dfrac{10}{s(s+1)}$



- The closed loop transfer function of the given system is:

$$\frac{C(s)}{R(s)}=\frac{10}{s^2+s+10}$$

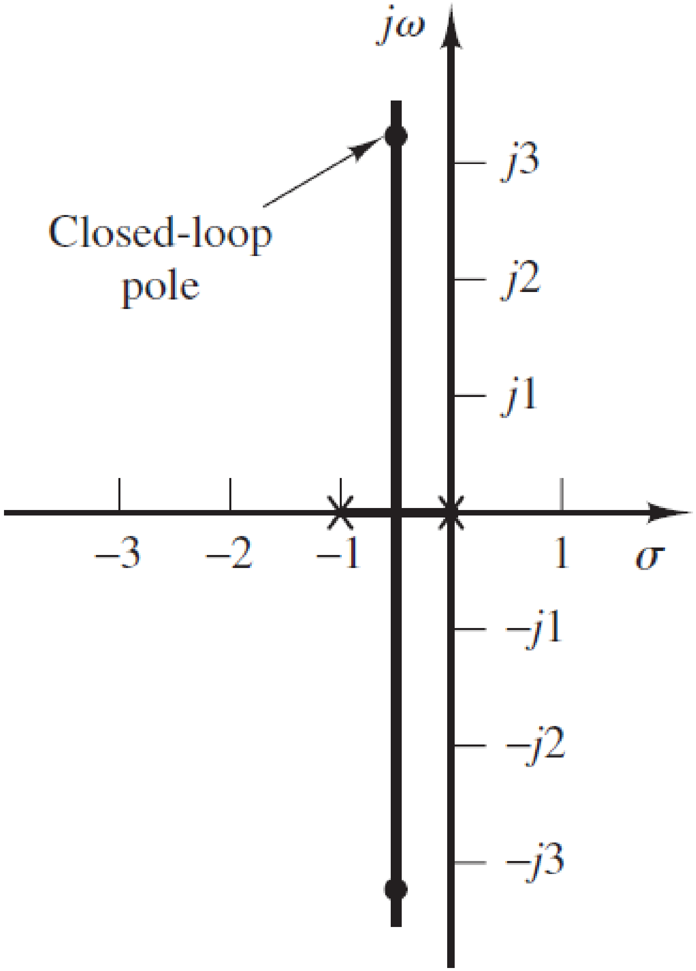
- The closed loop poles are $s=-0.5\pm j3.1225$

Step-1

- Determine the characteristics of given system using root loci.

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + s + 10}$$

- The damping ratio of the closed-loop poles is 0.158.
- The undamped natural frequency of the closed-loop poles is 3.1623 rad/sec.
- Because the damping ratio is small, this system will have a large overshoot in the step response and is not desirable.



Step-2

- From the performance specifications, determine the desired location for the dominant closed-loop poles.

- Desired performance Specifications are:
  - It is desired to have damping ratio 0.5 and undamped natural frequency 3 rad/sec.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{9}{s^2 + 3s + 9}$$

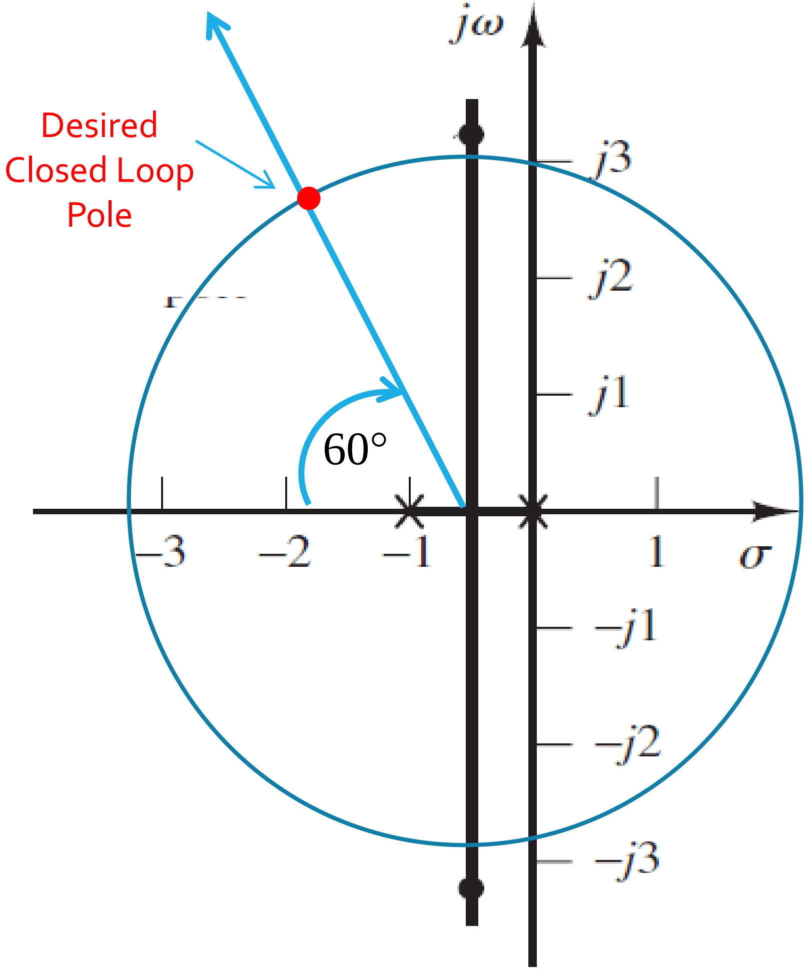
$$s = -1.5 \pm j2.5981$$

Step-2

- Alternatively desired location of closed loop poles can also be determined graphically
  - Desired ω_n= 3 rad/sec
  - Desired damping ratio= 0.5

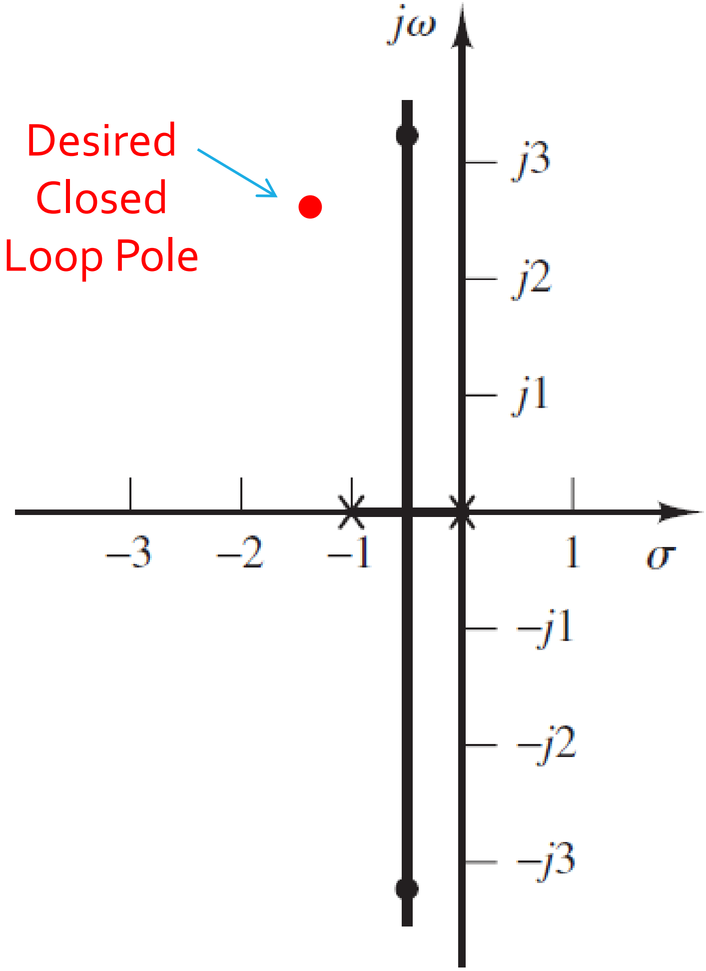
$$\theta = \cos^{-1}\zeta$$

$$\theta = \cos^{-1}(0.5) = 60°$$



Step-3

- From the root-locus plot of the uncompensated system ascertain whether or not the gain adjustment alone can yield the desired closed loop poles.



Step-3

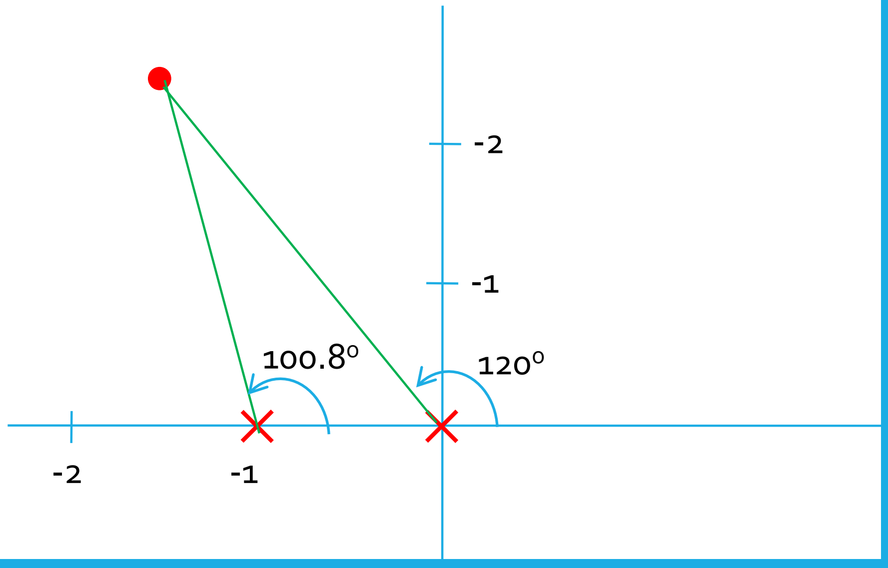
- If not, calculate the angle deficiency.
- To calculate the angle of deficiency apply Angle Condition at desired closed loop pole.

$$\theta_d = 180° - 120° - 100.8°$$

$$\theta_d = -40.89°$$

Desired Closed Loop Pole

$$s = -1.5 \pm j2.5981$$



Step-3

- Alternatively angle of deficiency can be calculated as.

$$\theta_d = 180^\circ + \angle \frac{10}{s(s+1)}\Bigg|_{s=-1.5+j2.5981}$$

Where $s = -1.5 \pm j2.5981$ are desired closed loop poles

$$\theta_d = 180^\circ + \angle 10 - \angle s\Big|_{s=-1.5+j2.5981} - \angle (s+1)\Big|_{s=-1.5+j2.5981}$$

$$\theta_d = 180^\circ - 120^\circ - 100.8^\circ$$

$$\theta_d = -40.89^\circ$$

Step-4

- This angle must be contributed by the lead compensator if the new root locus is to pass through the desired locations for the dominant closed-loop poles.

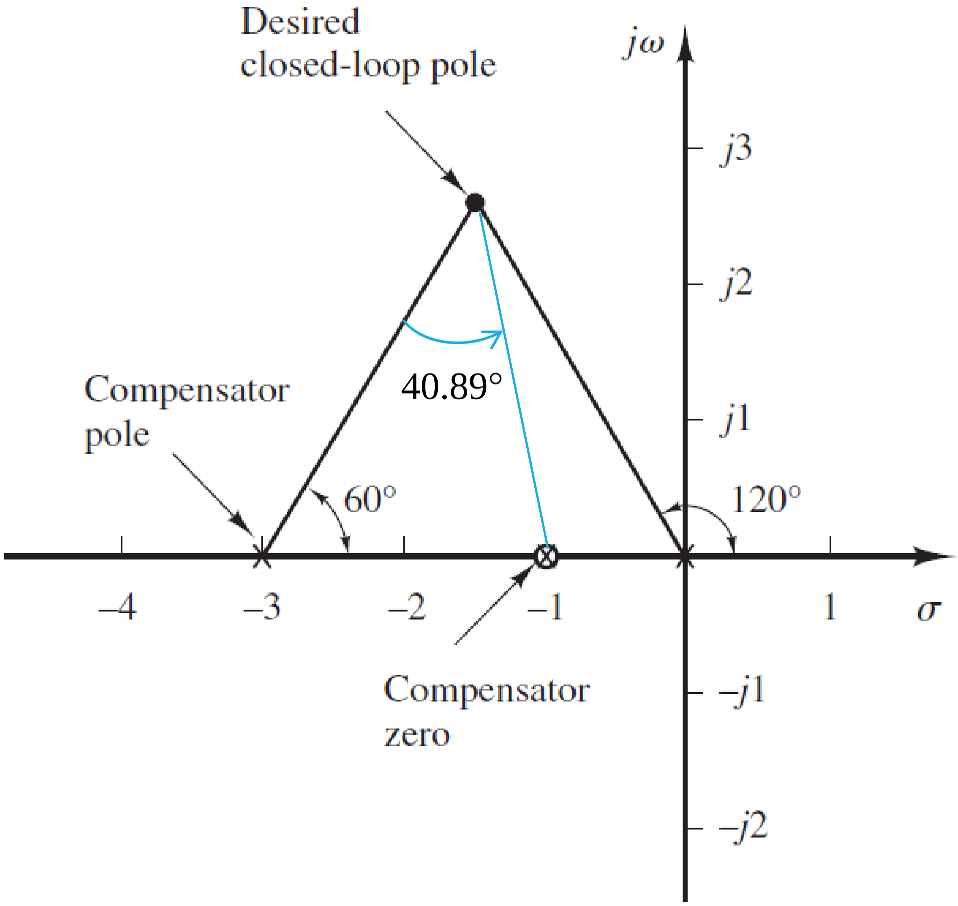
$$G_c(s) = K_c \alpha \frac{Ts + 1}{\alpha Ts + 1} = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}, \qquad (0 < \alpha < 1)$$

- Note that the solution to such a problem is not unique. There are infinitely many solutions.

Step-5

Solution-1

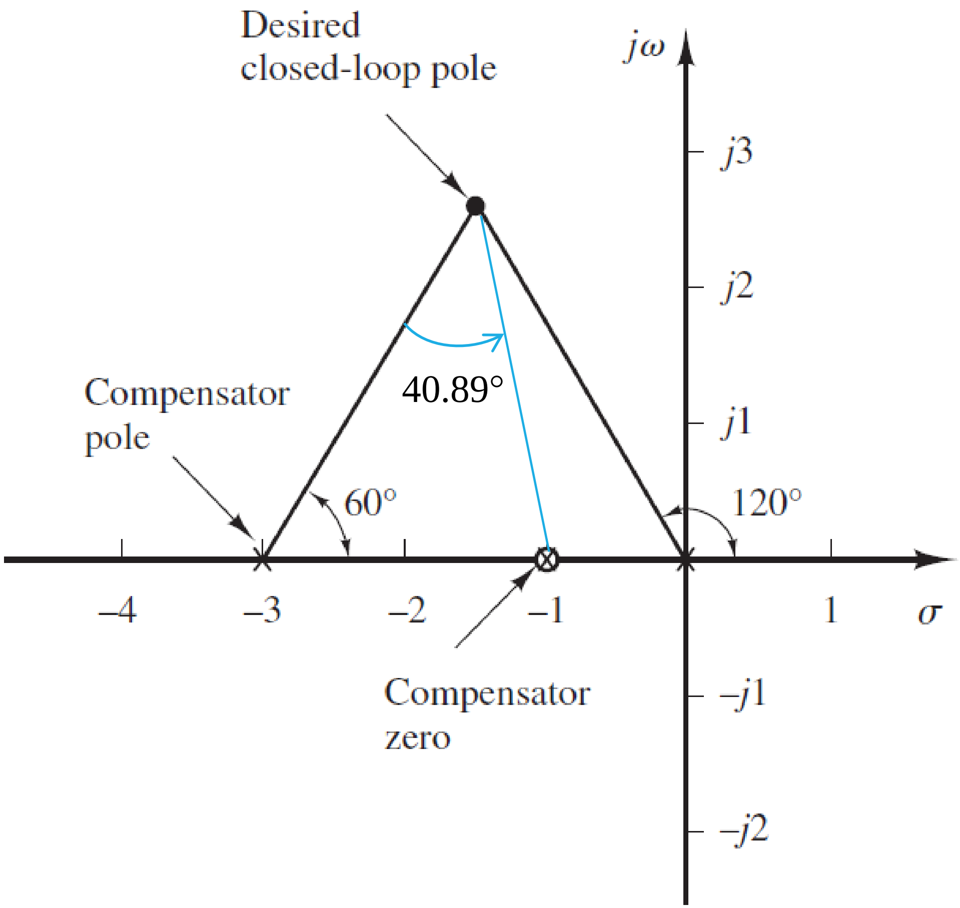
- **Solution-1**
  - If we choose the zero of the lead compensator at s = -1 so that it will cancel the plant pole at s =-1, then the compensator pole must be located at s =-3.



Step-5

Solution-1

- If static error constants are not specified, determine the location of the pole and zero of the lead compensator so that the lead compensator will contribute the necessary angle.



Step-5

Solution-1

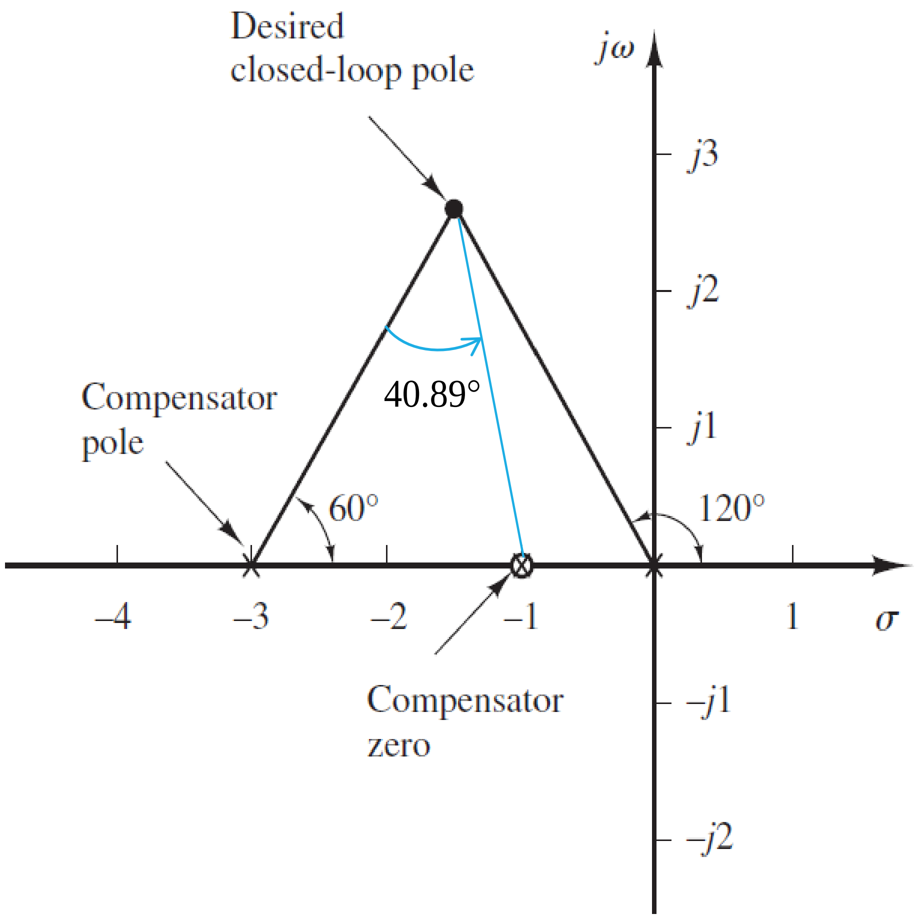
- The pole and zero of compensator are determined as

$$G_c(s) = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = K_c \frac{s+1}{s+3}$$

- The Value of α can be determined as

$$\frac{1}{T} = 1 \xrightarrow{\text{yields}} T = 1$$

$$\frac{1}{\alpha T} = 3 \xrightarrow{\text{yields}} \alpha = 0.333$$



Step-6

Solution-1

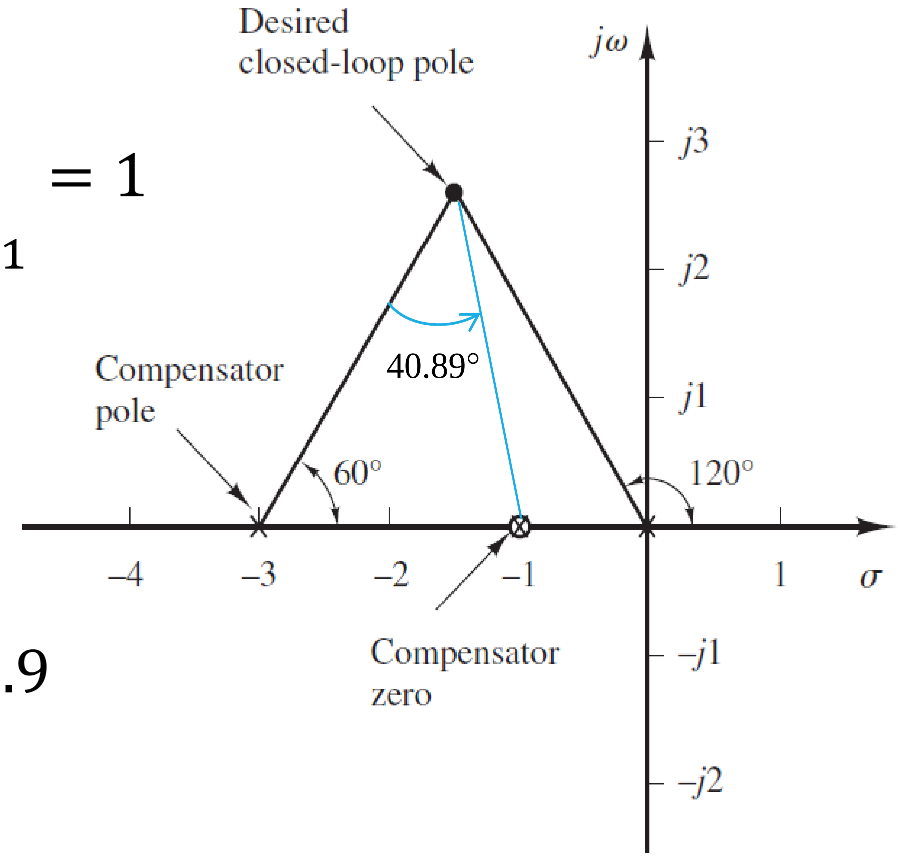
- The Value of K_c can be determined using magnitude condition.

$$\left| K_c \frac{(s+1)}{s+3} \frac{10}{s(s+1)} \right|_{s=-1.5+j2.5981} = 1$$

$$\left| K_c \frac{10}{s(s+3)} \right|_{s=-1.5+j2.5981} = 1$$

$$K_c = \left| \frac{s(s+3)}{10} \right|_{s=-1.5+j2.5981} = 0.9$$

$$G_c(s) = 0.9 \frac{s+1}{s+3}$$



Final Design Check

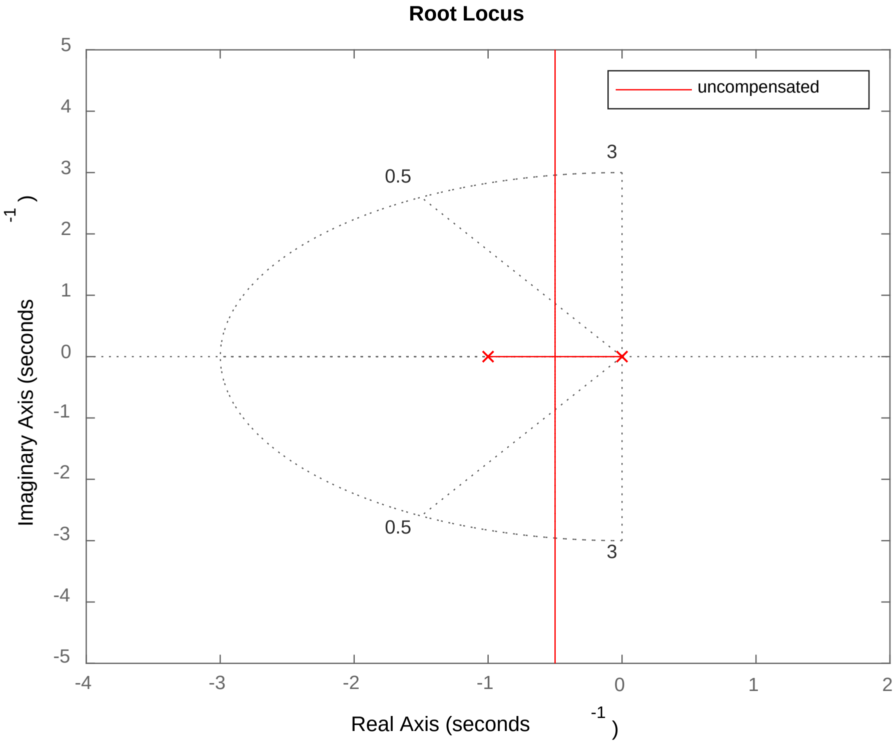
Solution-1

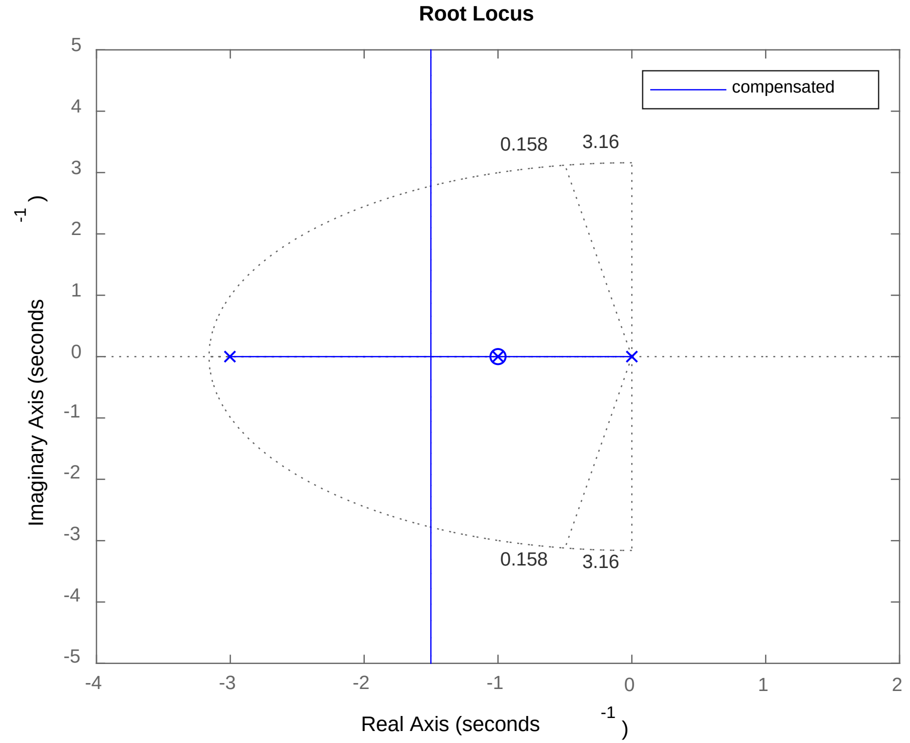
- The open loop transfer function of the designed system then becomes

$$G_c(s)G(s) = \frac{9}{s(s+3)}$$

- The closed loop transfer function of compensated system becomes.

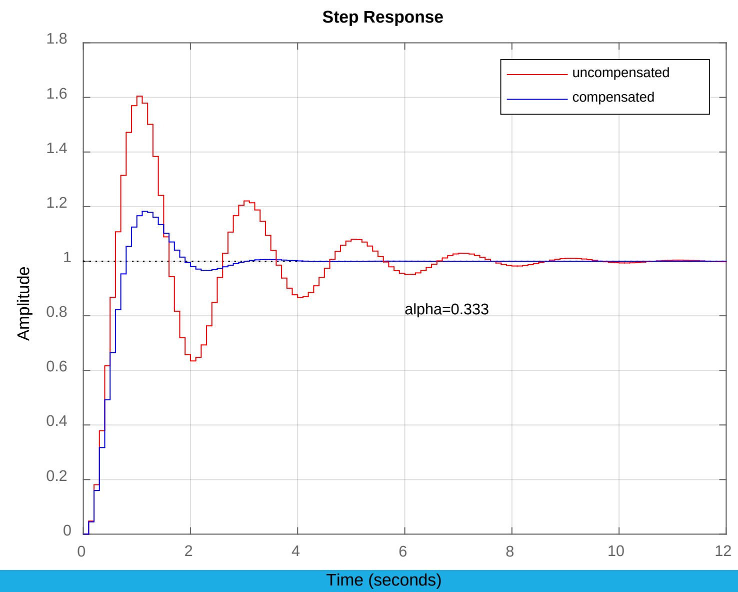
$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 3s + 9}$$





$$G(s) = \frac{10}{s(s+1)}$$

$$G_c(s)G(s) = \frac{9}{s(s+3)}$$



Final Design Check

Solution-1

- The static velocity error constant for original system is obtained as follows.

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{10}{s(s+1)}\right] = 10$$

- The steady state error is then calculated as

$$e_{ss} = \frac{1}{K_v} = \frac{1}{10} = 0.1$$

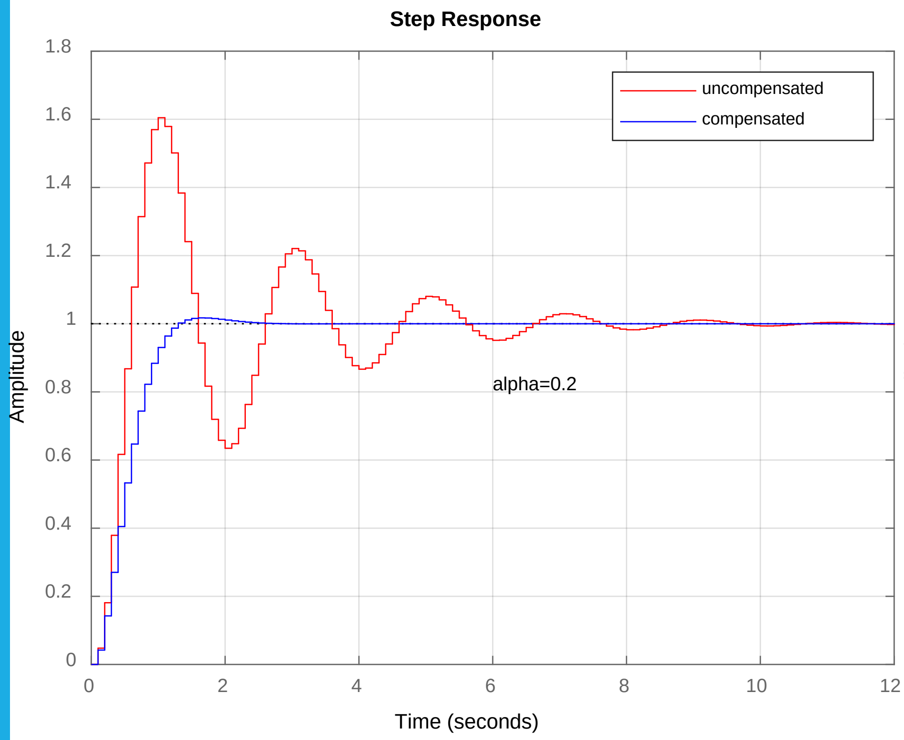
- The static velocity error constant for the compensated system can be calculated as

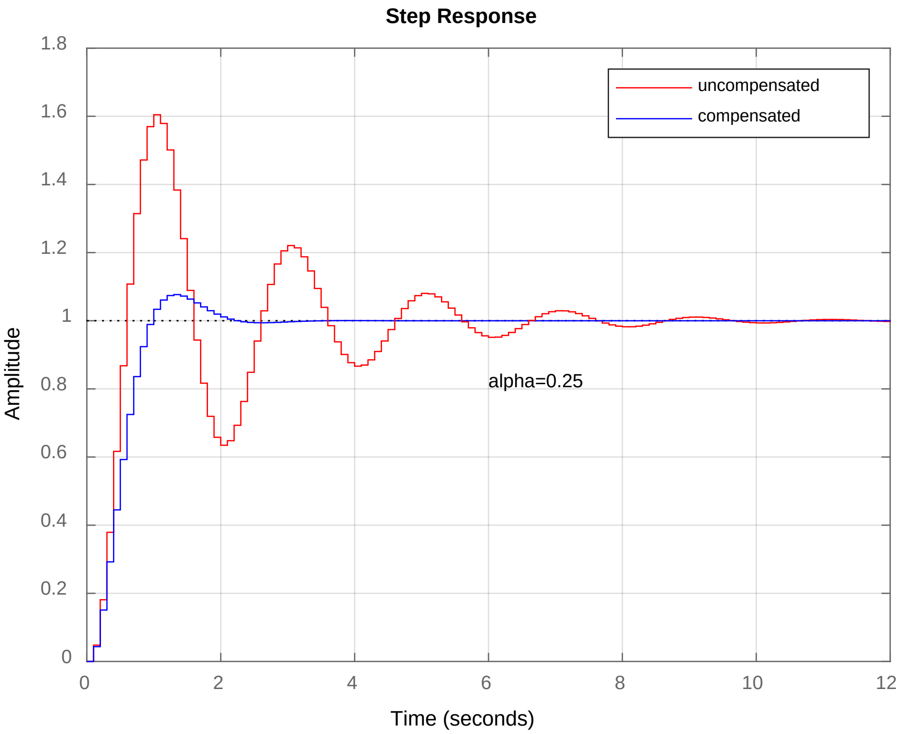
$$K_v = \lim_{s \to 0} sG_c(s)G(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{9}{s(s+3)}\right] = 3$$

- The steady state error is then calculated as

$$e_{ss} = \frac{1}{K_v} = \frac{1}{3} = 0.333$$


Step Response
uncompensated
compensated
alpha=0.2
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
2
4
6
8
10
12


Step Response
uncompensated
compensated
alpha=0.25
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
2
4
6
8
10
12