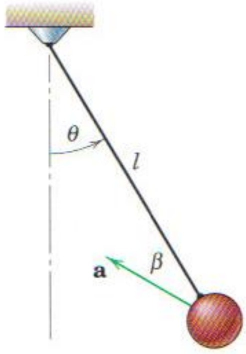
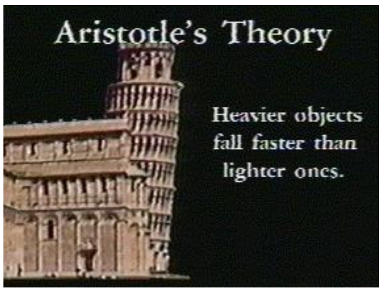
DYNAMICS FUNDAMENTALS

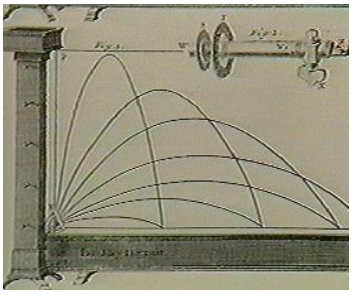
Historical Background:

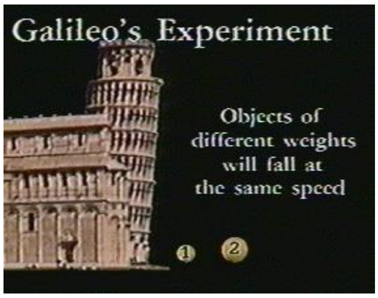
Galileo Galilei (1564-1642):

- Motion of the pendulum
- Bodies in free fall



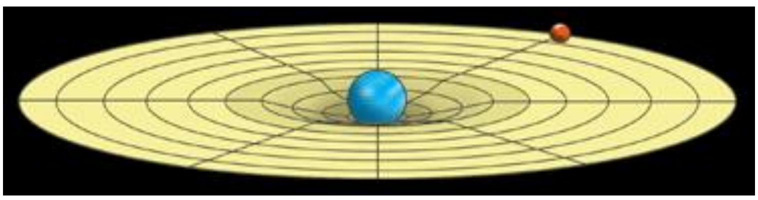


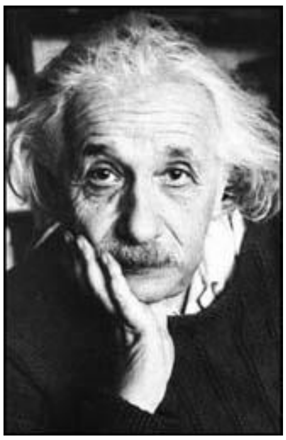




Albert Einstein (1879-1955)

Came up with a new idea describing the gravity (General Relativity).





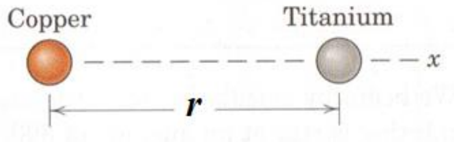
Newton (1642-1727)

Three Laws of motion and law of universal gravitation

Law I: A body remains at rest or continues to move with uniform velocity if there is no unbalanced force acting on it.

Law II: The acceleration of a body is proportional to the resultant force acting on it and is in the direction of this force ($\boldsymbol{F} = m\boldsymbol{a}$).

Law III: The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and collinear.



$$F = G\frac{m_1 m_2}{r^2}$$

F = mutual force of attraction between the two particles.
G = the constant gravitation.
m_1 & m_2 = masses of the two particles.
r = distance between centers of particles.

What is Dynamics?

Mechanics
Branch of physical sciences that deals with the state of rest or motion of bodies subjected to the action of forces

Statics
Deals with the action of forces on bodies at rest

$$\sum F = 0$$
$$\sum M = 0$$

Dynamics
Deals with the motion of bodies under the action of forces

$$\sum F \neq 0$$
$$\sum M \neq 0$$

Dynamics

Kinematics
Study the **<u>motion</u>** without reference to the forces that cause it

Displacement / Distance
Velocity / Speed
Acceleration / Deceleration

Including Time

Kinetics
Relates the action of **<u>forces</u>** on bodies to their resulting **<u>motions</u>**

Kinematics:

- Rectilinear kinematics: Continuous motion.
- Rectilinear kinematics: Graphic representation of the motion.
- Motion of projectile.
- Curvilinear motion.
- Relative - motion of two particles.
- Absolute dependent motion analysis of two particles.

Kinetics:

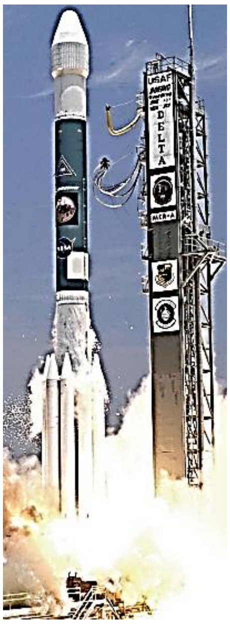
- Newton's Second Law of Motion.
- Work & Kinetic Energy.
- Potential Energy.
- Impulse-Momentum.
- Impact.

Part 1 / Kinematics of a particle:

Chapter 1 / Rectilinear kinematics: Continuous motion:

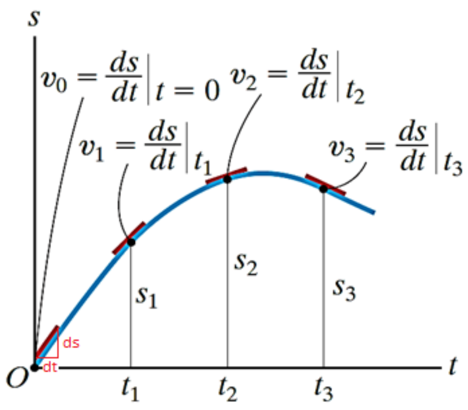






Chapter 2 /

Rectilinear kinematics : Graphic representation of the motion:



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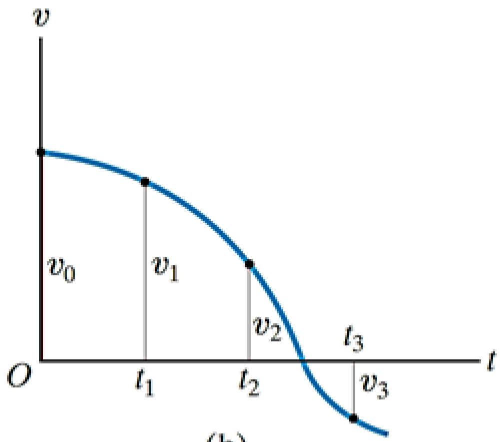

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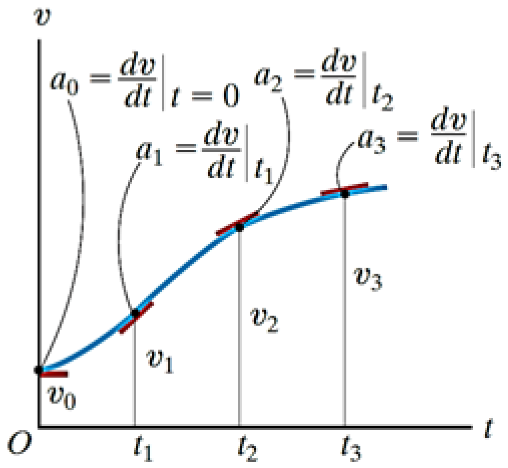

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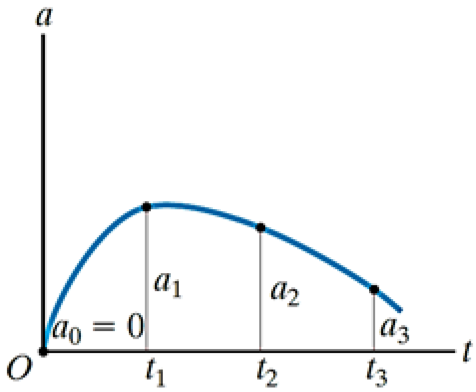

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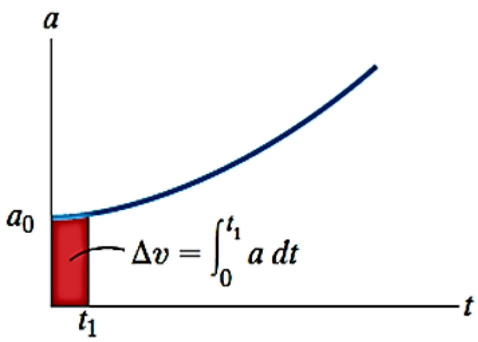

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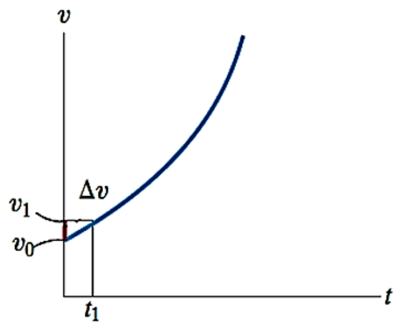

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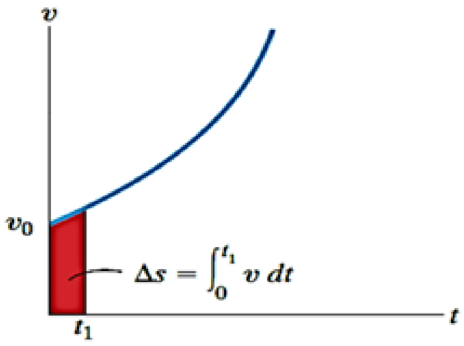
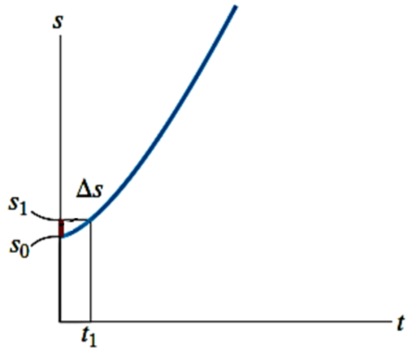
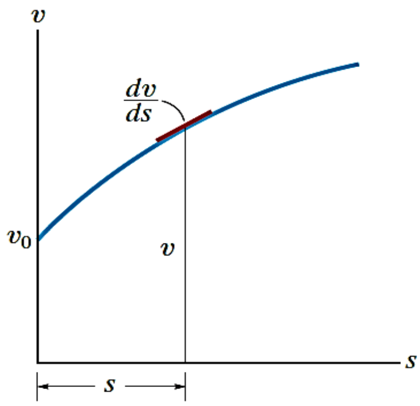

v
v_0
$\Delta s = \int_0^{t_1} v\, dt$
t_1
t

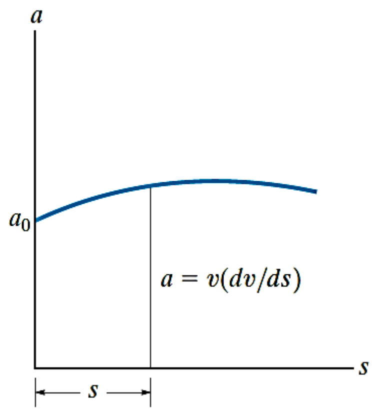
(a)


s
Δs
s_1
s_0
t_1
t

(b)

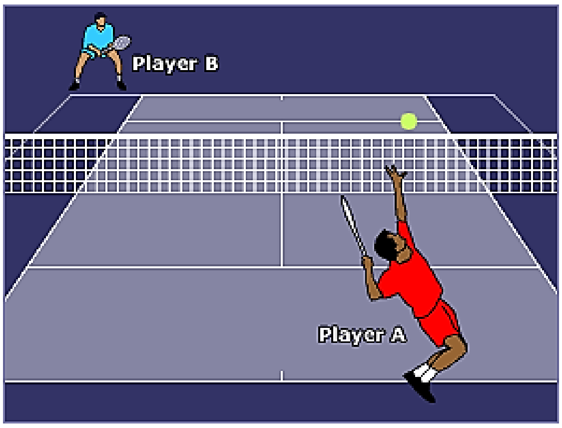

v
$\frac{dv}{ds}$
v_0
v
s
s

(a)


a
a_0
$a = v(dv/ds)$
s
s

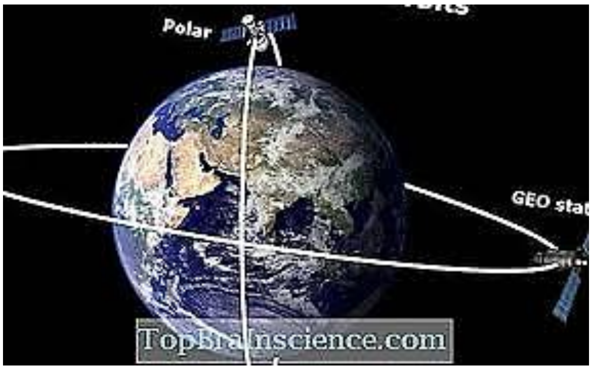
(b)

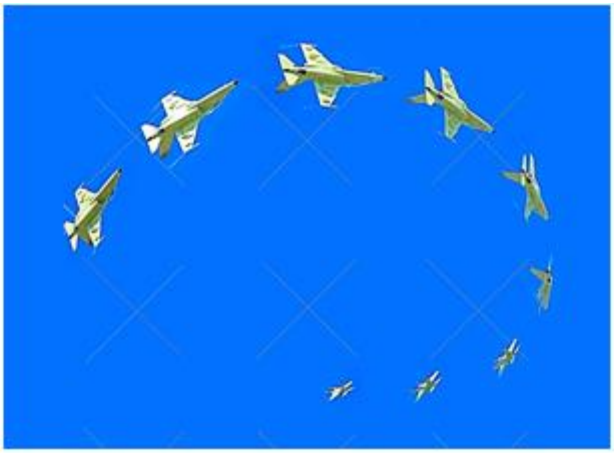
Chapter 3 / **Motion of projectiles:**

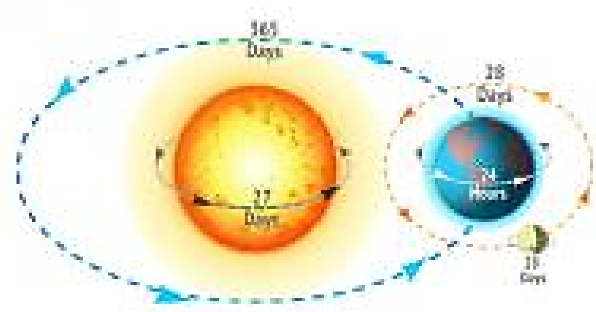



wikiHow

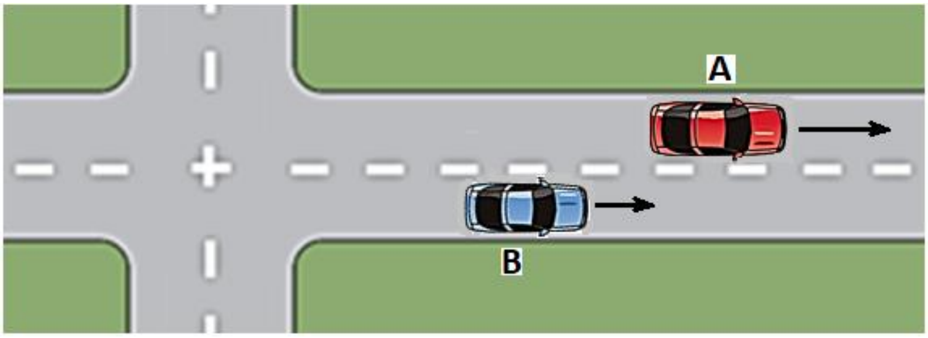
Chapter 4 / **Curvilinear motion:**

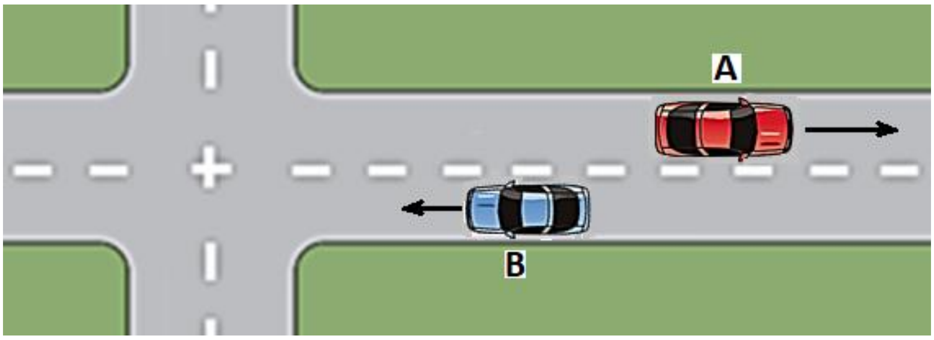


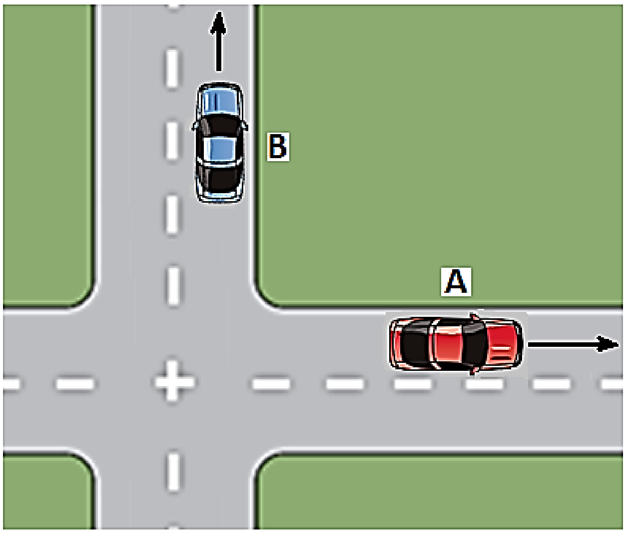




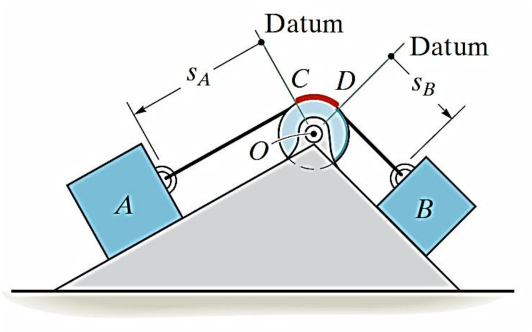
Chapter 5 / **Relative - motion of two particles:**

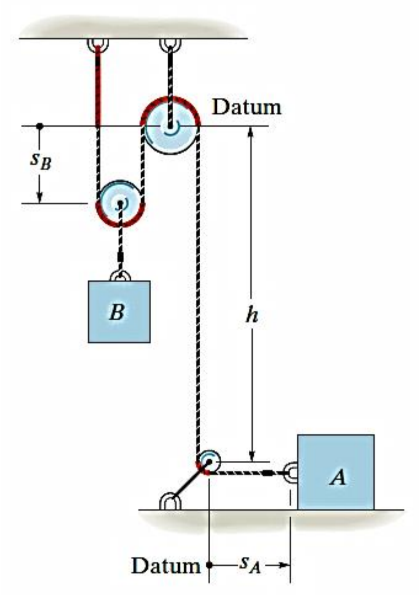


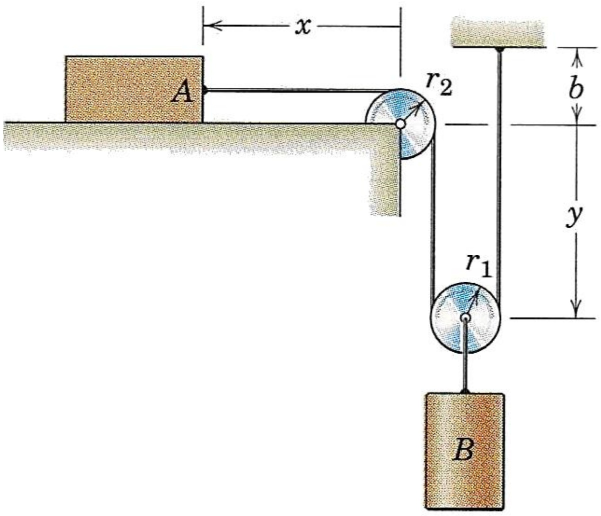


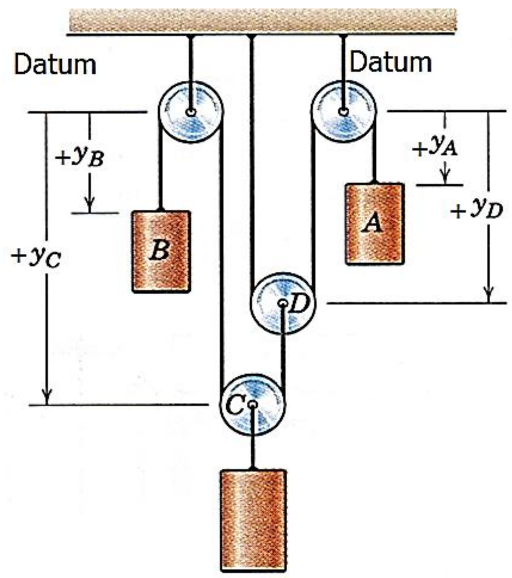


Chapter 6 / **Absolute dependent motion analysis of two particles:**









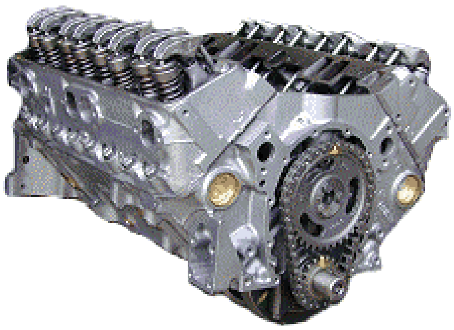
Why do we need to study dynamics ?

Dynamics principles are basic to the analysis and design of moving structures in:

- Automotive industries
- Aerospace industries
- Automatic control systems
- Turbines, pumps &machine tools.



- How do we decide how big to make the pistons?
- Where should they be placed in the engine block?
- How do we make the engine run smoothly?



Well, we could answer these questions by trial and error. But the 'errors' would be expensive exercises. Why not study the dynamics of engines and make some predictions instead!

Or suppose we want to build a robot;

- How do we decide how big to make the motors?
- How fast can we expect it to move from one place to another?
- How accurate will it be while it moves and stops?
- How many joints should it have and where?

Fundamental concepts:

- **Length (Space) :** needed to locate position of a point in space, & describe size of the physical system
    - Position in space is determined relative to some geometric reference system by means of linear and angular measurements.
    - This reference system may be:
        - Fixed in space: measurements are said to be absolute.
        - Moving in space: measurements are made relative to fixed reference system.

- **Time (t):** measure of succession of events.
- **Mass (m):** The quantity of matter in a body as well as the property which gives rise to gravitational attraction. Weight is related to the mass by: *mg.; where g* is the gravitational acceleration.
- **Force**: represents the action of one body on another characterized by its magnitude, direction of its action, and its point of application

Particles : A body of negligible dimensions (when its dimensions are irrelevant to the description of its motion or the action of the forces on it). Bodies of finite size, such as rockets, projectiles, or vehicles.

Rigid Bodies : A combination of large number of particles in which all particles remain at a fixed distance (practically) from one another before and after applying a load.

Basic vector agebra:

Scalars are quantities having only magnitude.

- Length or distance, speed, mass, time, work, energy, power, etc.

Vectors are quantities having both a magnitude and a direction.

- Position, displacement, velocity, acceleration, force, moment, impulse, momentum, etc.

Derived quantities related dynamics and their (SI) units:

Velocity = $\frac{\text{Displacement}}{\text{Time}}$	m/s
Speed = $\frac{\text{Distance}}{\text{Time}}$	m/s
Acceleration = $\frac{\text{Velocity}}{\text{Time}}$	m/s^2
Force = Mass × Acceleration	$kg.m/s^2$ = N (Newton)
Work = Energy = Force × Distance	N.m = J (Joule)
Power = $\frac{\text{Work}}{\text{Time}}$	J/s = W (Watt)
Impulse = Momentum = Force × Time	N.s

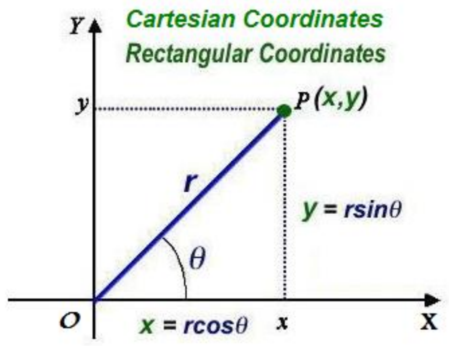
Angular quantities needed in dynamics:

Angular distance	$\theta\,° = \frac{\theta\times\pi}{180}$ rad.
Angular speed	$\dot{\theta} = \frac{\text{deg.}}{\text{s}} = \frac{\dot{\theta}\times\pi}{180}$ rad/s.
	$\dot{\theta}$ rpm $= \frac{2\pi\times\dot{\theta}}{60}$ rad/s.
Angular speed	$\ddot{\theta} = \frac{\text{deg.}}{\text{s}^2} = \frac{\ddot{\theta}\times\pi}{180}$ rad/s^2.

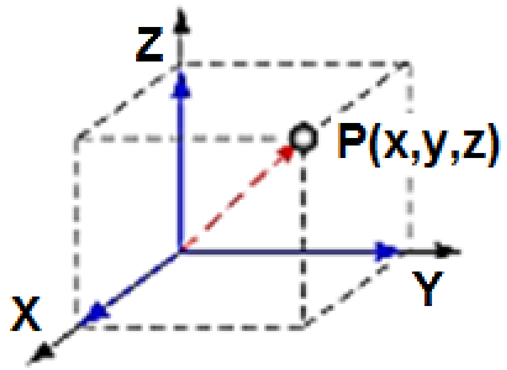
Coordinates system :

- **Cartesian coordinate system:**

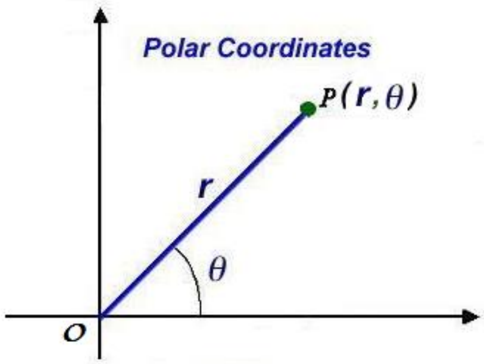
Two dimensions



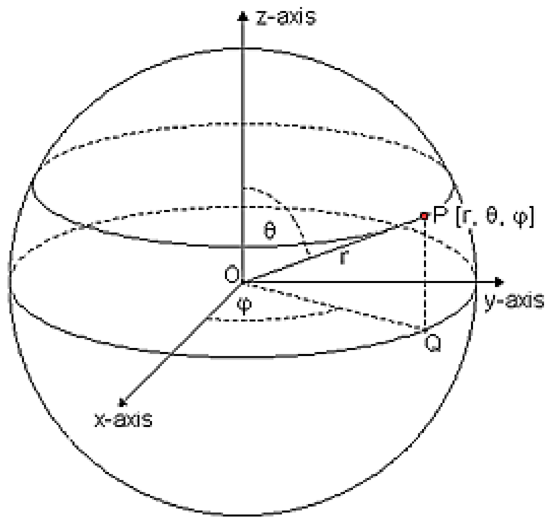
Three dimensions



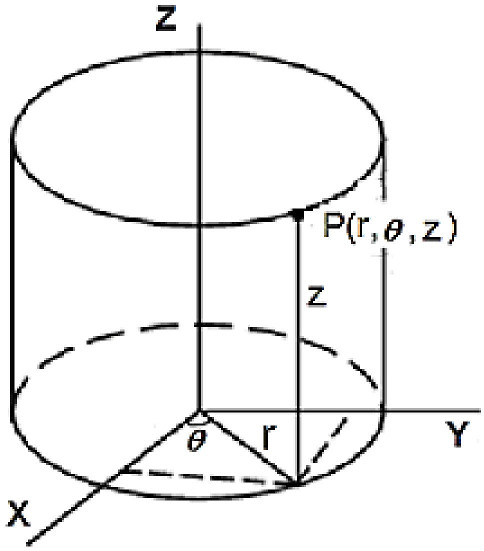
- **Polar coordinate system:**



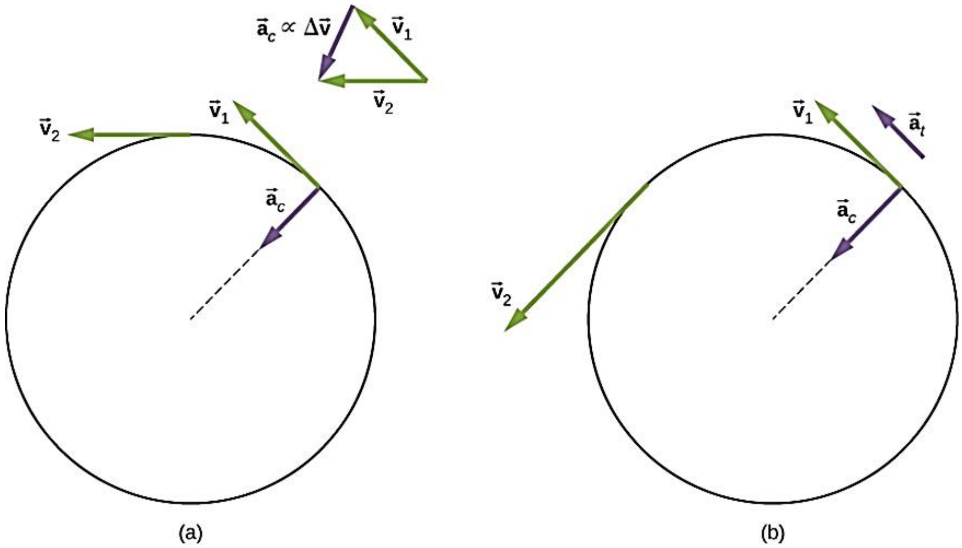
- **Spherical coordinate system:**



- **Cylindrical coordinate system:**



- **Normal - Tangential coordinate system:**

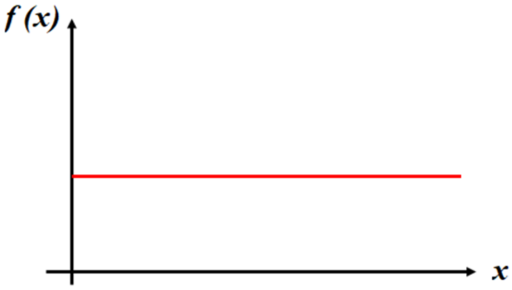


Math Revision:

Functions:

Zero order function
Constant value

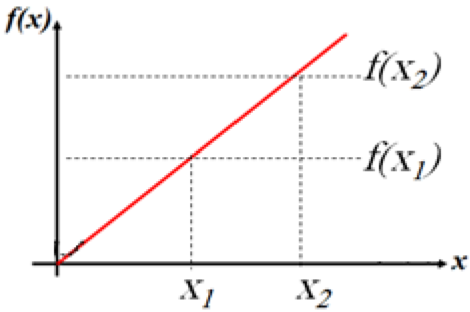
F(x) = c



$$Slope = m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

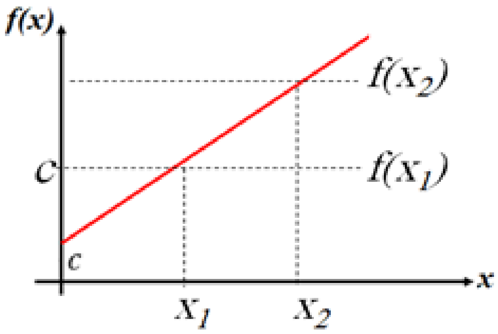
1st order function
Linearly Variable

F(x) = mx



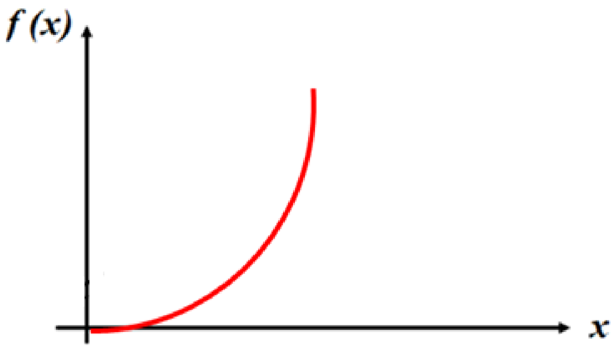
$$Slope = m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

F(x) = mx +c



2nd order function
Curvilinearly Variable

F(x) = x^2



Differential Calculus:

Derivatives Table

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$af(x)$	$af'(x)$	a^u	$(\ln a)a^u \dfrac{du}{dx}$
$u(x)+v(x)$	$u'(x)+v'(x)$	$\ln u$	$\dfrac{1}{u}\dfrac{du}{dx}$
$f(u)$	$f'(u)\dfrac{du}{dx}=\dfrac{df(u)}{du}\dfrac{du}{dx}$	$\log_a u$	$(\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$
a	0	$\sin u$	$\cos u\left(\dfrac{du}{dx}\right)$
$x^n \quad (n\neq 0)$	nx^{n-1}	$\cos u$	$-\sin u\dfrac{du}{dx}$
$u^n \quad (n\neq 0)$	$nu^{n-1}\dfrac{du}{dx}$	$\tan u$	$\sec^2 u\dfrac{du}{dx}$
uv	$u\dfrac{dv}{dx}+v\dfrac{du}{dx}$	$\sin^{-1} u$	$\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx} \quad \left(-\dfrac{\pi}{2}\leq \sin^{-1}u \leq \dfrac{\pi}{2}\right)$
$\dfrac{u}{v}$	$\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$	$\cos^{-1} u$	$\dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx} \quad (0\leq \cos^{-1}u\leq \pi)$
e^u	$e^u\dfrac{du}{dx}$	$\tan^{-1} u$	$\dfrac{1}{1+u^2}\dfrac{du}{dx} \quad \left(-\dfrac{\pi}{2}<\tan^{-1}u<\dfrac{\pi}{2}\right)$

Integral Table

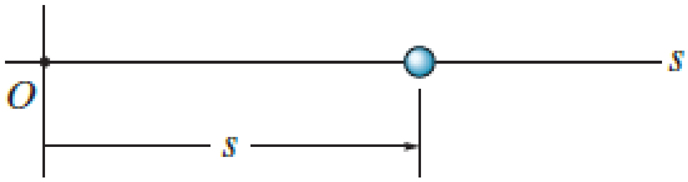
$f(x)$	$F(x)=\int f(x)dx$	$f(x)$	$F(x)=\int f(x)dx$
$af(x)$	$aF(x)$	$\cos^2 ax$	$\dfrac{1}{2}x+\dfrac{1}{4a}\sin 2ax$
$u(x)+v(x)$	$\int u(x)dx+\int v(x)dx$	$x\sin ax$	$\dfrac{1}{a^2}\sin ax-\dfrac{x}{a}\cos ax$
a	ax	$x\cos ax$	$\dfrac{1}{a^2}\cos ax+\dfrac{x}{a}\sin ax$
$x^n \quad (n\neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sin ax\cos ax$	$\dfrac{1}{2a}\sin^2 ax$
e^{ax}	$\dfrac{e^{ax}}{a}$	$\sin ax\cos bx$ for $a^2\neq b^2$	$-\dfrac{\cos(a-b)x}{2(a-b)}-\dfrac{\cos(a+b)x}{2(a+b)}$
$\dfrac{1}{x}$	$\ln x$	xe^{ax}	$\dfrac{e^{ax}}{a^2}(ax-1)$
$\sin ax$	$-\dfrac{1}{a}\cos ax$	$\ln x$	$x(\ln x-1)$
$\cos ax$	$\dfrac{1}{a}\sin ax$	$\dfrac{1}{ax^2+b}$	$\dfrac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\dfrac{a}{b}}\right)$
$\sin^2 ax$	$\dfrac{1}{2}x-\dfrac{1}{4a}\sin 2ax$		

Part 1
Kinematics of a particle

Chapter 1
Rectilinear kinematics: Continuous motion

Rectilinear motion: particle moving along a straight line.

1. **Position (S):** defined by positive or negative distance from a fixed origin on the line.

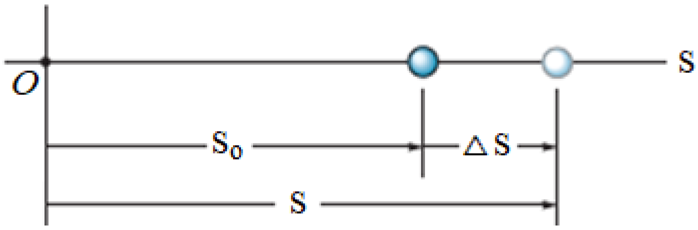


Position

The magnitude of (S) is the distance from (*O*) to the particle, usually measured in meters (m) or feet (ft), and the sense of direction is defined by the algebraic sign on (S).

2. **Displacement (ΔS):** The displacement of the particle is defined as the change in its position. For example, if the particle moves from one point to another, the displacement is:

$$\Delta s = s - s_o$$



- Δs is *positive* since the particle's final position is to the *right* of its initial position.
- Δs is *negative* since the particle's final position is to the *left* of its initial position.

3. **Distance (d):**

The distance is the total path length traveled by an object during its motion. It is a scalar quantity, so it only has magnitude and no direction. It is always non-negative, as it represents the total length traveled.

4. **Velocity (V):** If the particle moves through a displacement (ΔS) during the time interval (Δt), the **average velocity** of the particle during this time interval is:

$$\mathbf{V_{avg}} = \frac{\Delta S}{\Delta t}$$

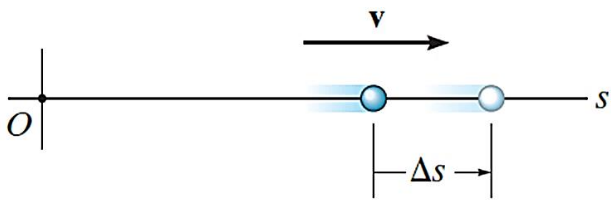
- If we take smaller and smaller values of (Δt), the magnitude of (ΔS) becomes smaller and smaller. Consequently, we get an **instantaneous velocity**:

$$+ \rightarrow \qquad v_{instant} = \lim_{\Delta t \to \mathbf{0}} \frac{\Delta s}{\Delta t}$$

- Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as **particle speed**.
- From the definition of a derivative,

$$v = \lim_{\Delta t \to \mathbf{0}} \frac{\Delta s}{\Delta t} = \frac{\mathbf{ds}}{\mathbf{dt}}$$

- The magnitude of the velocity is generally expressed in units of (m/s) or (ft/s).
- If the particle is moving to the right, the velocity is positive; whereas if it is moving to the left, the velocity is negative.

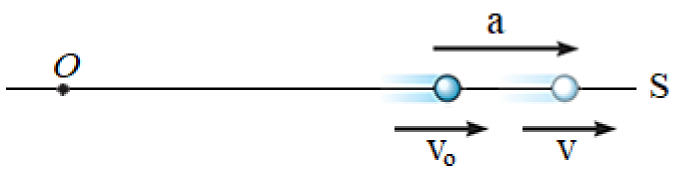


5. **Speed (V_{sp}):** Speed is the rate at which an object changes its position. It is a scalar quantity and is defined as the distance traveled divided by the time taken to travel that distance.

Mathematically, $V_{sp} = \frac{d}{t}$ $\qquad (V_{sp})_{avg} = \frac{d}{\Delta t}$

6. **Acceleration:** Provided the velocity of the particle is known at two points, the average acceleration of the particle during the time interval (Δt) is defined as

$$a_{avg} = \frac{\Delta V}{\Delta t} \qquad \text{when,} \quad \Delta v = v - v_o$$



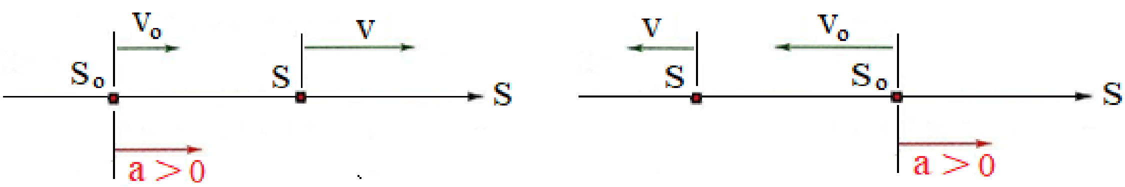
- From the definition of a derivative, $a = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt} = \frac{d^2S}{dt^2}$

• If the velocity is constant, then the acceleration is **zero**,

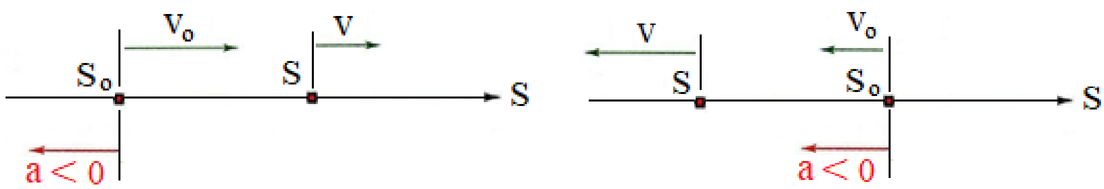
$$(\Delta v = v - v_o = 0)$$

- Instantaneous acceleration may be:

- **positive:** increasing positive velocity or decreasing negative velocity



- **negative:** decreasing positive velocity or increasing negative velocity



7- Equations of Motion:

Velocity:

$$v = \frac{dS}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

ds = v dt

$$\int_{s_o}^{s} ds = \int_0^t v\, dt$$

Acceleration:

$$a = \frac{dv}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

dv = a dt

$$\int_{v_o}^{v} dv = \int_0^t a\, dt$$

$$d\,t = \frac{ds}{v} = \frac{dv}{a}$$

$$a\, ds = v\, dv \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\int_{s_o}^{s} a\, ds = \int_{v_o}^{v} v\, dv$$

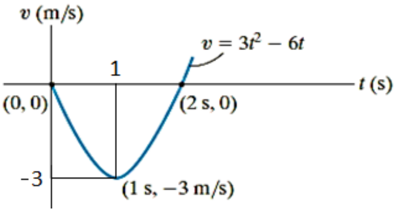
Example 1 - 1:

A particle moves along a horizontal path with a velocity of [v = (3t² - 6t) m/s], where (t) is the time in seconds. If it is initially located at the origin (*O*), determine the distance traveled in (3.5 s), and the particle's average velocity and average speed during the time interval.

Solution:

Coordinate System. Here positive motion is to the right, measured from the origin *O*, Fig. *a*.

Distance Traveled. Since v = f(t), the position as a function of time may be found by integrating v = ds/dt with $t = 0$, $s = 0$.



$+\longrightarrow$

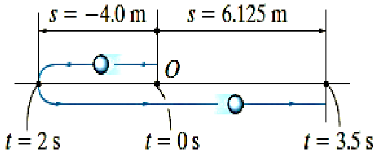
$$ds = v\,dt = (3t^2 - 6t)\,dt$$

$$\int_0^s ds = \int_0^t (3t^2 - 6t)\,dt$$

$$s = (t^3 - 3t^2)\text{ m} \quad \ldots\ldots \quad (1)$$

In order to determine the distance traveled in 3.5 s, it is necessary to investigate the path of motion. If we consider a graph of the velocity function, Fig. *b*, then it reveals that for 0 s *to* 2 s the velocity is *negative*, which means the particle is traveling to the *left*, and for $t > 2$ s the velocity is *positive*, and hence the particle is traveling to the *right*. Also, note that v = 0 at $t = 2$ s. The particle's position when $t = 0$, $t = 2$ s, and $t = 3.5$ s can be determined from Eq. 1. This yields:

$$s\big|_{t=0} = 0 \ , \quad s\big|_{t=2\,s} = -4\text{ m} \ , \quad s\big|_{t=3.5\,s} = 6.125\text{ m}$$



(a)

The path is shown in Fig. *a*. Hence, the distance traveled in 3.5 s is:

$$s_T = 4 + 4 + 6.125 = 14.125\text{ m} = 14.1\text{ m}$$

Velocity, The displacement from (t = 0) to (t = 3.5 s) is:

$$\Delta s = s\big|_{t=3.5\,s} - s\big|_{t=0} = 6.125 - 0 = 6.125\text{ m} \rightarrow$$

And so the average velocity is:

$$V_{avg} = \frac{\Delta S}{\Delta t} = \frac{6.125}{3.5 - 0} = 1.75\text{ m/s} \rightarrow$$

The average speed is defined in terms of the distance traveled (S_T).

$$(V_{sp})_{avg} = \frac{S_T}{\Delta t} = \frac{14.125}{3.5 - 0} = 4.04\text{ m/s}$$

Constant acceleration, ($a = a_c$):

When the acceleration is constant, each of the three kinematic equations:

$$a_c = \frac{dv}{dt} \qquad v = \frac{ds}{dt} \qquad a_c\,ds = v\,dv$$

can be integrated to obtain formulas that relate (a_c , v, *s*, and t).

Velocity as a function of time.

Integrate ($a_c = \frac{dv}{dt}$), assuming that initially ($v = v_o$) when (t = 0).

$$dv = a_c\,dt$$

$$\int_{v_o}^{v} dv = \int_0^t a_c\,dt = a_c \int_0^t dt$$

$$v - v_o = a_c\,t$$

$$v = v_o + a_c\,t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Position as a function of time.

Integrate ($v = \frac{ds}{dt} = v_o + a_c\,t$), assuming that initially ($s = s_o$) when (t = 0).

$$ds = (v_o + a_c\,t)\,dt$$

$$\int_{s_o}^{s} ds = \int_0^t (v_o + a_c\,t)\,dt$$

$$s - s_o = v_o\,t + \frac{1}{2} a_c\,t^2$$

$$s = s_o + v_o\,t + \frac{1}{2} a_c\,t^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Velocity as a function of position.

Either solve for (*t*) in Eq. (4) and substitute into Eq. (5), or integrate ($v\,dv = a_c\,ds$) assuming that initially ($v = v_o$) at ($s = s_o$).

Metod (1):

From Eq. (4) $\quad t = \frac{v - v_o}{a_c}$

Sub. in Eq. (5) $\quad s = s_o + v_o \left(\frac{v - v_o}{a_c}\right) + \frac{1}{2} a_c \left(\frac{v - v_o}{a_c}\right)^2$

$$s - s_o = \frac{v\,v_o - v_o 2}{a_c} + \frac{1}{2}\,\frac{v^2 - 2v\,v_o + v_o 2}{a_c} \qquad \times\, 2a_c$$

$$2a_c\,(s - s_o) = 2vv_o - 2v_o^2 + v^2 - 2vv_o + v_o^2$$

$$2a_c\,(s - s_o) = v^2 - v_o^2$$

$$v^2 = v_o^2 + 2a_c\,(s - s_o) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

Metod (2):

$$v\,dv = a_c\,ds$$

$$\int_{v_o}^{v} v\,dv = \int_{s_o}^{s} a_c ds = a_c \int_{s_o}^{s} ds$$

$$\frac{1}{2}\,v^2 - \frac{1}{2}\,{v_o}^2 = a_c\,(\,s - s_o\,)$$

$$\frac{1}{2}\,(\,v^2 - {v_o}^2\,) = a_c\,(\,s - s_o\,)$$

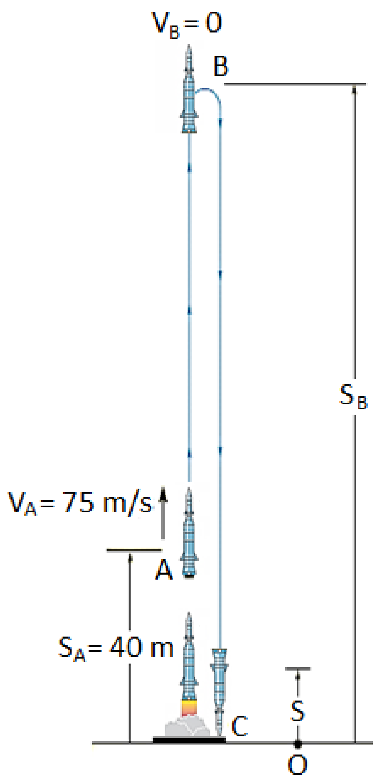
$$v^2 - {v_o}^2 = 2a_c\,(\,s - s_o\,)$$

$$v^2 = {v_o}^2 + 2a_c\,(\,s - s_o\,) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\,6\,)$$

Example 1 - 2:

During a test a rocket travels upward at (75 m/s), and when it is (40 m) from the ground its engine fails. Determine the maximum height (S_B) reached by the rocket and its speed just before it hits the ground. While in motion the rocket is subjected to a constant downward acceleration of (9.81 m/s^2) due to gravity. Neglect the effect of air resistance.

Solution:



Coordinate System. The origin *O* for the position coordinate *s* is taken at ground level with positive upward.

Maximum Height. Since the rocket is traveling *upward*, $v_A = +75$ m/s when $t = 0$. At the maximum height $s = s_B$ the velocity $v_B = 0$. For the entire motion, the acceleration is $a_c = -9.81$ m/s^2 (negative since it acts in the *opposite* sense to positive velocity or positive displacement). Since a_c is *constant* the rocket's position may be related to its velocity at the two points *A* and *B* on the path by using Eq. 6, namely,

$$(+\uparrow) \qquad v_B^2 = v_A^2 + 2\,a_c\,(s_B - s_A)$$

$$0 = (75)^2 + 2\,(-9.81)\,(s_B - 40)$$

$$s_B = 327 \text{ m}$$

Velocity. To obtain the velocity of the rocket just before it hits the ground, we can apply Eq. 6 between points *B* and *C*,

$$(+\downarrow) \qquad v_C^2 = v_B^2 + 2\,a_c\,(s_C - s_B)$$

$$= (0)^2 + 2\,(9.81)\,(0 - (-327))$$

$$v_C = 80.1 \text{ m/s} \downarrow$$

The negative root was chosen since the rocket is moving downward. Similarly, Eq. 6 may also be applied between points A and C, i.e.,

$$(+\downarrow) \qquad v_C^2 = v_A^2 + 2\,a_c\,(s_C - s_A)$$

$$= (75)^2 + 2\,(9.81)\,(0 - (-40))$$

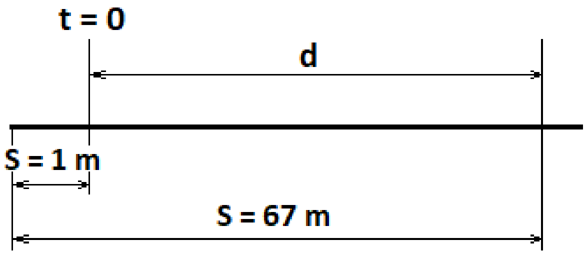
$$v_C = 80.1 \text{ m/s} \downarrow$$

NOTE: It should be realized that the rocket is subjected to a *deceleration* from *A* to *B* of 9.81 m/s^2, and then from *B* to *C* it is *accelerated* at this rate. Furthermore, even though the rocket momentarily comes to *rest* at *B* ($v_B = 0$) the acceleration at *B* is still 9.81 m/s^2 downward!

Example 1 - 3:

The acceleration of a particle as it moves along a straight line is given by [$a = (2t - 1)\ m/s^2$], where (t) is in seconds. If (s = 1 m) and (v = 2 m/s) when (t = 0), determine the particle's velocity and position when (t = 6 s). Also, determine the total distance the particle travels during this time period.

Solution:



$a = 2t - 1$

$a = \frac{dv}{dt}$

$dv = a\,dt$

$\int_2^v dv = \int_0^t (2t - 1)\,dt$

$v = t^2 - t + 2$

$v = \frac{ds}{dt}$

$ds = v\,dt$

$\int_1^s ds = \int_0^t (t^2 - t + 2)\,dt$

$s = \frac{1}{3}t^3 - \frac{1}{2}t^2 + 2t + 1$

When $t = 6$ s

$v = (6)^2 - (6) + 2$ $\qquad v = 32$ m/s

$s = \frac{1}{3}(6)^3 - \frac{1}{2}(6)^2 + 2(6) + 1$ $\qquad s = 67$ m

Since $v \neq 0$ for $0 \leq t \leq 6$ s, then:

$d = 67 - 1 = 66$ m

Example 1 - 4:

If a jeep has an initial velocity of (v_o = 12 ft/s) to the right, at (s_o = 0), determine its position when (t = 10 s), if ($a = 2\ ft/s^2$) to the lift.

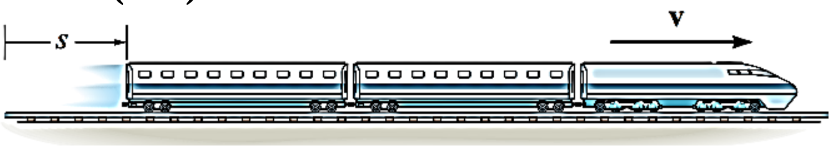
Solution:

$s = s_o + v_o\,t + \frac{1}{2}a_c\,t^2$

$= 0 + 12\,(10) + \frac{1}{2}(-2)(10)^2 = 20$ ft

Example 1 - 5:

When a train is traveling along a straight track at (2 m/s), it begins to accelerate at [$a = (60\, v^{-4})\ m/s^2$], where (v) is in (m/s). Determine its velocity (v) and the position (3 s) after the acceleration.



Solution:

$$a = \frac{dv}{dt} \quad \Rightarrow \quad dt = \frac{dv}{a}$$

$$\int_0^3 dt = \int_2^v \frac{dv}{60\, v^{-4}}$$

$$= \int_2^v \frac{1}{60}\, v^4\, dv$$

$$3 - 0 = \frac{1}{300} v^5 - \frac{1}{300} (2)^5$$

$$3 = \frac{1}{300} v^5 - \frac{32}{300} = \frac{1}{300} (v^5 - 32)$$

$$v = 3.93 \text{ m/s}$$

$$a\, ds = v\, dv \quad \Rightarrow \quad ds = \frac{v\, dv}{a} = \frac{1}{60} v^5\, dv$$

$$\int_0^s ds = \frac{1}{60} \int_2^{3.93} v^5\, dv$$

$$s = \frac{1}{60} \left(\frac{v^6}{6} \right) \Big|_2^{3.93}$$

$$= \frac{1}{60} \left(\frac{(3.93)^6}{6}\right) - \frac{1}{60} \left(\frac{(2)^6}{6}\right) = 9.98 \text{ m}$$

Example 1 - 6:

Traveling with an initial speed of (70 km/h), a car accelerates at ($6000\ km/h^2$) along a straight road. How long will it take to reach a speed of (120 km/h) ? Also, through what distance does the car travel during this time ?

Solution:

$$v = v_o + a_c\, t$$

$$120 = 70 + 6000\, t \quad \Rightarrow \quad 50 = 6000\, t$$

$$t = 8.33 \times 10^{-3} \text{ hr} = 30 \text{ s}$$

$$v^2 = v_o^2 + 2\, a_c\, (s - s_1)$$

$$(120)^2 = (70)^2 + 2\, (6000)\, (s - 0)$$

$$s = 0.792 \text{ km} = 792 \text{ m}$$

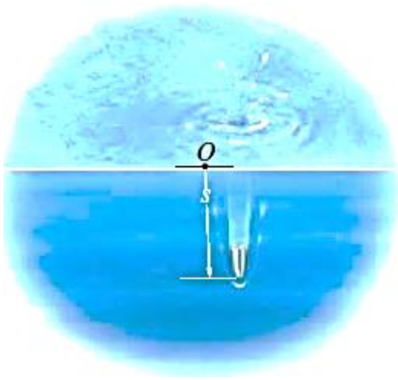
$$s = s_o + v_o\, t + \frac{1}{2} a_c\, t^2$$

$$s = 0 + (70)(8.33 \times 10^{-3}) + \frac{1}{2} (6000)(8.33 \times 10^{-3})^2$$

$$s = 0.5831 + 0.2082 = 0.7913 \text{ km} = 791 \text{ m}$$

Example 1 - 7:

A small projectile is fired vertically downward into a fluid medium with an initial velocity of (60 m/s). Due to the drag resistance of the fluid the projectile experiences a deceleration of [a = (– 0.4 v³) m/s²], where (v) is in (m/s). Determine the projectile's velocity and position (4 s) after it is fired.



Solution:

Velocity:

Since the motion is downward, the position coordinate is positive downward, with origin location at *O*.

$$+\downarrow \quad a = \frac{dv}{dt} = -0.4\,v^3$$

$$\int_{60}^{v} \frac{dv}{-0.4\,v^3} = \int_0^t dt \quad \Rightarrow \quad \int_{60}^{v} \frac{1}{-0.4}\,v^{-3}\,dv = \int_0^t dt$$

$$\frac{1}{-0.4} \times \frac{v^{-2}}{-2} \Big|_{60}^{v} = t - 0 \quad \Rightarrow \quad \frac{1}{0.8} \times \frac{1}{v^2} \Big|_{60}^{v} = t$$

$$\frac{1}{0.8}\left[\frac{1}{v^2} - \frac{1}{(60)^2}\right] = t \qquad \times 0.8$$

$$\frac{1}{v^2} - \frac{1}{(60)^2} = 0.8\,t \quad \Rightarrow \quad \frac{1}{v^2} = \frac{1}{(60)^2} + 0.8\,t$$

$$v^2 = \frac{1}{\left[\frac{1}{(60)^2} + 0.8\,t\right]} \quad \Rightarrow \quad v = \left\{\frac{1}{\sqrt{\frac{1}{60^2} + 0.8\,t}}\right\} \text{ m/s}$$

At (t = 4 s)
$$v = \frac{1}{\sqrt{\frac{1}{60^2} + 0.8\,(4)}} = \frac{1}{\sqrt{\frac{1}{60^2} + 3.2}} = 0.559 \text{ m/s} \downarrow$$

$$+\downarrow \quad v = \frac{ds}{dt} = \frac{1}{\sqrt{\frac{1}{60^2} + 0.8\,t}} = \left[\frac{1}{(60)^2} + 0.8\,t\right]^{-1/2}$$

Position:

$$\int_0^s ds = \int_0^t \left[\frac{1}{(60)^2} + 0.8\,t\right]^{-1/2} dt$$

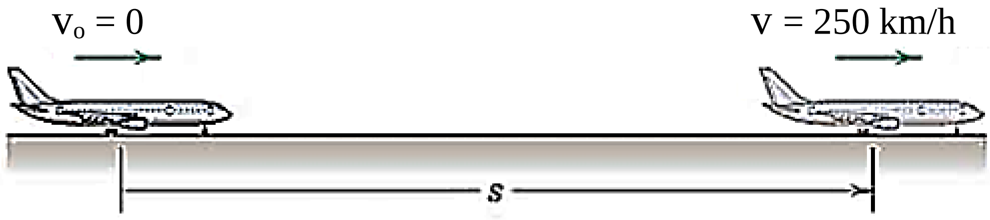
$$s = \frac{1}{1/2}\left[\frac{1}{(60)^2} + 0.8\,t\right]^{1/2} \frac{1}{0.8} \Big|_0^t = \frac{2}{0.8}\left[\frac{1}{(60)^2} + 0.8\,t\right]^{1/2} \Big|_0^t$$

$$s = \frac{1}{0.4}\left\{\left[\frac{1}{(60)^2} + 0.8\,t\right]^{1/2} - \frac{1}{60}\right\} = \frac{1}{0.4}\left\{\sqrt{\frac{1}{(60)^2} + 0.8\,t} - \frac{1}{60}\right\} \text{ m}$$

At (t = 4 s): $s = 4.43$ m

Example 1 - 8:

On its take-off roll, the airplane starts from rest and accelerates at (a = 1.8 m/s^2), determine the design length of runway required for the airplane to reach the take-off speed of (250 km/h).



Solution I:

$$v = 250 \text{ km/h} = \frac{250}{3.6} = 69.44 \text{ m/s}$$

$$v = v_o + a_c\, t$$

$$69.44 = 0 + 1.8\, t \quad \Rightarrow \quad t = \frac{69.44}{1.8} = 38.58 \text{ s}$$

$$s = s_o + v_o\, t + \frac{1}{2}\, a_c\, t^2$$

$$s = 0 + 0 + \frac{1}{2}(1.8)(38.58)^2 = 1340 \text{ m} = 1.34 \text{ km}$$

Solution II:

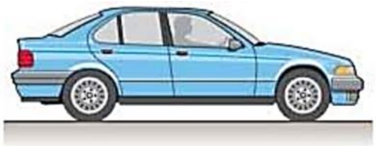
$$v^2 = v_o^2 + 2\, a_c\, (s - s_o)$$

$$(69.44)^2 = 0^2 + 2\,(1.8)\,(s - 0) \quad \Rightarrow \quad (69.44)^2 = 3.6\, s$$

$$s = 1340 \text{ m} = 1.34 \text{ km}$$

Example 1 - 9:

An automobile starting from rest, speeds up to (40 ft/s) with a constant acceleration of (4 ft/s^2), runs at this speed for a time, and finally comes to rest with a deceleration of (5 ft/s^2). If the total distance traveled is (1000 ft), find the total time required.



Solution:

t = 0

v = 0   a = 4 ft/s^2 v = 40 ft/s   a = 0   v = 40 ft/s   a = -5 ft/s^2 v = 0

+------------------------------+------------------------------------+---------------------------------+

A   B   C   D

First stage

$$v = v_o + a_c\, t$$

$40 = 0 + 4\, t_1 \quad \Rightarrow \quad t_1 = 10$ s

$$s = s_o + v_o\, t + \frac{1}{2} a_c\, t^2$$

$s_1 = 0 + 0 + \frac{1}{2}(4)(10)^2 = 200$ ft

Final stage

$$v = v_o + a_c\, t$$

$0 = 40 - 5\, t_3 \quad \Rightarrow \quad t_3 = 8$ s

$$s = s_o + v_o\, t + \frac{1}{2} a_c\, t^2$$

$s_3 = 0 + (40)(8) - \frac{1}{2}(5)(8)^2 = 160$ ft

Mid. stage

$s_2 = 1000 - 200 - 160 = 640$ ft

$$s = s_o + v_o\, t + \frac{1}{2} a_c\, t^2$$

$s_2 = 0 + 40\, t + \frac{1}{2}(0){t_2}^2 \quad \Rightarrow \quad s_2 = 40\, t_2$

$640 = 40\, t_2 \quad \Rightarrow \quad t_2 = 16$ s

Final result

$t_T = t_1 + t_2 + t_3 = 10 + 16 + 8 = 34$ s

t = 0   t =10 s   t =16 s   t =8 s

v = 0   a = 4 ft/s^2 v = 40 ft/s   a = 0   v = 40 ft/s   a = -5 ft/s^2 v = 0

+------------------------------+------------------------------------+---------------------------------+

A   B   C   D

S=200m   S=640m   S=160

Example 1 - 10:

A particle is moving along a straight line such that its position is defined by [$s = (10t^2 + 20)$ mm], where (t) is in seconds. Determine:

(a) the displacement of the particle during the time interval from (t = 1 s) to (t = 5 s).
(b) the average velocity of the particle during this time interval.
(c) the acceleration when (t = 1 s).

Solution:

$$s = 10\,t^2 + 20$$

(a) at (t = 1 s), $s = 10(1)^2 + 20 = 30$ mm

at (t = 5 s), $s = 10(5)^2 + 20 = 270$ mm

$$\Delta s = 270 - 30 = 240 \text{ mm}$$

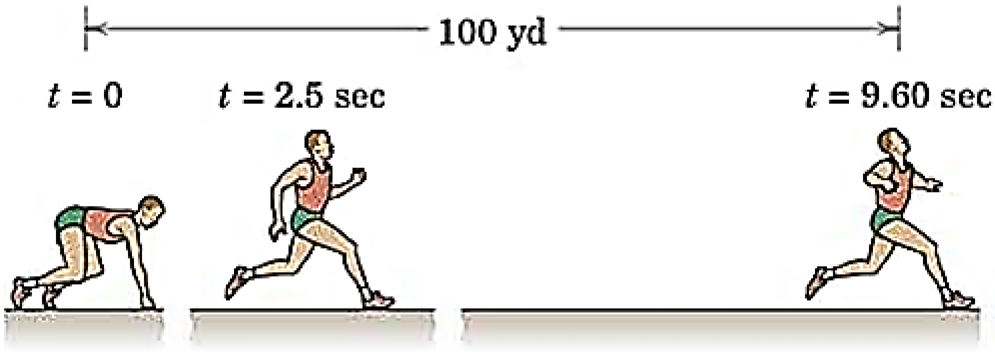
(b) $\Delta t = 5 - 1 = 4$ s

$$v_{avg} = \frac{\Delta s}{\Delta t} = \frac{240}{4} = 60 \text{ mm/s}$$

(c) $$a = \frac{d^2 s}{dt^2} = 20 \text{ mm/s}^2$$

Example 1 - 11:

A sprinter reaches his maximum speed (v_{max}) in (2.5 s) from rest with constant acceleration. He then maintains that speed and finishes the (100 yards) in the overall time of (9.6 s). Determine his maximum speed (v_{max}).



Solution:

$$v = v_o + a_c\, t$$

$$v_{max} = 0 + a_c\,(2.5) = 2.5\, a_c$$

$$s = s_o + v_o\, t + \frac{1}{2}\, a_c\, t^2$$

$$s = 0 + 0 + \frac{1}{2}\, a_c\,(2.5)^2 = 3.125\, a_c$$

$$s = s_o + v_o\, t + \frac{1}{2}\, a_c\,(2.5)^2$$

$$100 = s + v_{max}\,(\,9.6 - 2.5\,) + 0$$

$$v_{max} = \frac{100-s}{9.6-2.5} = \frac{100-3.125a_c}{7.1}$$

$$2.5\, a_c = \frac{100-3.125a_c}{7.1}$$

$$17.75\, a_c = 100 - 3.125\, a_c$$

$$20.875\, a_c = 100 \qquad \Rightarrow \qquad a_c = 4.79\ y/s^2$$

$$v_{max} = 2.5\, a_c = 2.5\,(4.79) = 11.976\ y/s = 11.976 \times 3 = 35.928\ ft/s$$

Example 1 - 12:

As a space probe carrier rocket is projected to a high altitude above the earth's surface, the variation of the acceleration of gravity with respect to altitude (y) must be taken into account. Neglecting air resistance, this acceleration is determined from the formula { $a = -g_o [R^2 / (R + y)^2]$ }, where (g_o) is the constant gravitational acceleration at sea level, (R) is the radius of the earth, and the positive direction is measured upward. If ($g_o = 9.81$ m/s^2) and (R = 6356 km), determine the minimum initial velocity (escape velocity) at which a rocket should be shot vertically from the earth's surface so that it does not fall back to the earth. Hint: This requires that (v = 0) as ($y \to \infty$).

Solution:

$$v\,dv = a\,dy$$

$$\int_v^0 v\,dv = \int_0^\infty -g_o \frac{R^2}{(R+y)^2}\,dy = -g_o R^2 \int_0^\infty \frac{dy}{(R+y)^2}$$

$$= -g_o R^2 \int_0^\infty (R+y)^{-2} dy$$

$$0 - \frac{v^2}{2} = -g_o R^2 \left(\frac{(R+y)^{-1}}{-1 \times 1} \right) \Big|_0^\infty$$

$$-\frac{v^2}{2} = g_o R^2 (R+y)^{-1} \Big|_0^\infty = \frac{g_o R^2}{R+y} \Big|_0^\infty$$

$$-\frac{v^2}{2} = \frac{g_o R^2}{R+\infty} - \frac{g_o R^2}{R+0} = -\frac{g_o R^2}{R} \quad \Rightarrow \quad \frac{v^2}{2} = \frac{g_o R^2}{R}$$

$$\frac{v^2}{2} = g_o R \quad \Rightarrow \quad v^2 = 2 g_o R$$

$$v = \sqrt{2 g_o R} = \sqrt{2(9.81)(6356)(10)^3} = 11167 \text{ m/s} = 11.2 \text{ km/s}$$

PROBLEMS:

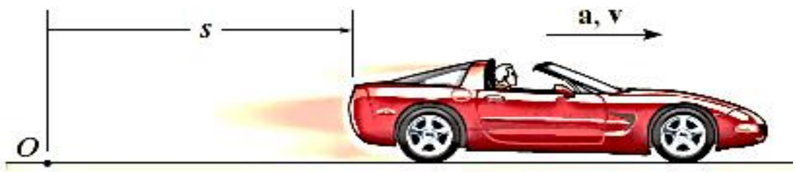
(1 - 1):

Starting from rest, a motorcycle moving in a straight line has an acceleration of [a = (2t – 6) m/s^2], where (t) in seconds. What is the motorcycle's velocity when (t = 6 s), and what is its position when (t = 11 s)?

Ans. : (v = 0) , (s = 80.7 m)

(1 - 2):

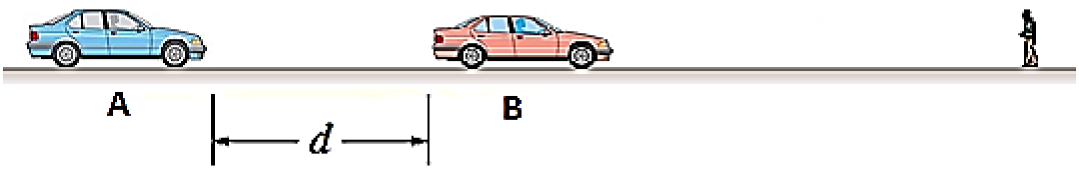
The car in the figure below moves in a straight line such that for a short time its velocity is defined by [v = ($3t^2$ + 2t) ft/s], where (t) is in seconds. Determine its position and acceleration when (t = 3 s). When (t = 0), (s = 0).



Ans. : (s = 36 ft) , (a = 20 ft/s^2)

(1 - 3):

Car (B) is traveling at distance (d) ahead of car (A). Both cars are traveling at (60 ft/s), when the driver of (B) suddenly applies the brakes causing his car to decelerate at (12 ft/s^2). It takes the driver of car (A) (0.75 s) to react (this is the normal reaction time for drivers). When he applies his brakes, he decelerates at (15 ft/s^2). Determine the minimum distance (d) between the cars so as to avoid a collision.



Ans. : (d = 16.9 ft)

(1 - 4):

The position of a bicycle along a straight line is given by [S = (1.5 t^3 – 13.5 t^2 + 22.5 t) ft], where (t) in seconds. Determine the position of the bicycle when (t = 6 s), and the total distance it travels during the (6) seconds time intervals.

Ans. : (s = – 27 ft) , (s_T = 69 ft)

(1 - 5):

A particle travels along a straight line with a constant acceleration. When (s = 4 ft), (v = 3 ft/s) and when (s = 10 ft), (v = 8 ft/s). Determine the velocity as a function of position.

Ans. : $v = \sqrt{9.166\, s - 27.664}$ ft/s

(1 - 6):

A bicycle travels along a straight line with a velocity [$v = (12 - 3t^2)$ m/s], where (t) is in seconds. When (t = 1 s), the bicycle is located (10 m) to the left of the origin. Determine the acceleration when (t = 4 s), the displacement from (t = 0) to (t = 10 s), and the distance the bicycle travels during this time period.

Ans. : ($a = -24$ m/s^2) , ($\Delta s = -880$ m) , ($s_T = 912$ m)

(1 - 7):

The velocity of a particle traveling in a straight line is given by [$v = (6t - 3t^2)$ m/s], where (t) is in seconds. If (s = 0) when (t = 0), determine the particle's deceleration and position when (t = 3 s). How far has the particle traveled during the (3 s) time interval, and what is its average speed?

Ans. : ($a = -12$ m/s^2) , ($s = 0$) , ($s_T = 8$ m) , ($(v_{sp})_{avg} = 2.67$ m/s)

(1 – 8):

A train starts from rest at station (A) and accelerates at (0.5 m/s^2) for (60 s). Afterwards it travels with a constant velocity for (15 min). It then decelerates at (1 m/s^2) until it is brought to rest at station (B). Determine the distance between the stations.

Ans. ($s = 28.4$ km)

Part 1
Kinematics of a particle

Chapter 2

Rectilinear kinematics : Graphic representation of the motion

When a particle has erratic or changing motion then its position, velocity, and acceleration cannot be described by a single continuous mathematical function along the entire path. Instead, a series of functions will be required to specify the motion at different intervals. For this reason, it is convenient to represent the motion as a graph. If a graph of the motion that relates any two of the variables (s, v, a, t) can be drawn, then this graph can be used to construct subsequent graphs relating two other variables since the variables are related by the differential relationships (v = ds/dt) , (a = dv/dt) , or (a ds = v dv). Several situations occur frequently.

The (s – t) , (v – t) , and (a – t) graphs.

To construct the (v – t) graph given the (s – t) graph, (Fig. 1– a) , the equation (v = ds/dt) should be used, since it relates the variables (s) and (t) to (v). This equation states that :

$$v = \frac{ds}{dt}$$

velocity = slop of (s – t) graph

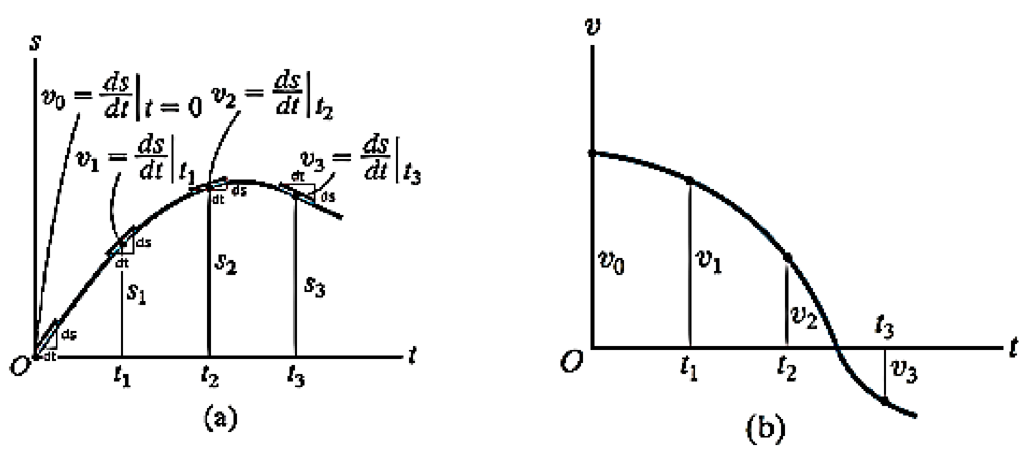


Fig. 1

For example, by measuring the slope on the (s – t) graph when ($t = t_1$), the velocity is (v_1), which is plotted in (Fig. 1 - b). The (v - t) graph can be constructed by plotting this and other values at each instant.

The (a – t) graph can be constructed from the (v - t) graph in a similar manner, (Fig. 2), since :

$$a = \frac{dv}{dt}$$

acceleration = slop of (v – t) graph

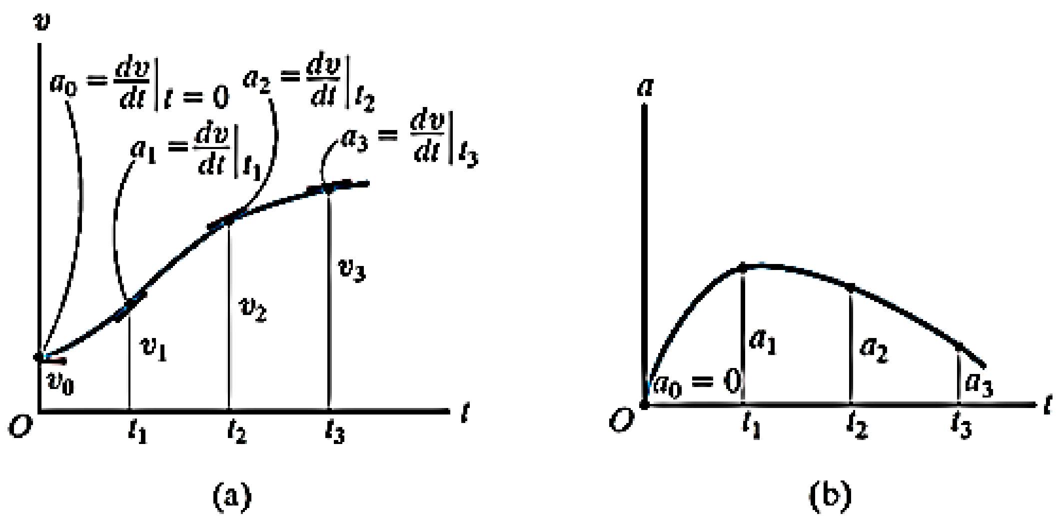


Fig. 2

Examples of various measurements are shown in (Fig. 2 - a) and plotted in (Fig. 2 - b).

If the (s – t) curve for each interval of motion can be expressed by a mathematical function [s = s(t)], then the equation of the (v – t) graph for the same interval can be obtained by differentiating this function with respect to time since (v = ds/dt). Likewise, the equation of the (a – t) graph for the same interval can be determined by differentiating [v = v(t)] since (a = dv/dt). Since differentiation reduces a polynomial of degree (n) to that of degree (n – 1), then if the (s – t) graph is parabolic (a second-degree curve), the (v – t) graph will be a sloping line (a first-degree curve), and the (a – t) graph will be a constant or a horizontal line (a zero-degree curve).

If the (a – t) graph is given, (Fig. 3 – a), the (v – t) graph may be constructed using ($a = dv/dt$), written as :

$$\Delta v = \int a\, dt$$

change in velocity = area under (a – t) graph

Hence, to construct the (v – t) graph, we begin with the particle's initial velocity (v_o) and then add to this small increments of area (Δv) determined from the (a – t) graph. In this manner successive points, ($v_1 = v_0 + \Delta v$), etc., for the (v – t) graph are determined, (Fig. 3-b). Notice that an algebraic addition of the area increments of the (a – t) graph is necessary, since areas lying above the (t) axis correspond to an increase in (v) ("positive" area), whereas those lying below the axis indicate a decrease in (v) ("negative" area).

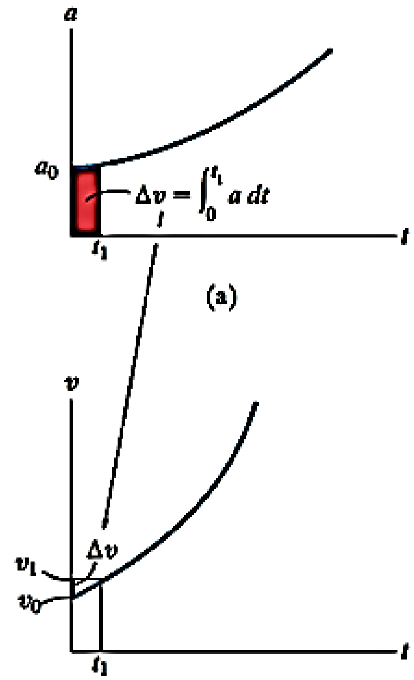


Fig. 3

Similarly, if the (v – t) graph is given, (Fig. 4 – a), it is possible to determine the (s – t) graph using ($v = ds / dt$), written as:

$$\Delta s = \int v\, dt$$

displacement = area under (v - t) graph

In the same manner as stated above, we begin with the particle's initial position (s_o) and add (algebraically) to this small area increments (Δs) determined from the (v – t) graph, (Fig. 4 – b).

If segments of the (a – t) graph can be described by a series of equations, then each of these equations can be integrated to yield equations describing the corresponding segments of the (v – t) graph. In a similar manner, the (s – t) graph can be obtained by integrating the equations which describe the segments of the (v – t) graph. As a result, if the (a – t) graph is linear (a first-degree curve), integration will yield a (v – t) graph that is parabolic (a second-degree curve) and an (s – t) graph that is cubic (third-degree curve).

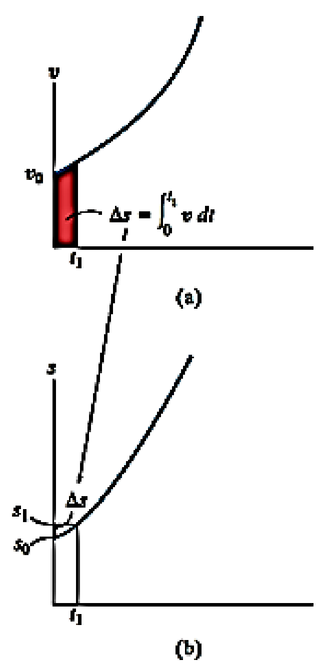


Fig. 4

The (v – s) and (a – s) Graphs.

If the (a – s) graph can be constructed, then points on the (v – s) graph can be determined by using (v dv = a ds).

Integrating this equation between the limits ($v = v_o$) at ($s = s_o$) and ($v = v_1$) at ($s = s_1$), we have :

$$\frac{1}{2}\left(v_1^{\,2} - v_o^{\,2} \right) = \int_{s_0}^{s_1} a\, ds$$

area under
(a – s) graph

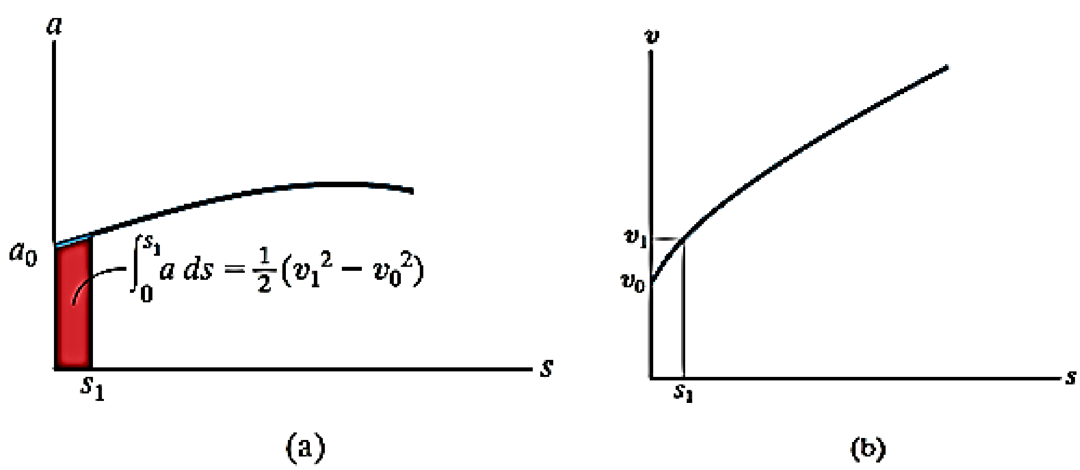


Fig. 5

Therefore, if the red area in (Fig. 5 – a) is determined, and the initial velocity (v_o) at ($s_o = 0$) is known, then:

$$v_1 = \left(2 \int_0^{s_1} a\, ds + v_0^{\,2} \right)^{1/2}$$

Fig. (5 – b). Successive points on the (v – s) graph can be constructed in this manner.

If the (v – s) graph is known, the acceleration (a) at any position (s) can be determined using (a ds = v dv), written as:

$$a = v\frac{dv}{ds}$$

acceleration = velocity times slop of (v – s) graph

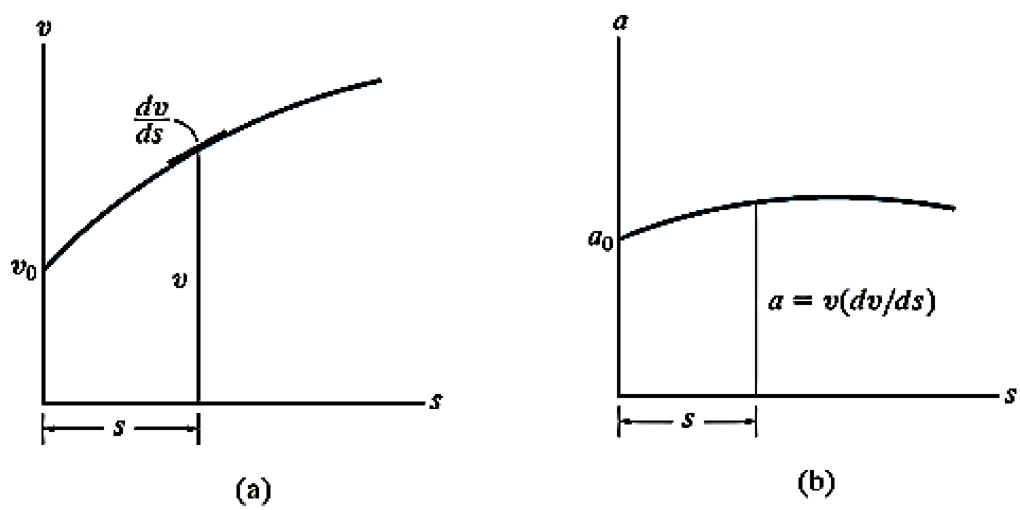


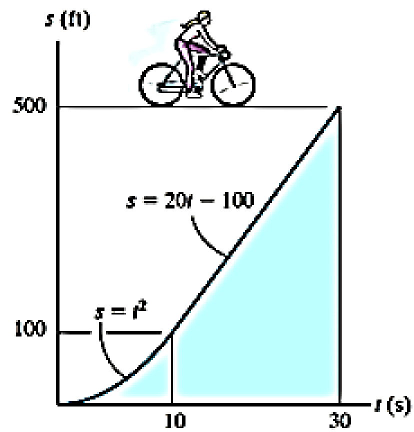
Fig. 6

Thus, at any point (s , v) in (Fig. 6 – a), the slope (dv / ds) of the (v – s) graph is measured. Then with (v) and (dv/ds) known, the value of a can be calculated, (Fig. 6 – b).

The (v – s) graph can also be constructed from the (a – s) graph, or vice versa, by approximating the known graph in various intervals with mathematical functions, (v = f(s)) or (a = g(s)), and then using (a ds = v dv) to obtain the other graph.

Example (2 - 1):

A bicycle moves along a straight road such that its position is described by the graph shown in (Fig. a.) Construct the (v - t) and (a – t) graphs for ($0 \le t \le 30$ s).



(a)

SOLUTION:

(v – t) Graph.

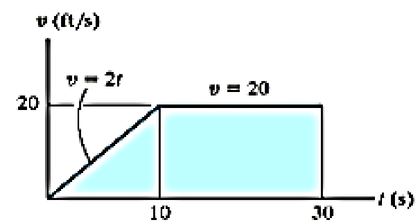
Since ($v = ds/dt$), the (v - t) graph can be determined by differentiating the equations defining the ($s - t$) graph, (Fig. *a.*) We have:

$0 \le t < 10$ s

$s = (t^2)$ ft $\Rightarrow$ $v = \dfrac{ds}{dt} = (2t)$ ft/s

$10 \text{ s} < t \le 30$ s

$s = (20t - 100)$ ft $\Rightarrow$ $v = \dfrac{ds}{dt} = (20)$ ft/s



(b)

The results are plotted in (Fig. **b**). We can also obtain specific values of (v) by measuring the *slope* of the ($s - t$) graph at a given instant. For example, at ($t = 20$ s), the slope of the ($s - t$) graph is determined from the straight line from (10 s) to (30 s), i.e.

$t = 20$ s, $\quad v = \dfrac{\Delta s}{\Delta t} = \dfrac{500-100}{30-10} = 20$ ft/s

(a – t) Graph.

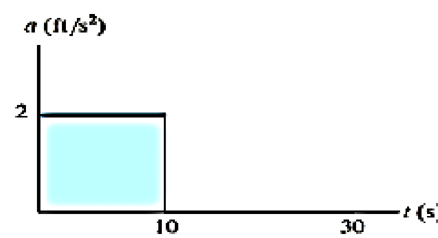
Since ($a = dv/dt$), the ($a - t$) graph can be determined by differentiating the equations defining the lines of the (v - t) graph.
This yields:

$0 \le t < 10$ s

$v = (2t)$ ft/s $\Rightarrow$ $a = \dfrac{dv}{dt} = 2$ ft/s^2

$10 \text{ s} < t \le 30$ s

$s = 20$ ft/s $\Rightarrow$ $a = \dfrac{dv}{dt} = 0$



(c)

The results are plotted in (Fig. *c*).

NOTE: Show that $a = 2$ ft/s2 when $t = 5$ s by measuring the slope of the v-t graph.

Example (2 - 2):

The car in (Fig. a) starts from rest and travels along a straight track such that it accelerates at (10 m/s²) for (10 s), and then decelerates at (2 m/s²). Draw the (v - t) and (s – t) graphs and determine the time (t ') needed to stop the car. How far has the car traveled?

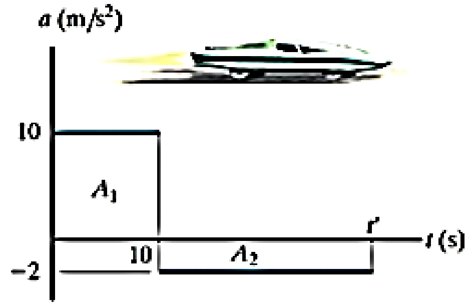


Fig. a

SOLUTION :

($v - t$) Graph.

Since (dv = a dt), the (v - t) graph is determined by integrating the straight-line segments of the ($a - t$) graph. Using the *initial condition* (v = 0) when ($t = 0$), we have:

$$0 \leq t < 10\ \text{s}, \quad \Rightarrow \quad a = 10\ \text{m/s}^2 \quad \Rightarrow \quad \int_0^v dv = \int_0^t 10\, dt \quad \Rightarrow \quad v = 10\, t$$

When $t = 10$ s, $\qquad v = 10(10) = 100$ m/s.

Using this as the *initial condition* for the next time period, we have:

$$10\ \text{s} < t \leq t', \quad \Rightarrow \quad a = (-2)\ \text{m/s}^2 \quad \Rightarrow \quad \int_{100}^v dv = \int_{10}^t -2\, dt$$

$$v - 100 = -2\, t - (-20)$$

$$v = (-2\, t + 120)\ \text{m/s}$$

When ($t = t'$) we require (v = 0).
This yields, (Fig. *b*).

$$0 = -2\, t' + 120$$

$$2\, t' = 120 \quad \Rightarrow \quad t' = 60\ \text{s}$$

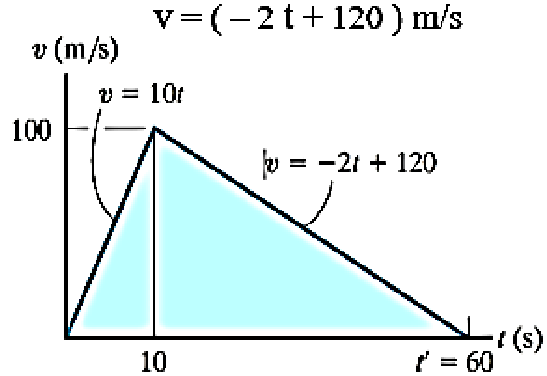


Fig. b

A more direct solution for (t') is possible by realizing that the area under the ($a - t$) graph is equal to the change in the car's velocity. We require ($v = 0 = A_1 + A_2$) , (Fig. *a*). Thus:

$$0 = 10\ \text{m/s}^2(10\ \text{s}) + (-2\ \text{m/s}_2)(t' - 10\ \text{s})$$

$$t' = 60\ \text{s}$$

(s–t) Graph.

Since (*ds* = v *dt*), integrating the equations of the (v - *t*) graph yields the corresponding equations of the (*s*–*t*) graph. Using the *initial condition* (*s* = 0) when (*t* = 0), we have:

$$0 \le t < 10\ s\ , \Rightarrow \quad v = (10\,t)\ m/s \Rightarrow \int_0^s ds = \int_0^t 10\,t\,dt \Rightarrow s = (5\,t^2)\ m$$

When (t = 10 s), $s = 5(10)^2 = 500$ m.

Using this *initial condition*:

$$10\ s < t \le 60\ s\ , \Rightarrow \quad v = (-2\,t + 120)\ m/s$$

$$\int_{500}^{s} ds = \int_{10}^{t} (-2\,t + 120)\,dt$$

$$s - 500 = -t^2 + 120\,t - [-(10)^2 + 120(10)]$$

$$s = -t^2 + 120\,t - 600$$

When (t' = 60 s), the position is :

$$s = -(60)^2 + 120(60) - 600$$
$$= 3000\ m$$

The (*s*–*t*) graph is shown in (Fig. *c*).

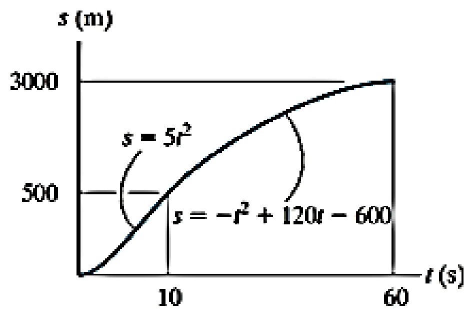


Fig. c

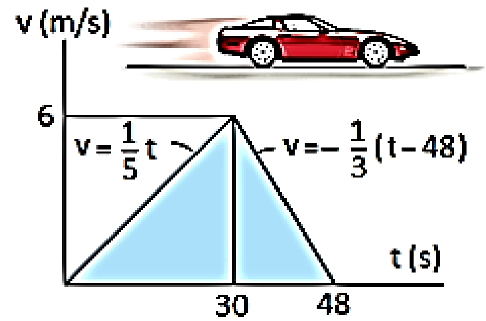
NOTE:

A direct solution for *s* is possible when (t' = 60 s), since the *triangular area* under the (v-*t*) graph would yield the displacement (C*s* = *s* - 0) from (*t* = 0) to (t' = 60 s).

Hence, (C*s* = 1/2 (60 s)(100 m/s) = 3000 m)

Example (2 - 3):

A car travels along a straight road with the speed shown by the (v – t) graph. Plot the (a – t) graph.



SOLUTION :

(a – t) Graph:

For ($0 \le t < 30$ s),

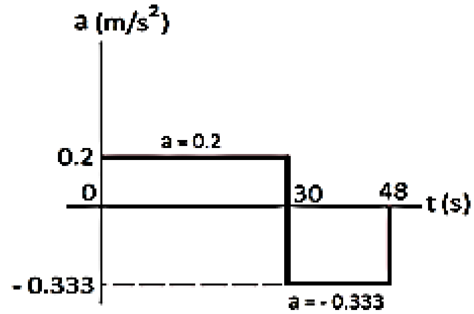
$$v = \left(\frac{1}{5} t \right) \text{ m/s}$$

$$a = \frac{dv}{dt} = \frac{1}{5} = 0.2 \text{ m/s}^2$$

For ($30 \text{ s} < t \le 48$ s),

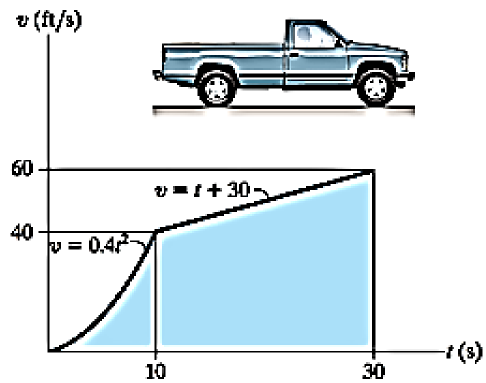
$$v = \frac{1}{3}(t - 48) \text{ m/s}$$

$$a = \frac{dv}{dt} = \frac{1}{3}(1) = -0.333 \text{ m/s}^2$$



Example (2 - 4):

The (v – t) graph for the motion of a pickup as it moves along a straight road is shown. Draw the (a – t) graph and determine the maximum acceleration during the (30 - s) time interval. The pickup starts from rest at (s = 0).



SOLUTION :

For ($t < 10$ s),

$$v = (0.4 t^2) \text{ ft/s}$$

$$a = \frac{dv}{dt} = (0.8 t) \text{ ft/s}^2$$

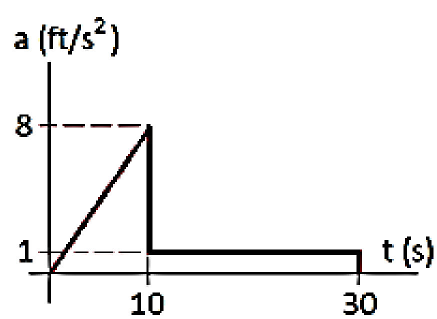
At ($t = 10$ s) $\quad a = 8 \text{ ft/s}^2$

For ($10 \text{ s} < t \le 30$ s),

$$v = (t + 30) \text{ ft/s}$$

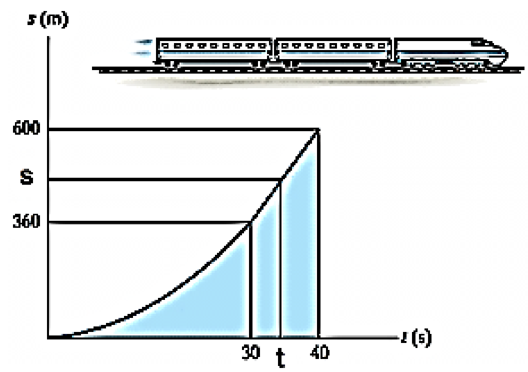
$$a = \frac{dv}{dt} = 1 \text{ ft/s}^2$$

$$a_{max} = 8 \text{ ft/s}^2$$



Example (2 - 5):

The (s - t) graph for a train has been experimentally determined. From the data, construct the (v – t) and (a – t) graphs for the motion, ($0 \le t \le 40$ s). For ($0 \le t \le 30$ s), the curve is [$s = (0.4\ t^2)$ m], and then it becomes straight for ($t \ge 30$ s).



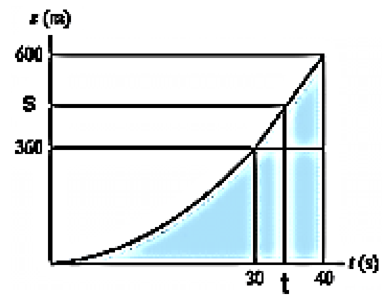
SOLUTION :

$0 \le t \le 30$ s ,

$$s = 0.4\,t^2$$

$$v = \frac{ds}{dt} = 0.8\,t$$

$$a = \frac{dv}{dt} = 0.8$$



$30 \le t \le 40$ s ,

$$\frac{s-360}{t-30} = \frac{600-360}{40-30}$$

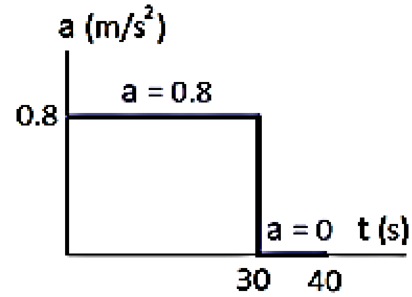
$$s - 360 = \left(\frac{600-360}{40-30}\right)(t-30)$$

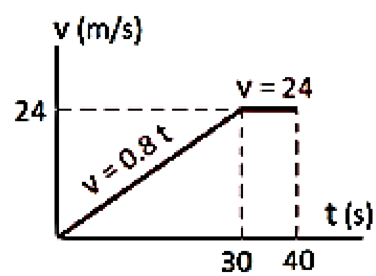
$$s = 24\,(t-30) + 360$$

$$s = 24\,t - 360$$

$$v = \frac{ds}{dt} = 24 \text{ m/s}$$

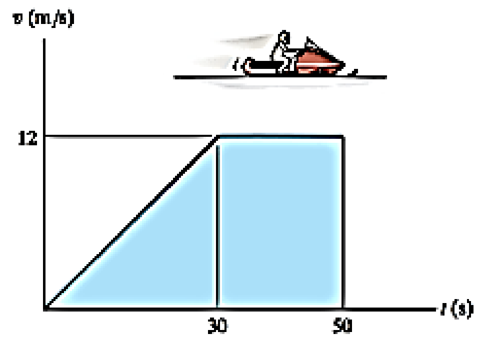
$$a = \frac{dv}{dt} = 0$$





<u>Example (2 - 6):</u>

The snowmobile moves along a straight course according to the (v – t) graph. Construct the (s – t) and (a – t) graphs for the same (50 s) time interval. When (t = 0), (s = 0).



SOLUTION :

(s – t) Graph:

The position function in terms of time (t) can be obtained by applying ($V = \frac{ds}{dt}$).

For time interval $(0 \le t < 30 \text{ s})$,

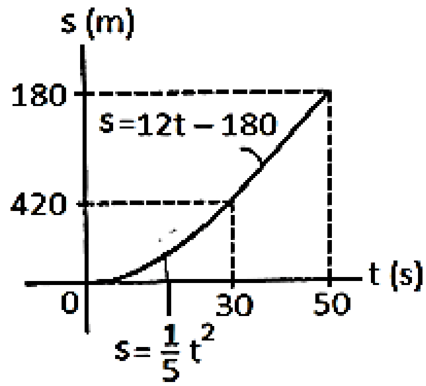
$$\frac{v}{t} = \frac{12}{30} \Rightarrow v = \frac{12}{30} t = (\frac{2}{5} t) \text{ m/s}$$

$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\, dt$$

$$\int_0^s ds = \int_0^t \frac{2}{5}\, t\, dt$$

$$s = (\frac{1}{5} t^2) \text{ m}$$

At $(t = 30 \text{ s})$, $\quad s = \frac{1}{5} (30)^2 = 180 \text{ m}$



For time interval $(30 \text{ s} < t \le 50 \text{ s})$,

$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\, dt$$

$$\int_{180}^s ds = \int_{30}^t 12\; dt$$

$$s - 180 = 12\,t - 12(30) \quad \Rightarrow \quad s = (12\,t - 180) \text{ m}$$

At $(t = 50 \text{ s})$, $\quad s = 12\,(50) - 180 = 420 \text{ m}$

(a – t) Graph:

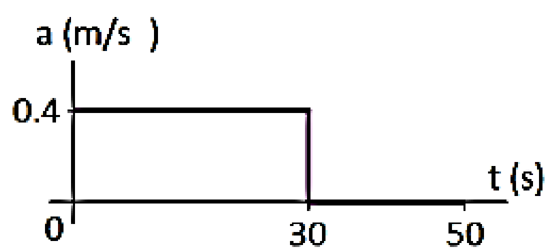
The acceleration function in terms of time t can be obtained by applying ($V = \frac{ds}{dt}$).

For time interval $(0 \le t < 30 \text{ s})$,

$$a = \frac{dv}{dt} = \frac{2}{5} = 0.4 \text{ m/s}^2$$

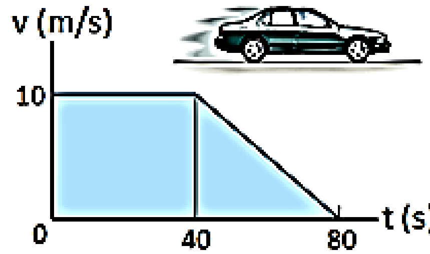
For time interval $(30 \text{ s} < t \le 50 \text{ s})$,

$$a = \frac{dv}{dt} = 0$$



<u>Example (2 - 7):</u>

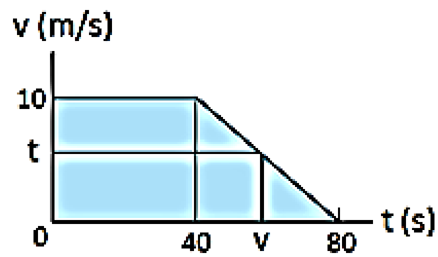
The velocity of a car is plotted as shown. Determine the total distance the car moves until it stops (t = 80 s) Construct the (a – t) graph.



SOLUTION :

Distance Traveled:

The total distance traveled can be obtained by computing the area under the (v – t) graph.



$$s = 10(40) + \frac{1}{2}(10)(80 - 40) = 600 \text{ m}$$

(a – t) Graph:

$(0 \leq t < 40 \text{ s})$,

$$v = 10$$

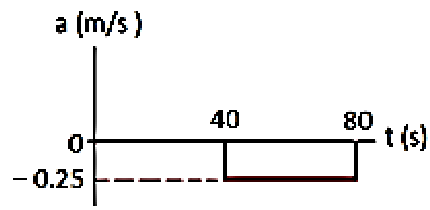
$$a = \frac{dv}{dt} = 0$$

$(40 \text{ s} < t \leq 80 \text{ s})$,

$$\frac{v - 10}{t - 40} = \frac{0 - 10}{80 - 40}$$

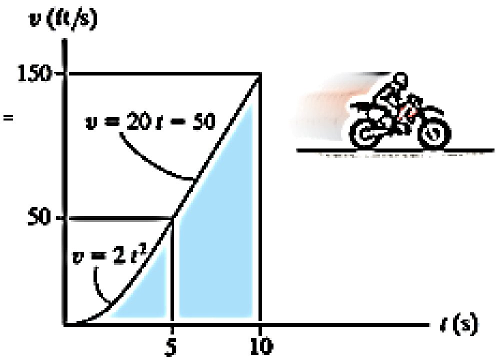
$$v = -\frac{1}{4}t + 20$$

$$a = \frac{dv}{dt} = -\frac{1}{4} = -0.25 \text{ m/s}^2$$



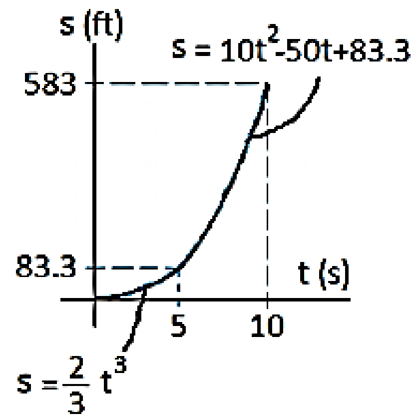
Example (2 - 8):

A motorcyclist travels along a straight road with the velocity described by the (v - t) graph. Construct the (s – t) and (a – t) graphs.



SOLUTION :

(s – t) Graph:



For ($0 \le t < 5$ s),

the initial condition is (s = 0) when (t = 0).

$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\,dt$$

$$\int_0^s ds = \int_0^t 2\,t^2\,dt$$

$$s = \left(\frac{2}{3}\,t^3\right)\text{ ft}$$

At (t = 5 s), $s = \frac{2}{3}\,(5)^3 = 83.3$ ft

For ($5\text{ s} < t \le 10$ s),

the initial condition is (s = 83.33 ft) when (t = 5 s).

$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\,dt$$

$$\int_{83.33}^s ds = \int_5^t (20\,t - 50)\,dt$$

$$s - 83.33 = (\,10\,t^2 - 50\,t\,) - [\,10\,(5)^2 - 50\,(5)\,]$$

$$s = 10\,t^2 - 50\,t - 250 + 250 + 83.33 = 10\,t^2 - 50\,t + 83.33$$

At (t = 10 s), $s = 10\,(10)^2 - 50\,(10) + 83.33 = 583.33$ ft

(a – t) Graph:

For ($0 \le t < 5$ s),

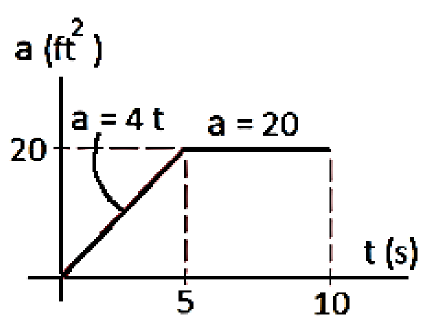
$$a = \frac{dv}{dt} = \frac{d}{dt}\,(\,2\,t^2\,) = 4\,t$$

At (t = 5 s), $a = 4\,(5) = 20$ ft/s²

For ($5\text{ s} < t \le 10$ s),

$$a = \frac{dv}{dt} = \frac{d}{dt}\,(\,20\,t - 50\,)$$

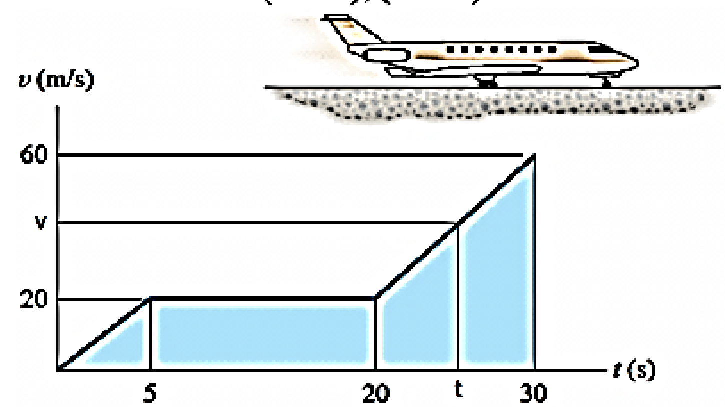
$$= 20 \text{ ft/s}^2$$



(b)

Example (2 - 9):

From experimental data, the motion of a jet plane while traveling along a runway is defined by the (v – t) graph. Construct the (s – t) and (a – t) graphs for the motion. When (t = 0), (s = 0).



SOLUTION:

(s – t) Graph:

For ($0 \le t < 5$ s), $v = \frac{20}{5} t = (4\,t)$ m/s

$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\,dt$$

$$\int_0^s ds = \int_0^t 4\,t\,dt \quad \Rightarrow \quad s = (2\,t^2)\ \text{m}$$

At (t = 5 s), $s = 2\,(5)^2 = 50$ m

For ($5\ \text{s} < t < 20$ s),

$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\,dt$$

$$\int_{50}^s ds = \int_5^t (20)\,dt$$

$$s - 50 = 20\,t - 20\,(5) = 20\,t - 50$$

At (t = 20 s), $s = 20\,(20) - 50 = 350$ m

For ($20\ \text{s} < t \le 30$ s), $\frac{v-20}{t-20} = \frac{60-20}{30-20} \quad \Rightarrow \quad v - 20 = \frac{60-20}{30-20}\ t - 20$

$$v - 20 = 4\,(t - 20) = (4\,t - 60)$$

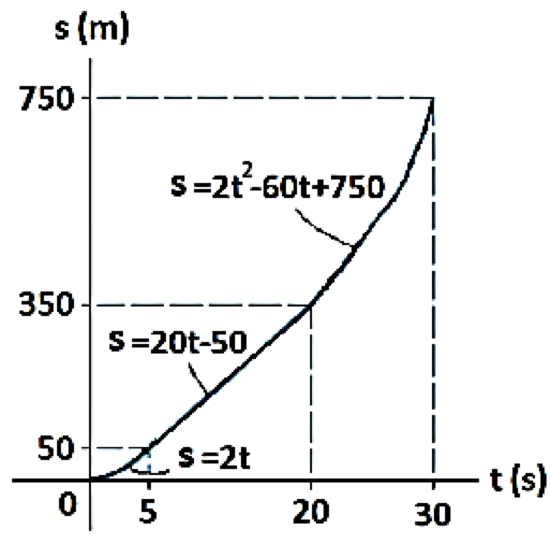
$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\,dt$$

$$\int_{350}^s ds = \int_{20}^t (4t - 60)\,dt$$

$$s - 350 = [\,2\,t^2 - 60\,t\,] - [\,2(20)^2 - 60(20)\,]$$

$$s = 2\,t^2 - 60\,t - 800 + 1200 + 350 = 2\,t^2 - 60\,t + 750$$

At (t = 30 s), $s = 2\,(30)^2 - 60\,(30) + 750 = 750$ m

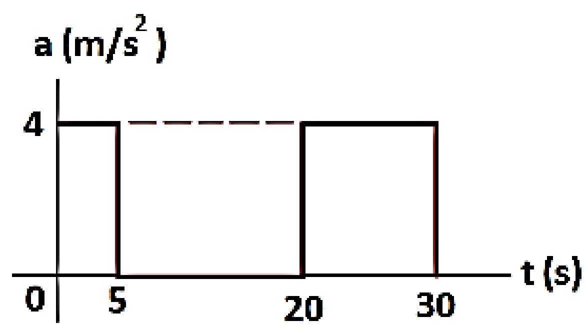


(a – t) Graph:

For ($0 \leq t < 5$ s), $a = \frac{dv}{dt} = \frac{d}{dt}(4t) = 4$ m/s

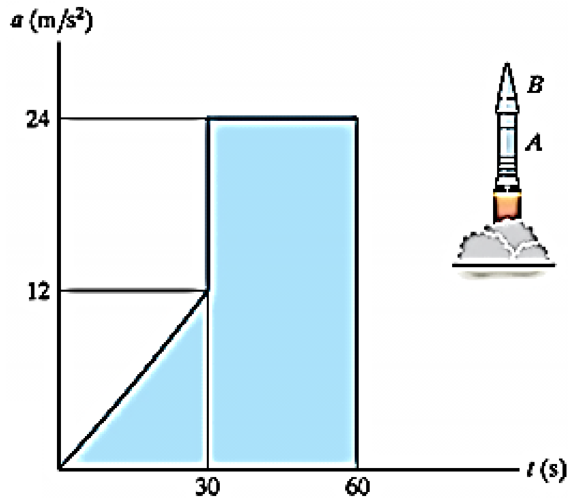
For (5 s < t < 20 s), $a = \frac{dv}{dt} = \frac{d}{dt}(20) = 0$

For (20 s < $t \leq 30$ s), $a = \frac{dv}{dt} = \frac{d}{dt}(4t - 60) = 4$ m/s



Example (2 - 10):

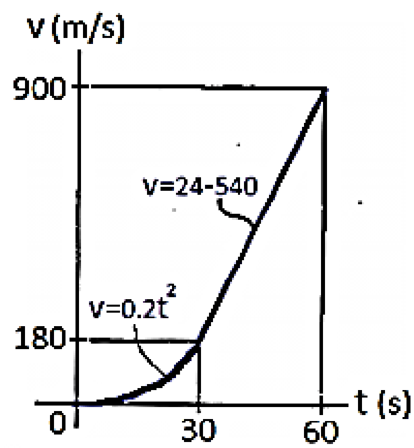
A two-stage rocket is fired vertically from rest at (s = 0) with the acceleration as shown. After (30 s) the first stage (A) burns out and the second stage (B) ignites. Plot the (v – t) and (s – t) graphs which describe the motion of the second stage (0 ≤ t ≤ 60 s).



SOLUTION:

(v – t) Graph:

For ($0 \le t < 30$ s), $a = \dfrac{12}{30} t = (0.4\,t)$ m/s².



$$a = \frac{dv}{dt} \quad \Rightarrow \quad dv = a\,dt$$

$$\int_o^v dv = \int_0^t (0.4\,t)\,dt$$

$$v = (0.2\,t^2)\text{ m/s}$$

At (t = 30 s), $v = 0.2\,(30)^2 = 180$ m/s

For ($30\text{ s} < t \le 60$ s), $a = 24$ m/s²

(v = 180 m/s) when (t = 30 s).

$$\int_{180}^v dv = \int_{30}^t 24\,dt$$

$$v - 180 = 24\,t - 24\,(30)$$

$$v = (24\,t - 540)\text{ m/s}$$

At (t = 60 s), $v = 24\,(60) - 540 = 900$ m/s

(s – t) Graph:

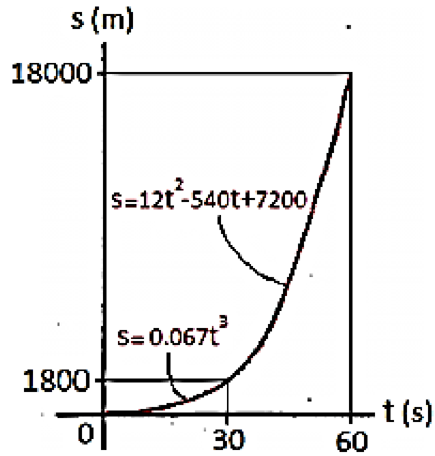
For ($0 \le t < 30$ s), (s = 0) when (t = 0).

$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v\,dt$$

$$\int_0^s ds = \int_0^t (0.2\,t^2)\,dt$$

$$s = \frac{0.2}{3}\,t^3 = (\,0.067\,t^3\,)\text{ m}$$

At (t = 30 s), $s = 0.067\,(30)^3 = 1800$ m



For ($30\text{ s} < t \le 60$ s),

(s = 1800 m) when (t = 30 s).

$$\int_{1800}^s ds = \int_{30}^t (\,24\,t - 540\,)\,dt$$

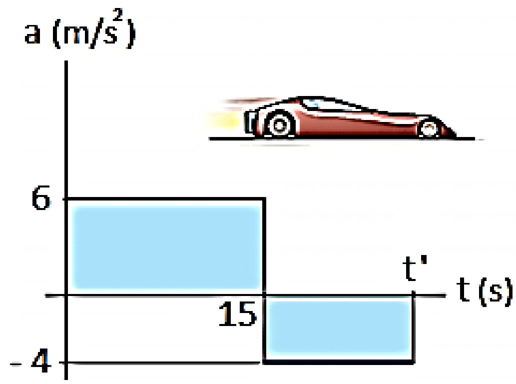
$$s - 1800 = (\,12\,t^2 - 540\,t\,) - [\,12\,(30)^2 - 540\,(30)\,]$$

$$s = (\,12\,t^2 - 540\,t + 7200\,)\text{ m}$$

At (t = 60 s), $s = 12\,(60)^2 - 540\,(60) + 7200 = 18000$ m

Example (2 - 11):

The jet car is originally traveling at a velocity of (10 m/s) when it is subjected to the acceleration shown. Determine the car's maximum velocity and the time (t ') when it stops. When (t = 0), (s = 0).



SOLUTION:

(v – t) Graph:

The (v – t) function can be determined by integrating (dv = a dt).

For ($0 \le t < 15$ s), (1 = 6 m/s^2). Using the initial condition (v = 10 m/s) at (t = 0).

$$a = \frac{dv}{dt} \quad \Rightarrow \quad dv = a\,dt$$

$$\int_{10}^{v} dv = \int_{0}^{t} 6\,dt$$

$$v - 10 = 6\,t \quad \Rightarrow \quad v = (\,6\,t + 10\,)\ \text{m/s}$$

At (t = 0), $v = 6(0) + 10 = 10$ m/s

The maximum velocity occurs when (t = 15 s)

At (t = 15 s), $v = 6(15) + 10 = 100$ m/s

For (15 s $\le t < t'$), (t = – 4 m/s^2). Using the initial condition (v = 100 m/s) at (t = 15 s).

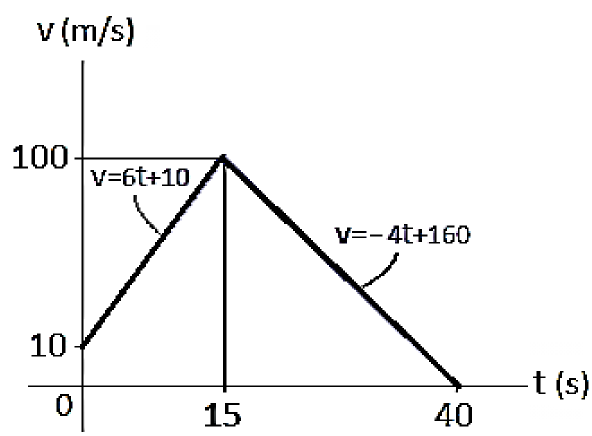
$$\int_{100}^{v} dv = \int_{15}^{t} -4\,dt$$

$$v - 100 = -4\,t - (-4 \times 15)$$

$$v = -4\,t + 60 + 100 \quad \Rightarrow \quad v = (\,-4\,t + 160\,)\ \text{m/s}$$

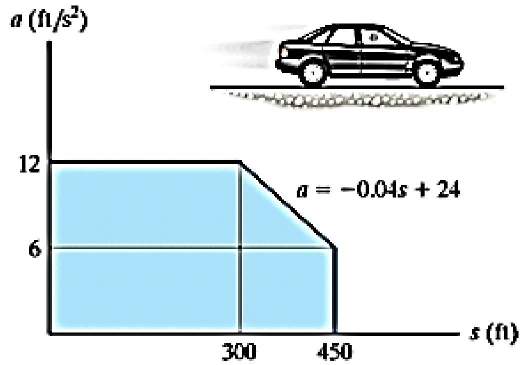
At (t = 15 s), $v = -4\,(15) + 160 = 100$ m/s

At (t = t') , (v = 0) $\Rightarrow$ $0 = (\,-4\,t' + 160\,)$ $\Rightarrow$ $t' = \dfrac{160}{4} = 40$ s



Example (2 - 12):

The car starts from rest at ($s = 0$) and is subjected to an acceleration shown by the (a – s) graph. Draw the (v – s) graph and determine the time needed to travel (200 ft).



SOLUTION:

For (s < 300 ft)

$$a\,ds = v\,dv$$
$$\int_0^s 12\,ds = \int_0^v v\,dv$$
$$12\,s = \frac{1}{2}\,v^2$$
$$v^2 = 24\,s \quad \Rightarrow \quad v = \sqrt{24\,s} = 4.9\,\sqrt{s}$$

At (s = 300 ft), $\quad v = 4.9\,\sqrt{300} = 84.85$ ft/s

For (300 ft < s < 450 ft)

$$a\,ds = v\,dv$$
$$\int_{300}^{s} (24 - 0.04s)\,ds = \int_{84.85}^{v} v\,dv$$
$$24\,s - 0.02\,s^2 \Big|_{300}^{s} = \frac{1}{2}\,v^2 \Big|_{84.85}^{v}$$
$$(\,24\,s - 0.02\,s^2\,) - [\,24\,(300) - 0.02\,(300)^2\,] = \frac{1}{2}\,(\,v^2 - 84.85^2\,)$$
$$(\,24\,s - 0.02\,s^2\,) - 5400 = \frac{1}{2}\,v^2 - 3600$$
$$24\,s - 0.02\,s^2 - 1800 = \frac{1}{2}\,v^2$$
$$v^2 = 48\,s - 0.04\,s^2 - 3600$$
$$v = \sqrt{0.04\,s^2 + 48\,s - 3600}$$

At (s = 450 ft), $\quad v = \sqrt{0.04\,(450)^2 + 48\,(450) - 3600} = 99.5$ ft/s

$$v = 4.9\,\sqrt{s} = 4.9\,(s)^{1/2}$$
$$\frac{ds}{dt} = 4.9\,(s)^{1/2}$$
$$\frac{ds}{dt} = \frac{4.9}{s^{-1/2}}$$

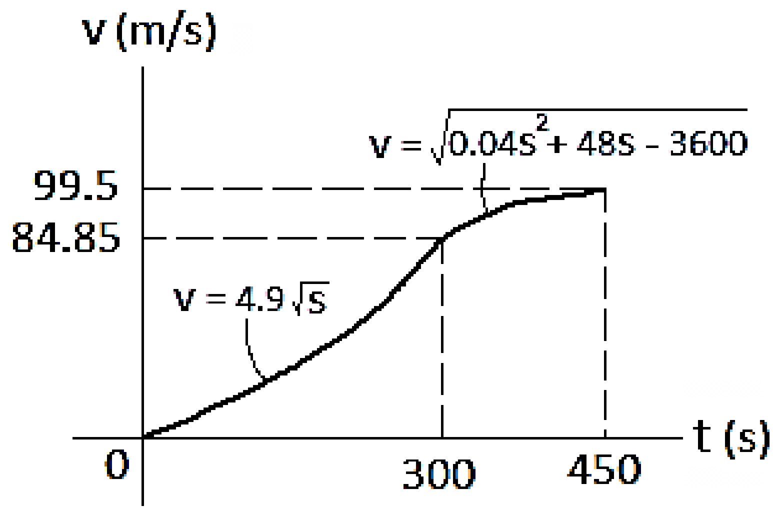
$$(s)^{-1/2}\,ds = 4.9\,dt$$

$$\int_0^{200} s^{-1/2}\,ds = \int_0^t 4.9\,dt$$

$$2\,(s)^{1/2}\Big|_0^{200} = 4.9\,t$$

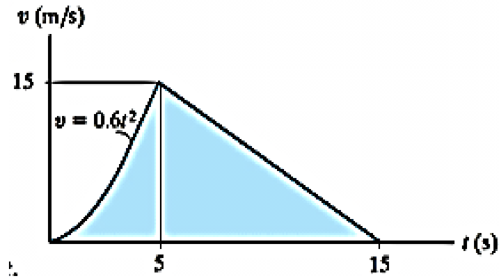
$$2\sqrt{200} - 0 = 4.9\,t$$

$$t = \frac{2\sqrt{200}}{4.9} = 5.77\ s$$



Example (2 - 13):

The (v – t) graph for the motion of a car as it moves along a straight road is shown. Draw the (s – t) and (a – t) graphs. Also determine the average speed and the distance traveled for the (15 - s) time interval. When (t = 0), (s = 0).



SOLUTION:

(s – t) Graph:

The (s – t) function can be determined by integrating (ds = v dt).

For ($0 \le t < 5$ s), ($v = 0.6\, t^2$). Using the initial condition (s = 0) at (t = 0).

$$ds = v\,dt$$
$$\int_0^s ds = \int_0^t (0.6\, t^2)\, dt$$
$$s = (0.2\, t^3)\ \text{m}$$

At (t = 5 s), $s = 0.2\,(5)^3 = 25$ m

For (5 s < $t \le 15$ s), $\frac{v-15}{t-5} = \frac{0-15}{15-5} \Rightarrow \frac{v-15}{t-5} = -1.5 \Rightarrow v - 15 = -1.5\,(t-5)$

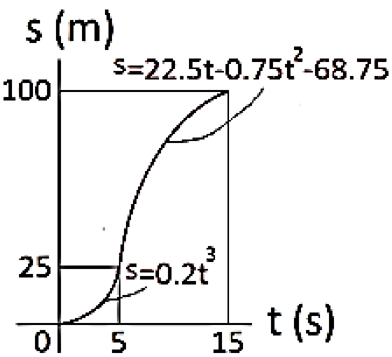
$v = -1.5\,t + 7.5 + 15 \Rightarrow v = -1.5\,t + 22.5 \Rightarrow v = 22.5 - 1.5\,t$.

The initial condition (s = 25 m) at (t = 5 s).

$$ds = v\,dt$$
$$\int_{25}^s ds = \int_5^t (22.5 - 1.5\,t)\, dt$$
$$s - 25 = 22.5\,t - 0.75\,t^2 - [\,22.5(5) - 0.75(5^2)\,]$$
$$s = 22.5\,t - 0.75\,t^2 - 93.75 + 25$$
$$s = 22.5\,t - 0.75\,t^2 - 68.75$$

At (t = 15 s),

$$s = 22.5(15) - 0.75(15^2) - 68.75 = 100\ \text{m}$$



The average speed is:

$$v_{avg} = \frac{s_T}{t} = \frac{100}{15} = 6.67\ \text{m/s}$$

(a – t) Graph:

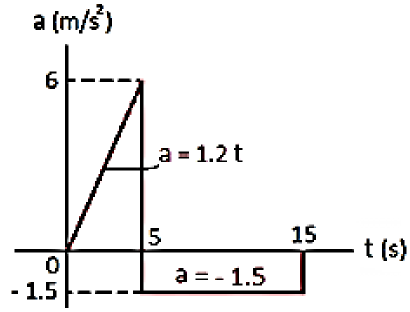
The (a – t) function can be determined by integrating

$$a = \frac{dv}{dt}$$

For ($0 \leq t < 5$ s),

$$a = \frac{d(0.6\,t^2)}{dt} = (\,1.2\,t\,)\ m/s^2$$

At (t = 5 s), $s = 1.2\,(5) = 6\ m/s^2$



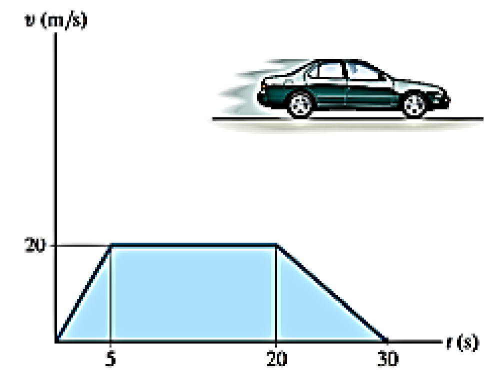
For ($0 \leq t < 5$ s),

$$a = \frac{d(22.5 - 1.5\,t)}{dt} = -\,1.5\ m/s^2$$

PROBLEMS:

(2 - 1):

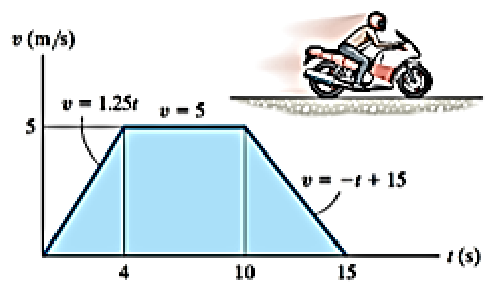
The (v – t) graph of a car while traveling along a road is shown. Draw the (s – t) and (a – t) graphs for the motion.



(2 - 2):

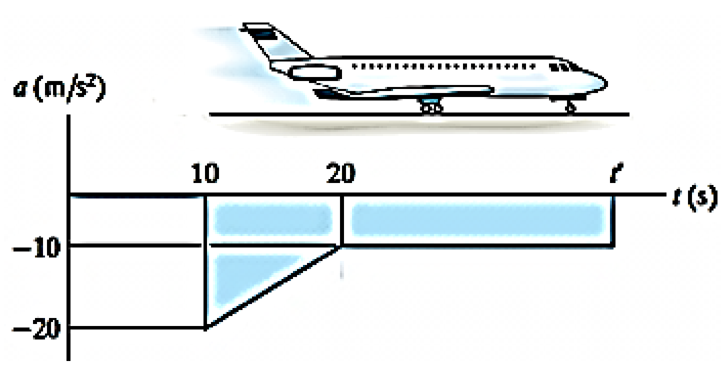
A motorcycle starts from rest at (s = 0) and travels along a straight road with the speed shown by the (v – t) graph. Determine the total distance the motorcycle travels until it stops when (t = 15 s). Also plot the (a–t) and (s–t) graphs.

Ans. : s = 52.5 m



(2 - 3):

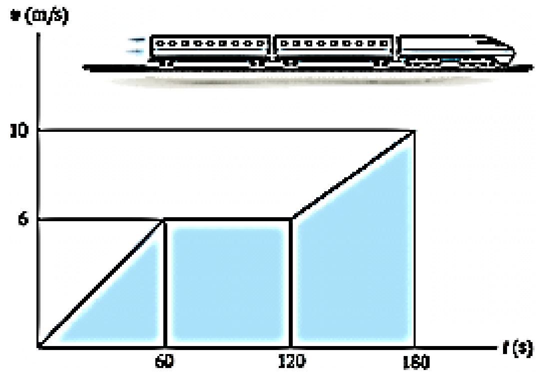
The motion of a jet plane just after landing on a runway is described by the (a – t) graph. Determine the time (t′) when the jet plane stops. Construct the (v – t) and (s – t) graphs for the motion. Here (s = 0), and (v = 300 ft/s) when (t = 0).



Ans. : t′ = 35 s

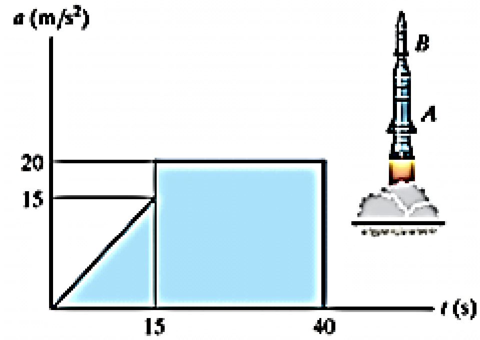
(2 - 4):

The (v – t) graph for a train has been experimentally determined. From the data, construct the (s – t) and (a – t) graphs for the motion for (0 ≤ t ≤ 180 s). When (t = 0), (s = 0).



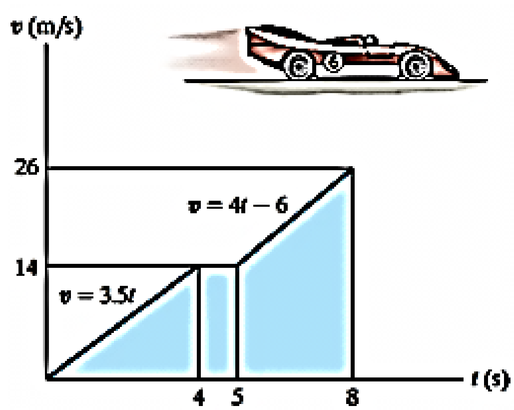
(2 - 5):

A two-stage rocket is fired vertically from rest with the acceleration shown. After (15 s) the first stage (A) burns out and the second stage (B) ignites. Plot the (v – t) and (s – t) graphs which describe the motion of the second stage for (0 ≤ t ≤ 40 s).



(2 - 6):

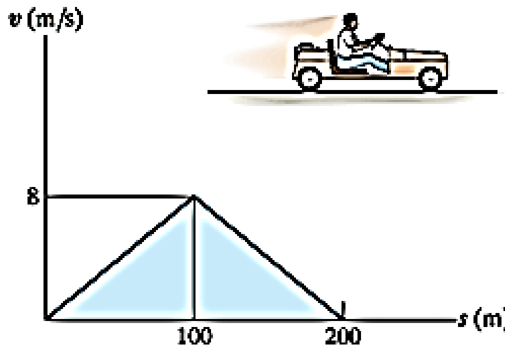
The race car starts from rest and travels along a straight road until it reaches a speed of (26 m/s) in (8 s) as shown on the (v – t) graph. The flat part of the graph is caused by shifting gears. Draw the (a – t) graph and determine the maximum acceleration of the car.



Ans. : For ($0 \le t < 4$ s), ($a = 3.5\ m/s^2$) , For ($4\ s \le t \le 5$ s), ($a = 0$)
For ($5\ s < t \le 8$ s), ($a = 4\ m/s^2$) , $a_{max} = 4\ m/s^2$

(2 - 7):

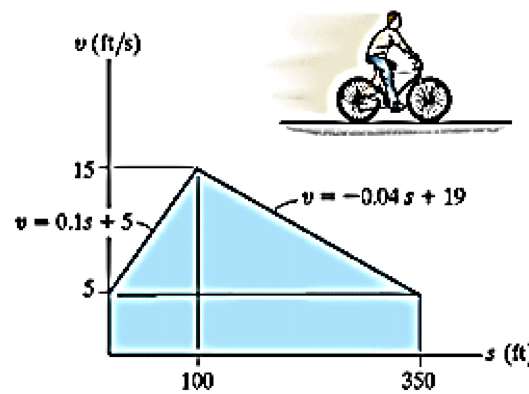
The (v – s) graph for a go-cart traveling on a straight road is shown. Determine the acceleration of the go-cart at (s = 50 m) and Draw the (s = 150 m). (a – s) graph.



Ans. : a = 0.32 m/s²

(2 - 8):

The (v - s) graph of a cyclist traveling along a straight road is shown. Construct the (a – s) graph.



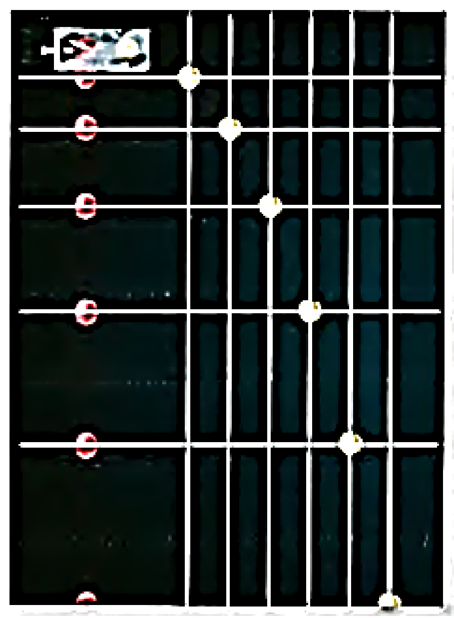
Ans. :

At (s = 0)	$a = 0.5\ ft/s^2$
At (s = 100 ft)	$a = 1.5\ ft/s^2$
At (s = 100 ft)	$a = -0.6\ ft/s^2$
At (s = 350 ft)	$a = -0.2\ ft/s^2$

Part 1
Kinematics of a particle

Chapter 3
Motion of projectiles

The free-flight motion of a projectile is often studied in terms of its rectangular components. To illustrate the kinematic analysis, consider a projectile launched at point (x_o , y_o), with an initial velocity of ($\mathbf{v}_o$), having components $(\mathbf{v}_o)_x$ and $(\mathbf{v}_o)_y$. When air resistance is neglected, the only force acting on the projectile is its weight, which causes the projectile to have a *constant downward acceleration* of approximately ($a_c = g = 9.81$ m/s^2) or ($g =$ 32.2 ft/s^2).

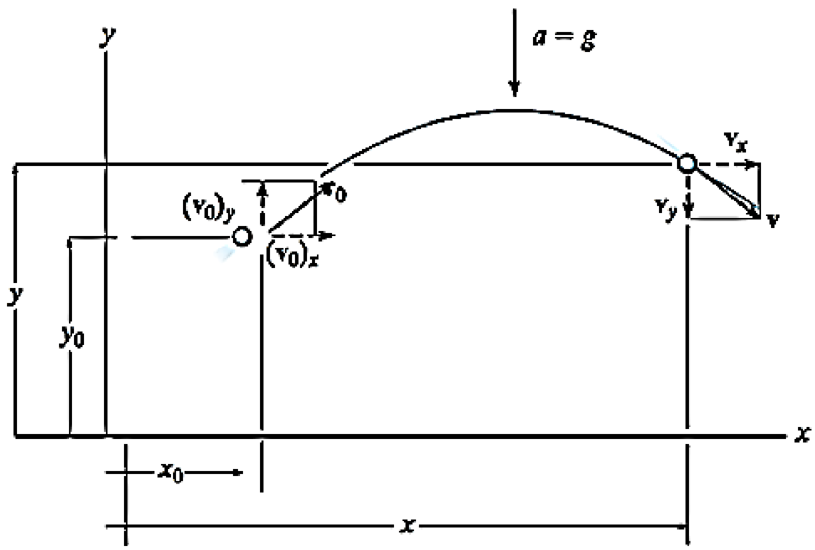


Each picture in this sequence is taken after the same time interval. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity when released.

Both balls accelerate downward at the same rate, and so they remain at the same elevation at any instant. This acceleration causes the difference in elevation between the balls to increase between successive photos.

Also, note the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction remains constant.

(R.C. Hibbeler)



Horizontal Motion.
Since $a_x = 0$, application of the constant acceleration equations, (4 – 6), yields:

$+\rightarrow$	$v = v_o + a_c\, t$	$v_x = (v_o)_x$
$+\rightarrow$	$s = s_o + v_o\, t + \frac{1}{2} a_c\, t^2$	$x = x_o + (v_o)_x\, t$
$+\rightarrow$	$v^2 = v_o^2 + 2\, a_c\, (\, s - s_o\,)$	$v_x = (v_o)_x$

Vertical Motion.
Since the positive *y* axis is directed upward, then $a_y = -\, g$. Applying Eqs. (4–6), we get:

$+\uparrow$	$v = v_o + a_c\, t$	$v_y = (v_o)_y - g\, t$
$+\uparrow$	$s = s_o + v_o\, t + \frac{1}{2} a_c\, t^2$	$y = y_o + (v_o)_y\, t - \frac{1}{2} g\, t^2$
$+\uparrow$	$v^2 = v_o^2 + 2\, a_c\, (\, s - s_o\,)$	$v_y^2 = (v_o)_y^2 - 2\, g\, (\, y - y_o\,)$

Recall that the last equation can be formulated on the basis of eliminating the time *t* from the first two equations, and therefore *only two of the above three equations are independent of one another.*

To summarize, problems involving the motion of a projectile can have at most three unknowns since only three independent equations can be written; that is, *one* equation in the *horizontal direction* and *two* in the *vertical direction.* Once $\mathbf{v}_x$ and $\mathbf{v}_y$ are obtained, the resultant velocity **v**, which is *always tangent* to the path, can be determined by the *vector sum*

Once thrown, the basketball follows a parabolic trajectory. (© R.C. Hibbeler

Gravel falling off the end of this conveyor belt follows a path that can be predicted using the equations of constant acceleration. In this way the location of the accumulated pile can be determined.
Rectangular coordinates are used for the analysis since the acceleration is only in the vertical direction. (© R.C. Hibbeler)

Procedure for Analysis

Coordinate System.

- Establish the fixed *x, y* coordinate axes and sketch the trajectory of the particle. Between any *two points* on the path specify the given problem data and identify the *three unknowns*. In all cases the acceleration of gravity acts downward and equals 9.81 m/s² or 32.2 ft/s². The particle's initial and final velocities should be represented in terms of their *x* and *y* components.
- Remember that positive and negative position, velocity, and acceleration components always act in accordance with their associated coordinate directions.

Kinematic Equations.

- Depending upon the known data and what is to be determined, a choice should be made as to which three of the following four equations should be applied between the two points on the path to obtain the most direct solution to the problem.

Horizontal Motion.

□ The *velocity* in the horizontal or *x* direction is *constant*, i.e., $v_x = (v_0)_x$, and
$x = x_o + (v_o)_x t$

Vertical Motion.

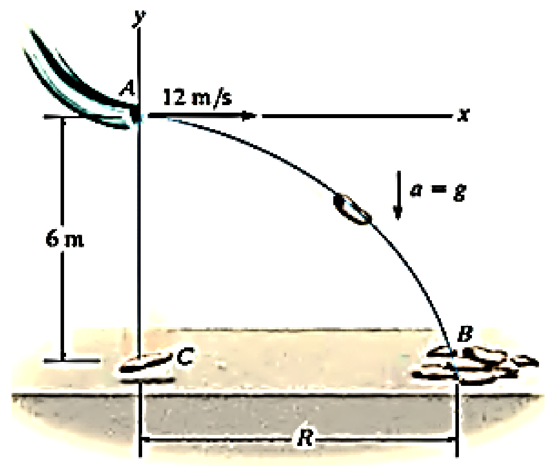
□ In the vertical or y direction only two of the following three equations can be used for solution.

$$v_y = (v_o)_y - g t$$
$$y = y_o + (v_o)_y t - 1/2 \, g \, t^2$$
$$v_y^2 = (v_o)_y^2 - 2g(y - y_o)$$

For example, if the particle's final velocity v_y is not needed, then the first and third of these equations will not be useful.

Example 3 - 1:

A sack slides off the ramp, shown in the figure, with a horizontal velocity of (12 m/s). If the height of the ramp is (6 m) from the floor, determine the time needed for the sack to strike the floor and the range (R) where sacks begin to pile up.



SOLUTION:

Coordinate System:

The origin of coordinates is established at the beginning of the path, point *A*. The initial velocity of a sack has components $(v_A)_x = 12$ m/s and $(v_A)_y = 0$. Also, between points A and B the acceleration is $a_y = -9.81$ m/s^2. Since $(v_B)_x = (v_A)_x = 12$ m/s, the three unknowns are $(v_B)_y$, *R*, and the time of flight (t_{AB}). Here we do not need to determine $(v_B)_y$.

Vertical Motion:

The vertical distance from *A* to *B* is known, and therefore we can obtain a direct solution for t_{AB} by using the equation:

$$y_B = y_A + (v_A)_y\, t_{AB} - \frac{1}{2}\, g\, (t_{AB})^2$$

$$-6 = 0 + 0 - \frac{1}{2}\,(9.81)\,(t_{AB})^2$$

$$t_{AB} = 1.11 \text{ s}$$

Horizontal Motion:

Since t_{AB} has been calculated, *R* is determined as follows:

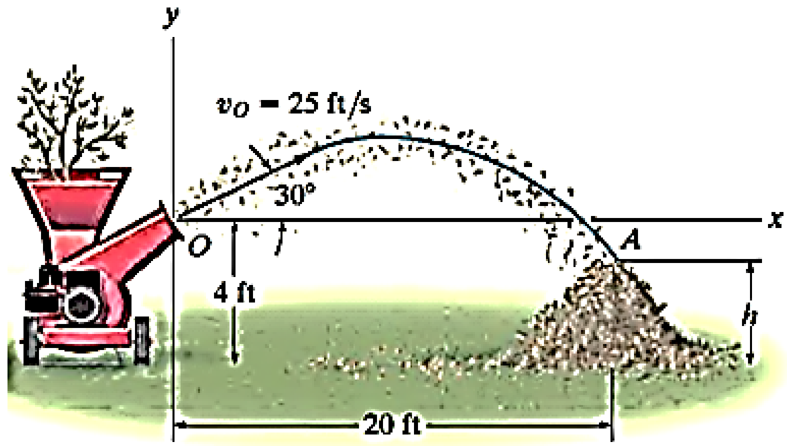
$$x_B = x_A + (v_A)_x\, t_{AB}$$

$$R = 0 + 12\,(1.11) = 13.3 \text{ m}$$

NOTE: The calculation for t_{AB} also indicates that if a sack were released *from rest* at *A*, it would take the same amount of time to strike the floor at *C*.

Example 3 - 2:

The chipping machine is designed to eject wood chips at ($v_o = 25$ ft/s) as shown in the figure below. If the tube is oriented at (30°) from the horizontal, determine how high (h) the chips strike the pile if at this instant they land on the pile (20 ft) from the tube.



SOLUTION:

Coordinate system:

When the motion is analyzed between points O and A, the three unknowns are the height h, time of flight t_{OA}, and vertical component of velocity $(v_A)_y$. [Note that $(v_A)_x = (v_O)_x$]. With the origin of coordinates at O, the initial velocity of a chip has components of

$$(v_o)_x = (25 \cos 30°) = 21.65 \text{ ft/s} \rightarrow$$
$$(v_o)_y = (25 \sin 30°) = 12.5 \text{ ft/s} \uparrow$$

Also, $(v_A)_x = (v_o)_x = 21.65$ ft/s , and $g = -32.2$ ft/s².

Since we do not need to determine $(v_A)_y$, we have:

Horizontal motion:

$+\rightarrow$
$$x_A = x_o + (v_o)_x\, t_{OA}$$
$$20 = 0 + (21.65)\, t_{OA}$$
$$t_{OA} = 0.9238 \text{ s}$$

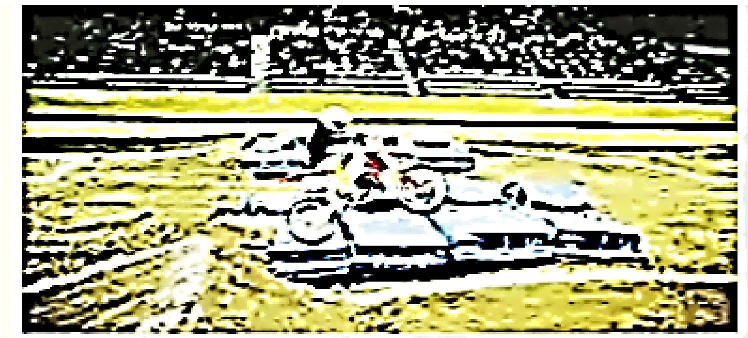
Vertical motion:

$+\uparrow$
$$y_A = y_o + (v_o)_y\, t_{OA} - \frac{1}{2}\, g\, t_{OA}^2$$
$$-(4 - h) = (h - 4) = 0 + (12.5)(0.9238) - \frac{1}{2}(32.2)(0.9238)^2$$
$$h = 1.81 \text{ ft}$$

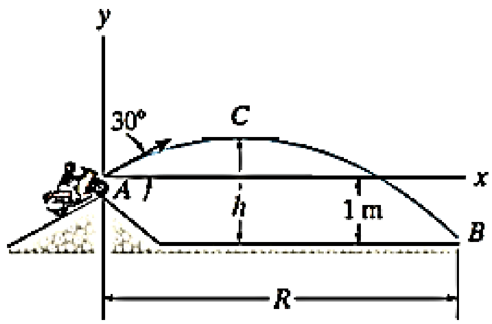
Note: We can determine $(v_A)_y$ by using [$(v_A)_y = (v_o)_y - g\, t_{OA}$].

Example 3 - 3:

The track for this racing event was designed so that riders jump off the slope at (30°) from a height of (1 m). During a race it was observed that the rider shown in Figure, remained in mid air for (1.5 s). Determine the speed at which he was traveling off the ramp, the horizontal distance he travels before striking the ground, and the maximum height he attains. Neglect the size of the bike and rider.



SOLUTION:



Coordinate System:

The origin of the coordinates is established at A. Between the end points of the path *AB* the three unknowns are the initial speed v_A, range R, and the vertical component of velocity $(v_B)_y$.

Vertical Motion:

Since the time of flight and the vertical dista

$+\uparrow$

$$y_B = y_A + (v_A)_y\, t_{AB} - \frac{1}{2}\, g\, t_{AB}^2$$

$$-1 = 0 + v_A \sin 30° \,(1.5) - \frac{1}{2}\,(9.81)\,(1.5)^2$$

$$v_A = 13.38 \text{ m/s}$$

Horizontal Motion:

The range *R* can now be determined.

$+\rightarrow$

$$x_B = x_A + (v_A)_x\, t_{AB}$$

$$R = 0 + 13.38 \cos 30° \,(1.5) = 17.4 \text{ m}$$

In order to find the maximum height *h* we will consider the path *AC*. Here the three unknowns are the time of flight t_{AC}, the horizontal distance from *A* to *C*, and the height *h*. At the maximum height $(v_C)_y = 0$, and since v_A is known, we can determine *h directly* without considering t_{AC} using the following equation.

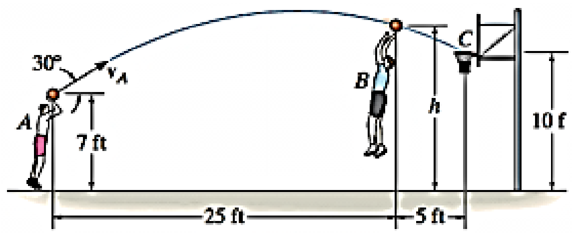
$$(v_C)_y^2 = (v_A)_y^2 - 2\,g\,[\,y_C - y_A\,]$$

$$0 = (\,13.38 \sin 30°\,)^2 - 2\,(9.81)\,[\,(h-1) - 0\,]$$

$$h = 3.28 \text{ m}$$

Example 3 - 4:

The basketball passed through the hoop even though it barely cleared the hands of the player (B) who attempted to block it. Neglecting the size of the ball, determine the magnitude of its initial velocity (v_A) and the height (h) of the ball when it passes over player (B).



SOLUTION:

(+→) $x = x_o + (v_o)_x\ t$

$30 = 0 + v_A \cos 30\ t_{AC}$

$t_{AC} = \dfrac{30}{0.866\ v_A} = \dfrac{34.64}{v_A}$ ………………… (1)

(+↑) $y = y_o + (v_o)_y - 0.5\ g\ t^2$

$10 = 7 + v_A \sin 30\ t_{AC} - 0.5\ (\ 32.2\ )\ t_{AC}^2$

$16.1\ (\ t_{AC}\ )^2 - 0.5\ v_A\ t_{AC} + 3 = 0$ ………….. (2)

Sub. Eq. (1) in (2)

$16.1\ (\dfrac{34.64}{v_A})^2 - 0.5\ v_A \dfrac{34.64}{v_A} + 3 = 0$

$\dfrac{19318.87}{v_A^2} - 14.32 = 0 \quad \Rightarrow \quad \dfrac{19318.87}{v_A^2} = 14.32$

$v_A^2 = \dfrac{19318.87}{14.32} = 1349.08 \quad \Rightarrow \quad v_A = 36.73$ ft/s

$t_{AC} = \dfrac{34.64}{36.73} = 0.943$ s

(+→) $x = x_o + (v_o)_x\ t$

$25 = 0 + 36.73 \cos 30°\ t_{AB}$

$t_{AB} = 0.786$ s

(+↑) $y = y_o + (v_o)_y - 0.5\ g\ t^2$

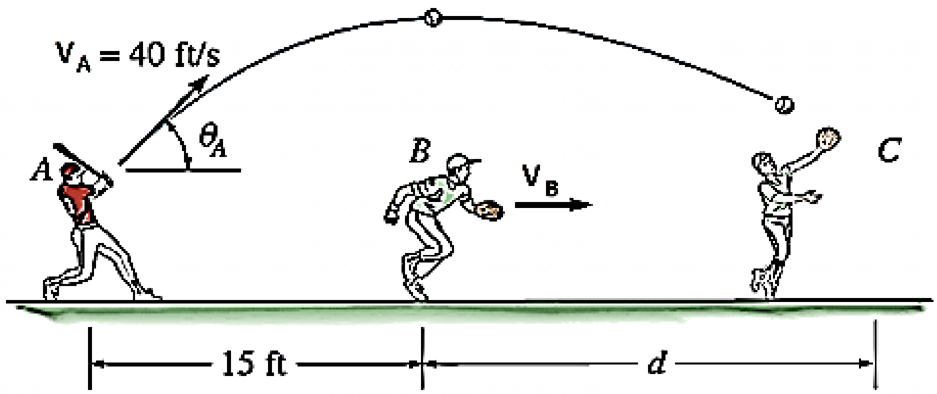
$h = 7 + 36.73 \sin 30°\ t_{AB} - \frac{1}{2}(32.2)(\ t_{AB})^2$

$= 7 + 36.73 \sin 30°\ (0.786) - \frac{1}{2}(32.2)\ (0.786)^2$

$= 11.5$ ft

Example 3 - 5:

The baseball player (A) hits the baseball at (v_A = 40 ft/s) and (θ_A = 60°) from the horizontal. When the ball is directly overhead of player (B) he begins to run under it. Determine the constant speed at which (B) must run and the distance (d) in order to make the catch at the same elevation at which the ball was hit.



SOLUTION:

y - motion: Here, $(v_o)_y = 40 \sin 60° = 34.64$ ft/s , [$y_o = 0$] and [$y = 0$].

$+\uparrow$

$$y = y_o + (v_o)_y t - \frac{1}{2} g t^2$$

$$0 = 34.64 t - \frac{1}{2} (32.2) t^2$$

$$t = 2.152 \text{ s}$$

x - motion: Here, $(v_o)_x = 40 \cos 60° = 20$ ft/s , [$x_o = 0$] and [$X = R$].

$+\rightarrow$

$$x = x_o + (v_o)_x t$$

$$R = 0 + 20 (2.152) = 43.03 \text{ ft}$$

The distance for which player (B) must travel in order to catch the baseball is:

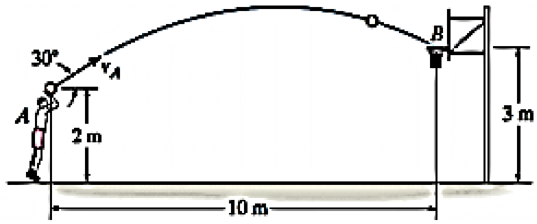
$$d = R - 15 = 43.03 - 15 = 28.03 \text{ ft}$$

Player (B) is required to run at a same speed as the horizontal component of velocity of the baseball in order to catch it.

$$v_B = 20 \text{ ft/s}$$

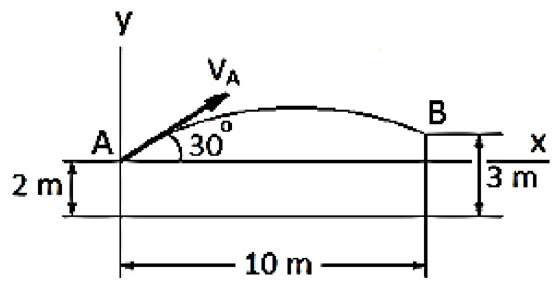
Example 3 - 6:

Neglecting the size of the ball, determine the magnitude of initial velocity (v_A) of the basketball and its velocity when it passes through the basket.



SOLUTION:

x - motion: Here, $(v_A)_x = v_A \cos 30°$, [$x_A = 0$] and [$x_B = 10$ m]. Thus,



$+\rightarrow$

$$x_B = x_A + (v_A)_x\, t$$

$$10 = 0 + (v_A \cos 30°)\, t$$

$$t = \frac{10}{v_A \cos 30}$$

$$t = \frac{10}{0.866\ v_A} = \frac{11.55}{v_A} \quad \ldots\ldots (1)$$

Also, $(v_B)_x = (v_A)_x = v_A \cos 30°$

$$(v_B)_x = 0.866\ v_A \quad \ldots\ldots. (2)$$

y - motion: Here, $(v_A)_y = v_A \sin 30°$, [$y_A = 0$] and [$y_B = 3 - 2 = 1$ m], and ($g = 9.81$ m/s^2).

$+\uparrow$

$$y_B = y_A + (v_A)_y\, t - \frac{1}{2}\, g\, t^2$$

$$1 = 0 + (v_A \sin 30°)\, t - \frac{1}{2}\,(9.81)\, t^2$$

$$4.905\, t^2 - 0.5\, v_A\, t + 1 = 0 \quad \ldots\ldots. (3)$$

Also,

$+\uparrow$

$$(v_B)_y = (v_A)_y - g\, t$$

$$(v_B)_y = v_A \sin 30° - (9.81)\, t$$

$$(v_B)_y = 0.5\, v_A - 9.81\, t \quad \ldots\ldots. (4)$$

Substitute Eq. (1) in Eq. (3)

$$4.905\left(\frac{11.55}{v_A}\right)^2 - 0.5\, v_A\, \frac{11.55}{v_A} + 1 = 0$$

$$\frac{654.34}{v_A^2} - 5.775 + 1 = 0$$

$$\frac{654.34}{v_A^2} = 4.775$$

$$v_A{}^2 = \frac{654.34}{4.775} = 137.03$$

$$v_A = 11.705 \text{ m/s}$$

Substitute in Eq. (1)

$$t = \frac{11.55}{v_A} = \frac{11.55}{11.705} = 0.987 \text{ s}$$

Substitute these results in Eq. (2) and Eq. (4)

$$(v_B)_x = 0.866\,(11.705) = 10.14 \text{ m/s} \rightarrow$$

$$(v_B)_y = 0.5\,(11.705) - 9.81\,(0.987) = -3.83 = 3.83 \text{ m/s} \downarrow$$

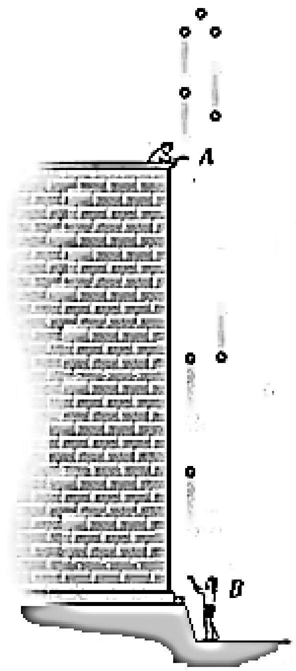
Thus, the magnitude and direction of (v_B) is:

$$v_B = \sqrt{(v_B)_x{}^2 + (v_B)_y{}^2} = \sqrt{(10.14)^2 + (3.83)^2} = 10.8 \text{ m/s}$$

$$\theta_B = \tan^{-1}\frac{(v_B)_y}{(v_B)_x} = \tan^{-1}\frac{3.83}{10.14} = 20.7^\circ$$

Example 3 - 7:

A ball (A) is thrown vertically upward from the top of a (30 m) high building with an initial velocity of (5 m/s). At the same instant another ball (B) is thrown upward from the ground with an initial velocity of (20 m/s). Determine the height from the ground and the time at which they pass.



SOLUTION:

Origin at roof:

Ball A:

$+\uparrow$

$$y = y_o + (v_o)_y\, t - \frac{1}{2}\, g\, t^2$$

$$h = 30 + 5\,t - \frac{1}{2}\,(9.81)\,t^2$$

$$h = 30 + 5\,t - 4.905\,t^2 \quad \ldots\ldots \quad (1)$$

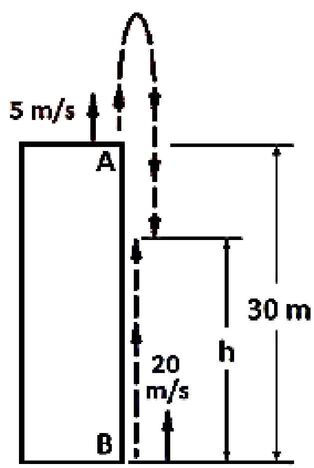
Ball B:

$+\uparrow$

$$y = y_o + (v_o)_y\, t - \frac{1}{2}\, g\, t^2$$

$$h = 0 + 20\,t - \frac{1}{2}\,(9.81)\,t^2$$

$$h = 0 + 20\,t - 4.905\,t^2 \quad \ldots\ldots \quad (2)$$



$$h = 0 \quad + 20\,t - 4.905\,t^2$$

$$h = 30 \quad + 5\,t \quad - 4.905\,t^2$$

--- Subtraction

$$0 = -30 + 15\,t$$

$$15\,t = 30 \qquad \Rightarrow \qquad t = 2\ \text{s}$$

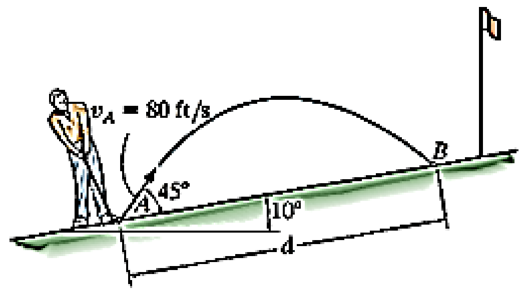
Sub. in (1)

$$h = 30 + 5\,t - 4.905\,t^2$$

$$h = 30 + 5\,(2) - 4.905\,(2)^2 = 20.38\ \text{m}$$

Example 3 - 8:

A golf ball is struck with a velocity of (80 ft/s) as shown. Determine the distance (d) to where it will land.



SOLUTION:

x - motion: $(v_o)_x = 80 \cos 55° = 45.89$ ft/s

$x_o = 0 \qquad x = d \cos 10°$

$(+ \rightarrow)$ $x = x_o + (v_o)_x\, t$

$d \cos 10° = 0 + 45.89\, t$

$0.985\, d = 45.89\, t$ ……………… (1)

y - motion: $(v_o)_y = 80 \sin 55° = 65.53$ ft/s

$y_o = 0 \qquad y = d \sin 10°$

$(+ \uparrow)$ $y = y_o + (v_o)_y\, t + \frac{1}{2} (a_c)_y\, t^2$

$d \sin 10° = 0 + 65.53\, t + \frac{1}{2} (-32.2)\, t^2$

$0.174\, d = 65.53\, t - 16.1\, t^2$ ……………… (2)

From Eq. (1): $d = \frac{45.89\, t}{0.985} = 46.6\, t$

Sub. in Eq. (2) $0.174\,(46.6\, t) = 65.53\, t - 16.1\, t^2$

$8.1\, t = 65.53\, t - 16.1\, t^2$

$16.1\, t^2 - 57.43\, t = 0$

$t\,(16.1\, t - 57.43) = 0$

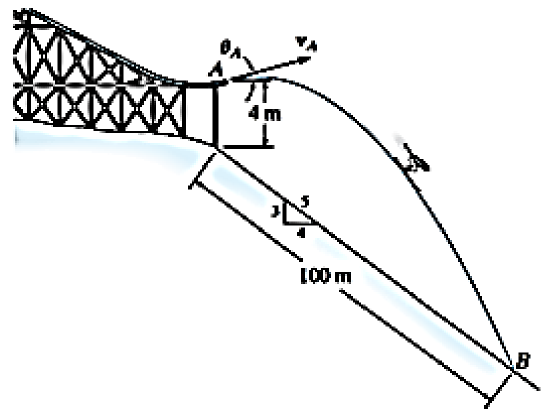
$t = 0 \quad$ or $\quad 16.1\, t - 57.43 = 0$

$16.1\, t = 57.43 \quad \Rightarrow \quad t = \frac{57.43}{16.1} = 3.57$ s

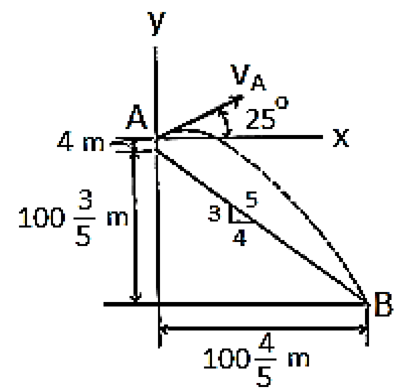
$d = 46.6\, t = 46.6\,(3.57) = 166.2$ ft

Example 3 - 9:

It is observed that the skier leaves the ramp (A) at an angle ($\theta_A = 25^\circ$) with the horizontal. If he strikes the ground at (B), determine his initial speed (v_A) and the time of flight (t_{AB}).



SOLUTION:



$(+\rightarrow)$ $\qquad x = x_o + (v_o)_x\, t$

$$100\,\frac{4}{5} = 0 + v_A \cos 25^\circ\, t_{AB}$$

$$80 = 0.9\,(v_A)\,(t_{AB}) \quad \ldots\ldots \quad (1)$$

$(+\uparrow)$ $\qquad y = y_o + (v_o)_y\, t - \frac{1}{2}\, g\, t^2$

$$-\left(4 + 100\,\frac{3}{5}\right) = 0 + v_A \sin 25^\circ\, t_{AB} - \frac{1}{2}\,(9.81)\, t_{AB}^{\,2}$$

$$-64 = 0.42\,(v_A)\,(t_{AB}) - 4.905\, t_{AB}^{\,2} \quad \ldots\ldots\ldots \quad (2)$$

From Eq. (1): $\qquad v_A = \dfrac{80}{0.9\, t_{AB}} = \dfrac{88.9}{t_{AB}}$

Sub. in Eq. (2) $\qquad -64 = 0.42\left(\dfrac{88.9}{t_{AB}}\right)(t_{AB}) - 4.905\, t_{AB}^{\,2}$

$$-64 = 37.3 - 4.905\, t_{AB}^{\,2}$$

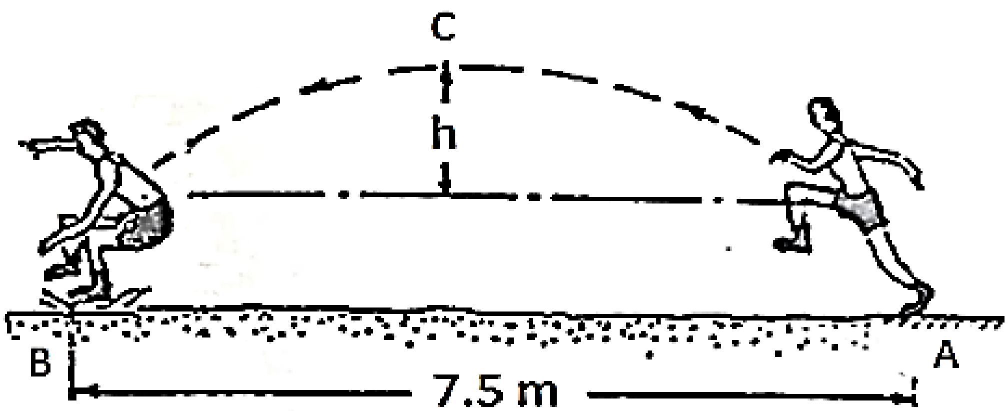
$$4.905\, t_{AB}^{\,2} = 101.3 \quad \Rightarrow \quad t_{AB}^{\,2} = \frac{101.3}{4.905} = 20.66$$

$$t_{AB} = 4.55 \text{ s}$$

$$v_A = \frac{88.9}{4.55} = 19.56 \text{ m/s}$$

Example 3 - 10:

The horizontal component of the initial velocity (v_o)$_x$ for the jumping player shown in the figure is (10 m/s). Calculate the vertical component of his initial velocity (v_o)$_y$ at point (A) for obtaining a horizontal distance of (7.5 m), and find the height (h).



SOLUTION - I:

A → B

$x = x_o + (v_o)_x\, t$

$7.5 = 0 + 10\, t$

$t = 0.75$ s

A → C

$v_y = (v_o)_y - g\, t$

$0 = (v_o)_y - (9.81)\,(0.375)$

$(v_o)_y = 3.68$ m/s

$y = y_o + v_o\, t - \frac{1}{2}\, g\, t^2$

$h = 0 + (3.68)\,(0.375) - \frac{1}{2}\,(9.81)\,(0.375)^2$

$h = 0.69$ m

SOLUTION - I:

A → B

$x = x_o + (v_o)_x\, t$

$7.5 = 0 + 10\, t$

$t = 0.75$ s

$v_y = (v_o)_y - g\, t$

$(v_B)_y = (v_A)_y - g\, t$

$(v_B)_y = -\,(v_A)_y$

$-\,(v_A)_y = (v_A)_y - g\, t$

$-\,(v_A)_y - (v_A)_y = (9.81)\,(0.75)$

$-\,2(v_A)_y = -\,7.32$

$(v_A)_y = 3.67$ m/s

A → C

$v_y^2 = (v_o)_y^{\,2} - 2\, g\,(\, y - y_o\,)$

$(v_C)_y^2 = (v_A)_y^2 - 2g\,(\, y_C - y_A\,)$

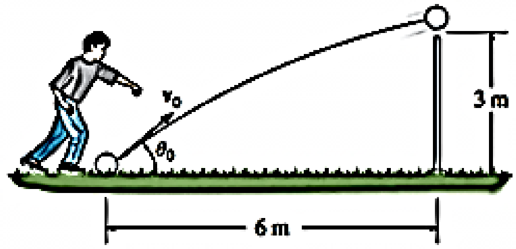
$0 = (3.67)^2 - 2 \times 9.81\,(\, h - 0\,)$

$0 = 13.46 - 19.62\, h$

$h = 0.68$ m

Example 3 - 11:

Determine the minimum initial velocity (v_o) and the corresponding angle (θ_o) at which the ball must be kicked in order for it to just cross over the (3 m) high fence.



SOLUTION:

Coordinate System:

The coordinate system will be set so that its origin coincides with the ball's initial position.

x - motion: Here, [$(v_o)_x = v_o \cos\theta$] , ($x_o = 0$) and ($x = 6$ m). Thus,

$+\rightarrow$

$$x = x_o + (v_o)_x t$$
$$6 = 0 + (v_o \cos\theta)\, t$$
$$t = \frac{6}{v_0 \cos\theta} \quad \ldots\ldots\ldots\ldots\ldots\ (1)$$

y - motion: Here, [$(v_o)_y = v_o \sin\theta$] , ($a_y = -g = -9.81$ m/s^2) and ($y_o = 0$). Thus,

$+\uparrow$

$$y = y_o + (v_o)_y t - \frac{1}{2} g t^2$$
$$3 = 0 + (v_o \sin\theta)\, t - \frac{1}{2}(9.81)\, t^2$$
$$3 = (v_o \sin\theta)\, t - 4.905\, t^2 \quad \ldots\ldots\ldots\ldots\ldots\ (2)$$

Sub. Eq. (1) in Eq. (2)

$$3 = v_0 (\sin\theta) \frac{6}{v_0 \cos\theta} - 4.905 \left(\frac{6}{v_0 \cos\theta}\right)^2$$
$$3 = (\sin\theta) \frac{6}{\cos\theta} - \frac{176.58}{v_0^2 \cos^2\theta}$$
$$3 - (\sin\theta) \frac{6}{\cos\theta} = - \frac{176.58}{v_0^2 \cos^2\theta}$$
$$\left(3 - 6 \frac{\sin\theta}{\cos\theta}\right) v_0^2 \cos^2\theta = -176.58$$
$$\left(3 \cos^2\theta - 6 \frac{\sin\theta}{\cos\theta} \cos^2\theta\right) v_0^2 = -176.58$$
$$3 (\cos^2\theta - 2 \sin\theta \cos\theta)\, v_0^2 = -176.58$$
$$3 (\cos^2\theta - \sin 2\theta)\, v_0^2 = -176.58 \qquad \times -1$$
$$3 (\sin 2\theta - \cos^2\theta)\, v_0^2 = 176.58$$

$$v_0{}^2 = \frac{176.58}{3\,(\sin 2\theta - cos^2\theta)} = \frac{58.86}{(\sin 2\theta - cos^2\theta)}$$

$$v_0 = \sqrt{\frac{58.86}{(\sin 2\theta - cos^2\theta)}} \quad \text{-----------------------------} \quad (3)$$

From Eq. (3), we notice that (v_o) is minimum when [$f(\theta) = \sin 2\theta - \cos^2\theta$] is maximum. This requires ($\frac{d\,f(\theta)}{d\theta} = 0$)

$$f(\theta) = \sin 2\theta - \cos^2\theta = 2\sin\theta\,\cos\theta - \cos^2\theta$$

$$\frac{d\,f(\theta)}{d\theta} = 2\,[\,\sin\theta(-\sin\theta) + \cos\theta\,(\,\cos\theta)\,] - [\,2\cos\theta\,(-\sin\theta\,)\,]$$

$$= 2\,(-\sin^2\theta + \cos^2\theta\,) - (-2\sin\theta\,\cos\theta)$$

$$= 2\,(\cos^2\theta - \sin^2\theta\,) + 2\sin\theta\,\cos\theta = 2\cos 2\theta + \sin 2\theta$$

$$2\cos 2\theta + \sin 2\theta = 0$$

$$\sin 2\theta = -2\cos 2\theta \qquad \div \cos 2\theta$$

$$\tan 2\theta = -2$$

$$2\theta = 116.57° \quad \Rightarrow \quad \theta = 58.28°$$

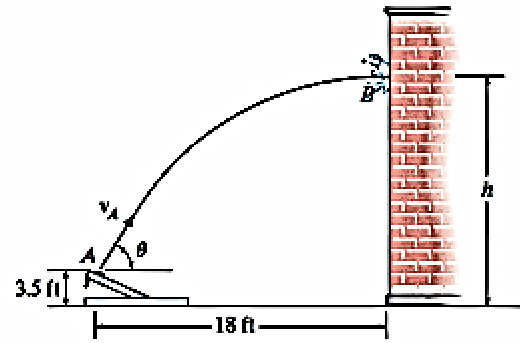
Substituting the result of (θ) into Eq. (2), we have:

$$(v_o)_{min} = \sqrt{\frac{58.86}{\sin 116.57 - cos^2 58.28}} = 9.76 \text{ m/s}$$

PROBLEMS:

(3 - 1):

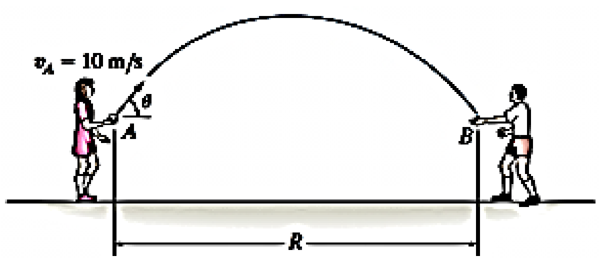
The catapult is used to launch a ball such that it strikes the wall of the building at the maximum height of its trajectory. If it takes (1.5 s) to travel from (A) to (B), determine the velocity (v_A) at which it was launched, the angle of release (θ), and the height (h).



Ans. : ($v_A = 49.8$ ft/s) , (h = 39.7 ft)

(3 - 2):

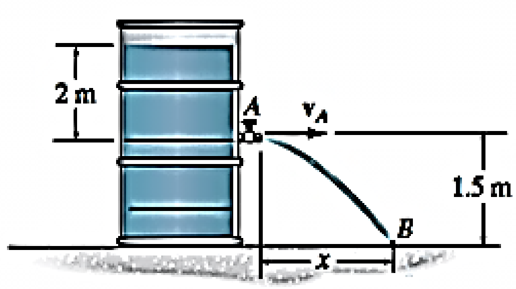
The girl at (A) can throw a ball at (v_A = 10 m/s). Calculate the maximum possible range ($R = R_{max}$) and the associated angle (θ) at which it should be thrown. Assume the ball is caught at (B) at the same elevation from which it is thrown.



Ans. : (R = 102 m) , ($\theta = 45°$)

(3 - 3):

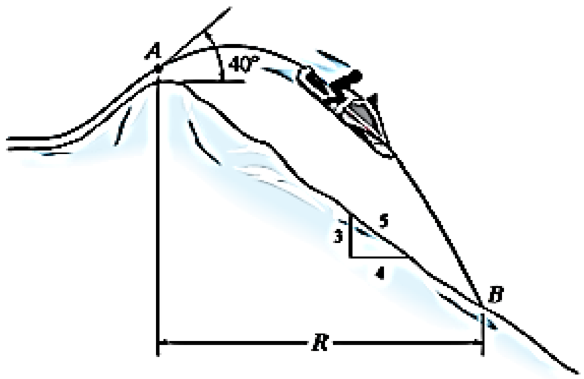
The velocity of the water jet discharging from the orifice can be obtained from ($v = \sqrt{2gh}$, where (h = 2m) is the depth of the orifice from the free water surface. Determine the time for a particle of water leaving the orifice to reach point (B) and the horizontal distance (x) where it hits the surface.



Ans. : ($t_A = 0.553$ s) , (x = 3.46 m)

(3 - 4):

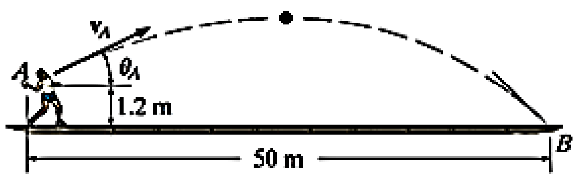
The snowmobile is traveling at (10 m/s) when it leaves the embankment at (A). Determine the time of flight from (A) to (B) and the range (R) of the trajectory.



Ans. : (t = 2.48 s) , (R = 19 m)

(3 - 5):

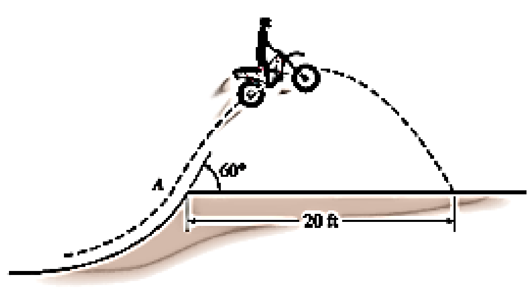
It is observed that the time for the ball to strike the ground at (B) is (2.5 s). Determine the speed (v_A) and angle (θ_A) at which the ball was thrown.



Ans. : ($\theta_A = 30.5°$) , ($v_A = 23.2$ m/s)

(3 - 6):

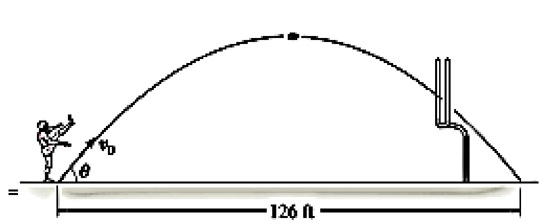
During a race the dirt bike was observed to leap up off the small hill at (A) at an angle of (60°) with the horizontal. If the point of landing is (20 ft) away, determine the approximate speed at which the bike was traveling just before it left the ground. Neglect the size of the bike for the calculation.



Ans. : ($v_A = 27.3$ ft/s)

(3 - 7):

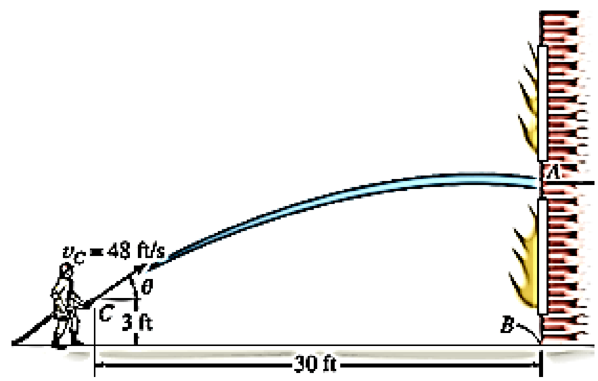
From a videotape, it was observed that a pro football player kicked a football (126 ft) during a measured time of (3.6) seconds. Determine the initial speed of the ball and the angle (θ) at which it was kicked.



Ans. : ($v_o = 67.7$ ft/s) , ($\theta = 58.9°$)

(3 - 8):

Determine the maximum height on the wall to which the firefighter can project water from the hose, if the speed of the water at the nozzle is ($v_C = 48$ ft/s).



Ans. : (h = 11.1 ft)

Part 1
Kinematics of a particle

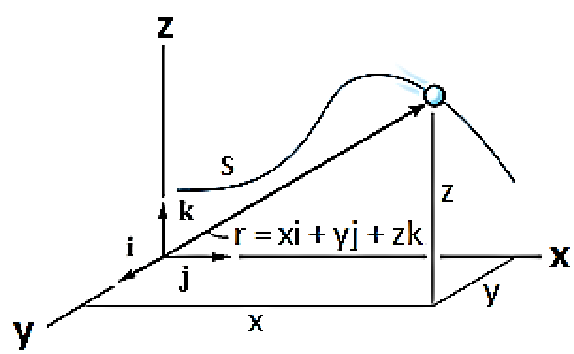
Chapter 4
Curvilinear motion

Curvilinear motion: Rectangular components

Occasionally the motion of a particle can best be described along a path that can be expressed in terms of its (x, y, z) coordinates.

Position:

If the particle is at point (x, y, z) on the curved path *s* shown in the figure, then its location is defined by the *position vector*

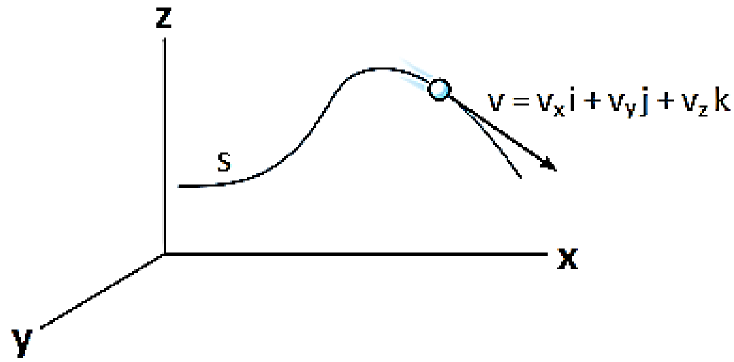


Position

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \mathbf{r} = \sqrt{\mathbf{x^2 + y^2 + z^2}}$$

Velocity:

The first time derivative of **r** yields the velocity of the particle. Hence,



Velocity

$$\mathbf{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}(\,\mathrm{x}\mathrm{i}\,) + \frac{d}{dt}(\,\mathrm{y}\mathrm{j}\,) + \frac{d}{dt}(\,\mathrm{z}\mathrm{k}\,)$$

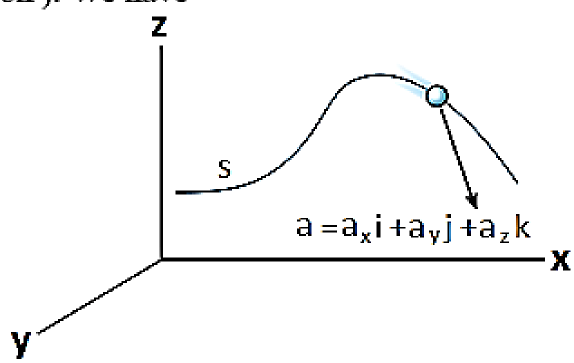
$$\mathbf{v} = \frac{d\boldsymbol{r}}{dt} = \mathrm{v_x}\mathbf{i} + \mathrm{v_y}\,\mathbf{j} + \mathrm{v_z}\mathbf{k}$$

$$\mathbf{v_x} = \dot{\mathbf{x}} \quad , \quad \mathbf{v_y} = \dot{\mathbf{y}} \quad , \quad \mathbf{v_z} = \dot{\mathbf{z}}$$

$$\mathbf{v} = \sqrt{\mathbf{v_x}^2 + \mathbf{v_y}^2 + \mathbf{v_z}^2}$$

Acceleration:

The acceleration of the particle is obtained by taking the first of the velocity (or the second derivative of the position). We have



Acceleration

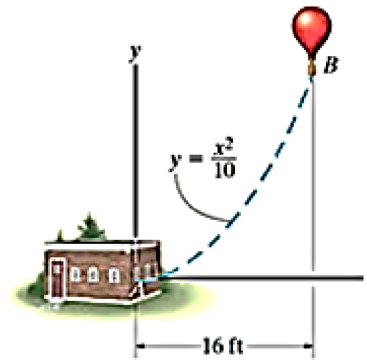
$$\mathbf{a} = \frac{\mathbf{dv}}{\mathbf{dt}} = \mathrm{a_x}\mathbf{i} + \mathrm{a_y}\,\mathbf{j} + \mathrm{a_z}\mathbf{k}$$

$$\mathbf{a_x} = \dot{\mathbf{v}}_\mathbf{x} = \ddot{\mathbf{x}} \;,\; \mathbf{a_y} = \dot{\mathbf{v}}_\mathbf{y} = \ddot{\mathbf{y}} \;,\; \mathbf{a_z} = \dot{\mathbf{v}}_\mathbf{z} = \ddot{\mathbf{z}}$$

$$\mathbf{a} = \sqrt{\mathbf{a_x}^2 + \mathbf{a_y}^2 + \mathbf{a_z}^2}$$

Example 4 - 1:

At any instant the horizontal position of the weather balloon in (Fig. a) is defined by [x = (8t) ft], where (t) is in seconds. If the equation of the path is ($y = x^2/10$), determine the magnitude and direction of the velocity and the acceleration when (t = 2 s).



a

SOLUTION:

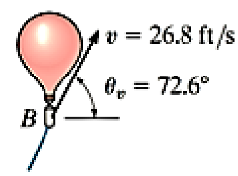
Velocity. The velocity component in the *x* direction is

$$v_x = \dot{x} = \frac{d}{dt}(8\,t) = 8 \text{ ft/s} \rightarrow$$

When (t = 2 s), $\quad x = 8\,(2) = 16$ ft

$$v_y = \dot{y} = \frac{d}{dt}\left(\frac{x^2}{10}\right) = \frac{2\,x\,\dot{x}}{10} = \frac{2\,(16)(8)}{10} = 25.6 \text{ ft/s} \uparrow$$

When (t = 2 s), the magnitude of velocity is therefore



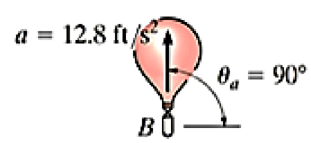
(b)

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(8)^2 + (25.6)^2} = 26.8 \text{ m/s}$$

The direction is tangent to the path, Fig. *b*, where

$$\theta_v = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{25.6}{8} = 72.6°$$

Acceleration. The relationship between the acceleration components is determined using the chain rule. We have:



(c)

$$a_x = \dot{v}_x = \frac{d}{dt}(8) = 0$$

$$a_y = \dot{v}_y = \frac{d}{dt}\left(\frac{2\,x\,\dot{x}}{10}\right) = \frac{2}{10}(x\,\ddot{x} + \dot{x}\,\dot{x})$$

$$= \frac{2}{10}(x\,a_x + v_x^2)$$

$$= \frac{2}{10}[\,(16)(0) + (8)^2 = 12.8 \text{ ft/s}^2 \uparrow$$

Thus,

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(0)^2 + (12.8)^2} = 12.8 \text{ m/s}^2$$

The direction of **a**, as shown in Fig. *c*, is:

$$\theta_a = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\frac{12.8}{0} = 90°$$

NOTE: It is also possible to obtain v_y and a_y by first expressing $y = f(t) = (8t)2/10 = 6.4t2$ and then taking successive time derivatives.

Example 4 - 2:

For a short time, the path of the plane in the figure is described by [$y = (0.001x^2)$ m]. If the plane is rising with a constant upward velocity of (10 m/s), determine the magnitudes of the velocity and acceleration of the plane when it reaches an altitude of ($y = 100$m).

SOLUTION:

When (y = 100 m),

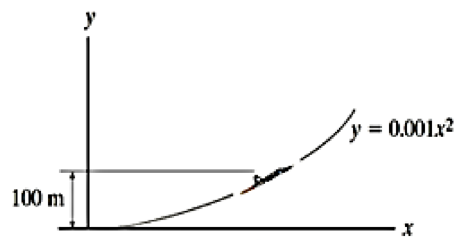
Then, $100 = 0.001x^2 \quad \Rightarrow \quad x = 316.2$ m.

Also, due to constant velocity ($v_y = 10$ m/s), so:

$y = v_y\, t; \qquad 100 \text{ m} = (10 \text{ m/s})\, t \qquad t = 10 \text{ s}$

Velocity. Using the chain rule to find the relationship between the velocity components, we have

$$y = 0.001\, x^2$$
$$v_y = \dot{y} = \frac{d}{dt}(0.001\, x^2) = (0.002\, x)\, \dot{x}$$
$$v_y = 0.002\, x\, v_x$$
$$10 = 0.002\, (316.2)\, (v_x)$$
$$v_x = 15.81 \text{ m/s}$$



(a)

The magnitude of the velocity is therefore

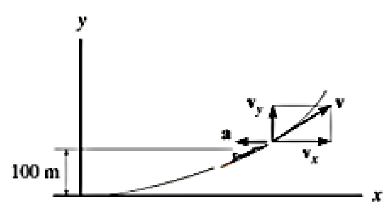
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(15.81)^2 + (10)^2} = 18.7 \text{ m/s}$$

Acceleration. Using the chain rule, the time derivative of the velocity gives the relation between the acceleration components.

$$a_y = \dot{v}_y = (0.002\, \dot{x})\, \dot{x} + 0.002\, x\, (\ddot{x})$$
$$= 0.002\, (v_x^2 + x\, a_x)$$

When, ($x = 316.2$ m), ($\dot{v}_x = 15.81$ m/s^2), $v_y = a_y = 0$

$$0 = 0.002\, [\, (15.81)^2 + (316.2\, a_x)\,]$$
$$a_x = -\,0.791 \text{ m/s}^2$$



(b)

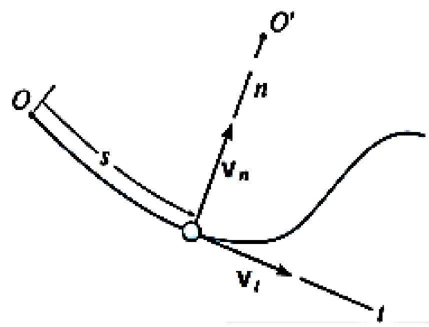
The magnitude of the plane's acceleration is therefore:

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.791)^2 + (0)^2} = 0.791 \text{ m/s}^2$$

Curvilinear motion: Normal and tangential components

Coordinate system:

- Provided the path of the particle is known, we can establish a set of (n) and (t) coordinates having a fixed origin, which is coincident with the particle at the instant considered.
- The positive tangent axis acts in the direction of motion and the positive normal axis is directed toward the path's center of curvature.



Velocity:

- The particle's velocity is always tangent to the path.
- The magnitude of velocity is found from the time derivative of the path function.

$$v = \dot{s}$$

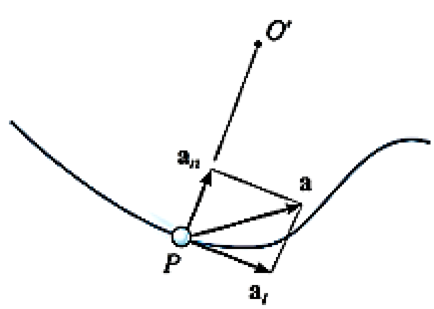
Tangential acceleration:

- The tangential component of acceleration is the result of the time rate of change in the magnitude of velocity. This component acts in the positive (s) direction if the particle's speed is increasing or in the opposite direction if the speed is decreasing.
- The relations between at (v) , (t) , and (s) are the same as for rectilinear motion, namely

$$a_t = \dot{v} \qquad a_t \, ds = v \, dv$$

- If (a_t) is constant, [$a_t = (a_t)_c$], the above equations, when integrated, yield:

$$v = v_o + (a_t)_c \, t$$
$$s = s_o + v_o \, t + \frac{1}{2} (a_t)_c \, t^2$$
$$v^2 = v_o^2 + 2 \, (a_t)_c \, (s - s_o)$$



Normal acceleration:

- The normal component of acceleration is the result of the time rate of change in the direction of the velocity. This component is always directed toward the center of curvature of the path, i.e., along the positive (n – axis).
- The magnitude of this component is determined from

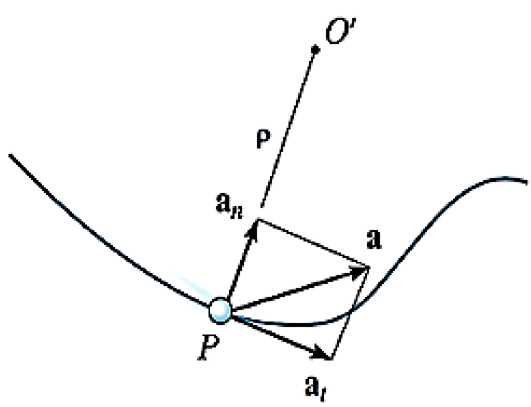
$$a_n = \frac{v^2}{\rho}$$

ρ : radius of curvature.

- If the path is expressed as (y = f(x)), the radius of curvature (r) at any point on the path is determined from the equation

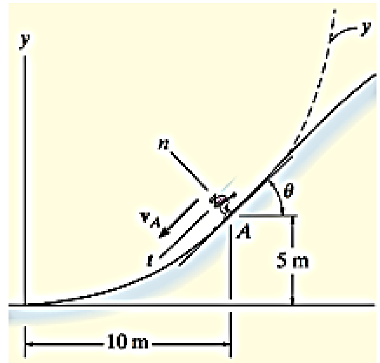
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

The derivation of this result is given in any standard calculus text.



Example 4 - 3:

When the skier reaches point (A) along the parabolic path in the figure, he has a speed of (6 m/s) which is increasing at (2 m/s²). Determine the direction of his velocity and the direction and magnitude of his acceleration at this instant. Neglect the size of the skier in the calculation.



SOLUTION:

Velocity:

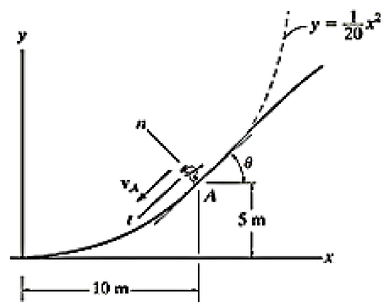
$$y = \frac{1}{20}x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{10}x$$

at point (A), $\quad x = 10$ m, $\quad \dfrac{dy}{dx} = 1$

$\theta = \tan^{-1} 1 = 45°$ with the (x-axis).

Therefore,

$$v = 6 \text{ m/s} \quad 45° \;\swarrow$$

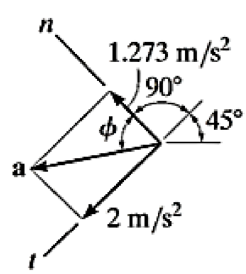


Acceleration:

$$\frac{dy}{dx} = \frac{1}{10}x \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{10}$$

$$\rho = \frac{[1+\left(\frac{dy}{dx}\right)^2]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{[1+\left(\frac{1}{10}x\right)^2]^{3/2}}{\left|\frac{1}{10}\right|}$$

At (x = 10 m), $\quad \rho = 28.28$ m



$$a_n = \frac{v^2}{\rho} = \frac{6^2}{28.28} = 1.273 \text{ m/s}^2$$

$$a_t = 2 \text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{2^2 + 1.273^2} = 2.37 \text{ m/s}^2$$

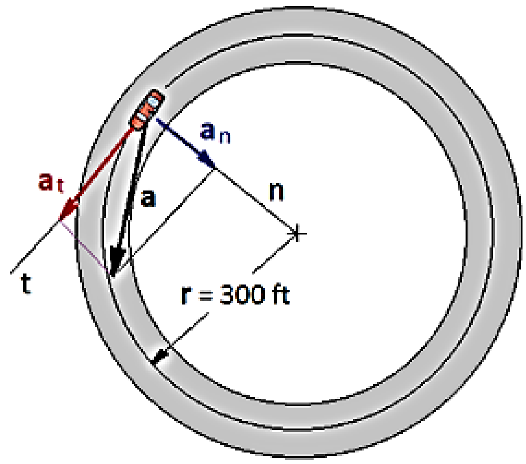
$$\phi = \tan^{-1} \frac{2}{1.273} = 57.5°$$

Thus , $\quad 45° + 90° + 57.5° - 180° = 12.5°$

So that: $\quad a = 2.37 \text{ m/s}^2 \quad 12.5° \;\swarrow$

Example 4 - 4:

A race car travels around the horizontal circular track that has a radius of (300 ft). If the car increases its speed at a constant rate of (7 ft/s²), starting from rest, determine the time needed for it to reach an acceleration of (8 ft/s²). What is its speed at this instant?



SOLUTION:

Acceleration:

$$a = \sqrt{a_t^2 + a_n^2}$$

$$a_t = 7 \text{ ft/s}^2 \qquad a_n = \frac{v^2}{\rho}$$

$$v = v_o + (a_t)_c\, t = 0 + 7\,t = 7\,t$$

$$a_n = \frac{v^2}{\rho} = \frac{(7t)^2}{300} = 0.163\, t^2 \text{ ft/s}^2$$

The time needed for the acceleration to reach (8 ft/s²) is therefore

$$a = \sqrt{a_t^2 + a_n^2}$$

$$8 = \sqrt{(7)^2 + (0.163\, t^2)^2}$$

$$64 = 49 + 0.0266\, t^4$$

$$t = \sqrt[4]{\frac{15}{0.0266}} = 4.87 \text{ s}$$

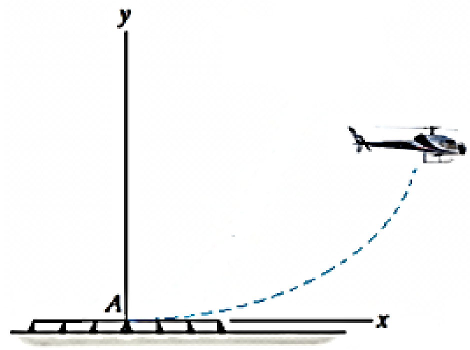
Velocity:

The speed at time ($t = 4.87$ s) is:

$$v = 7\,t = 7\,(4.87) = 34.1 \text{ ft/s.}$$

Example 4 - 5:

The flight path of the helicopter as it takes off from (A) is defined by the parametric equations [$x = (2\,t^2)$ m] and [$y = (0.04\,t^3)$ m], where (t) is the time in seconds. Determine the distance the helicopter is from point (A) and the magnitudes of its velocity and acceleration when (t = 10 s).



SOLUTION:

$x = 2\,t^2 \qquad y = 0.04\,t^3$

At (t = 10 s), $x = 200\text{ m} \qquad y = 40\text{ m}$

$$d = \sqrt{(200)^2 + (40)^2} = 204\text{ m}$$

$$v_x = \frac{dx}{dt} = 4\,t$$

$$a_x = \frac{dv_x}{dt} = 4$$

$$v_y = \frac{dy}{dt} = 0.12\,t^2$$

$$a_y = \frac{dv_y}{dt} = 0.24\,t$$

At (t = 10 s),

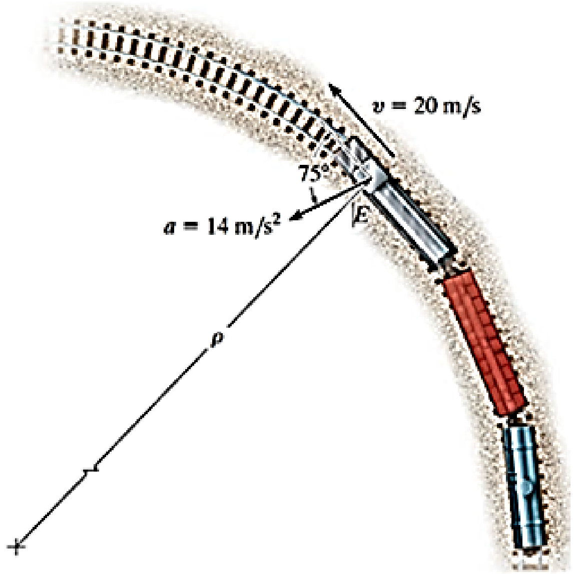
$v_x = 40$ m/s , $v_y = 12$ m/s , $a_x = 4$ m/s^2 , $a_y = 2.4$ m/s^2

$$v = \sqrt{(40)^2 + (12)^2} = 41.8\text{ m/s}$$

$$a = \sqrt{(4)^2 + (2.4)^2} = 4.66\text{ m/s}^2$$

Example 4 - 6:

At a given instant the train engine at (E) has a speed of (20 m/s) and an acceleration of (14 m/s²) acting in the direction shown. Determine the rate of increase in the train's speed and the radius of curvature (ρ) of the path.



SOLUTION:

$a_t = 14 \cos 75° = 3.62 \text{ m/s}^2$

$a_n = 14 \sin 75° = 13.52 \text{ m/s}^2$

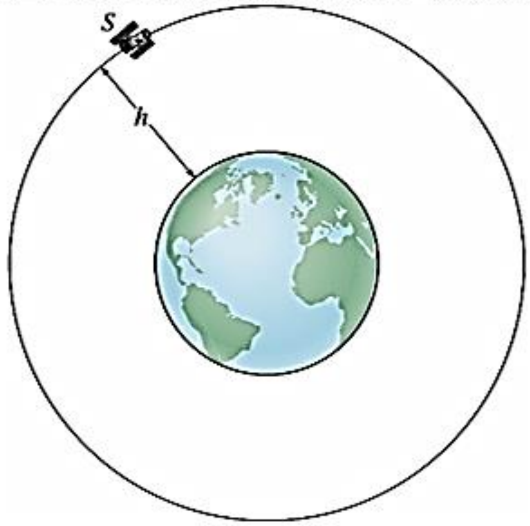
$$a_n = \frac{v^2}{\rho}$$

$$13.52 = \frac{(20)^2}{\rho} = \frac{400}{\rho}$$

$$\rho = \frac{400}{13.52} = 29.6 \text{ m}$$

Example 4 - 7:

The satellite (s) travels around the earth in a circular path with a constant speed of (20 Mm/h). If the acceleration is (2.5 m/s²), determine the altitude (h). Assume the earth's diameter to be (12713 km).



SOLUTION:

$$v = 20\ \text{Mm/h} = \frac{20\times10^6}{3600} = 5.56\times10^3\ \text{m/s}$$

Since $a_t = \dfrac{dv}{dt} = 0$ then, $a = a_n = 2.5\ \text{m/s}^3$

$$a_n = \frac{v^2}{\rho} \quad \Rightarrow \quad \rho = \frac{v^2}{a_n} = \frac{(5.56\times10^3)^2}{2.5} = 12.35\times10^6\ \text{m}$$

The radius of the earth is:

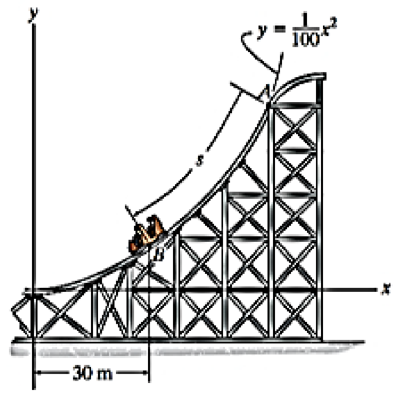
$$\frac{12713\times10^3}{2} = 6.36\times10^6\ \text{m}$$

Hince,

$$h = 12.35\times10^6 - 6.36\times10^6 = 5.99\times10^6\ \text{m} = 5.99\ \text{Mm}$$

Example 4 - 8:

When the roller coaster is at (B), it has a speed of (25 m/s), which is increasing at (a_t = 3 m/s^2). Determine the magnitude of the acceleration of the roller coaster at this instant and the direction angle it makes with the (x – axis).



SOLUTION:

Radius of curvature:

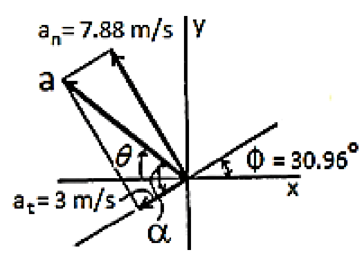
$$y = \frac{1}{100}x^2$$
$$\frac{dy}{dx} = \frac{1}{50}x$$
$$\frac{d^2y}{dx^2} = \frac{1}{50}$$
$$\rho = \frac{[1+\left(\frac{dy}{dx}\right)^2]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{[1+\left(\frac{1}{50}x\right)^2]^{3/2}}{\left|\frac{1}{50}\right|}$$

At (x = 30 m), ρ = 79.3 m

Acceleration:

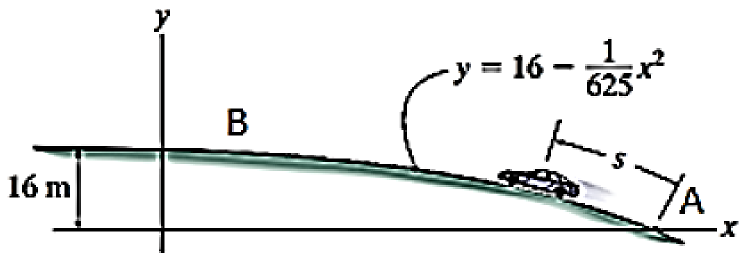
$$a_t = \dot{v} = 3 \text{ m/s}^2$$
$$a_n = \frac{{v_B}^2}{\rho} = \frac{(25)^2}{79.3} = 7.881 \text{ m/s}^2$$
$$a = \sqrt{{a_t}^2 + {a_n}^2} = \sqrt{(3)^2 + (7.881)^2}$$
$$= 8.43 \text{ m/s}^2$$

$$\phi = \tan^{-1}\frac{dy}{dx} = \tan^{-1}\frac{1}{50}x$$
$$= \tan^{-1}\frac{1}{50}(30) = 30.96^\circ$$
$$\alpha = \tan^{-1}\frac{a_n}{a_t} = \tan^{-1}\frac{7.881}{3} = 69.16^\circ$$
$$\theta = \alpha - \phi = 69.16^\circ - 30.96^\circ$$
$$= 38.2^\circ$$



Example 4 - 9:

If the car passes point (A) with a speed of (20 m/s) and begins to increase its speed at a constant rate of (a_t = 0.5 m/s^2), determine the magnitude of the car's acceleration at (B) when (s = 100 m).



SOLUTION:

Velocity:

The speed of the car at (B) is:

$$v_B^2 = v_A^2 + 2\,a_t\,(s_B - s_A)$$
$$v_B^2 = (20)^2 + 2\,(0.5)\,(100 - 0)$$
$$v_B = 22.361 \text{ m/s}$$

Radius of Curvature:

$$y = 16 - \frac{1}{625}x^2$$
$$\frac{dy}{dx} = -3.2 \times 10^{-3}\,x$$
$$\frac{d^2y}{dx^2} = -3.2 \times 10^{-3}$$
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1+\left(-3.2\times10^{-3}\,x\right)^2\right]^{3/2}}{\left|-3.2\times10^{-3}\right|}$$

At (x = 0), $\rho = 312.5$ m

Acceleration:

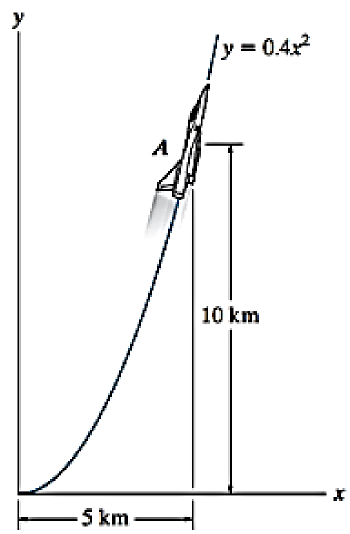
$$a_t = \dot{v} = 0.5 \text{ m/s}^2$$
$$a_n = \frac{v_B^2}{\rho} = \frac{(22.361)^2}{312.5} = 1.6 \text{ m/s}^2$$

The magnitude of the car's acceleration at (B) is:

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(0.5)^2 + (1.6)^2} = 1.68 \text{ m/s}^2$$

Example 4 - 10:

The jet plane travels along the vertical parabolic path. When it is at point (A) it has a speed of (200 m/s), which is increasing at the rate of (0.8 m/s²). Determine the magnitude of acceleration of the plane when it is at point (A).



SOLUTION:

Radius of Curvature:

$$y = 0.4\,x^2$$

$$\frac{dy}{dx} = 0.8\,x$$

$$\frac{d^2y}{dx^2} = 0.8$$

$$\rho = \frac{\left[\,1+\left(\frac{dy}{dx}\right)^2\,\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{[\,1+(4)^2\,]^{3/2}}{|0.8|} = 87.62\text{ km}$$

Acceleration:

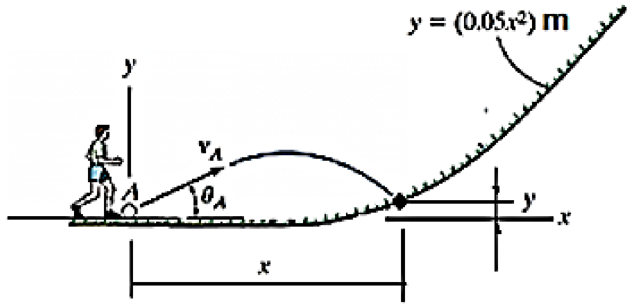
$$a_t = 0.8\text{ m/s}^2$$

$$a_n = \frac{{v_B}^2}{\rho} = \frac{(200)^2}{87.62\times10^3} = 0.457\text{ m/s}^2$$

$$a = \sqrt{{a_t}^2 + {a_n}^2} = \sqrt{(0.8)^2 + (0.457)^2} = 0.921\text{ m/s}^2$$

Example 4 - 11:

The ball at (A) is kicked with a speed ($v_A = 25$ m/s) and at an angle ($\theta_A = 30^o$). Determine the point (x , y) where it strikes the ground. Assume the ground has the shape of a parabola as shown.



SOLUTION:

$(v_A)_x = 25 \cos 30 = 21.65$ m/s

$(v_A)_y = 25 \sin 30 = 12.5$ m/s

$+\rightarrow$ $x = x_o + v_o\, t$

$x = 0 + 21.65\, t$

$x = 21.65\, t$ ………………………. (1)

$+\uparrow$ $y = y_o + v_o\, t - 0.5\, g\, t^2$

$y = 0 + 12.5\, t - 0.5\,(9.81)\, t^2$

$y = 12.5\, t - 4.905\, t^2$ ……………..….. (2)

$y = 0.05\, x^2$ ……………………..………. (3)

From Eq. (1) $t = \dfrac{x}{21.65}$

Sub. in Eq. (2) $y = 12.5\,(\dfrac{x}{21.65}) - 4.905\,(\dfrac{x}{21.65})^2$

$y = 0.5774\, x - 0.01\, x^2$

Sub. in Eq. (3) $0.05\, x^2 = 0.5774\, x - 0.01\, x^2$

$0.06\, x^2 = 0.5774\, x$

$0.06\, x = 0.5774$

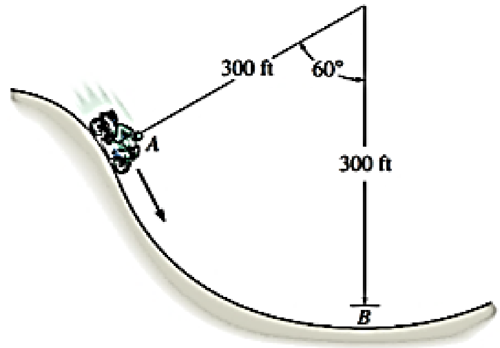
$x = 9.623$ m

$y = 0.05\,(9.623)^2 = 4.63$ m

The point is: (9.623 , 4.63)

Example 4 - 12:

When the motorcyclist is at (A), he increases his speed along the vertical circular path at the rate of [a = (0.3 t) ft/s²], where (t) is in seconds. If he starts from rest at (A), determine the magnitudes of his velocity and acceleration when he reaches (B).



SOLUTION:

$$\int_0^v dv = \int_0^t 0.3\,t\,dt$$

$$v = 0.15\,t^2$$

$$\int_0^s ds = \int_0^t 0.15\,t^2\,dt$$

$$s = 0.05\,t^3$$

$$s = \frac{1}{6}\,[\,2\pi\rho\,] = \frac{1}{6}\,[\,2\pi(300)\,] = \frac{\pi}{3}\,(300) = 100\,\pi$$

$$0.05\,t^3 = 100\,\pi \quad \Rightarrow \quad t^3 = \frac{100\,\pi}{0.05} = 2000\,\pi$$

$$t = 18.453\ \text{s}$$

$$v = 0.15\,t^2 = 0.15\,(18.453)^2 = 51.1\ \text{ft/s}$$

$$a_t = 0.3\,t = 0.3\,(18.453) = 5.536\ \text{ft/s}^2$$

$$a_n = \frac{v^2}{\rho} = \frac{51.1^2}{300} = 8.7\ \text{ft/s}^2$$

$$a = \sqrt{a_t^{\,2} + a_n^{\,2}} = \sqrt{(5.536)^2 + (8.7)^2} = 10.3\ \text{ft/s}^2$$

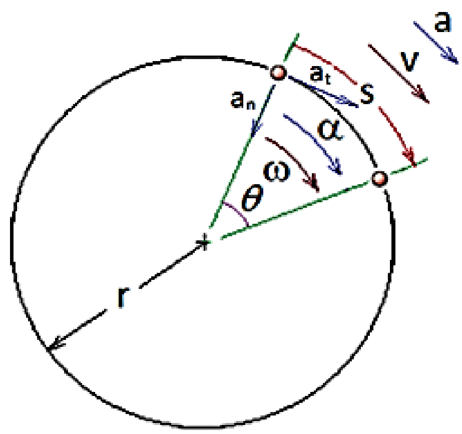
Circular motion:

Circular motion is a special and important case of plane curvilinear motion, where the radius of curvature (ρ) becomes the constant radius of the circle (r).

Components of velocity and acceleration of circular motion become:

(θ): Angular displacement.
(ω): Angular velocity.
(α): Angular acceleration.

$$v = r\,\omega$$
$$a_n = \frac{v^2}{\rho} = r\,\omega^2 = v\,\omega$$
$$a_t = \frac{dv}{dt} = r\,\alpha$$
$$\omega = \frac{d\theta}{dt}$$
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$



Common examples of circular motion:

The flywheel on a stationary machine, the pulleys on axles, the satellites in circular orbits, the car traveling in a circular path of a given radius, the centrifuge.

Angular displacement (θ):

Angular displacement is the central angle formed by the arc of motion, and is measured in the radial unit, and it is measured in the radial unit (radian).

1 radian = 57.3°.

Angular velocity (ω):

Angular velocity is the time rate of change in angular displacement, and it is measured by (rad/s) or (rpm) revolutions per minuet.

Angular acceleration (α):

Angular acceleration is the time rate of change in angular velocity, and it is measured by (rad/s^2).

Equations of circular motion:

The equations describing circular motion are similar to the linear motion equations, but the symbols used are the symbols for circular motion, (θ, ω , α) instead of (s , v , a).

Circular motion	**Linear motion**
$\omega = \frac{d\theta}{dt}$	$v = \frac{ds}{dt}$
$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$	$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
$\omega\, d\omega = \alpha\, d\theta$	$v\, dv = a\, ds$
$\omega = \omega_o + \alpha\, t$	$v = v_o + a\, t$
$\theta = \theta_o + \omega_o\, t + \frac{1}{2}\alpha\, t^2$	$s = s_o + v_o\, t + \frac{1}{2} a\, t^2$
$\omega^2 = \omega_o^2 + 2\,\alpha\,(\theta - \theta_o)$	$v^2 = v_o^2 + 2\,a\,(s - s_o)$
$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$, $\Delta\theta = \theta - \theta_o$, $\Delta t = t - t_o$	$v_{avg} = \frac{\Delta S}{\Delta t}$, $\Delta S = S - S_o$

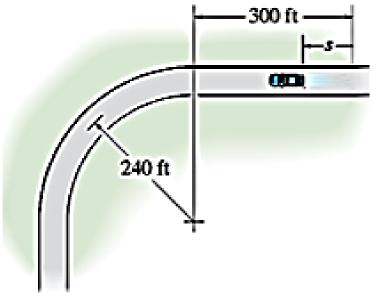
θ : Final angular displacement.
θ_o : Initial angular displacement.
ω : Final angular velocity.
ω_o : Initial angular velocity.
α : Angular acceleration.
ω_{avg} : Average angular velocity.
t : Final time

s : Final linear displacement.
s_o : Initial linear displacement.
v : Final linear velocity.
v_o : Initial linear velocity.
a : Linear acceleration.
v_{avg} : Average linear velocity.
t_o : Initial time

PROBLEMS:

(4 - 1):

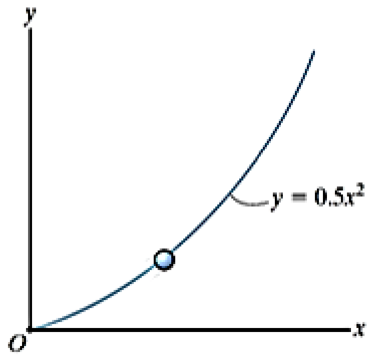
The automobile is originally at rest at (s = 0). If its speed is increased by [($\dot{v} = 0.05\ t^2$) ft/s²], where (t) is in seconds, determine the magnitudes of its velocity and acceleration when (t = 18 s).



Ans. : (v = 97.2 ft/s) , (a = 42.6 ft/s²)

(4 - 2):

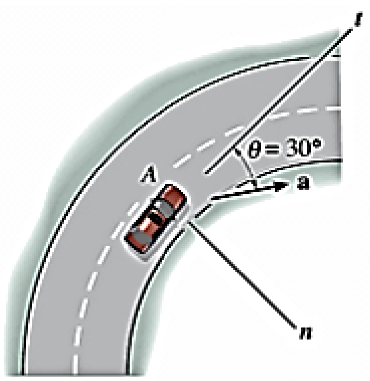
The particle travels along the path defined by the parabola ($y = 0.5\ x^2$). If the component of velocity along the (x - axis) is [v_x = (5t) ft/s], where (t) is in seconds, determine the particle's distance from the origin (*O*) and the magnitude of its acceleration when (t = 1 s). When (t = 0), (x = 0), (y = 0).



Ans. : (d = 4 ft) , (a = 37.8 ft/s²)

(4 - 3):

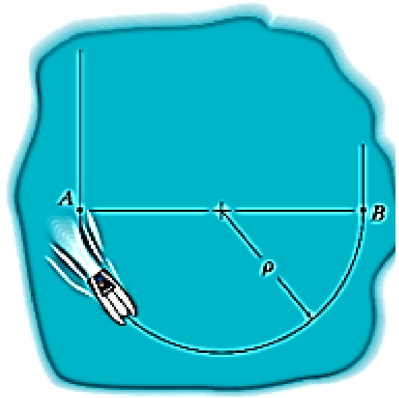
The automobile has a speed of (80 ft/s) at point (A) and an acceleration (a) having a magnitude of (10 ft/s²), acting in the direction shown. Determine the components of acceleration at point (A) and the radius of curvature of the path.



Ans. : (a_t = 8.66 ft/s²) , (a_n = 5 ft/s²) , (ρ = 1280 ft)

(4 - 4):

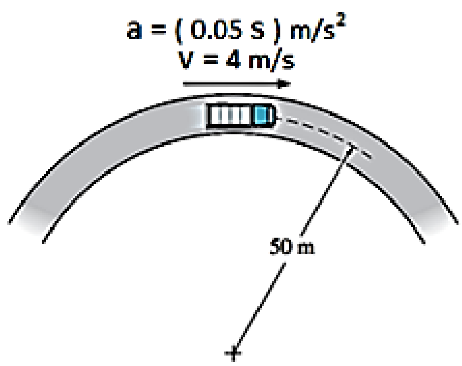
The speedboat travels at a constant speed of (15 m s) while making a turn on a circular curve from (A) to (B). If it takes (45 s) to make the turn, determine the magnitude of the boat's acceleration during the turn.



Ans. : ($a = 1.05\ m/s^2$)

(4 - 5):

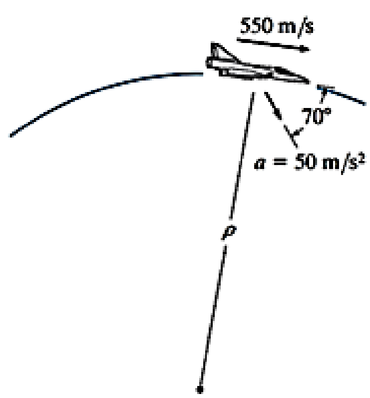
The truck travels in a circular path having a radius of (50 m) at a speed of (v = 4 m/s). For a short distance from (s = 0), its speed is increased by [$a = (0.05\ s)\ m/s^2$) where (s) is in meters. Determine its speed and the magnitude of its acceleration when it has moved (s = 10 m).



Ans. : ($v = 4.58\ m/s$) , ($a = 0.653\ m/s^2$)

(4 - 6):

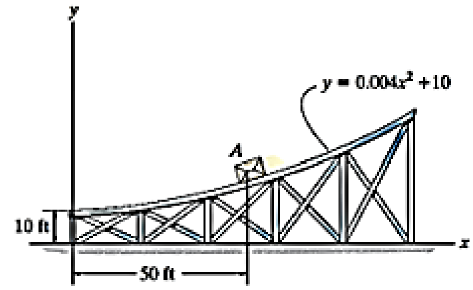
At a given instant the jet plane has a speed of (550 m/s) and an acceleration of ($50\ m/s^2$) acting in the direction shown. Determine the radius of curvature (ρ) of the path.



Ans. : ($\rho = 6.44\ km$)

(4 - 7):

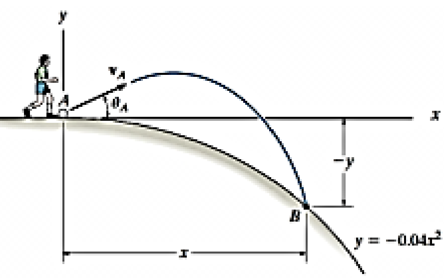
If the speed of the box at point (A) on the track is (30 ft/s) which is increasing at the rate of ($a = 5\ ft/s^2$) , determine the magnitude of the acceleration of the box at this instant.



Ans. : ($a = 7.63\ ft/s^2$)

(4 - 8):

The ball at (A) is kicked with a speed ($v_A = 80$ ft/s) and at an angle ($\theta_A = 30°$). Determine the point (x , –y) where it strikes the ground. Assume the ground has the shape of a parabola as shown.



The point is: (13.3 , − 7.09)

Part 1
Kinematics of a particle

Chapter 5

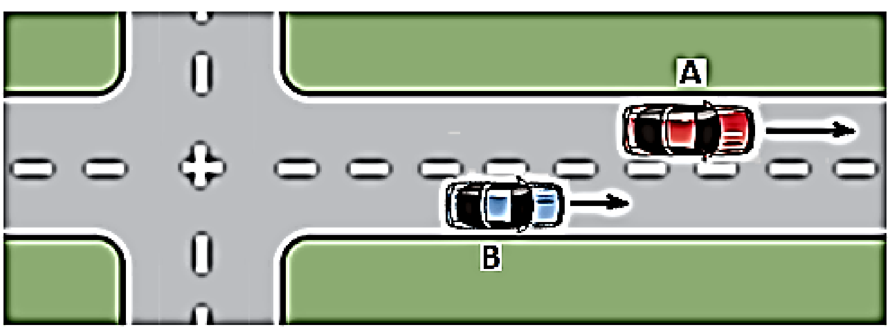
Relative - motion of two particles

Relative - motion of two particles using translating axes

In the previous chapters of this part, we have described particle motion using coordinates referred to a fixed reference axes. The displacements, velocities, and accelerations so determined are termed ***absolute***. It is not always possible or convenient, however, to use fixed set of axes to describe or measure motion. In addition, there are many engineering problems for which the analysis of motion is simplified by using measurements made with respect to moving reference system. These measurements, when combined with the absolute motion of the moving coordinate system, enable us to determine the absolute motion in question. This approach is called a ***relative – motion*** analysis.

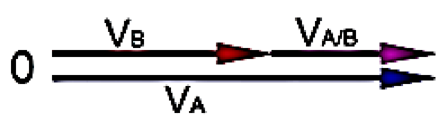
In the same direction:

Fig. 1



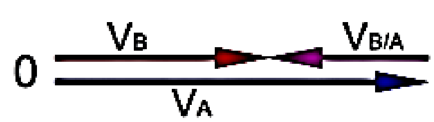
$\mathbf{V}_{A/B} = \mathbf{V}_A - \mathbf{V}_B$, $\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{A/B}$

$\mathbf{V}_{B/A} = \mathbf{V}_B - \mathbf{V}_A$, $\mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{B/A}$



$\mathbf{a}_{A/B} = \mathbf{a}_A - \mathbf{a}_B$, $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{A/B}$

$\mathbf{a}_{B/A} = \mathbf{a}_B - \mathbf{a}_A$, $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$



$\mathbf{v}_A$: *absolute* velocity *of* **A**
$\mathbf{v}_B$: *absolute* velocity *of* **B**
$\mathbf{a}_A$: *absolute* acceleration *of* **A**
$\mathbf{a}_B$: *absolute* acceleration *of* **B**

$\mathbf{v}_{A/B}$: velocity of **A** relative to **B**
$\mathbf{v}_{B/A}$: velocity of **B** relative to **A**
$\mathbf{a}_{A/B}$: acceleration of **A** relative to **B**
$\mathbf{a}_{B/A}$: acceleration of **B** relative to **A**

In opposite direction:

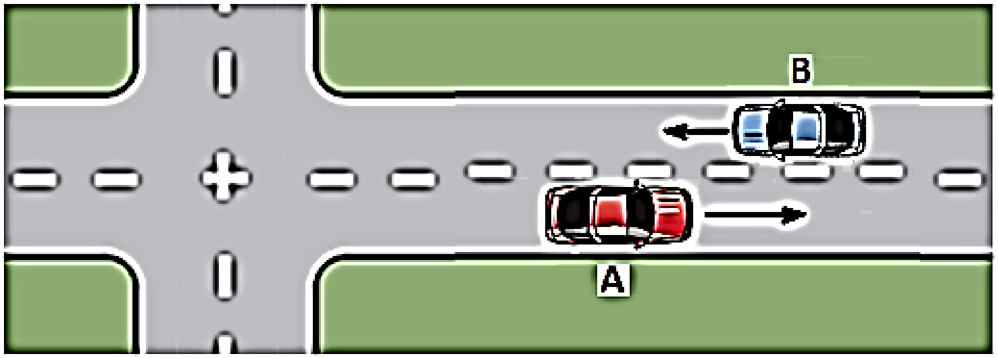


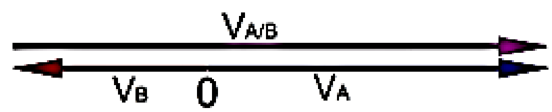
Fig. 2

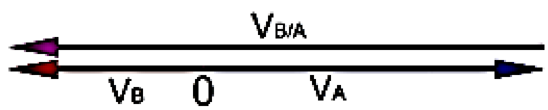
$V_{A/B} = V_A + V_B$

$V_{B/A} = V_B + V_A$

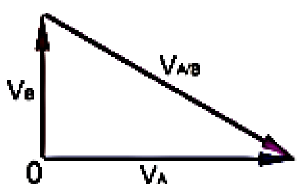
$a_{A/B} = a_A + a_B$

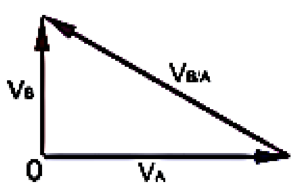
$a_{B/A} = a_B + a_A$





In perpendicular direction:





$V_{A/B} = V_{B/A}$

$= \sqrt{V_A^2 + V_B^2}$

$a_{A/B} = a_{B/A}$

$= \sqrt{a_A^2 + a_B^2}$

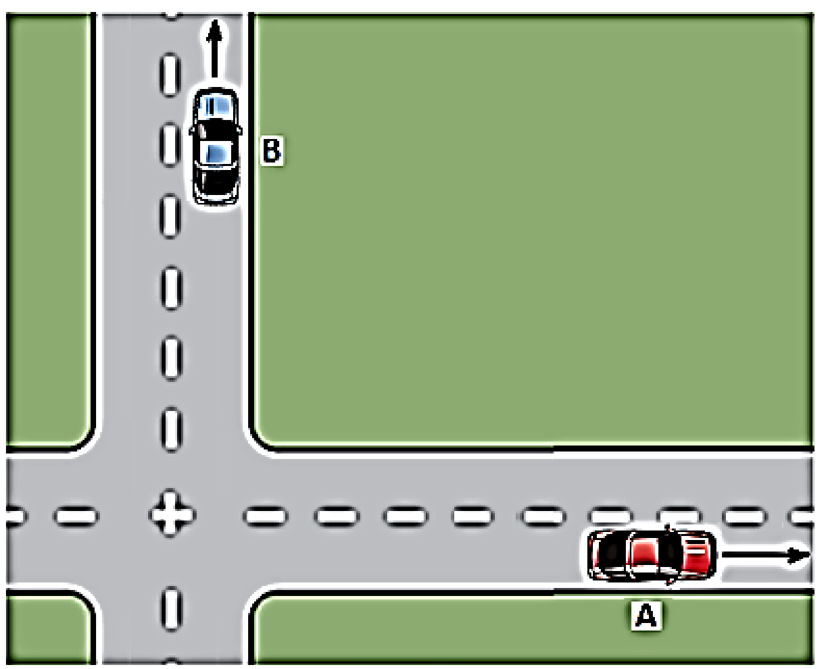
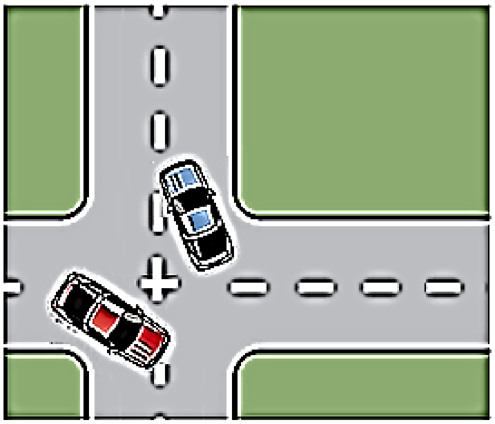
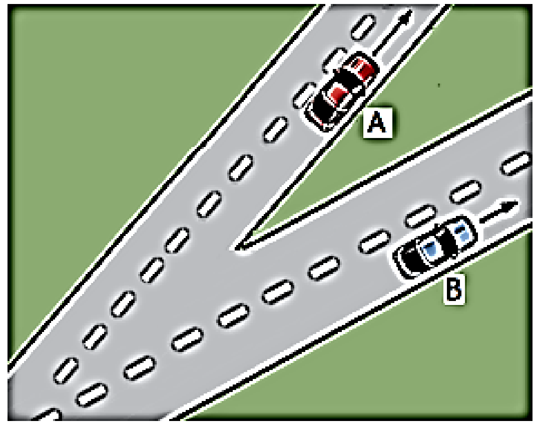


Fig. 3

Vector Representation:





Consider two particles (A) and (B) which may have separate curvilinear motions in a given plane or in parallel planes as in shown figure.

$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$

$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$

$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$

$\mathbf{r}_A$: *absolute position of* **A**

$\mathbf{r}_B$: absolute position of **B**

$\mathbf{r}_{A/B}$: position **A** relative to **B** or **A** with respect to **B**

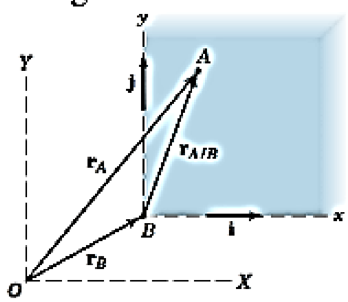


Fig. 4

Position:

Consider particles (A) and (B), which move along the arbitrary paths shown in (Fig. 5). The *absolute position* of each particle, ($\mathbf{r}_A$) and ($\mathbf{r}_B$), is measured from the common origin (*O*) of the fixed (x, y, z) reference frame. The origin of a second frame of reference (x', y', z') is attached to and moves with particle (A).

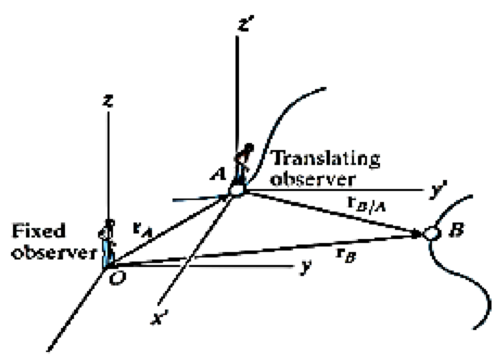


Fig. 5

The axes of this frame are *only permitted to translate* relative to the fixed frame. The position of (B) measured relative to (A) is denoted by the *relative-position vector* ($\mathbf{r}_{B/A}$). Using vector addition, the three vectors shown in (Fig. 5) can be related by the equation.

Velocity:

An equation that relates the velocities of the particles is determined by taking the time derivative of the last equation; i.e.,

$$\mathbf{v_B} = \mathbf{v_A} + \mathbf{v_{B/A}}$$

Here ($\mathbf{v_B} = d\, r_B / dt$) and ($\mathbf{v_A} = d\, r_A / dt$) refer to *absolute velocities*, since they are observed from the fixed frame; whereas the *relative velocity* ($v_{B/A} = d\, r_{B/A} / dt$) is observed from the translating frame. It is important to note that since the x', y', z' axes translate, the *components* of ($r_{B/A}$) will *not* change direction and therefore the time derivative of these components will only have to account for the change in their magnitudes. Therefore states that the velocity of (B) is equal to the velocity of *A* plus (vectorially) the velocity of "*B* with respect to A," as measured by the *translating observer* fixed in the x', y', z' reference frame.

Acceleration.

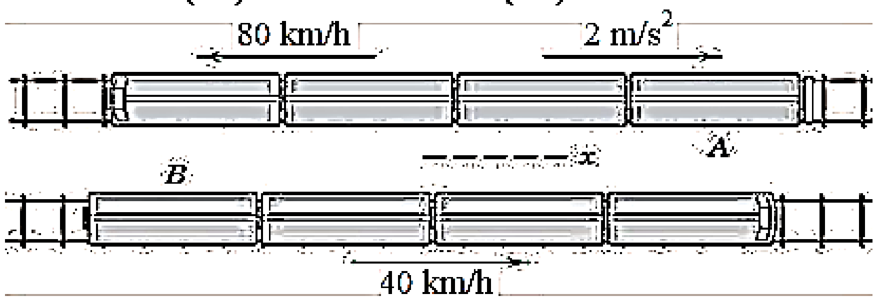
The time derivative of above equation yields a similar vector relation between the *absolute* and *relative accelerations* of particles (A) and (B).

$$a_B = a_A + a_{B/A}$$

Here ($a_{B/A}$) is the acceleration of (B) as seen by the observer located at (A) and translating with the (x', y', z') reference frame.

Example 5 - 1:

Rapid – transit trains (A) and (B) travel on parallel tracks. Train (A) has a speed of (80 km/h) and is slowing at the rate of (2 m/s²), while train (B) has a constant speed of (40 km/h). Determine the velocity and acceleration of train (B) relative to train (A).



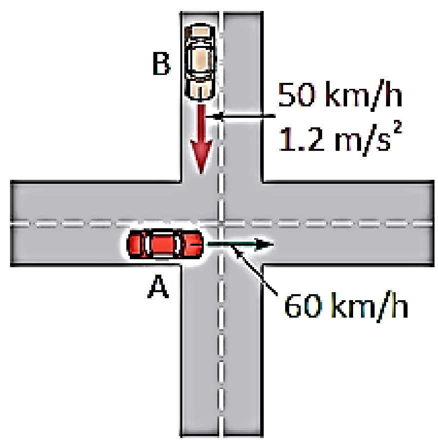
SOLUTION:

$$v_{B/A} = v_B + v_A = 40 + 80 = 120 \text{ km/h}$$

$$a_{B/A} = a_B + a_A = 0 + (-2) = -2 \text{ m/s}^2$$

Example 5 - 2:

If the car (A) is traveling east at a constant velocity of (60 km/h), and the car (B) is traveling south at a velocity of (50 km/h) and acceleration of (1.2 m/s²). Determine the velocity and acceleration of car (B) relative to car (A).



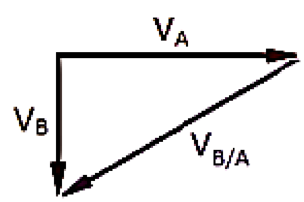
SOLUTION:

$$v_{A/B} = v_{B/A} = \sqrt{v_A^{\,2} + v_B^{\,2}}$$

$$= \sqrt{(60)^2 + (50)^2} = 78.1 \text{ km/h}$$

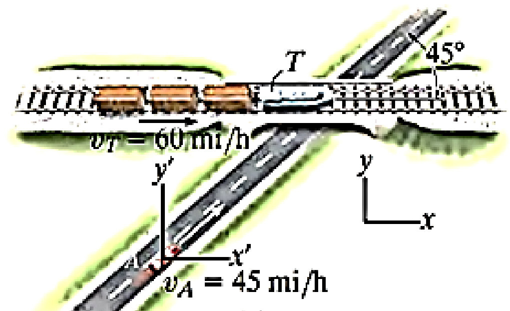
$$a_{A/B} = \sqrt{a_A^{\,2} + a_B^{\,2}}$$

$$= \sqrt{(0)^2 + (1.2)^2} = 1.2 \text{ m/s}^2$$



Example 5 - 3:

train travels at a constant speed of (60 mi/h) and crosses over a road as shown in the figure. If the automobile (A) is traveling at (45 mi/h) along the road, determine the magnitude and direction of the velocity of the train relative to the automobile.



(a)

SOLUTION I:

Vector Analysis. The relative velocity $\mathbf{v}_{T/A}$ is measured from the translating x', y' axes attached to the automobile. It is determined from $\mathbf{v}_T = \mathbf{v}_A + \mathbf{v}_{T/A}$. Since $\mathbf{v}_T$ and $\mathbf{v}_A$ are known in *both* magnitude and direction, the unknowns become the *x* and *y* components of $\mathbf{v}_{T/A}$. Using the *x, y* axes in we have :

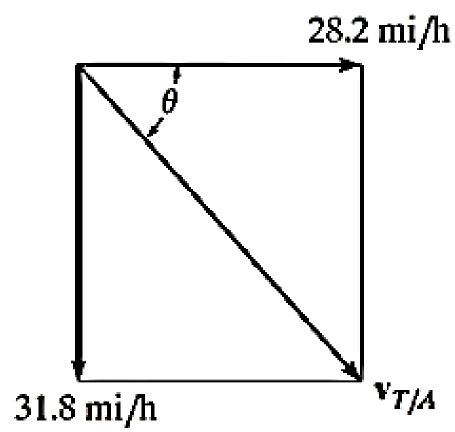
Using the *x, y* axes in we have :

$$v_T = v_A + v_{T/A}$$

$$60i = (45 \cos 45^\circ\, i + 45 \sin 45^\circ\, j) + v_{T/A}$$

$$v_{T/A} = 60i - (45 \cos 45^\circ\, i + 45 \sin 45^\circ\, j)$$

$$v_{T/A} = \{28.2i - 31.8j\} \text{ mi/h}$$



The magnitude of $\mathbf{v}_{T/A}$ is thus

$$v_{T/A} = \sqrt{28.2^2 + 31.8^2} = 42.5 \text{ mi/h}$$

From the direction of each component, the direction of ($\mathbf{v}_{T/A}$) is:

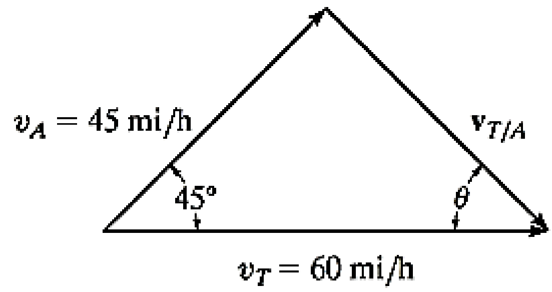
$$\tan\theta = \frac{(v_{T/A})_y}{(v_{T/A})_x} = \frac{31.8}{28.2} = 1.128$$

$$\theta = \tan^{-1} 1.128 = 48.5^\circ \;\measuredangle$$

SOLUTION II:

Scalar Analysis:

The unknown components of ($v_{T/A}$) can also be determined by applying a scalar analysis. We will assume these components act in the positive (x) and (y) directions. Thus,



$$v_T = v_A + v_{T/A}$$

$$\begin{bmatrix} 60\text{ mi/h} \\ \rightarrow \end{bmatrix} = \begin{bmatrix} 45\text{ mi/h} \\ \measuredangle 45° \end{bmatrix} + \begin{bmatrix} (v_{T/A})_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (v_{T/A})_y \\ \uparrow \end{bmatrix}$$

Resolving each vector into its *x* and *y* components yields

$$\overset{+}{\rightarrow} \quad 60 = 45\cos 45° + (v_{T/A})_x + 0$$

$$+\uparrow \quad 0 = 45\sin 45° + 0 + (v_{T/A})_y$$

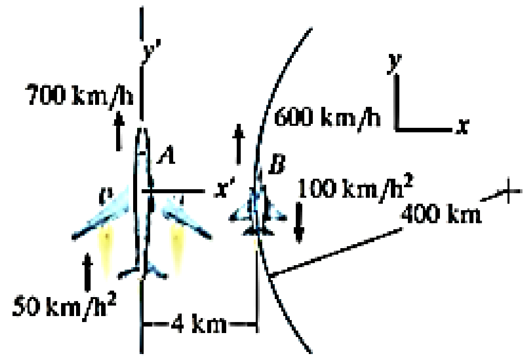
Solving, we obtain the previous results,

$$(v_{T/A})_x = 28.2\text{ mi/h} = 28.2\text{ mi/h} \rightarrow$$

$$(v_{T/A})_y = -31.8\text{ mi/h} = 31.8\text{ mi/h} \downarrow$$

Example 5 - 4:

Plane (A) in the figure is flying along a straight - line path, whereas plane (B) is flying along a circular path having a radius of curvature of (r_B = 400 km). Determine the velocity and acceleration of (B) as measured by the pilot of (A).



SOLUTION:

Velocity:

The origin of the x and y axes are located at an arbitrary fixed point. Since the motion relative to plane A is to be determined, the translating frame of reference x ', y ' is attached to it,

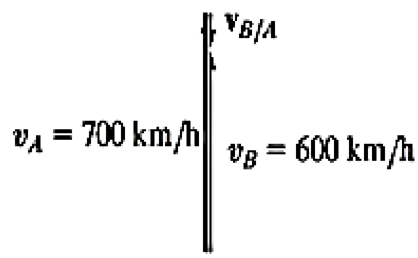
Applying the relative-velocity equation in scalar form since the velocity vectors of both planes are parallel at the instant shown, we have:

$+\uparrow \qquad v_B = v_A + v_{B/A}$

$600 \text{ km/h} = 700 \text{ km/h} + v_{B/A}$

$v_{B/A} = 600 - 700 \quad v_{B/A} = -100 \text{ km/h}$

$= 100 \text{ km/h} \downarrow$



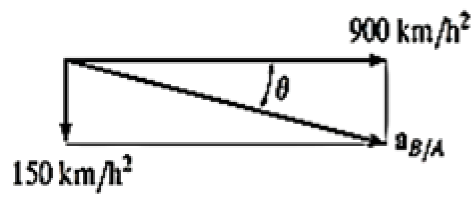
Acceleration. Plane *B* has both tangential and normal components of acceleration since it is flying along a *curved path*. the magnitude of the normal component is:

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(600)^2}{400} = 900 \text{ km/h}^2$$

Applying the relative-acceleration equation gives

$$a_B = a_A + a_{B/A}$$

$$900i - 100j = 50j + a_{B/A}$$



Thus,

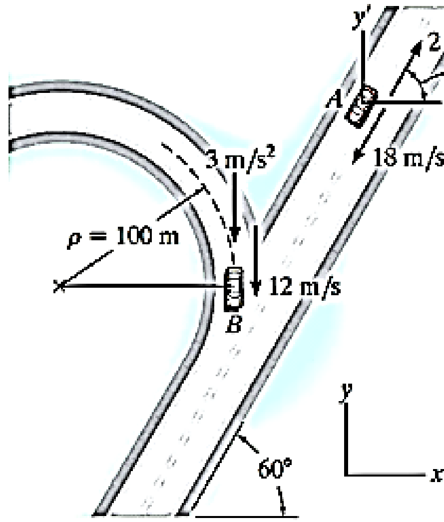
$$a_{B/A} = \{ 900i - 150j \} \text{ km/h}^2$$

$$a_{B/A} = \sqrt{900^2 + (-150)^2} = 912 \text{ km/h}^2$$

$$\theta = \tan^{-1}\frac{150}{900} = 9.46°$$

Example 5 - 5:

At the instant shown in the figure, cars (A) and (B) are traveling with speeds of (18 m/s) and (12 m/s), respectively. Also at this instant, (A) has a decrease in speed of (2 m/s²), and (B) has an increase in speed of (3 m/s²). Determine the velocity and acceleration of *B* with respect to (A).



SOLUTION:

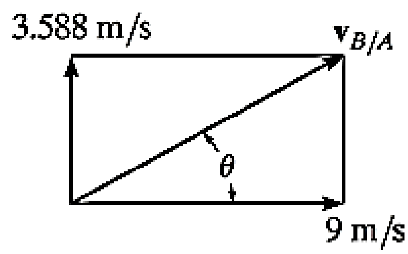
Velocity:

The fixed *x, y* axes are established at an arbitrary point on the ground and the translating *x', y'* axes are attached to car A. Why? The relative velocity is determined from $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$. What are the two unknowns? Using a Cartesian vector analysis, we have

$$v_B = v_A + v_{B/A}$$

$$-12j = (-18\cos 60^\circ\, i - 18\sin 60^\circ\, j) + v_{B/A}$$

$$v_{B/A} = 9i + 3.588j \text{ m/s}$$



Thus, $\quad v_{B/A} = \sqrt{9^2 + 3.588^2} = 9.69 \text{ m/s}$

Noting that $v_{B/A}$ has +i and +j components, its direction is:

$$\theta = \tan^{-1}\frac{3.588}{9} = 21.7^\circ$$

Acceleration:

Car *B* has both tangential and normal components of acceleration. Why? The magnitude of the normal component is:

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(12)^2}{100} = 1.44 \text{ m/s}^2$$

Applying the equation for relative acceleration yields

$$a_B = a_A + a_{B/A}$$

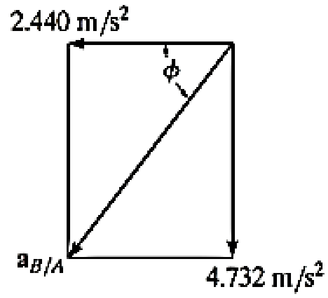
$$(-1.440i - 3j) = (2\cos 60^\circ\, i + 2\sin 60^\circ\, j) + a_{B/A}$$

$a_{B/A} = \{ -2.440\mathbf{i} - 4.732\mathbf{j} \}\ \text{m/s}^2$

Here $a_{B/A}$ has $-\mathbf{i}$ and $-\mathbf{j}$ components. Thus,

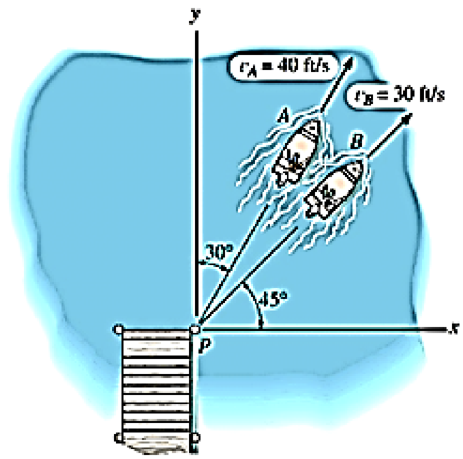
$$a_{B/A} = \sqrt{2.44^2 + 4.732^2} = 5.32\ \text{m/s}^2$$

$$\phi = \tan^{-1}\frac{4.732}{2.44} = 62.7^\circ$$



Example 5 - 6:

Two boats leave the pier (P) at the same time and travel in the directions shown. If (v_A = 40 ft/s) and (v_B = 30 ft/s), determine the velocity of boat (A) relative to boat (B). How long after leaving the pier will the boats be (1500 ft) apart ?



SOLUTION:

Relative Velocity:

$$v_A = v_B + v_{A/B}$$

$$40 \sin 30^\circ\mathbf{i} + 40 \cos 30^\circ\mathbf{j} = 30 \cos 45^\circ\mathbf{i} + 30 \sin 45^\circ\mathbf{j} + v_{A/B}$$

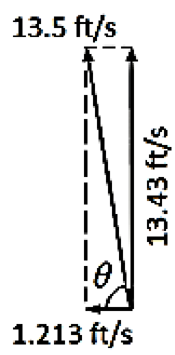
$$v_{A/B} = \{ -1.213\mathbf{i} + 13.43\mathbf{j} \}\ \text{ft/s}$$

Thus, the magnitude of the relative velocity $v_{A/B}$ is:

$$v_{A/B} = \sqrt{(1.213)^2 + (13.43)^2}$$
$$= 13.48\ \text{ft/s} = 13.5\ \text{ft/s}$$

And its direction is:

$$\theta = \tan^{-1}\frac{13.43}{1.213} = 84.8^\circ$$

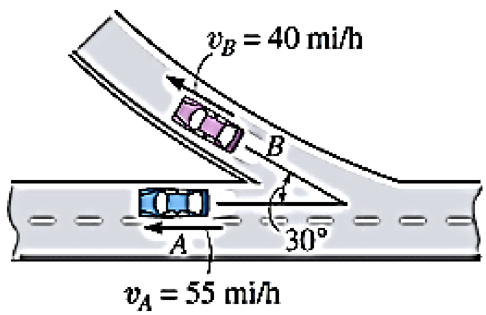


One can obtained the time t required for boats (A) and (B) to be (1500 ft) apart by noting that boat (B) is at rest and boat (A) travels at the relative speed ($v_{A/B}$ = 13.48 ft/s) for a distance of (1500 ft).Thus :

$$t = \frac{1500}{v_{A/B}} = \frac{1500}{13.48} = 111.26\ \text{s} = 1.85\ \text{min}$$

Example 5 - 7:

At the instant shown, cars (A) and (B) are traveling at speeds of (55 mi/h) and (40 mi/h), respectively. If (B) is increasing its speed by (1200 mi/h^2), while (A) maintains a constant speed, determine the velocity and acceleration of (B) with respect to (A). Car (B) moves along a curve having a radius of curvature of (0.5 mi).



SOLUTION:

$$v_B = -40\cos 30^\circ\, i + 40\sin 30^\circ\, j = \{-34.64\, i + 20\, j\}\ \text{mi/h}$$

$$v_A = \{-55\, i\}\ \text{mi/h}$$

$$v_{B/A} = v_B - v_A = \{-34.64\, i + 20\, j\} - \{-55\, i\}$$

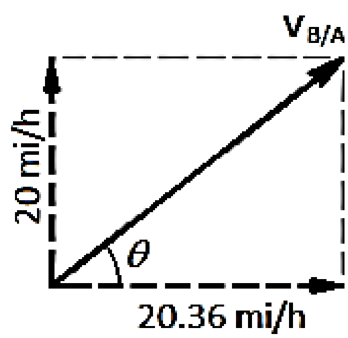
$$= \{20.36\, i + 20\, j\}\ \text{mi/h}$$

$$v_{B/A} = \sqrt{(20.36)^2 + (20)^2} = 28.5\ \text{mi/h}$$

$$\theta = \tan^{-1}\frac{20}{20.36} = 44.5^\circ$$

$$(a_B)_n = \frac{v_B{}^2}{\rho} = \frac{(40)^2}{0.5} = 3200\ \text{mi/h}^2$$

$$(a_B)_t = 1200\ \text{mi/h}^2$$



$$a_B = (3200\cos 60^\circ - 1200\cos 30^\circ)\, i + (3200\sin 60^\circ + 1200\sin 30^\circ)\, j$$

$$= \{560.77\, i + 3371.28\, j\}\ \text{mi/h}^2$$

$$a_A = 0$$

$$a_{B/A} = a_B - a_A = \{560.77\, i + 3371.28\, j\} - 0$$

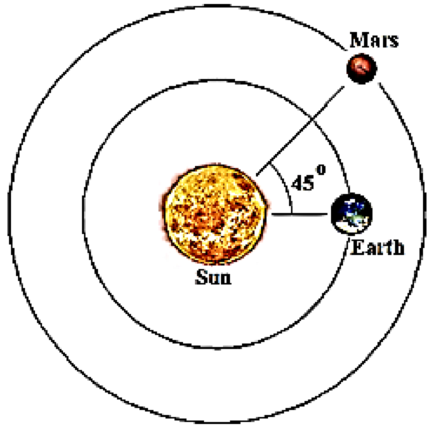
$$= \{560.77\, i + 3371.28\, j\}\ \text{mi/h}^2$$

$$a_{B/A} = \sqrt{(560.77)^2 + (3371.28)^2} = 3418\ \text{mi/h}^2$$

$$\theta = \tan^{-1}\frac{3371.28}{560.77} = 80.6^\circ$$

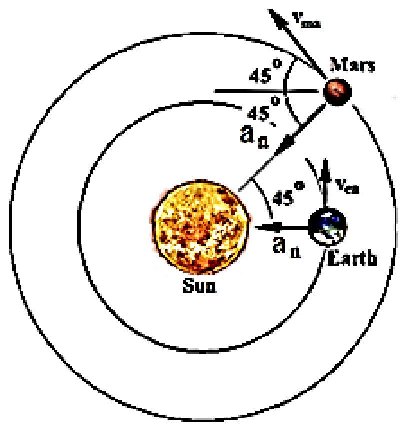
Example 5 - 8:

The positions of the Earth and Mars in their orbits around the sun are shown in the figure, where the Earth rotates around the sun at constant speed (30 km/s) and is far from the center of the sun (150 Gm), and Mars rotates around the sun at constant speed (24 km/s) and far from the center of the sun (228 Gm). Determine the magnitude and direction of the velocity and acceleration of Mars relative to the Earth.



SOLUTION:

Relative Velocity:



$$v_{ma} = v_{ea} + v_{ma/ea}$$

$$-24 \cos 45° \, i + 24 \sin 45° \, j = 30 \, j + v_{ma/ea}$$

$$v_{ma/ea} = \{ 16.97 \, i - 13.03 \, j \} \text{ km/s}$$

$$v_{ma/ea} = \sqrt{(16.97)^2 + (-13.03)^2}$$

$$= 21.4 \text{ km/s}^2$$

$$\theta_v = \tan^{-1} \frac{13.03}{16.97} = 37.5°$$

Relative acceleration:

- Earth acceleration:

$$a_t = 0 \qquad a = a_n = \frac{v_{ea}{}^2}{\rho} = \frac{(30\times10^3)^2}{150\times10^9} = 6 \times 10^{-3} \text{ m/s}^2$$

- Mars acceleration:

$$a_t = 0 \qquad a = a_n = \frac{v_{ma}{}^2}{\rho} = \frac{(24\times10^3)^2}{228\times10^9} = 2.53 \times 10^{-3} \text{ m/s}^2$$

$$a_{ma} = a_{ea} + a_{ma/ea}$$

$$-2.53 \times 10^{-3} \cos 45° \, i - 2.53 \times 10^{-3} \sin 45° \, j = -6 \times 10^{-3} \, i + a_{ma/ea}$$

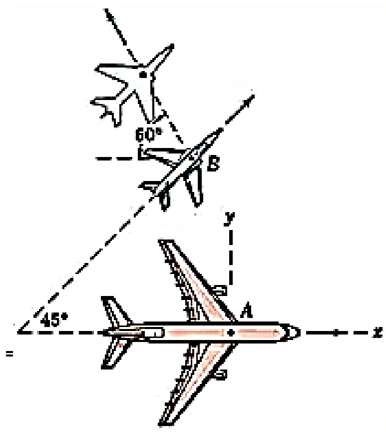
$$v_{ma/ea} = \{ 3.47 \times 10^{-3} \, i - 2.53 \times 10^{-3} \, j \} \text{ m/s}^2$$

$$v_{ma/ea} = \sqrt{(3.47 \times 10^{-3})^2 + (-2.53 \times 10^{-3})^2} = 4.29 \times 10^{-3} \text{ m/s}^2$$

$$\theta_a = \tan^{-1} \frac{2.53\times10^{-3}}{3.47\times10^{-3}} = 36.1°$$

Example 5 - 9:

Passenger in the jet transport (A) flying east at a speed of (800 km/h) observe a second jet plane (B) that passes under the transport in horizontal flight. Although the nose of (B) is pointed in the (45°) northeast direction, plane (B) appears to the passengers in (A) to be moving away from the transport at the (60°) angle as shown. Determine the true velocity of (B).



SOLUTION:

$$v_B = v_A + v_{B/A}$$

Vector Algebra:

$$v_A = 800\ i\ \text{km/h} \qquad v_B = \{ (v_B \cos 45°)\ i + (v_B \sin 45°)\ j \}\ \text{km/h}$$

$$v_{B/A} = \{ (- v_{B/A} \cos 60°)\ i + (v_{B/A} \sin 60°)\ j \}\ \text{km/h}$$

$$\{(v_B \cos 45°)\ i + (v_B \sin 45°)\ j\} = 800\ i + \{(- v_{B/A} \cos 60°)\ i + (v_{B/A} \sin 60°)\ j\}$$

i – terms, $v_B \cos 45° = 800 - v_{B/A} \cos 60°$

$0.707\ v_B = 800 - 0.5\ v_{B/A}$ .... (1)

j – terms, $v_B \sin 45° = v_{B/A} \sin 60°$

$0.707\ v_B = 0.866\ v_{B/A}$ .... (2)

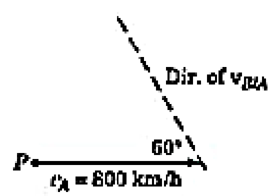
From (2) $v_B = \dfrac{0.866}{0.707} v_{B/A} = 1.225\ v_{B/A}$

Sub. in (1) $0.707\ (1.225\ v_{B/A}) = 800 - 0.5\ v_{B/A}$

$0.866\ v_{B/A} = 800 - 0.5\ v_{B/A}$

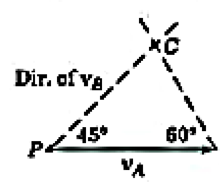
$1.366\ v_{B/A} = 800 \Rightarrow v_{B/A} = 585.65$ km/h

$v_B = 1.225\ (585.65) = 717.4$ km/h



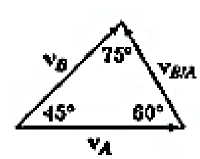
Graphical: $v_{B/A} = 585.65$ km/h

$v_B = 1.225\ (585.65) = 717.4$ km/h



Trigonometric: $\dfrac{v_{B/A}}{\sin 45°} = \dfrac{v_A}{\sin 75°}$

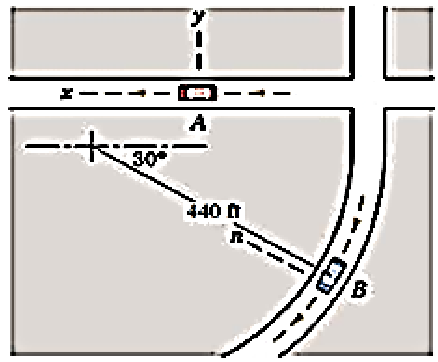
$v_{B/A} = 800 \dfrac{\sin 45°}{\sin 75°} = 585.65$ km/h



$$\frac{v_B}{\sin 60°} = \frac{v_A}{\sin 75°} \Rightarrow v_B = 800 \frac{\sin 60°}{\sin 75°} = 717.4\ \text{km/h}$$

Example 5 - 10:

Car (A) is accelerated in the direction of its motion at the rate of (3 ft/s²). Car (B) is rounding a curve of (440 ft) radius at a constant speed of (30 mi/h). Determine the velocity and acceleration which car (B) appears to have to an observer in car (A) if car (A) has reached a speed of (45 mi/h) for the position represented.



SOLUTION:

Velocity:

$v_B = v_A + v_{B/A}$

$$v_A = 45\,\frac{5280}{3600} = 66 \text{ ft/s}$$

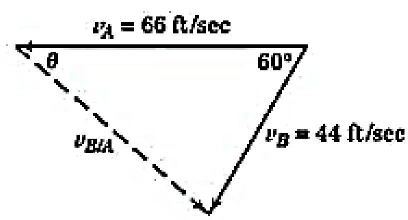
$$v_B = 30\,\frac{5280}{3600} = 44 \text{ ft/s}$$

$$v_{B/A} = \sqrt{v_A^2 + v_B^2 - 2(v_A v_B)\cos 60}$$

$$= \sqrt{(66)^2 + (44)^2 - 2(66)(44)\cos 60} = 58.2 \text{ ft/s}$$

$$\frac{v_{B/A}}{\sin 60^\circ} = \frac{v_B}{\sin\theta} \Rightarrow \frac{58.2}{\sin 60^\circ} = \frac{44}{\sin\theta}$$

$$\sin\theta = \frac{44 \sin 60^\circ}{58.2} \Rightarrow \theta = 40.9^\circ$$



Acceleration:

$a_B = a_A + a_{B/A}$

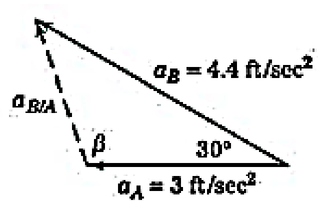
$v_B = \text{constant} \Rightarrow (a_B)_t = 0$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(44)^2}{440} = 4.4 \text{ ft/s}^2 = a_B$$

$$(a_{B/A})_x = 4.4 \cos 30^\circ - 3 = 0.81 \text{ ft/s}^2$$

$$(a_{B/A})_y = 4.4 \sin 30^\circ = 2.2 \text{ ft/s}^2$$

$$a_{B/A} = \sqrt{(0.81)^2 + (2.2)^2} = 2.34 \text{ ft/s}^2$$

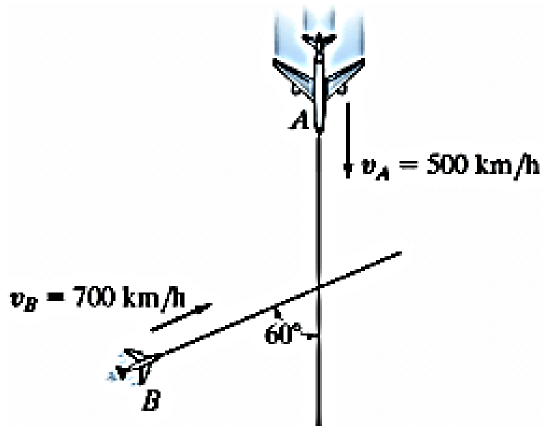


$$\frac{4.4}{\sin\beta} = \frac{2.34}{\sin 30^\circ} \qquad \sin\beta = \frac{4.4 \sin 30^\circ}{2.34} \Rightarrow \beta = 110.2^\circ$$

PROBLEMS:

(5 - 1):

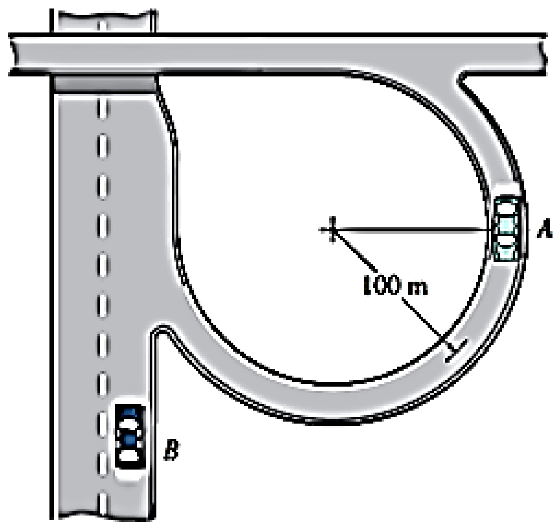
Two planes, (A) and (B), are flying at the same altitude. If their velocities are (v_A = 500 km/h) and (v_B = 700 km/h) such that the angle between their straight-line courses is (θ = 60°), determine the velocity of plane (B) with respect to plane (A).



Ans. : ($v_{B/A}$ = 1044 km/h) , (θ = 54.5°)

(5 - 2):

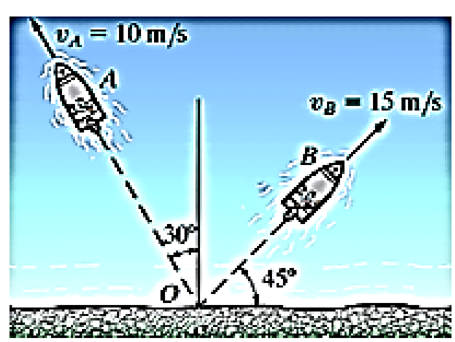
At the instant shown, car (A) has a speed of (20 km/h), which is being increased at the rate of (300 km/h^2) as the car enters an expressway. At the same instant, car (B) is decelerating at (250 km/h^2) while traveling forward at (100 km/h). Determine the velocity and acceleration of (A) with respect to (B).



Ans. : ($v_{A/B}$ = 120 km/h)
($a_{A/B}$ = 4000 km/h^2)

(5 - 3):

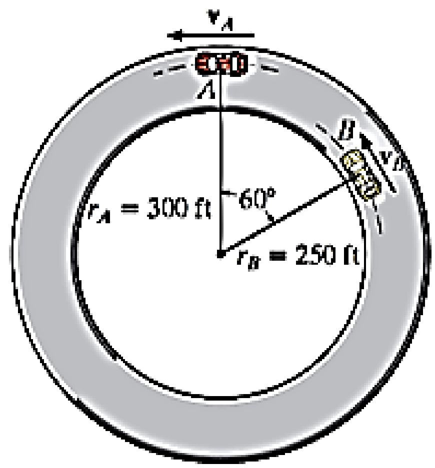
Two boats leave the shore at the same time and travel in the directions shown. If (v_A = 10 m/s) and (v_B = 15 m/s), determine the velocity of boat (A) with respect to boat (B). How long after leaving the shore will the boats be (600 m) apart?



Ans. : ($v_{A/B}$ = 15.7 m/s) , (t = 38.15 s)

(5 - 4):

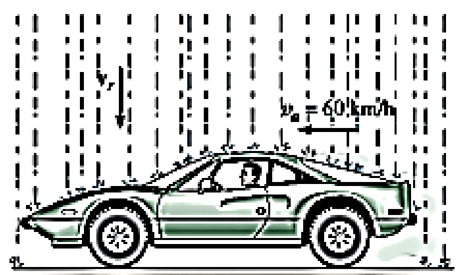
Cars (A) and (B) are traveling around the circular race track. At the instant shown, (A) has a speed of (90 ft/s) and is increasing its speed at the rate of (15 ft/s^2), whereas (B) has a speed of (105 ft/s) and is decreasing its speed at (25 ft/s^2).
Determine the relative velocity and relative acceleration of car (A) with respect to car (B) at this instant.



Ans. : ($v_{A/B}$ = 98.4 ft/s)
($a_{A/B}$ = 19.8 m/s^2)

(5 - 5):

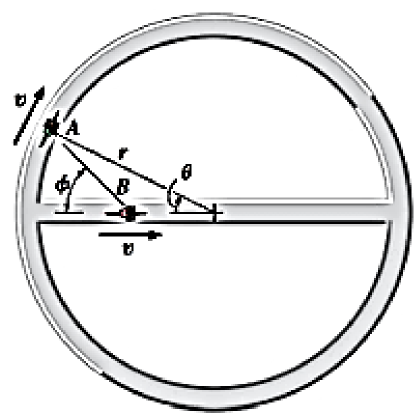
A passenger in an automobile observes that raindrops make an angle of (30°) with the horizontal as the auto travels forward with a speed of (60 km/h). Compute the terminal (constant) velocity (v_r) of the rain if it is assumed to fall vertically.



Ans. : (v_r = 34.6 km/h)

(5 - 6):

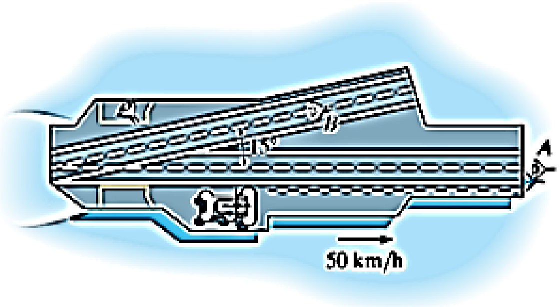
The two cyclists and travel at the same constant speed (v). Determine the speed of (A) with respect to (B) if (A) travels along the circular track, while (B) travels along the diameter of the circle.



Ans. : ($v_{A/B} = v\sqrt{2(1 - sin\theta)}$)

(5 - 7):

An aircraft carrier is traveling forward with a velocity of (50 km/h). At the instant shown, the plane at (A) has just taken off and has attained a forward horizontal air speed of (200 km/h), measured from still water.

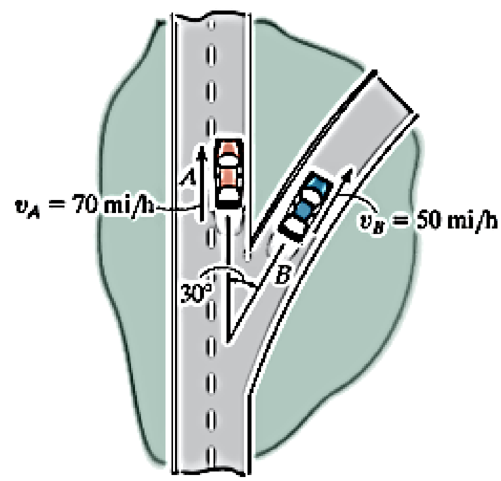


If the plane at (B) is traveling along the runway of the carrier at (175 km/h) in the direction shown, determine the velocity of (A) with respect to (B).

Ans. : ($v_{A/B} = 49.1$ km/h) , ($\theta = 67.2°$)

(5 - 8):

At the instant shown, cars (A) and (B) travel at speeds of (70 mi/h) and (50 mi/h), respectively. If (B) is increasing its speed by (1100 mi/h^2), while (A) maintains a constant speed, determine the velocity and acceleration of (B) with respect to (A). Car (B) moves along a curve having a radius of curvature of (0.7 mi).



Ans. : ($v_{B/A} = 36.6$ mi/h)

($a_{B/A} = 3737$ mi/h^2)

Part 2 - Kinetics
Chapter 1
Newton's Second Law
The Equation of Motion

- Kinetics is the study of the relations between the unbalanced forces and the changes in motion that they produce.
- Newton's 2^{nd} law states that the particle will accelerate when it is subjected to unbalanced forces. The acceleration of the particle is always in the direction of the applied forces.
- Newton's 2^{nd} law is also known as the equation of motion.
- To solve the equation of motion, the choice of an appropriate coordinate systems depends on the type of motion involved.
- Two types of problems are encountered when applying this equation:
  - The acceleration of the particle is either specified or can be determined directly from known kinematic conditions. Then, the corresponding forces, which are acting on the particle, will be determined by direct substitution.
  - The forces acting on the particle are specified, then the resulting motion will be determined. Note that, if the forces are constant, the acceleration is also constant and is easily found from the equation of motion. However, if the forces are functions of time, position, or velocity, the equation of motion becomes a differential equation which must be integrated to determine the velocity and displacement.
- In general, there are three general approaches to solve the equation of motion: the direct application of Newton's 2^{nd} law, the use of the work & energy principles, and the impulse and momentum method.

Free-body Diag — Kinetic Diagram

Note: The equation of motion has to be applied in such way that the measurements of acceleration are made from a Newtonian or inertial frame of reference. This coordinate does not rotate and is either fixed or translates in a given direction with a constant velocity (zero acceleration).

Rectilinear Motion ------------------------------------

$$\begin{cases} \sum F_x = m\, a_x \\ \sum F_y = 0 \end{cases}$$

Curvilinear Motion

Rectangular Coordinates

$$\begin{cases} \sum F_x = m\, a_x \\ \sum F_y = m\, a_y \\ a = a_x\, i + a_y\, j \\ |a| = \sqrt{a_x^2 + a_y^2} \\ \sum F = \sum F_x\, i + \sum F_y\, j \\ |\sum F| = \sqrt{\sum F_x^2 + \sum F_y^2} \end{cases}$$

(n - t) Coordinates

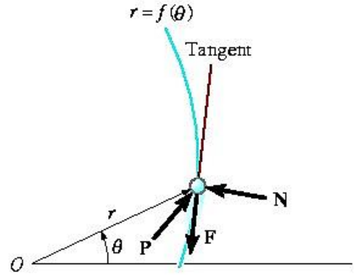
$$\begin{cases} \sum F_t = m\, a_t \\ \sum F_n = m\, a_n \\ a = a_t\, e_t + a_n\, e_n \\ |a| = \sqrt{a_t^2 + a_n^2} \\ \sum F = \sum F_t\, e_t + \sum F_n\, e_n \\ |\sum F| = \sqrt{\sum F_t^2 + \sum F_n^2} \end{cases}$$

Polar Coordinates

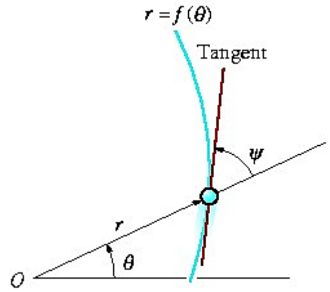
$$\begin{cases} \sum F_r = m\, a_r \\ \sum F_\theta = m\, a_\theta \\ a = a_r\, e_r + a_\theta\, e_\theta \\ |a| = \sqrt{a_r^2 + a_\theta^2} \\ \sum F = \sum F_r\, e_r + \sum F_\theta\, e_\theta \\ |\sum F| = \sqrt{\sum F_r^2 + \sum F_\theta^2} \end{cases}$$

In Polar Coordinates

- Consider the force (**P**) that causes the particle to move along a path (r = f(θ)).

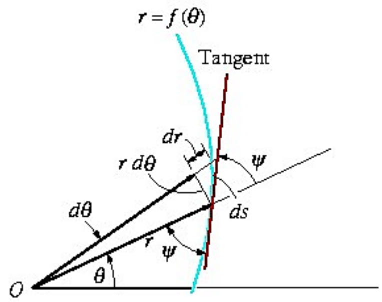


- The *normal force* (**N**) which the path exerts on the particle is always *perpendicular to the tangent of the path.*
- Frictional force (**F**) always acts along the tangent in the opposite direction of motion.

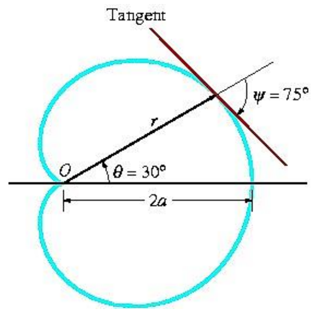


- The *directions* of (**N)** and (**F)** can be specified relative to the radial coordinate by using the angle (ψ), which is defined between the *extended* radial line and the tangent to the curve.

$$\tan \psi = \frac{r}{dr/d\theta}$$



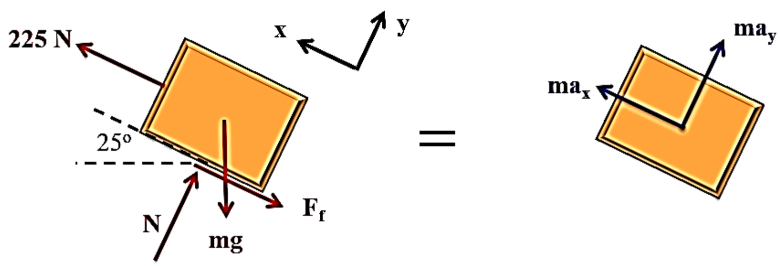
- If (ψ) is positive, it is measured from the extended radial line to the tangent in a (CCW) sense or in the positive direction (θ).
- If it is negative, it is measured in the opposite direction to positive (θ).



Free Body Diagrams and Kinetic Diagrams

The free body diagram is the same as you have done in statics; we will add the kinetic diagram in our dynamic analysis.

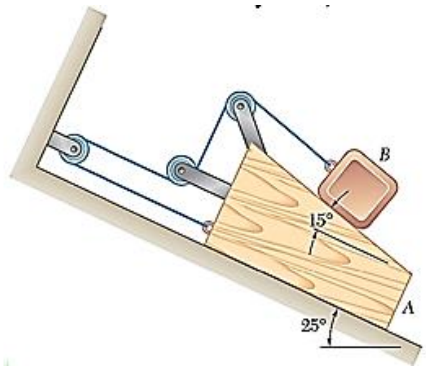
1. Isolate the body of interest (free body).
2. Draw your axis system (e.g., Cartesian, polar, path).
3. Add in applied forces (e.g., weight, [225 N] pulling force).
4. Replace supports with forces (e.g., normal force).
5. Draw appropriate dimensions (usually angles for particles).



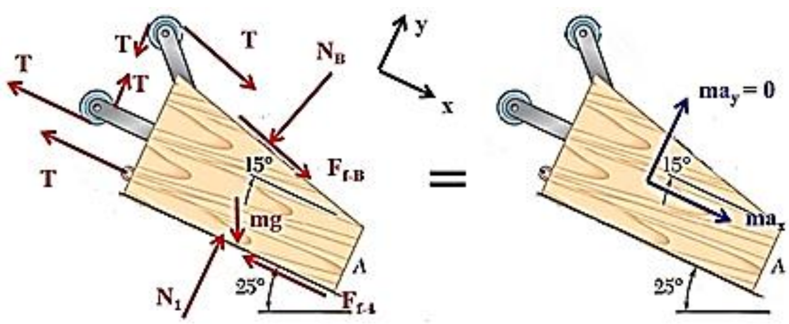
$$\sum \mathbf{F} \quad = \quad \mathbf{m\,a}$$

Example 2 - 1 - 1:

Draw the (FBD) and (KD) for block (A) [note that the massless, frictionless pulleys are attached to block (A) and should be included in the system].



1. Isolate body.
2. Axes.
3. Applied forces.
4. Replace supports with forces.
5. Dimensions (already drawn).
6. Kinetic diagram.



Example 2 - 1 - 2:

A body at rest of mass (10 kg) is exerted by a force of (20 N) for a period of (5 s). find:

a – Resulted acceleration.

b – Velocity after (5 s).

SOLUTION:

$$\sum F = m\,a$$
$$20 = 10\,a$$
$$a = 2\ m/s^2$$

$$v = v_o + a\,t$$
$$= 0 + (2)(5)$$
$$= 10\ m/s$$

Example 2 - 1 - 3:

Calculate the mass of a body on which a force of magnitude (20 N) was applied, which caused it to accelerate at (5 m/s^2).

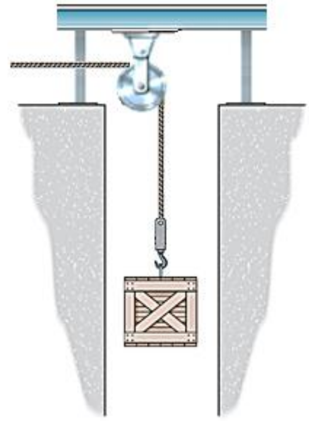
SOLUTION:

$$\sum F = m\,a$$
$$20 = 5\,m$$
$$m = \frac{20}{5} = 4\ kg$$

<u>Example 2 - 1 - 4:</u>

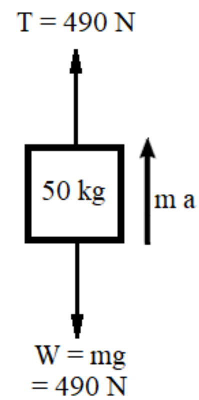
A box of mass (50 kg) suspended at the end of a rope. Find the acceleration of the box if the tension in the rope is:

a - (490 Newtons).
b - (240 Newtons).
c - (890 Newtons).



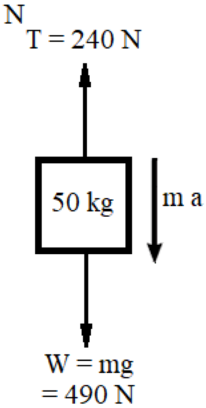
SOLUTION:

$$W = mg = 50 \times 9.8 = 490 \text{ N}$$
$$\sum F = T - W$$



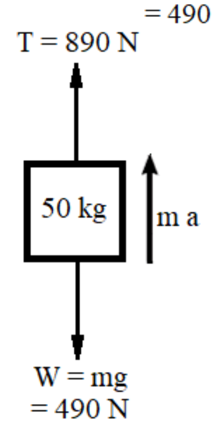
a- $W = T = 490$ N

$$\sum F = T - W = 490 - 490 = 0$$
$$\sum F = m\,a$$
$$0 = 50\,a$$
$$a = 0 \qquad \text{(Static state)}$$



b- $W = 490$ N , $T = 240$ N

$$\sum F = T - W = 240 - 490 = -250 \text{ N} = 250 \text{ N} \downarrow$$
$$\sum F = m\,a$$
$$250 = 50\,a$$
$$a = \frac{250}{50} = 5 \text{ m/s}^2 \downarrow$$

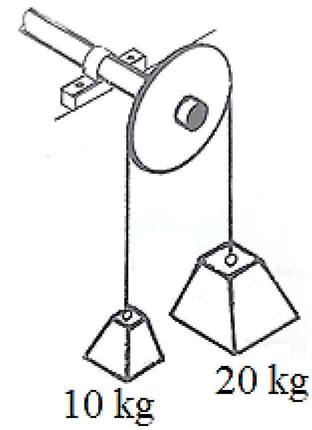


c- $W = 490$ N , $T = 890$ N

$$\sum F = T - W = 890 - 490 = 400 \text{ N}$$
$$\sum F = m\,a$$
$$400 = 50\,a$$
$$a = \frac{400}{50} = 8 \text{ m/s}^2 \uparrow$$

<u>Example 2 - 1 - 5:</u>

In the figure, the rope passes on a completely smooth pulley and is suspended to one end (10 kg) and to the other end (20 kg), calculate the approximate acceleration of the system and the tension in the rope.



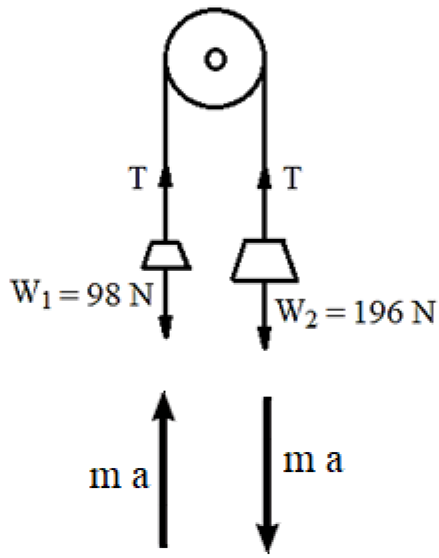
SOLUTION:

$$W_1 = 10 \times 9.8 = 98 \text{ N}$$
$$W_2 = 20 \times 9.8 = 196 \text{ N}$$

Lift side:

$$\sum F = T - W_1 = m\,a$$
$$T - 98 = 10\,a$$
$$T = 10\,a + 98 \quad \ldots\ldots.. (1)$$

Right side:



$$\sum F = W_2 - T = m\,a$$
$$196 - T = 20\,a$$
$$T = 196 - 20\,a \quad \ldots\ldots.. (2)$$

$$10\,a + 98 = 196 - 20\,a$$
$$30\,a = 98$$
$$a = 3.27 \text{ m/s}^2$$

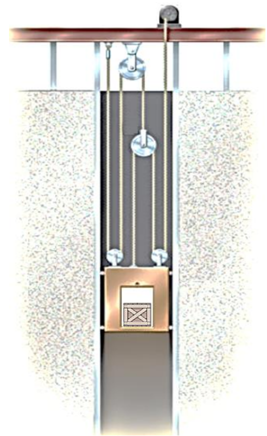
Sub. in Eq. (1):

$$T = 10\,a + 98 = 10\,(3.27) + 98$$
$$= 130.7 \text{ N}$$

<u>Example 2 - 1 - 6:</u>

A box of mass (30 kg) placed on the floor of an elevator. Calculate the force that the box exerts on the elevator floor in the following cases:

- **a- When the elevator moves up at a constant velocity.**
- **b - When the elevator moves up with an acceleration of ($1.2\ m/s^2$).**
- **c - When the elevator moves down with an acceleration of ($1.2\ m/s^2$).**



SOLUTION:

$$W = m\,g = 30 \times 9.8 = 294\ N$$

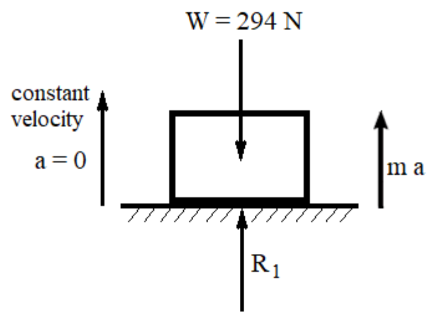
$$\sum F = m\,a$$

a- $a = 0$ (constant velocity)

$R_1 - W = m\,a$

$R_1 - 294 = 0$

$R_1 = 294\ N$

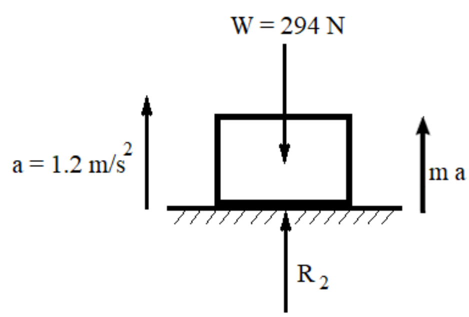


b- $a = 1.2\ m/s^2$

$R_2 - W = m\,a$

$R_2 - 294 = 30 \times 1.2$

$R_2 = 36 + 294 = 330\ N$

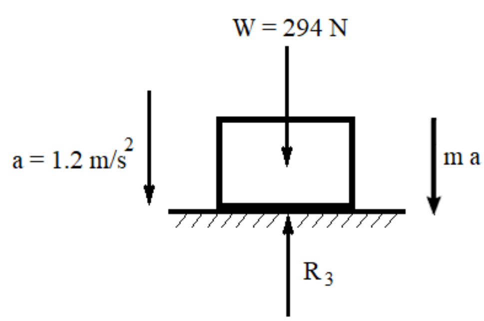


c- $a = 1.2\ m/s^2$

$R_3 - W = -(m\,a)$

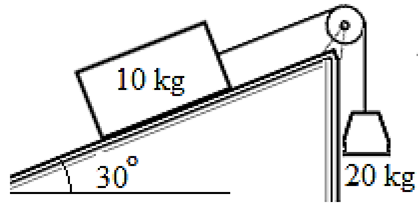
$R_3 - 294 = -(30 \times 1.2)$

$R_3 = -36 + 294 = 258\ N$



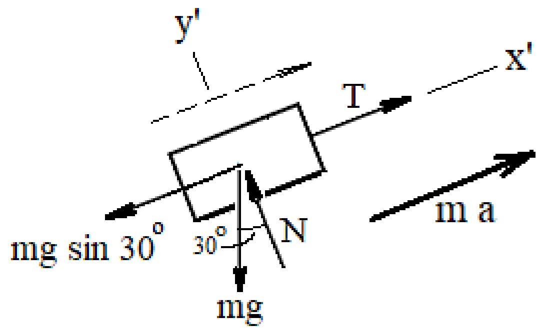
Example 2 - 1 - 7:

In the figure, the weights (10 kg) and (20 kg) are tied with a rope that passes the bug of the pulley. Ignore the friction. Find the tension force in the rope and the acceleration of the system.



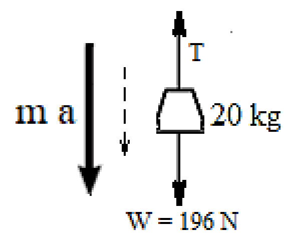
SOLUTION:

For (10 kg) mass:



$$\sum F_{x'} = m\,a$$
$$T - mg \sin 30° = m\,a$$
$$T - (10)(9.8) \sin 30° = 10\,a$$
$$T - 49 = 10\,a$$
$$T = 10\,a + 49 \quad \ldots\ldots\ldots\ldots \quad (1)$$

For (20 kg) mass:



$$W = mg = (20)(9.8) = 196\ N$$
$$\sum F_y = m\,a$$
$$196 - T = 20\,a$$
$$T = 196 - 20\,a \quad \ldots\ldots\ldots\ldots \quad (2)$$

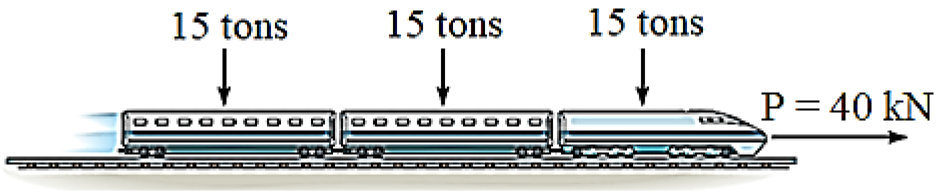
$$196 - 20\,a = 10\,a + 49$$
$$196 - 49 = 10\,a + 20\,a$$
$$147 = 30\,a$$
$$a = \frac{147}{30} = 4.9\ m/s^2$$

Sub. in Eq. (1):

$$T = 10\,a + 49 = 10(4.9) + 49 = 98\ N$$

Example 2 - 1 - 8:

A train consisting of three cars, each car weighing (15 tons). The first cart works as a machine and exerts a pulling force of (40 kN) on the rail. The frictional resistance of each cart on the rail is (1 kN). Find the acceleration of the train and the tension in the connections between the carts.



SOLUTION:

$$\sum F_x = m\,a$$

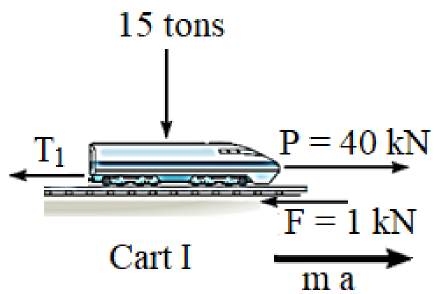
Cart I: $P - F - T1 = m\,a$

$40000 - 1000 - T_1 = 15000\,a$

$39000 - T_1 = 15000\,a$ .... (1)

Cart II: $T_1 - 1000 - T_2 = 15000\,a$ .... (2)

Cart III: $T_2 - 1000 = 15000\,a$ .... (3)



From Eq. (3) $T_2 = 15000\,a + 1000$

Sub. in Eq. (2)

$T_1 - 1000 - (15000\,a + 1000) = 15000\,a$

$T_1 - 2000 - 15000\,a = 15000\,a$

$T_1 = 30000\,a + 2000$

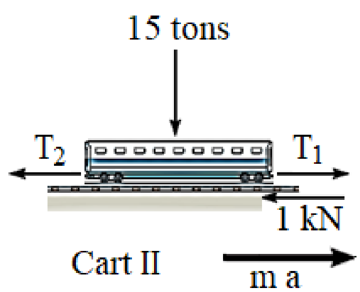
Sub. in Eq. (1)

$39000 - (30000\,a + 2000) = 15000\,a$

$39000 - 30000\,a - 2000 = 15000\,a$

$37000 = 45000\,a$

$$a = \frac{37000}{45000} = 0.822 \text{ m/s}^2$$

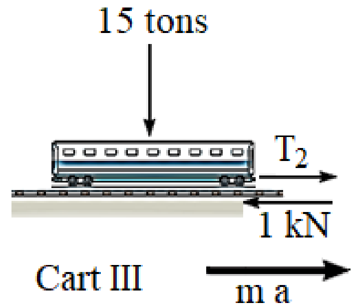


$T_1 = 30000 (0.822) + 2000$

$= 26666 \text{ N} = 26.67 \text{ kN}$

$T_2 = 15000 (0.822) + 1000$

$= 13333 \text{ N} = 13.33 \text{ kN}$



<u>Example 2 - 1 - 9:</u>

A man of mass (75 kg) is lifted by a rope hanging from a helicopter that is above him. If the mass of the hook is (10 kg), find the tension in the rope when the man is lifting:

a - at a constant velocity.

b - with a constant acceleration of (0.5 m/s^2).

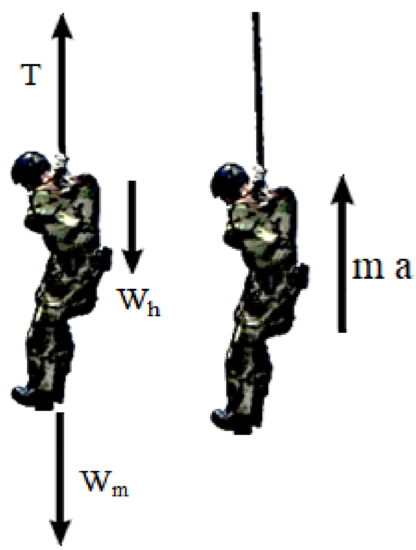
Ignore the rope mass.

SOLUTION:

$W_m = 75 \times 9.8 = 735$ N
$W_h = 10 \times 9.8 = 98$ N

a- at a constant velocity, $a = 0$

$\sum F_y = m\,a$
$T - W_m - W_h = m\,a$
$T - 735 - 98 = 0$
$T = 833$ N



b- at ($a = 0.5$ m/s^2)

$\sum F_y = m\,a$
$T - W_m - W_h = m\,a$
$T - 735 - 98 = (75 + 10) (0.5)$
$T = 42.5 + 735 + 98 = 875.5$ N

Example 2 - 1 - 10:

A machine of mass (4 tons) was lifted vertically to the top by a chain, a distance of (3 m) during (4 s) of rest. Assuming the acceleration is constant, find the tension in the chain.

SOLUTION:

$s = s_o + v_o + 0.5\ a\ t^2$

$3 = 0 + 0 + 0.5\ a\ (4)^2$

$3 = 8\ a \quad \Rightarrow \quad a = \frac{3}{8} = 0.375\ m/s^2$

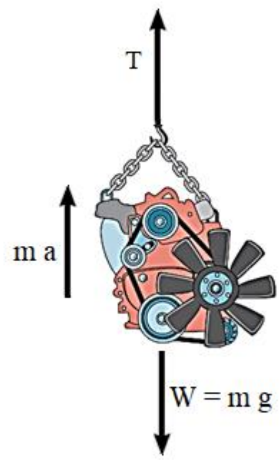
$W = 4000 \times 9.8 = 39200\ N$

$\sum F_y = m\ a$

$T - W = m\ a$

$T - 39200 = 4000 \times 0.375$

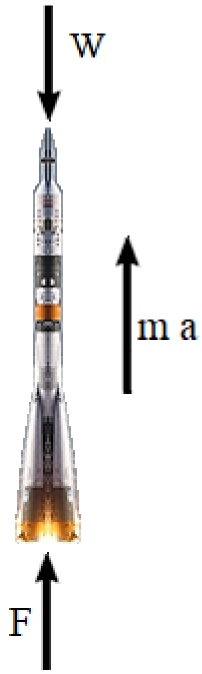
$T = 1500 + 39200 = 40700\ N = 40.7\ kN$



<u>Example 2 - 1 - 11:</u>

During the test of a vertically upward rocket, the thrust of the engine was (5 kN) for a period of (15 s). If you know that the minimum vertical acceleration of the rocket on which the engine will be installed is (65 m/s^2), what is the largest mass of the rocket? neglect the engine mass.

SOLUTION:



$$\sum F_y = m\,a$$
$$F - W = m\,a$$
$$F - m\,g = m\,a$$
$$F = m\,g + m\,a = m\,(g + a)$$
$$5000 = m\,(9.8 + 65)$$
$$5000 = 74.8\,m$$
$$m = \frac{5000}{74.8} = 66.8\ \text{kg}$$

PROBLEMS:

(2 - 1 - 1):

A person whose mass is (80 kg) is located in a building at a height of (10 m) above the ground. The person was forced to go down on a rope with a maximum tension force of (650 N). Calculate the minimum acceleration he must have on the rope so that it does not break.

$a = 1.7\ m/s^2 \downarrow$

(2 - 1 - 2):

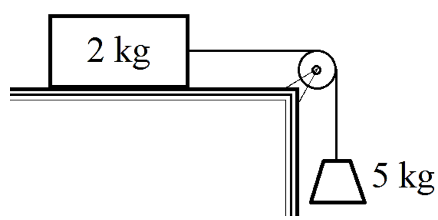
A body traveling towards the east at a speed of (30 m/s), was subjected to the effect of a force of (250 N) towards the north for a period of (2 s). If the mass of the body is (50 kg), what is its final velocity?

$v = 31.62\ m/s$

$\theta = 18.435°$

(2 - 1 - 3):

In the system of weights shown in the figure, the horizontal surface is smooth (frictionless), and the friction in the pulley is negligible. Find acceleration of the system and tension in the rope.



$a = 7\ m/s^2$ $T = 14\ N$

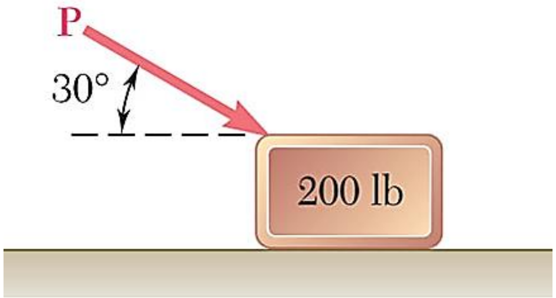
(2 - 1 - 4):

An air-to-air missile with a mass of (500 kg) was launched horizontally from a fighter aircraft. If the acceleration of the missile is (90 m/s^2) relative to the aircraft, what is the thrust of the missile engine?

F = 45 kN

(2 - 1 - 5):

A (200 lb) block rests on a horizontal plane. Find the magnitude of the force (P) required to give the block an acceleration of (10 ft/s^2) to the right. Neglect the friction between the block and plane.



P = 71.7 lb

(2 - 1 - 6):

A car of mass (1 ton) is traveling at a speed of (30 km/h) in a straight line. Calculate the magnitude of resistance that must be applied by the brakes to stop the car within (75 m).

F = 6 kN ←

(2 - 1 - 7):

A spacecraft landed on the surface of the moon with a vertical deceleration of (1 m/s^2). If the mass of the spacecraft is (13×10^3 kg) and the gravitational acceleration on the moon is (1.67 m/s^2). Find the thrust for the landing machine during this stage.

T = 8.7 kN

CHAPTER TWO
WORK AND ENERGY AND POWER PRINCIPLES

In previous chapter, Newton's 2nd law ($\sum F = ma$) was applied to various problems of particle motion to establish the instantaneous relationship between the net force acting on a particles or bodies and the resulting acceleration of this particle or bodies. To get the velocity and displacement, the appropriate kinematics equations may be applied.

There are two general classes of problems in which the cumulative effects of the unbalanced forces acting on a particle are of interest:

- Integration of the forces with the respect to the displacement of the particle. This leads to the equations of work and energy.
- Integration of the forces with the respect to the time. This leads to the equations of impulse and momentum.

Incorporation of the results of these integrations directly into the governing equations of motion makes it unnecessary to solve directly for the acceleration.

Conservation of Energy:

A conservative force does work that is independent of its path. Two examples are the weight of a particle and the spring force.

Friction is a nonconservative force since the work depends upon the length of the path. The longer the path, the more work done.

Mechanical energy consists of kinetic energy (KE) and gravitational and elastic potential energies (PE). According to the conservation of energy, this sum is constant and has the same value at any position on the path. If only gravitational and spring forces cause motion of the particle, then the conservation of energy equation can be used to solve problems involving these conservative forces, displacement, and velocity.

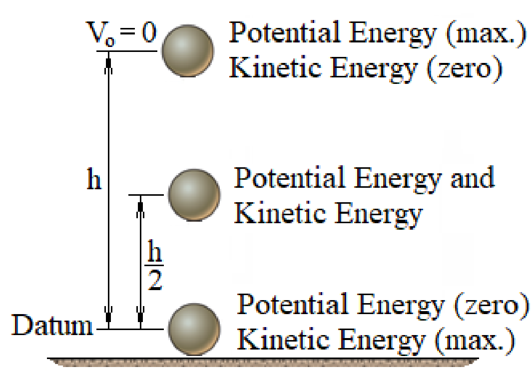


Fig. (2-2-1) Conservation of Energy

$$KE_1 + PE_1 = KE_2 + PE_2 \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}1)$$

The Principle of Work, Energy and power:

Work and kinetic energy are closely related concepts in physics and engineering, often described within the framework of the Work-Energy theorem.

Work:

Work is defined as the process of transferring energy to or from an object by means of a force acting over a distance.

In physics, work is defined as the product of the force applied to an object and the displacement of the object in the direction of the force.

The work done by a force is given by:

$U = (F \cos\theta).d$

$U = F \cdot d \cdot \cos\theta$ ……… (2-2-2)

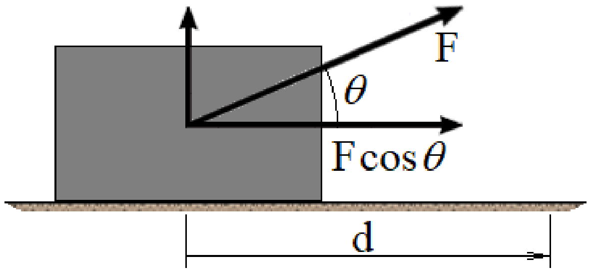


Fig. (2-2-2) The work

Where:

U : is the work done, (measured in joules, J).
F : is the magnitude of the force applied, (measured in newtons, N).
d : is the displacement of the object in the direction of the force, (measured in meters, m).
θ : is the angle between the applied force and the direction of displacement.

A force does work when it undergoes a displacement along its line of action. If the force varies with the displacement, then the work is:

$U = \int (F \cos\theta)\, dS$ ………… (2-2-3)

Graphically, this represents the area under the (F - S) diagram.

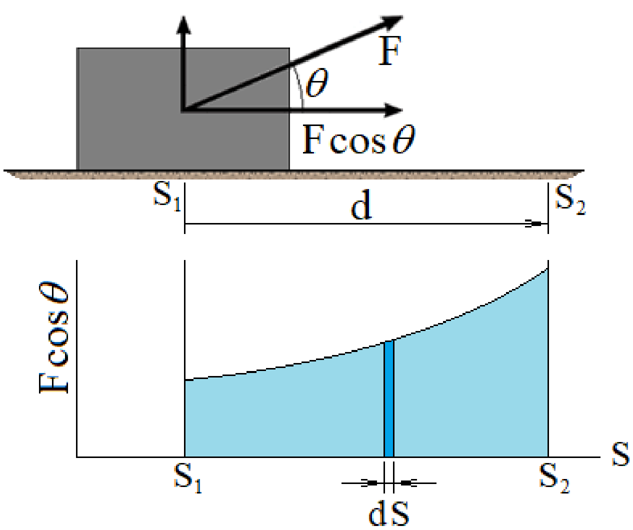


Fig. (2-2-3) Graphical representation of the work

Key Points about Work:

- Work is scalar quantity.
- Work in (SI unit) is Joule (J); where: 1 J = 1 N.m
- Work is done only when the force has a component in the direction of the displacement. If the force is perpendicular to the displacement ($\theta = 90°$), no work is done.
- Work can be positive or negative depending on the direction of the force relative to the displacement. Positive work adds energy to the object, while negative work removes energy.
- During a finite movement of the point of application of a force, the force does an amount work equal to:

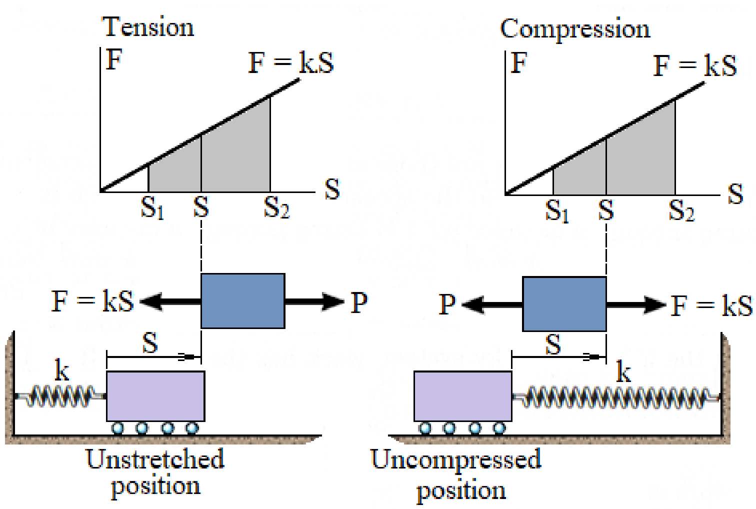


Fig. (2-2-4) The work in states of spring

$$U = \int F\, dS$$

$$U_{i\text{-}f} = \int_{x_i}^{x_f} F\, dS = \int_{x_i}^{x_f} (kS)\, dS = \frac{1}{2} k \left(S_f^{\,2} - S_i^{\,2} \right) \quad \ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}4)$$

$U_{i\text{-}f}$: is the area under the curve (the shaded area).
is the integration the force equation relative to distance.

Note:
The spring force is (+ ve) if the spring in tension.
The spring force is (– ve) if the spring in compression.

Example (2–2–1):

A block of (10 kg) mass is pulled (3 m) along a horizontal surface by a force of (50 N) as shown in Fig. (Ex. 2-2-1). Determine the work done by the force.

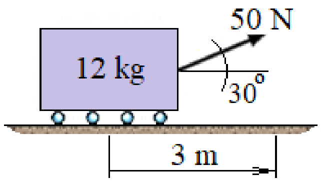


Fig. (Ex. 2-2-1)

Solution:

$$U = F \cdot d \cdot \cos\theta = 50 \times 3 \times \cos 30 = 129.9 \text{ J}$$

Example (2–2–2):

A block of (15 kg) mass is pulled (4 m) along a rough horizontal surface by a force of (60 N) as shown in Fig. (Ex. 2-2-2). Determine the work done by the force. The coefficient of kinetic friction between the contacting surfaces is ($\mu_k = 0.3$).

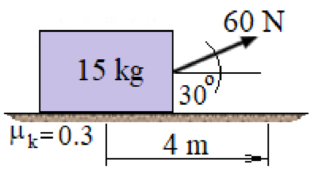
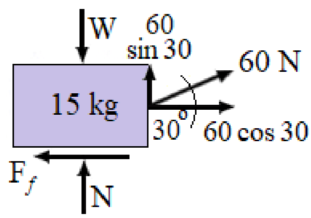


Fig. (Ex. 2-2-2)

Solution:



$$W = mg = 15 \times 9.81 = 147.15 \text{ N}$$
$$N = W - F_y = 147.15 - 60 \sin 30 = 117.15 \text{ N}$$
$$F_f = \mu_k N = 0.3 \times 117.15 = 35.145 \text{ N}$$

Work due to the load:

$U_F = F_x \,.\, d = 60 \cos 30 \times 4 = 207.846$ J (positive)

Work due to the friction:

$U_f = F_f \,.\, d = 35.145 \times 4 = 140.58$ J (negative)

Total work:

$$U_T = U_F - U_f = 207.846 - 140.58 = 67.266 \text{ J}$$

Example (2–2–3):

A store filled with water is pulled to the top from the bottom of a well using a motor and a rope at a constant speed of (2 m/s) for (10 s) as shown in Fig. (Ex. 2-2-3). If the mass of the store and water is (10 kg), Determine the work done by the motor.

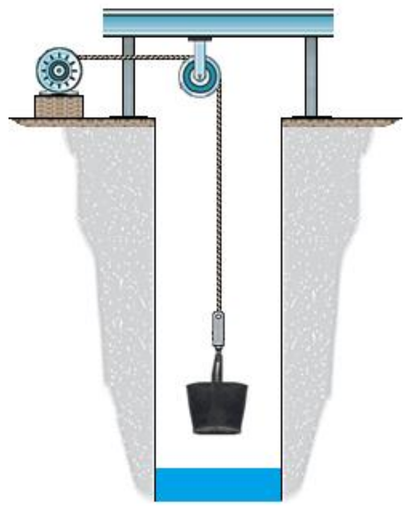


Fig. (Ex. 2-2-3)

Solution:

$$W = mg = 10 \times 9.81 = 98.1\ N$$

$$S = S_o + V_o t + \frac{1}{2} a t^2$$

$$S = 0 + (2 \times 10) = 20\ m = h$$

$$T = W = 98.1\ N$$

Work due to the rope tension:

$$U_T = F.T = 98.1 \times 20 = 1962\ J$$
$$= 1.962\ kJ \qquad (\text{positive})$$

Work due to the weight:

$$U_W = F.W = 98.1 \times 20 = 1962\ J = 1.962\ kJ \qquad (\text{negative})$$

Total work:

Kinetic Energy:

Kinetic energy is the energy possessed by an object due to its motion. It is a fundamental concept in physics that relates to the energy of an object in motion. It is given by the equation:

$$KE = \frac{1}{2} mV^2 \quad \ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}5)$$

Where:

KE : is the kinetic energy, (measured in joules, J).

m : is the mass of the object, (measured in kilograms, kg).

V : is the velocity of the object, (measured in meters per second, m/s).

Key Points about Kinetic Energy:

- Kinetic energy is a scalar quantity, meaning it has magnitude but no direction.
- It depends on both the mass of the object and the square of its velocity.
- Doubling the velocity of an object will quadruple its kinetic energy.

The Work-Energy Theorem

The Work-Energy Theorem states that the net work done on an object is equal to the change in its kinetic energy:

$$U_{i\text{-}f} = \int_1^2 F.dr = \int_1^2 ma \,.\, dr = \int_1^2 ma_t \,.\, ds$$

But: $$a_t = \frac{dV}{dt} = \frac{dV}{dS}\frac{dS}{dt} = V\frac{dV}{ds}$$

$$a_t \, dS = V \, dV$$

$$U_{i\text{-}f} = \int_{V_1}^{V_2} mV \,.\, dV = \frac{1}{2} m \left(V_2^2 - V_1^2 \right)$$

$$U_{i\text{-}f} = KE_2 - KE_1 = \Delta \, KE$$

or $$KE_1 + U_{i\text{-}f} = KE_2 \quad \ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}6)$$

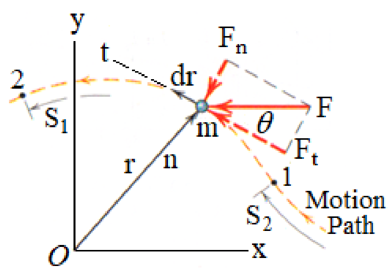


Fig. (2-2-5) Work-Energy relationship

This theorem connects the concepts of work and kinetic energy and can be used to solve a variety of problems where forces cause changes in the motion of objects.

Practical Applications:

- Braking Systems: In vehicles, the work done by friction forces (brakes) reduces the kinetic energy of the vehicle, bringing it to a stop.
- Sports and Athletics: Understanding how much work is required to accelerate a sprinter or how the kinetic energy of a thrown ball changes during flight.
- Machinery: Calculating the energy required to move parts of a machine or how much energy is transferred when a machine part is set into motion.

<u>Example (2–2–4):</u>

The car shown in Fig. (Ex. 2-2-4) with a mass of (1.2 tons) is traveling at a speed of (108 km/h). Determine its kinetic energy.

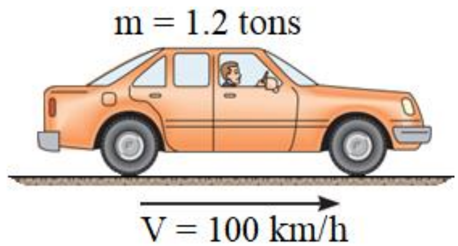


Fig. (Ex. 2-2-4)

Solution:

$$V = \frac{108}{3.6} = 30 \text{ m/s}$$

$$KE = \frac{1}{2} mV^2 = \frac{1}{2} (1200) (30)^2 = 54000 \text{ J} = 540 \text{ kJ}$$

<u>Example (2–2–5):</u>

A (10 kg) block is sliding down an inclined surface at a speed of (20 m/s), as shown in Fig. (Ex. 2-2-5). Determine its kinetic energy.

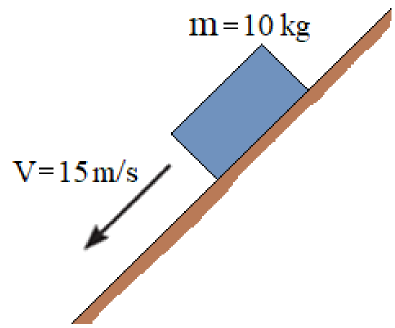


Fig. (Ex. 2-2-5)

Solution:

$$KE = \frac{1}{2} mV^2 = \frac{1}{2} (10) (20)^2 = 2000 \text{ J} = 2 \text{ kJ}$$

<u>Example (2–2–6):</u>

The motorcycle shown in Fig. (Ex. 2-2-6) is traveling at a speed of (72 km/h). If it has a kinetic energy of (20 kJ), determine the combined mass of the motorcycle and its rider.

Fig. (Ex. 2-2-6)

Solution:

$$V = \frac{72}{3.6} = 20 \text{ m/s}$$

$$KE = \frac{1}{2} mV^2$$

$$20000 = \frac{1}{2} \times m \times (20)^2$$

$$200 \text{ m} = 20000 \quad \Rightarrow \quad m = 100 \text{ kg}$$

Potential Energy:

Potential energy is a fundamental concept in physics that relates to the stored energy an object possesses due to its position or condition. It is one of the two main types of mechanical energy, potential energy is always associated with forces that can move an object or system into a more stable or lower energy state.

Potential energy is the energy stored in an object due to its position, condition, or configuration. It has the potential to be converted into kinetic energy (the energy of motion) or other forms of energy. The most common types of potential energy include:

- **Gravitational Potential Energy:**

  Energy stored in an object because of its height above the ground. The higher the object, the more gravitational potential energy it has:

$$PE_g = \pm Wh \quad \ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}7)$$

$$PE_g = \pm mgh \quad \ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}8)$$

Where:

PE : is the Potential energy, (measured in joules, J).

m : is the mass of the object, (measured in kilograms, kg).

g : is the acceleration due to gravity, (measured in meters per second square, m/s^2).

± h : is the height above or below the datum, (measured in meters, m).

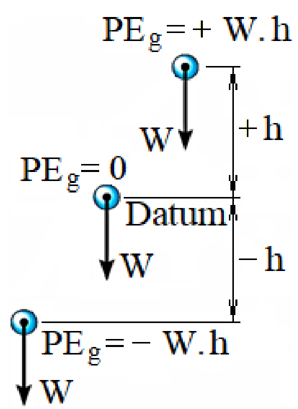


Fig. (2-2-6) Gravitational Potential Energy

- **Elastic Potential Energy:**

  Energy stored in objects that can be stretched or compressed, like springs or rubber bands. For a spring, the elastic potential energy is given by:

$$PE_e = \pm \frac{1}{2} kS^2 \quad \ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}9)$$

Where:

PE : is the Potential energy, (measured in joules, J).

k : is the spring constant (a measure of stiffness).

±S : is the displacement from the spring's equilibrium position, (measured in meters, m).

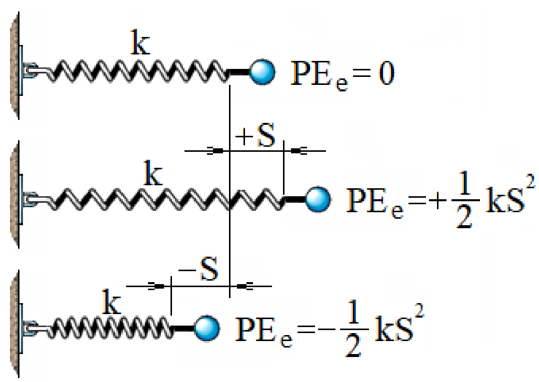


Fig. (2-2-7) Elastic Potential Energy

- **Gravitational-Elastic Potential Energy:**

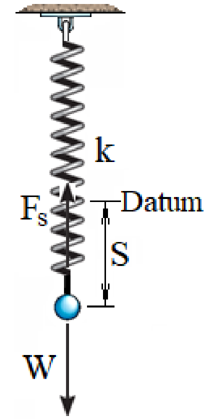


Fig. (2-2-8) – Gravitational-Elastic Potential Energy

$$PE = PE_g + PE_e \quad \ldots\ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}10)$$

$$PE = -Wh + \frac{1}{2}kS^2 \quad \ldots\ldots\ldots\ldots \quad (2\text{-}2\text{-}11)$$

Example (2–2–7):

Fig. (Ex. 2-2-7) shows a person of (75 kg) mass climbing a flight of stairs. Calculate his potential energy before starting to climb the stairs and at stairs (5), (10) and (17).

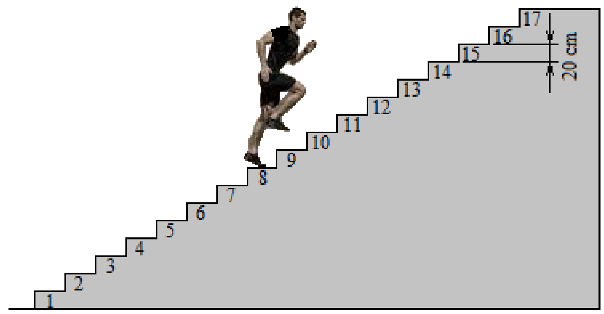


Fig. (Ex. 2-2-7)

Solution:

$$PE = mgh$$
$$PE_o = mgh_o = 75 \times 9.81 \times 0 = 0$$
$$PE_5 = mgh_5 = 75 \times 9.81 \times (5 \times 0.2) = 735.75 \text{ J}$$
$$PE_{10} = mgh_{10} = 75 \times 9.81 \times (10 \times 0.2) = 1471.5 \text{ J}$$
$$PE_{17} = mgh_{17} = 75 \times 9.81 \times (17 \times 0.2) = 2501.55 \text{ J}$$

Example (2–2–8):

A block of (3kg) is placed on top of a spring, then pushed down to the position shown in Fig. (Ex. 2-2-8). If it is then released, determine the maximum height h to which it will rise.

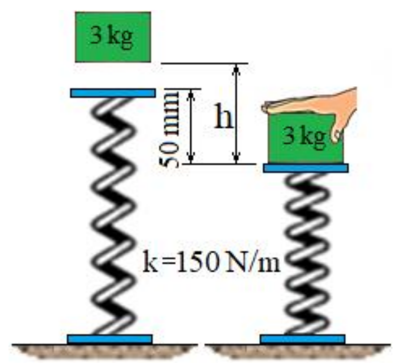


Fig. (Ex. 2-2-8)

Solution:

Conservation of Energy:

$$W = mg = 3 \times 9.81 = 29.43 \text{ N}$$
$$KE_1 + PE_1 = KE_2 + PE_2$$
$$KE_1 + [(PE_g)_1 + (PE_e)_1] = KE_2 + [(PE_g)_2 + (PE_e)_2]$$
$$\frac{1}{2} mV_1^2 + [(Wh)_1 + (\frac{1}{2} kS^2)_1] = \frac{1}{2} mV_2^2 + [(Wh)_2 + (\frac{1}{2} kS^2)_2]$$
$$0 + [0 + \frac{1}{2} (150)(0.05)^2] = 0 + [29.43h + 0]$$
$$29.43h = 0.1875$$
$$h = 0.0064 \text{ m} = 6.4 \text{ mm}$$

Example (2–2–9):

A person throws a weight of (50 N) from a height of (8 m) as shown in the Fig. (Ex. 2-2-9). Calculate its kinetic energy and potential energy at the moment it is thrown, at a height of (4 m), and at the moment it almost touches the ground. Neglect air resistance to gravity.

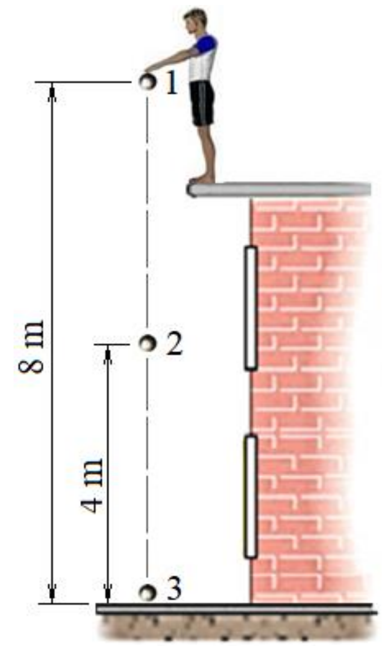


Fig. (Ex. 2-2-9)

Solution:

$y = y_o + (V_o)_y\, t - \frac{1}{2} gt^2$

$y_2 = y_1 + (V_1)_y\, t - \frac{1}{2} gt^2$

$4 = 8 + 0 - \frac{1}{2} (9.81)\, t^2$

$4.905t^2 = 4 \quad \Rightarrow \quad t = 0.903 \text{ s}$

$V_y = (V_o)_y - gt$

$V_{2y} = (V_1)_y - gt$

$V_{2y} = 0 - (9.81\times 0.903) = - 12.53 = 8.86 \text{ m/s} \downarrow$

$y_3 = y_1 + (V_1)_y\, t - \frac{1}{2} gt^2$

$0 = 8 + 0 - \frac{1}{2} (9.81)\, t^2$

$4.905t^2 = 8 \quad \Rightarrow \quad t = 1.277 \text{ s}$

$V_{3y} = (V_1)_y - gt$

$V_{3y} = 0 - (9.81\times 1.277) = - 12.53 = 12.53 \text{ m/s} \downarrow$

$m = \frac{W}{g} = \frac{50}{9.81} = 5.1 \text{ kg}$

$KE = \frac{1}{2} mV^2$

$KE_1 = \frac{1}{2} mV_1^2 = \frac{1}{2} \times 5.1 \times (0)^2 = 0$

$KE_2 = \frac{1}{2} mV_2^2 = \frac{1}{2} \times 5.1 \times (8.86)^2 = 200 \text{ J}$

$KE_1 = \frac{1}{2} mV_1^2 = \frac{1}{2} \times 5.1 \times (12.53)^2 = 400 \text{ J}$

$PE = Wh$

$PE_1 = Wh_1 = 50 \times 8 = 400 \text{ J}$

$PE_2 = Wh_2 = 50 \times 4 = 200 \text{ J}$

$PE_3 = Wh_3 = 50 \times 0 = 0$

Power (P):

- In physics, power is the rate at which work is done or energy is transferred. It is typically measured in watts (W). The formula is:

$$P = \frac{U}{t} \quad \dots\dots\dots\dots\dots \quad (2\text{-}2\text{-}12)$$

- In engineering, Power refers to the output or capability of a machine, engine, or system to perform work, often measured in horsepower (hp).

$$P = \frac{dU}{dt} = F\,\frac{dr}{dt} \quad \Rightarrow \quad P = F.V \quad \dots\dots\dots\dots\dots \quad (2\text{-}2\text{-}13)$$

- Power is scalar and its SI unit is *watt* (W), where

1 W = 1 J/s
1 hp = 550 ft.Ib/s = 33000 ft.Ib/min
1 hp = 746 W = 0.746 kW

Mechanical Efficiency (η_m):

Efficiency is a measure of how effectively a system, process, or machine converts input into useful output. It is often expressed as a percentage and is calculated as the ratio of useful output to total input. The general formula for efficiency is:

$$\left.\begin{aligned} \text{Efficiency} &= \frac{\text{Useful Output}}{\text{Total Input}} \times 100\,\% \\ \eta_m &= \frac{P_{output}}{P_{input}} \\ \eta_m &= \frac{E_{output}}{E_{input}} \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots \quad (2\text{-}2\text{-}14)$$

$$\eta_m < 1$$

Example (2–2–10):

A machine of mass (0.6 tons) was lifted to a height of (10 m) in (5 s), as shown in Fig. (Ex. 2-2-10). Calculate the required power.

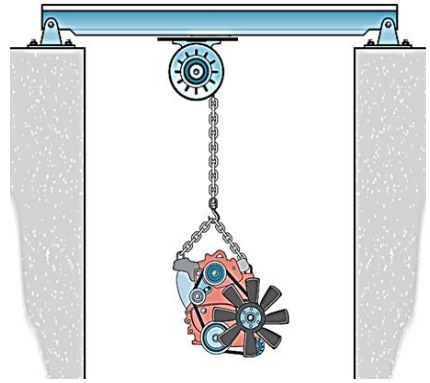


Fig. (Ex. 2-2-10)

Solution:

$$U = F.h = m.g.h = 600 \times 9.81 \times 10 = 58860\ J = 58.86\ kJ$$

$$P = \frac{U}{t} = \frac{58860}{5} = 11772\ W = 11.772\ kW$$

Example (2–2–11):

A car engine shown in Fig. (Ex. 2-2-11). provides a constant force of (1 kN) to keep the car moving at a constant velocity of (72 km/h). Calculate the required power.

Fig. (Ex. 2-2-11)

Solution:

$$V = \frac{72}{3.6} = 20\ m/s$$

$$P = F.V = 1000 \times 20 = 20000\ W = 20\ kW$$

Example (2–2–12):

A motor applies a torque of (100 N·m) to a rotating disc at an angular velocity of (20 rad/s) as shown in Fig. (Ex. 2-2-12). Calculate the required power.

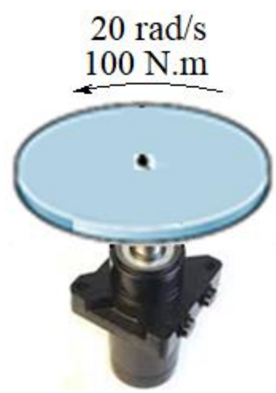


Fig. (Ex. 2-2-12)

Solution:

$$P = \tau \cdot \omega$$

$$P = 100 \times 20 = 2000\ W = 2\ kW$$