Vector Functions

Vectors functions

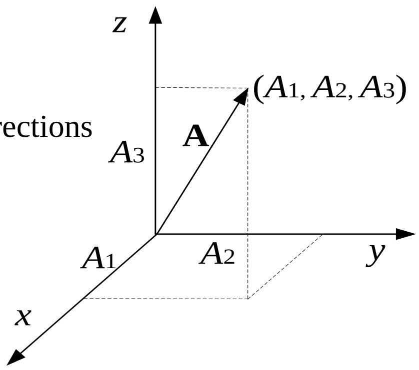
Definition:

Scalars: are quantities having only a magnitude. Length, mass, temperature etc

Vectors: are quantities having both a magnitude and a direction. Force, velocity, acceleration etc

Vectors in Cartesian Coordinate System:

$A = A_1 i + A_2 j + A_3 k$



i, j, and k are unit vectors pointing in the positive *x*, *y*, and *z* directions

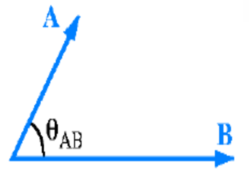
A_1, A_2 and A_3 are called *x*, *y*, and *z* component of vector A

Magnitude of A: $|\mathbf{A}| = A = \sqrt{A_1^2 + A_2^2 + A_3^2}$

Direction of A: $\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}}{\sqrt{A_1^2 + A_2^2 + A_3^2}}$

Dot Product: **A.B=|A| |B| cosθ**

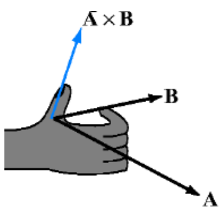
If two nonzero vectors A . B = 0, then cos Θ = 0 , Θ = 90 , Perpendicular



Cross Product: $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta$

If two nonzero vectors A × B = 0, then

sinΘ = 0 , Θ = 0°or 180° Parallel



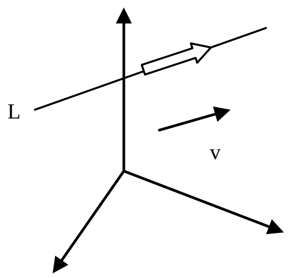
Lines and Planes in Space

In the **plane**, **a line** is determined by a **point** and a **number giving the slope** of the line.

In **space** a **line** is determined by a **point** and a **vector** giving the direction of the line.

Equation for a line

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $v = v_1 i + v_2 j + v_3 \mathrm{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to v.



The standard parameterization equation of the line through $P_0(x_0, y_0, z_0)$ ***parallel*** to $v = v_1 i + v_2 i + v_3 \mathrm{k}$ is:

$$x = x_0 + tv1 \quad , \quad y = y_0 + tv2 \quad , \quad z = z_0 + tv3 \quad , \quad -\infty \prec t \prec \infty$$

and $(x, y, z) = (x_0 + tv \ , \ y_0 + tv \ , \ z_0 + tv)$

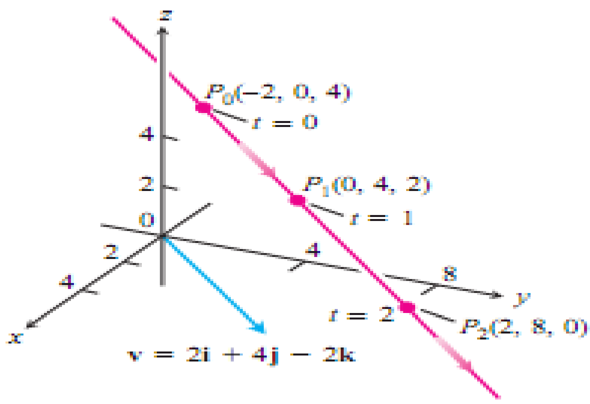
Ex.:

Find the parametric equations for the line through $(-2,0,4)$ parallel to $v = 2i + 4j - 2\mathrm{k}$.

Solution :

With $P_0(x_0, y_0, z_0)$ equal to $(-2,0,4)$ and $v = v_1 i + v_2 j + v_3 \mathrm{k}$ equal to $v = 2i + 4j - 2\mathrm{k}$

$x = -2 + 2t \quad , \quad y = 4t \quad , \quad z = 4 - 2t$



Ex.: Find the equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$.

Solution:

The vector $\overrightarrow{PQ} = 4i - 3j + 7k$ is parallel to the line and equation with $(x_0, y_0, z_0) = (-3,2,-3)$ give

$$x = -3 + 4t \quad , \quad y = 2 - 3t \quad , \quad z = -3 + 7t$$

We could have choose $Q(1,-1,4)$

$$x = 1 + 4t \quad , \quad y = -1 - 3t \quad , \quad z = 4 + 7t$$

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to v is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty,$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

The vector form Equation above for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector $\mathbf{v}$.

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$
$$= \mathbf{r}_0 + t|\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Initial position — Time — Speed — Direction

In other words, the position of the particle at time t is its initial position plus its distance moved (speed × time) in the direction $\mathbf{v}/|\mathbf{v}|$ of its straight-line motion.

EXAMPLE

A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

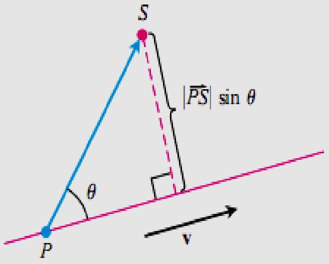
gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}\,(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \left\langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \right\rangle.\end{aligned}$$

After 10 sec of flight from the origin toward (1, 1, 1), the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of (60 ft/sec)(10 sec) = 600 ft, which is the length of the vector $\mathbf{r}(10)$.



The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}|\sin\theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\overrightarrow{PS}$ in the direction of a vector normal to the line (Figure). In the notation of the figure, the absolute value of the scalar component is, $|\overrightarrow{PS}|\sin\theta$, which is $\dfrac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$.

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|}$$

EXAMPLE Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1-1)\mathbf{i} + (1-3)\mathbf{j} + (5-0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

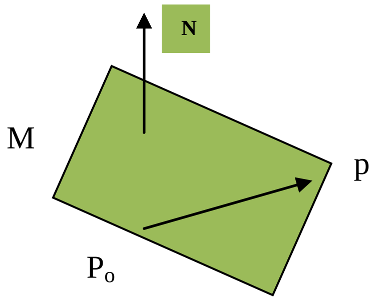
Equation (5) gives

$$d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An equation for a Plane in space

A plane in space is determined by knowing a ***point*** on the plane and its "***tilt***" or orientation. This "tilt" is defined ***by specifying a vector*** that is perpendicular or normal to the plane.

Suppose that plane **M** passes through a point $P_0(x_0, y_0, z_0)$ and is ***normal*** to the nonzero vector $N = Ai + Bj + Ck$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is ***orthogonal*** to **N.**



Thus, the plane through $P_0(x_0, y_0, z_0)$ ***normal*** to $N = Ai + Bj + Ck$ has equation:

$$\boldsymbol{N} \cdot \overrightarrow{P_0P} = \mathbf{0} \rightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or $$Ax + By + Cz = D \quad , \quad \text{where} \quad D = Ax_0 + By_0 + Cz_0$$

Ex.:

Find an equation for the plane through $P_0(-3,0,7)$ perpendicular to

$N = 5i + 2j - k$.

Solution

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$
$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0$$
$$5x + 15 + 2y - z + 7 = 0$$
$$5x + 2y - z = -22$$

Notice in this example how the components of $n = 5i + 2j - k$ become the coefficients of x , y *and* z in equation $5x + 2y - z = -22$. The vector $n = Ai + Bj + Ck$ is normal to the plane $Ax + By + Cz = D$.

Example :

Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$ and $C(0,3,0)$.

Solution :

We find a vector ***normal*** to the plane and use it with one of the point to write an equation for the plane.

$\overrightarrow{AB}$ = (2-0) **i** + (0-0) **j** + (0-1) **K**

$\overrightarrow{AC}$ = (0-0) **i** + (3-0) **j** + (0-1) **K**

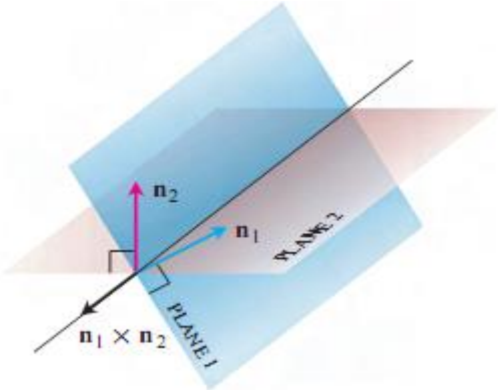
The cross product: between the vectors AB and AC is

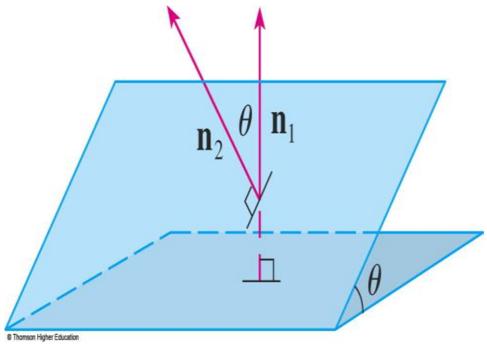
$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3i + 2j + 6k$$ is ***normal*** to the plane.

$$3(x-0)+2(y-0)+6(z-1)=0$$
$$3x+2y+6z=6$$

<u>Lines of intersection</u>

- Two lines are parallel if and only if they have the same direction.
- Two planes are parallel if and only if their normal's are parallel.
- The planes that are ***not*** parallel ***intersect in a line.***





<u>Example :</u>

Find a vector parallel to the line of intersection of the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

Solution

The line of intersection of two planes is perpendicular to both ***planes' normal*** vectors n_1 and n_2 and therefore parallel to $n_1\times n_2$. i.e. $n_1\times n_2$ is a vector parallel to the planes' line of intersection.

$$n_1\times n_2=\begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14i + 2j + 15k$$

EXAMPLE Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution We find a vector parallel to the line and a point on the line

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14i + 2j + 15k \quad \therefore \text{ vector parallel to the line}$$

To find point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3,$$
$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for y and z. The different choices would simply give different parametrizations of the same line.

EXAMPLE Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$
$$8 + 6t - 4t + 6 + 6t = 6$$
$$8t = -8$$
$$t = -1.$$

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$

The Distance from a Point to a Plane

If P is a point on a plane with normal $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$. That is, the distance from S to the plane is

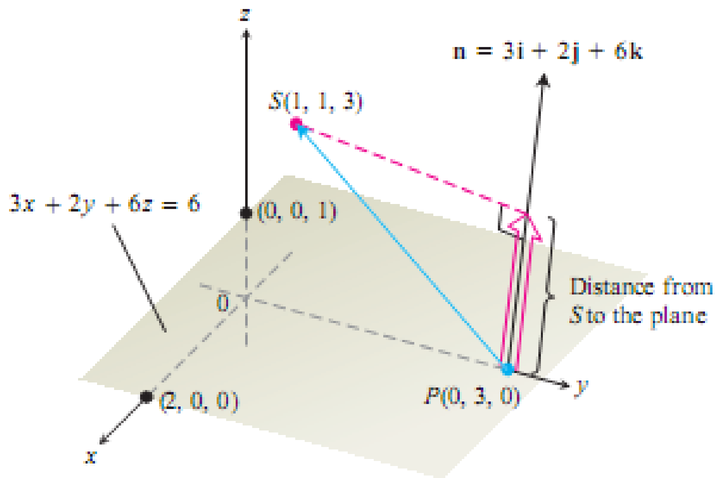
$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right|$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.

EXAMPLE Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\mathbf{n}$ normal to the plane (Figure 12.41). The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$



The distance from S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y-intercept $(0, 3, 0)$, then

$$\overrightarrow{PS} = (1-0)\mathbf{i} + (1-3)\mathbf{j} + (3-0)\mathbf{k}$$

$$= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$

$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

The distance from S to the plane is

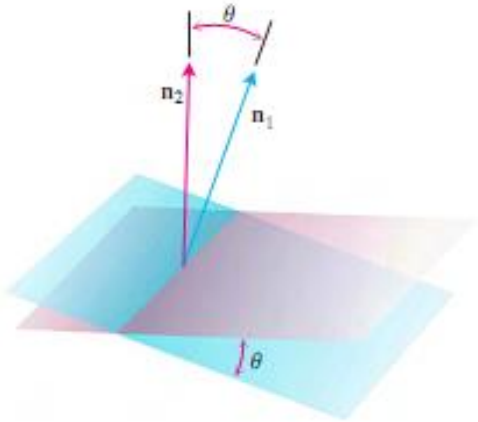
$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \quad \text{length of proj}_{\mathbf{n}} \overrightarrow{PS}$$

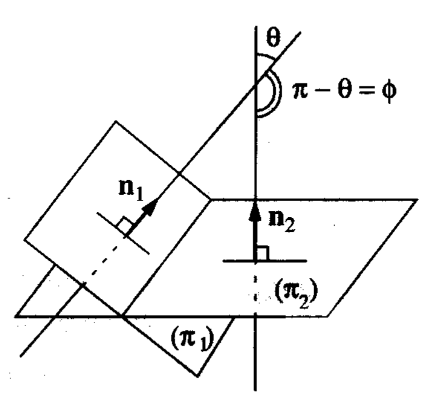
$$= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right|$$

$$= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}.$$

angles between planes

The angle between two intersecting planes is defined to be the angle determined by ***their normal vectors***. $n_1 \cdot n_2 = |n_1||n_2|\cos\theta$,$\Theta=\cos^{-1}\left(\frac{n1.n2}{|n1||n2|}\right)$





Example:

Find the angle between the planes $3x-6y-2z=15$ and $2x+y-2z=5$

Solution

The vectors $n_1 = 3i-6j-2\text{k}$ and $n_2 = 2i+j-2\text{k}$ are normals to the planes.

The angle between them is

$$\theta = \cos^{-1}\left(\frac{n_1 \cdot n_2}{|n_1||n_2|}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right)$$

Vector -valued functions and motion in space

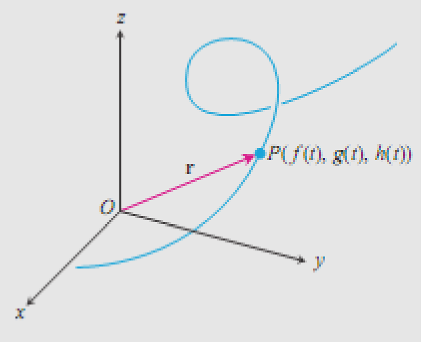
When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I. \tag{1}$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space that we call the particle's **path**. The equations and interval in Equation (1) **parametrize** the curve. A curve in space can also be represented in vector form. The vector

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \tag{2}$$

from the origin to the particle's **position** $P(f(t), g(t), h(t))$ at time t is the particle's **position vector** (Figure). The functions f, g, and h are the **component functions (components)** of the position vector. We think of the particle's path as the **curve traced by r** during the time interval I. Figure displays several space curves



The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

Equation (2) defines **r** as a vector function of the real variable t on the interval I. More generally, a **vector function** or **vector-valued function** on a domain set D is a rule that assigns a vector in space to each element in D.

Derivative of the vector –valued function:

DEFINITION Derivative

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at t** if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

Then, a vector function r(t) is differentiable if it is ***differentiable at every point*** of its domain. The curve traced by ***r*** is ***smooth*** if (***dr /dt***) is **continuous** and ***never 0***, that is, if ***f, g, and h,*** have continuous first derivatives that are not simultaneously ***0***.

DEFINITIONS Velocity, Direction, Speed, Acceleration

If **r** is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of **v** is the **direction of motion**, the magnitude of **v** is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

We can express the velocity of a moving particle as the product of its speed and direction:

$$\text{Velocity} = |\mathbf{v}|\left(\frac{\mathbf{v}}{|\mathbf{v}|}\right) = (\text{speed})(\text{direction}).$$

Example:

Find the position, **velocity**, **speed**, **acceleration** and **scalar acceleration** for the given value of t.

a) $\vec{r}(t) = t^2 i - t^3 j$ at t=2 , t = 0.

Position: $\vec{r}(2) = 4i - 8j$

velocity: $\vec{v}(t) = 2ti - 3t^2 j \rightarrow \vec{v}(2) = 4i - 12j$

speed: $|\vec{v}(2)| = \sqrt{4^2 + (-12)^2} = \sqrt{160} = 4\sqrt{10}$

acceleration: $\vec{a}(t) = 2i - 6tj \rightarrow \vec{a}(2) = 2i - 12j$

Scalar acceleration $|\vec{a}(2)| = \sqrt{2^2 + (-12)^2} = \sqrt{148} = 2\sqrt{37}$

b) At t = 0, H.W.

DEFINITION Length of a Smooth Curve

The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Arc Length Formula

$$L = \int_a^b |\mathbf{v}|\, dt$$

Example :

1.In example 1 part a. Find the distance along the curve from t=2 to t=5
Solution: distance is arc length :

$$d=\int_{2}^{5}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_{2}^{5}\sqrt{(2t)^2+\left(-3t^2\right)^2}\,dt=\int_{2}^{5}\sqrt{4t^2+9t^4}\,dt=\int_{2}^{5}t\sqrt{4+9t^2}\,dt$$

$$Let\ u=4+9t^2,\ du=18tdt\to d=\frac{1}{18}\int_{40}^{229}u^{1/2}dt=\frac{1}{18}\cdot\frac{u^{3/2}}{3/2}\Bigg|_{40}^{229}=$$

$$\frac{229\sqrt{229}-80\sqrt{10}}{27}\approx 118.978393$$

Example :

The velocity of a particle moving in space is $v(t)=i+2tj+2k$.
Find the particle's position as a function of t , if r = i - j at t =0.

v(t) = i + 2t j + 2 k

r(t) = $\int v(t)dt$ = $ti+2\frac{t^2}{2}j+2tk+c$

r(0) = c = i - j

r(t) = t i + t^2 + 2t k + I - j

r(t) = (t + 1) i + (t^2 – 1) j + 2t k

H.W. :

A particle move through 3-space in such away that it's velocity is

$$v(t)=i+tj+t^2k$$

Find the coordinates of the particle at time t = 1, given that the particle is at the point (-1,2,4) at time t = 0.

Tangent Lines to Smooth Curves :

The tangent line to a smooth curve **r (t) = f(t) i + g(t) j+ h(t) k** , at $\mathbf{t = t_0}$ is the line that passes through the point $\mathbf{(f(t_0) , g(t_0) , h(t_0))}$ parallel to $\mathbf{V(t_0)}$, ***$V(t_0)$ is the curve's velocity vector*** .

Example :

Find parametric equations for the line that is tangent to the given curve at the given parameter value at $\mathbf{t = t_0}$,

$$\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}, \quad t_0 = 0$$

Solution :

$$\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}$$

$$\Rightarrow \quad \mathbf{v}(t) = (\cos t)\mathbf{i} + (2t + \sin t)\mathbf{j} + e^t\mathbf{k} ;$$

$$t_0 = 0 \Rightarrow \mathbf{v}(t_0) = \mathbf{i} + \mathbf{k} \text{ and}$$

$$\mathbf{r}(t_0) = P_0 = (0, -1, 1) \Rightarrow$$

$$x = 0 + t = t, y = -1, \text{and } z = 1 + t ;$$

are parametric equations of the tangent line

H.W.

Find parametric equations for the line that is tangent to the given curve at the given parameter value at $\mathbf{t = t_0}$

1. $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 5t\mathbf{k}, \quad t_0 = 4\pi$
2. $\mathbf{r}(t) = (a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + bt\mathbf{k}, \quad t_0 = 2\pi$
3. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad t_0 = \frac{\pi}{2}$

Unit Tangent Vector T

DEFINITION Unit Tangent Vector

The **unit tangent vector** of a smooth curve $\mathbf{r}(t)$ is

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

The unit tangent vector ***T*** is a differentiable function of ***t*** whenever ***v*** is a differentiable function of ***t***. **T** is one of three unit vectors in a traveling reference frame that is used to describe the motion of space vehicles and other bodies traveling in three dimensions.

EXAMPLE Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k}$$

and

$$|\mathbf{v}| = \sqrt{9 + 4t^2}.$$

Thus,

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9 + 4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9 + 4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9 + 4t^2}}\mathbf{k}.$$

<u>H.W.</u>

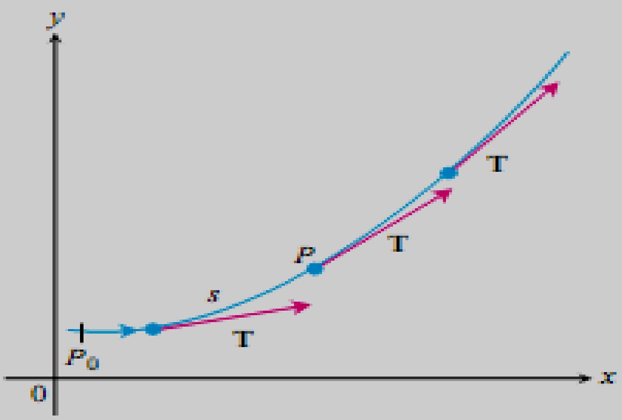
Find the unit tangent vector to the curve at $t = 2$ **,** $\mathbf{r}(t) = 4t\,\mathbf{i} - 3t^2\mathbf{j}$

Curvature and the Unit Normal Vector N :

In this section we will study how a curve turns or bends. We look first at curves in the **coordinate plane**, and then at **curves in space**.

Curvature of a *Plane* Curve :

As a particle moves along a smooth curve in the plane, **T = dr / ds**, turns as the curve bends. Since **T** is a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The rate at which **T** turns per unit of length along the curve is called the ***curvature*** (Figure below). The traditional symbol for the curvature function is the Greek letter ("kappa"), K .



As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P.

If $|d\mathbf{T}/ds|$ is large, **T** turns sharply as the particle passes through P, and the curvature at P is large. If $|d\mathbf{T}/ds|$ is close to zero, **T** turns more slowly and the curvature at P is smaller.

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Example :

Find the curvature of the vector function given below :

r(t) = (**a cos(t)**) i + (**a sin(t)**) j

Solution :

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2} = |a| = a.$$

Since $a > 0$, $|a| = a$.

From this we find

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

$$\frac{d\mathbf{T}}{dt} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$$

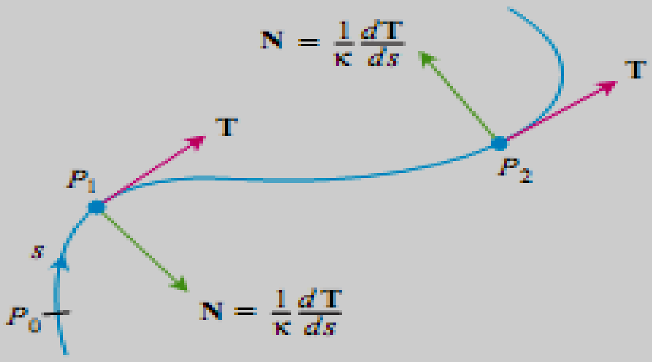
$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1.$$

Hence, for any value of the parameter t,

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a}.$$

Unit Normal Vector :

Among the vectors orthogonal to the unit tangent vector **T** is one of particular significance because ***it points in the direction in which the curve is turning***. Since **T** has constant length (namely, 1), the derivative (**dT/ ds**) is orthogonal to **T** . Therefore, if we divide (**dT / ds**) by its length we obtain a ***unit vector N orthogonal*** to **T**,



The vector $d\mathbf{T}/ds$, normal to the curve, always points in the direction in which **T** is turning. The unit normal vector **N** is the direction of $d\mathbf{T}/ds$.

DEFINITION Principal Unit Normal

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

From the definition above, the ***principal normal vector N*** will point toward the concave side of the curve, The formula that enables us to find N without having to find **s** and **k** is :

Formula for Calculating N

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Example :

Find ***T*** and ***N*** for $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}$:

Solution We first find **T**:

$$\mathbf{v} = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}.$$

From this we find

$$\frac{d\mathbf{T}}{dt} = -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

and

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

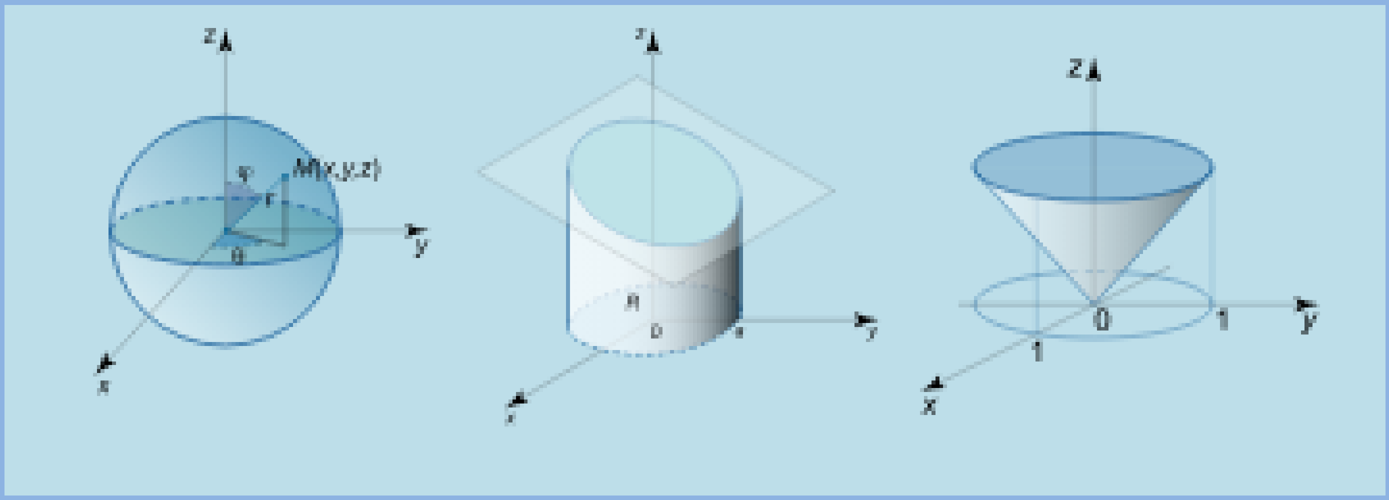
$$= -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}.$$

Notice that $\mathbf{N}\cdot\mathbf{T} = 0$, *verifying that* ***N*** *is orthogonal to* ***T***.

H.W.

Find the unit normal vector to the curve at **t = 2** , $\mathbf{r}(t) = 4t\,\mathbf{i} - 3t^2\,\mathbf{j}$.

Multiple Integrals



(Multiple integrals)

Introduction :

In ***multiple integral*** we consider the integral of a function of ***two variables*** $f(x, y)$ over a region in the ***plane*** and the integral of a function of ***three variables*** $f(x, y, z)$ over a region in ***space***.

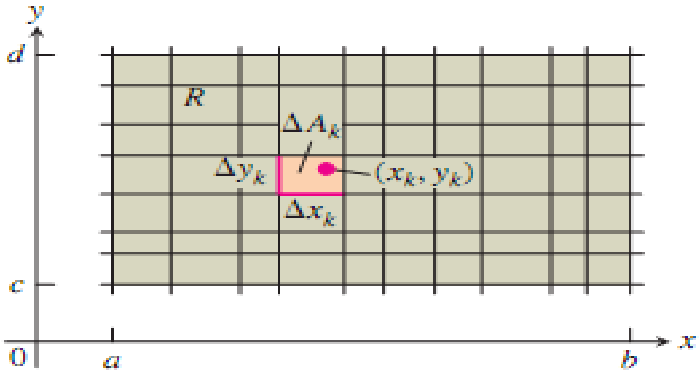
Basic Integration Rules

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
$\frac{d}{dx}[C]=0$	$\int 0dx=C$
$\frac{d}{dx}[kx]=k$	$\int kdx=kx+C$
$\frac{d}{dx}[kf(x)]=kf'(x)$	$\int kf(x)dx=k\int f(x)dx+C$
$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$	$\int [f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx+C$
$\frac{d}{dx}[x^n]=nx^{n-1}$	$\int x^n dx=\frac{x^{n+1}}{n+1}+C,\ n\neq -1$
$\frac{d}{dx}[\sin x]=\cos x$	$\int \cos xdx=\sin x+C$
$\frac{d}{dx}[\cos x]=-\sin x$	$\int \sin xdx=-\cos x+C$
$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int \sec^2 xdx=\tan x+C$
$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int \sec x\tan xdx=\sec x+C$
$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int \csc^2 xdx=-\cot x+C$
$\frac{d}{dx}[\csc x]=-\csc x\cot x$	$\int \csc x\cot xdx=-\csc x+C$

1 - Double(*Iterated, repeated*) integrals over a r*ectangle regions* :

Definition of Double Integrals in rectangular region:

In the previous (in class one) we defined the definite integral of a continuous function $f(x)$ over an interval [a, b] as a limit of Riemann sums. In this section we extend this idea to define the ***double integral*** of a **continuous function** of **two** variables $f(x, y)$ over a bounded rectangle region **R** in the **plane**. In both cases the integrals are limits of approximating ***Riemann sums***. ***The Riemann sums*** for the integral of a single-variable function $f(x)$ are obtained by **partitioning** a finite interval into **thin** subintervals, **multiplying** the **width** of each subinterval by the value of ***f*** at a point $\mathbf{C_k}$ inside that subinterval, and then **adding** together all the products. A similar method of partitioning, multiplying, and summing is used to construct double integrals.



Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \Delta y_k$.

$R: \quad a \le x \le b, \quad c \le y \le d.$

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\iint_R f(x, y)\,dA \qquad \text{or} \qquad \iint_R f(x, y)\,dx\,dy.$$

Double Integrals as Volumes :

When $f(x, y)$ is a positive function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$ (Figure 15.2). Each term $f(x_k, y_k)\Delta A_k$ in the sum $S_n = \sum f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k. The sum S_n thus approximates what we want to call the total volume of the solid. We *define* this volume to be

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x, y)\, dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.

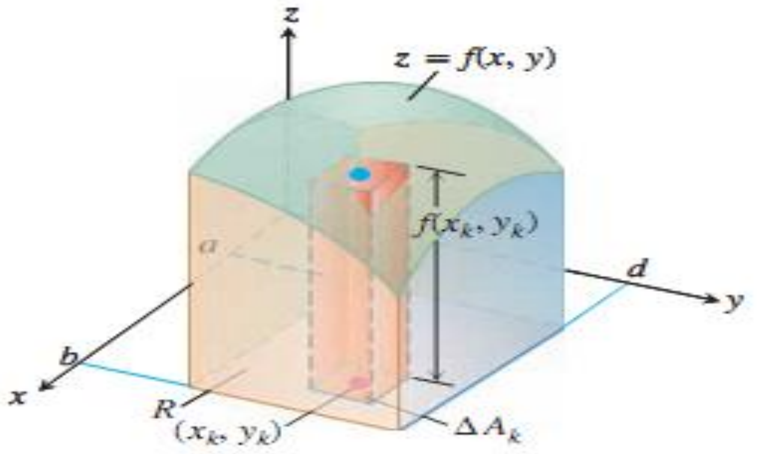
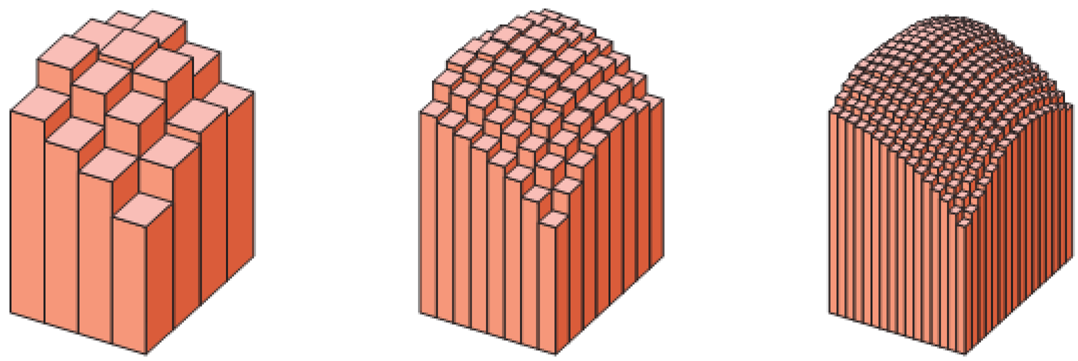


Fig. M. Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R.



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

FIGURE 15.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Fig. M

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Example :

Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region R: $0 \le x \le 2$, $0 \le y \le 1$ in the xy-plane.

Solution :

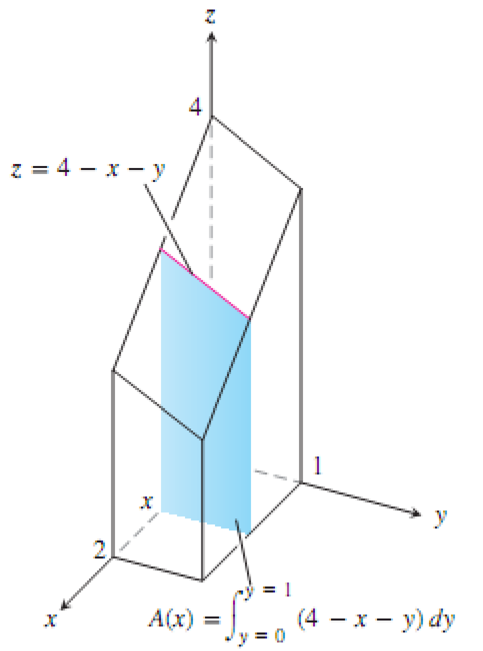
If we apply the method of slicing , with slices perpendicular to the x-axis then the volume is

$$\int_{x=0}^{x=2} A(x)\, dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 1$$

where $A(x)$ is the cross-sectional area at x. For each value of x, we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 2$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y.
Combining Equations (1) and (2), we see that the volume of the entire solid is



To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

$$\text{Volume} = \int_{x=0}^{x=2} A(x)\, dx = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\, dy \right) dx$$

$$= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx$$

$$= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = 5.$$

If we just wanted to write a formula for the volume, without carrying out any of the integrations, we could write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\, dy\, dx.$$

What meaning of iterated(repeated) integrals?

The expression on the right, called an **iterated** or **repeated integral**, says that the volume is obtained by integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then integrating the resulting expression in x with respect to x from $x = 0$ to $x = 2$. The limits of integration 0 and 1 are associated with y, so they are placed on the integral closest to dy. The other limits of integration, 0 and 2, are associated with the variable x, so they are placed on the outside integral symbol that is paired with dx.

H.W :

calculated the volume in the previous example by slicing with planes perpendicular to the y-axis(Solve with draw).

EXAMPLE Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: \; 0 \le x \le 2, \; -1 \le y \le 1.$$

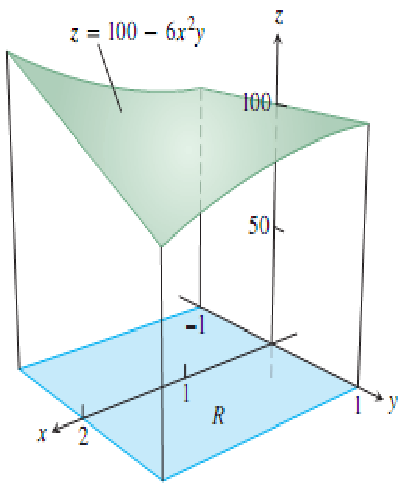


Fig. a The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R (Example 1).

Solution Fig. a displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{0}^{2} (100 - 6x^2y)\, dx\, dy = \int_{-1}^{1} \Big[100x - 2x^3y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$

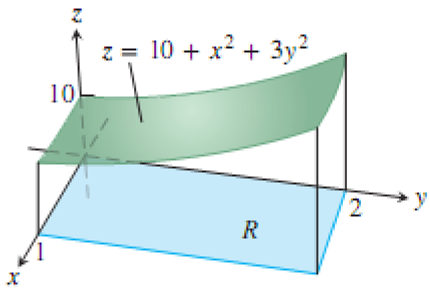
Reversing the order of integration gives the same answer:

$$\int_{0}^{2}\int_{-1}^{1} (100 - 6x^2y)\, dy\, dx = \int_{0}^{2} \Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1} dx$$

$$= \int_{0}^{2} [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_{0}^{2} 200\, dx = 400.$$

H .W . Find the volume of the region bounded above by the ellipitical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1$, $0 \le y \le 2$.



<u>Double Integrals over *General* Regions :</u>

In this section we define and evaluate double integrals over bounded regions in the **plane** which are more general than rectangles. These double integrals are also evaluated as ***iterated*** integrals, with the ***main practical problem*** being that of ***determining the limits of integration***. Since the region of integration may ***have boundaries other than line segments*** parallel to the coordinate axes, <u>***the limits of integration often involve variables, not just constants***</u>.

<u>Double Integrals over Bounded, Nonrectangular Regions :</u>

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region **R**, such as the one in Figure.

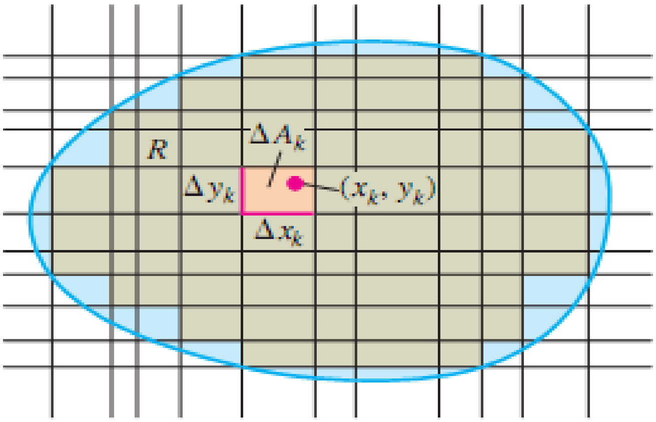


FIGURE A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$

As the norm of the partition forming S_n goes to zero, $\|P\| \to 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity. If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is called the **double integral** of $f(x, y)$ over R:

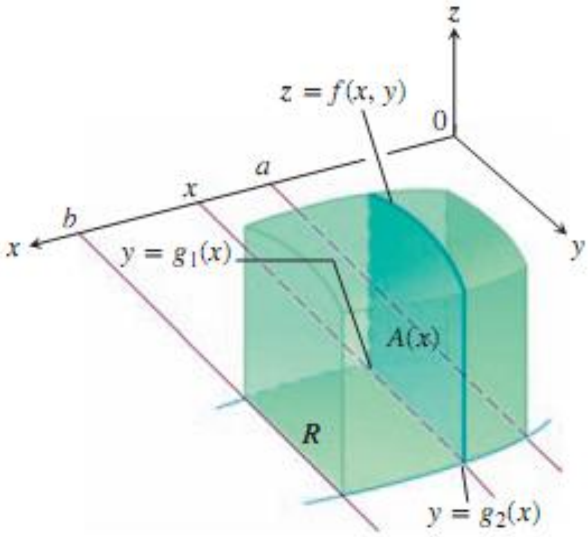
$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = \iint_R f(x, y)\, dA.$$

Volumes :

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

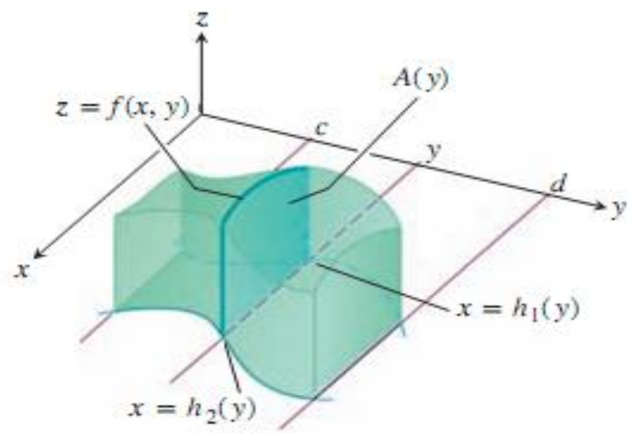
1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:
$$\int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$



The volume of the solid shown here is
$$\int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

For a given solid, Theorem 2 says we can calculate the volume as in Figure _____, or in the way shown here. Both calculations have the same result.

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Finding Limits of Integration

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Using Vertical Cross-sections When faced with evaluating $\iint_R f(x, y)\,dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch*. Sketch the region of integration and label the bounding curves (Figure a).
2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants) (Figure b).
3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral shown here (see Figure c) is

$$\iint_R f(x, y)\,dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\,dy\,dx.$$

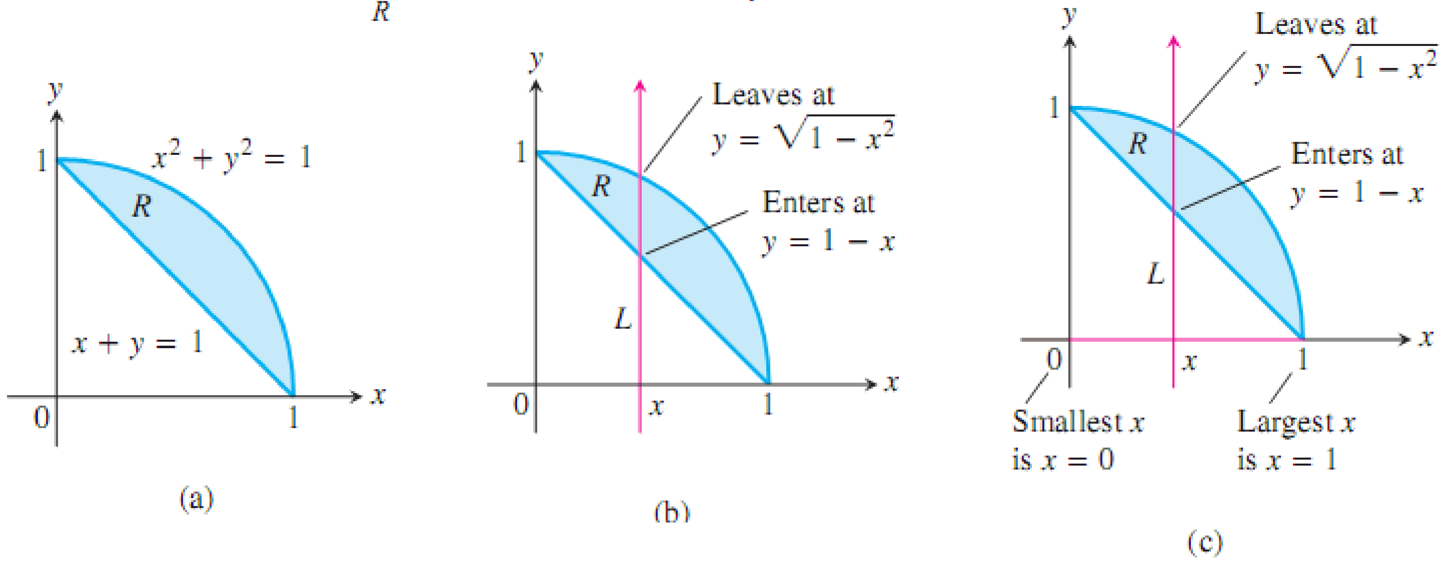


FIGURE Finding the limits of integration when integrating first with respect to y and then with respect to x.

Using Horizontal Cross-sections To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3 (see Figure). The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

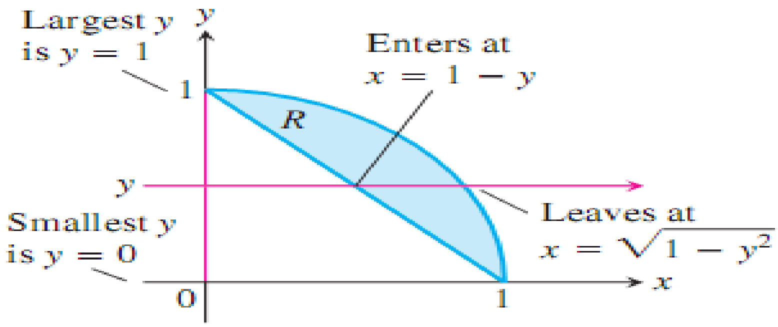


FIGURE Finding the limits of integration when integrating first with respect to x and then with respect to y.

Reverse Order of Integration

EXAMPLE Sketch the region of integration for the integral

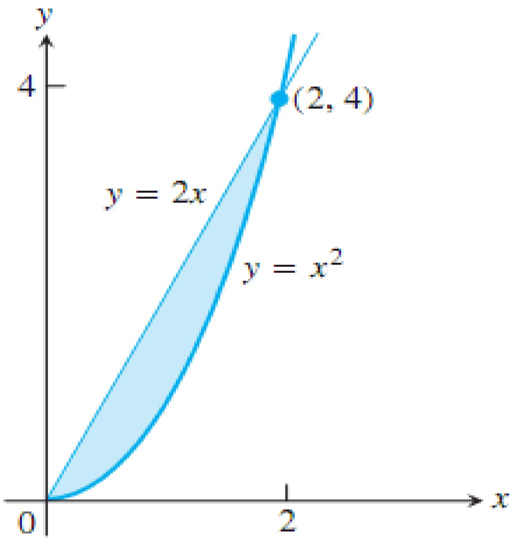
$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

Solution The region of integration is given by the inequalities $x^2 \le y \le 2x$ and $0 \le x \le 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure a).

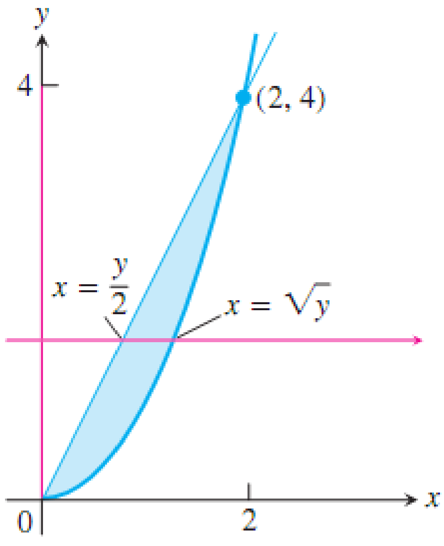
To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8.



(a)



(b)

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$ (any number c)

2. *Sum and Difference:*

$$\iint\limits_R (f(x,y) \pm g(x,y))\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA$$

3. *Domination:*

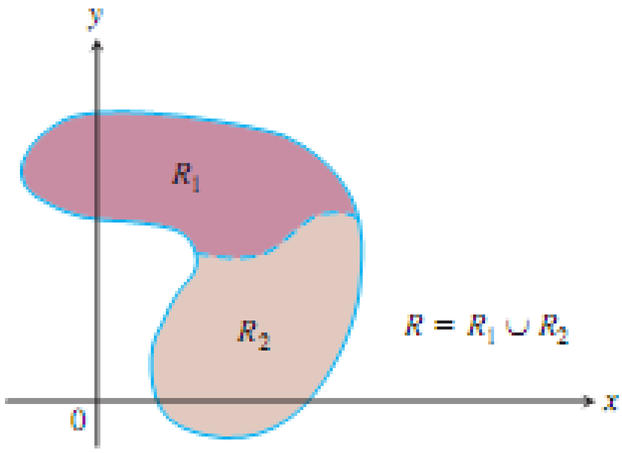
(a) $\iint\limits_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

(b) $\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

4. *Additivity:* $\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$

if R is the union of two nonoverlapping regions R_1 and R_2

Property 4 assumes that the region of integration R is decomposed into nonoverlapping regions R_1 and R_2 with boundaries consisting of a finite number of line segments or smooth curves. Figure illustrates an example of this property.

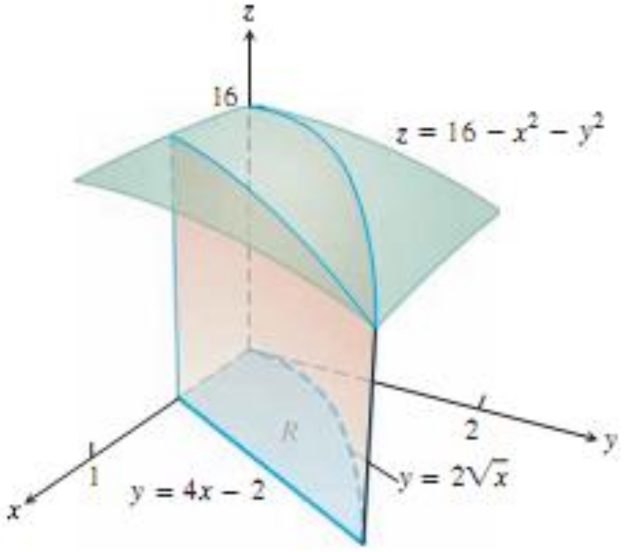


$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA$$

FIGURE : The Additivity Property for rectangular regions holds for regions bounded by smooth curves.

H.W :

Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



"Area" by Double Integration :

In this section we show how to use double integrals to calculate the **areas** of bounded regions in **the plane**.

Areas of Bounded Regions in the Plane:

If we take ***f* (x, y) = 1** in the definition of the double integral over a region **R** in the preceding section, the Riemann sums reduce to :

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \sum_{k=1}^{n} \Delta A_k .$$

This is simply the sum of the areas of the small rectangles in the partition of **R**, and approximates what we would like to call the area of **R**.

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA .$$

To evaluate the integral in the definition of area, we integrate the constant function ***f* (x, y) = 1** over the **R**egion " **R** ".

EXAMPLE Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure), noting where the two curves intersect at the origin and $(1, 1)$, and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 \Big[y \Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6} .$$

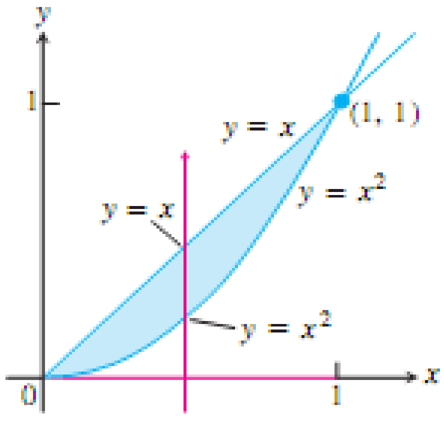


FIGURE The region in Example

H.W Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Double Integrals in Polar Form :

Integrals are sometimes ***easier*** to evaluate if we **change** to polar coordinates. This section shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by ***polar equations***.

Integrals in Polar Coordinates :

When we defined the double integral of a function over a region R in the x-y plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have ***either constant x-values or constant y-values***. In polar coordinates, the natural shape is a "***polar rectangle***" whose sides have ***constant r- and θ -values***.

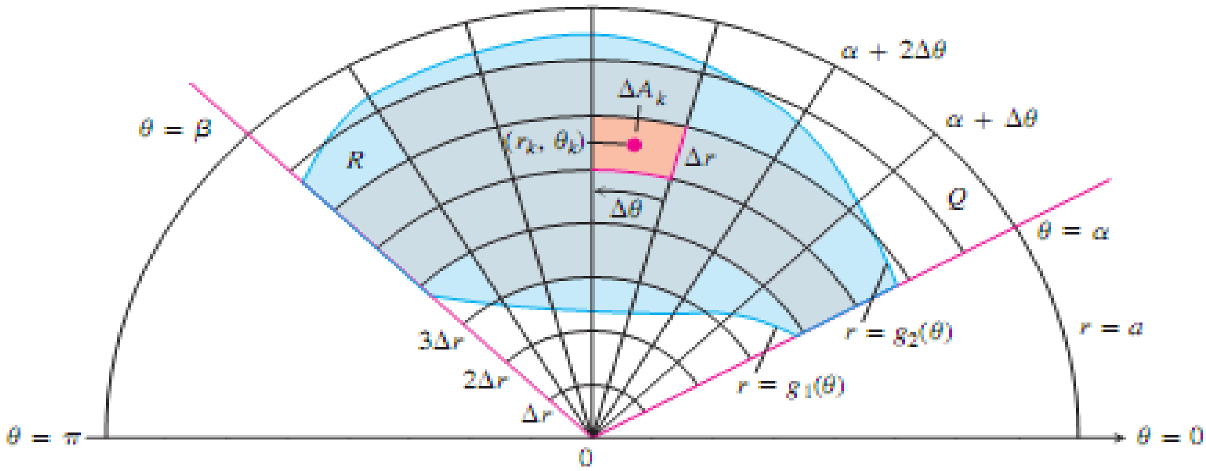


FIGURE The region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, is contained in the fan-shaped region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$. The partition of Q by circular arcs and rays induces a partition of R.

A version of **Fubini's Theorem** says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and as ***θ.***

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta)\, dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

1. *Sketch.* Sketch the region and label the bounding curves (Figure a).
2. *Find the r-limits of integration*. Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis (Figure b).
3. *Find the θ-limits of integration*. Find the smallest and largest θ-values that bound R. These are the θ-limits of integration (Figure c). The polar iterated integral is

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r, \theta)\, r\, dr\, d\theta.$$

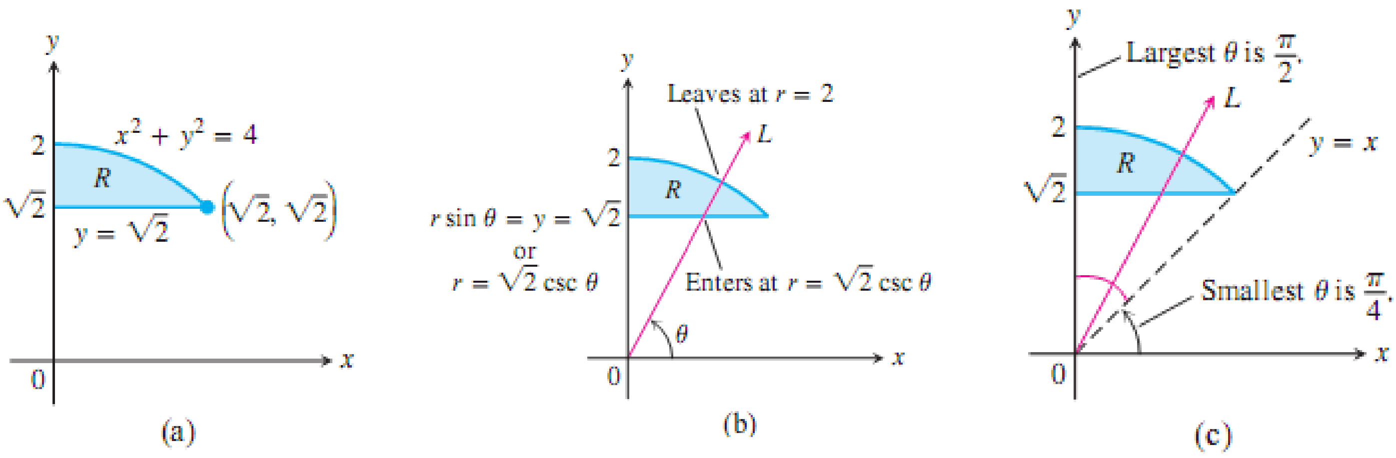


FIGURE Finding the limits of integration in polar coordinates.

EXAMPLE Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

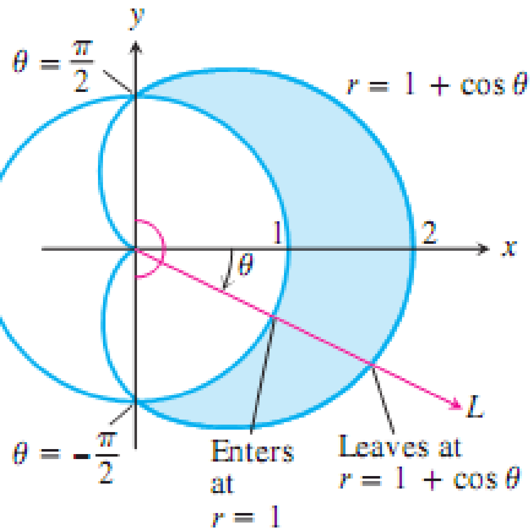


FIGURE Finding the limits of integration in polar coordinates for the region in Example

“ Area “ in Polar Coordinates :

If ***f(r, θ)*** is the constant function whose value is 1, then the integral of *f* over R is the area of R.

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta.$$

EXAMPLE Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration (Figure) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

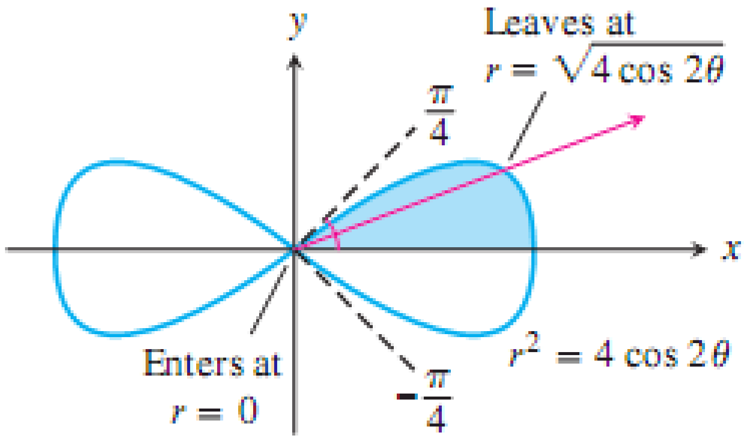


FIGURE To integrate over the shaded region, we run r from 0 to $\sqrt{4\cos 2\theta}$ and θ from 0 to $\pi/4$ (Example).

Changing Cartesian Integrals into Polar Integrals coordinates:

The procedure for changing a Cartesian integral $\iint_R f(x, y)\, dx\, dy$ into a polar integral has two steps.

- First substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\, dy$ by $r\, dr\, d\theta$ in the Cartesian integral
- Then supply polar limits of integration for the boundary of R.

The Cartesian integral then becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

EXAMPLE Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx,$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$ (Figure).

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r\cos\theta, y = r\sin\theta$ and replacing $dy\, dx$ by $r\, dr\, d\theta$ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e - 1)\, d\theta = \frac{\pi}{2}(e - 1).$$

The r in the $r\, dr\, d\theta$ was just what we needed to integrate e^{r^2}. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral.

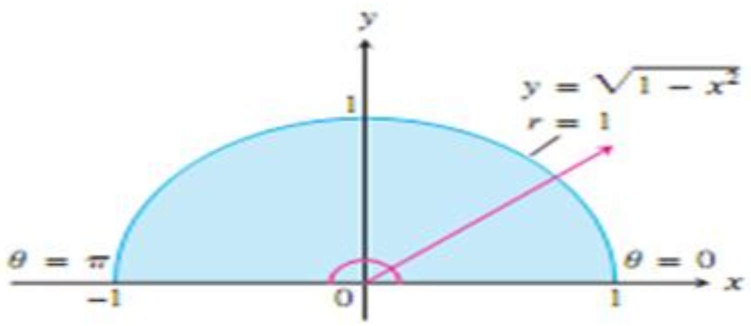


FIGURE The semicircular region in Example is the region

$0 \le r \le 1, \quad 0 \le \theta \le \pi.$

EXAMPLE Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Solution Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalites $0 \le y \le \sqrt{1 - x^2}$ and $0 \le x \le 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant. (See Figure , first quadrant.) Substituting the polar coordinates $x = r\cos\theta, y = r\sin\theta, 0 \le \theta \le \pi/2$ and $0 \le r \le 1$, and replacing $dx\,dy$ by $r\,dr\,d\theta$ in the double integral, we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

Why is the polar coordinate transformation so effective here? One reason is that $x^2 + y^2$ simplifies to r^2. Another is that the limits of integration become constants.

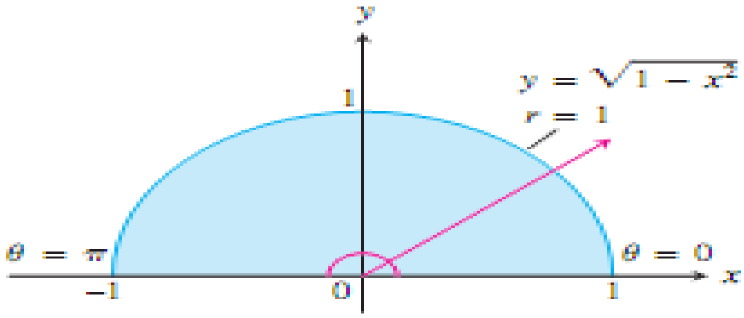


FIGURE The semicircular region in Example is the region
$0 \le r \le 1, \quad 0 \le \theta \le \pi.$

H.W : Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

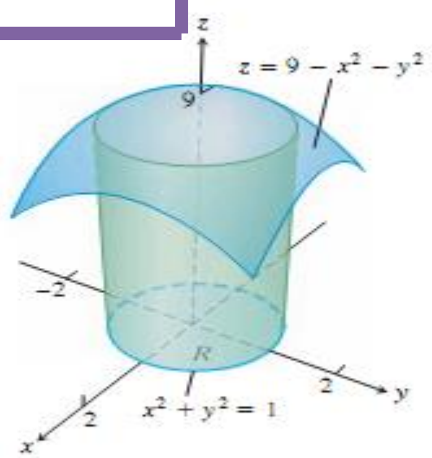


FIGURE The solid region in Example

Triple Integrals in Rectangular Coordinates :

We use triple integrals to calculate the ***volumes*** of ***three-dimensional*** shapes. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions.

Triple Integrals :

If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes (Figure). We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k. \qquad (1)$$

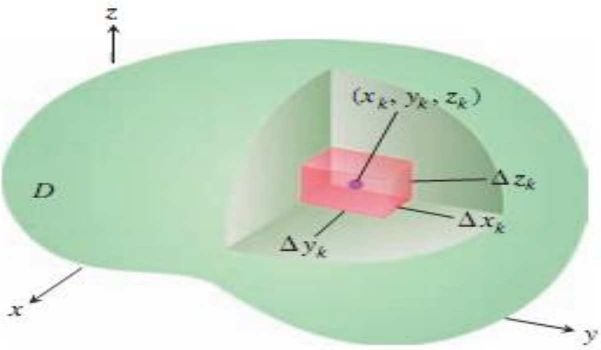


FIGURE Partitioning a solid with rectangular cells of volume ΔV_k.

Volume of a Region in Space

If F is the constant function whose value is 1, then the sums in Equation (1) reduce to

$$S_n = \sum F(x_k, y_k, z_k)\, \Delta V_k = \sum 1 \cdot \Delta V_k = \sum \Delta V_k.$$

As Δx_k, Δy_k, and Δz_k approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D. We therefore define the volume of D to be the triple integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV.$$

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

Finding Limits of Integration in the Order *dz dy dx*

We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.

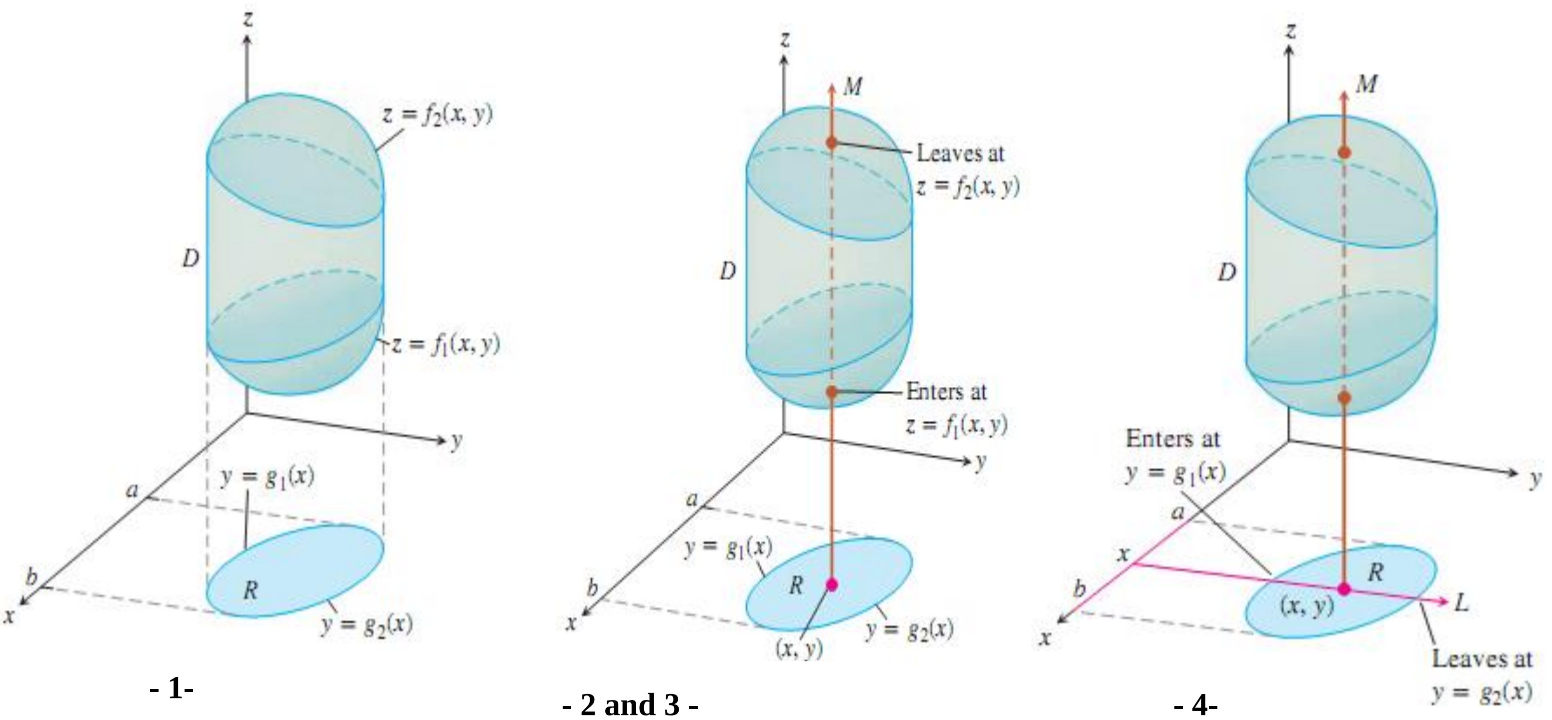
To evaluate

$$\iiint_D F(x, y, z)\, dV$$

over a region D, integrate first with respect to z, then with respect to y, and finally with respect to x.

1. *Sketch*. Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.
2. *Find the z-limits of integration*. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.
3. *Find the y-limits of integration*. Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.
4. *Find the x-limits of integration*. Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$



- 1-   - 2 and 3 -   - 4-

EXAMPLE Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint_D dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over D. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

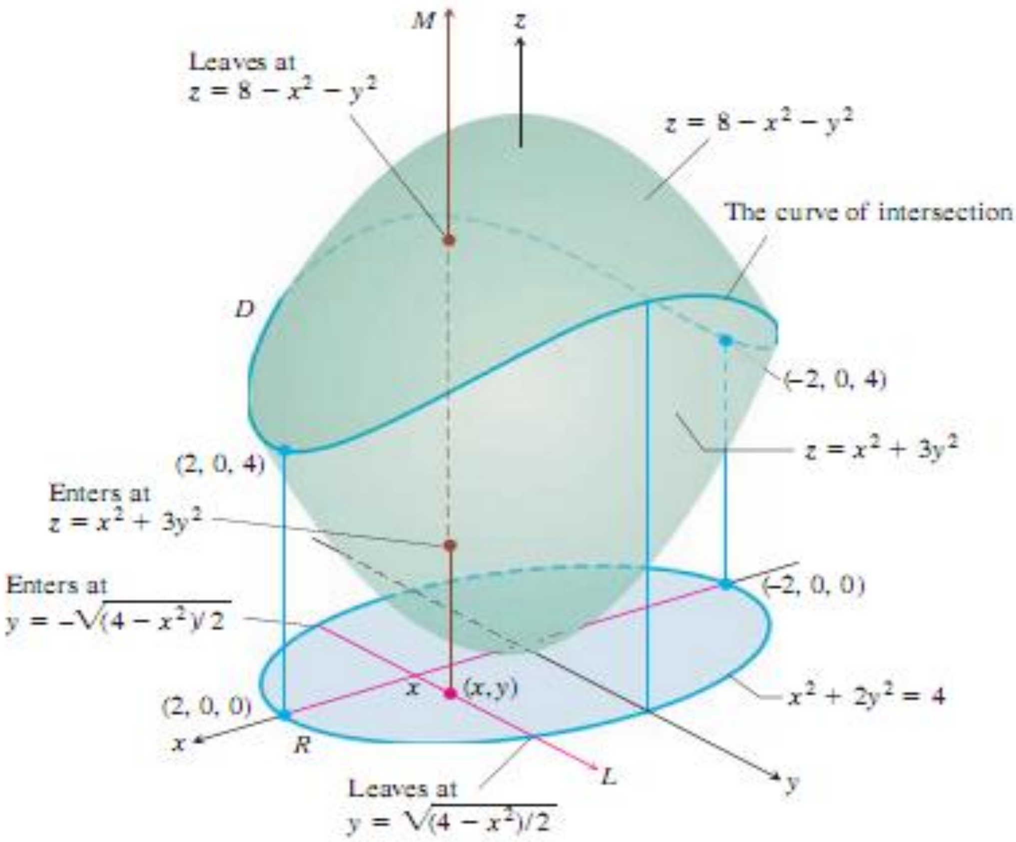


FIGURE The volume of the region enclosed by two paraboloids, calculated in Example

Next we find the y-limits of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$V = \iiint_D dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}(8 - 2x^2 - 4y^2)\,dy\,dx$$

$$= \int_{-2}^{2}\left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2}\left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right) dx$$

$$= \int_{-2}^{2}\left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4 - x^2)^{3/2}\,dx$$

$$= 8\pi\sqrt{2}.$$ After integration with the substitution $x = 2\sin u$

Triple Integrals in Cylindrical and Spherical Coordinates :

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section.

Integration in Cylindrical Coordinates :

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z), as shown in Figure .

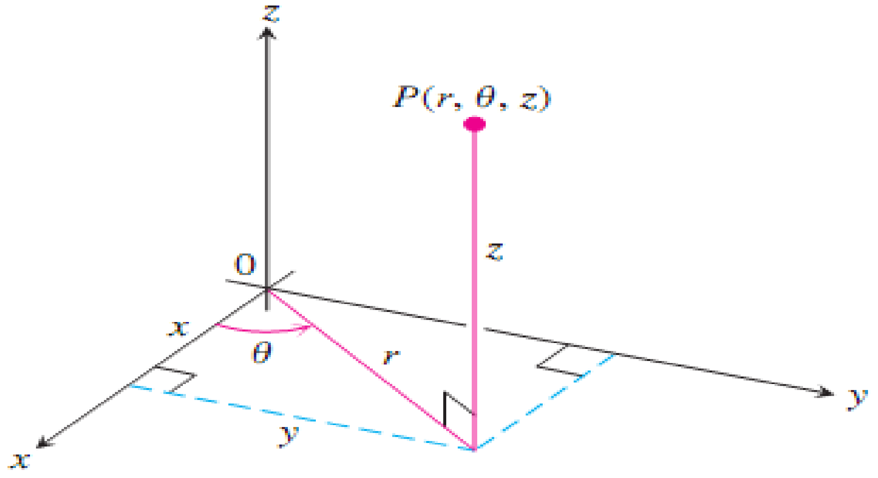


FIGURE The cylindrical coordinates of a point in space are r, θ, and z.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

The values of $x, y, r,$ and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\iiint_D f\,dV = \iiint_D f\,dz\,r\,dr\,d\theta.$$

How to Integrate in Cylindrical Coordinates

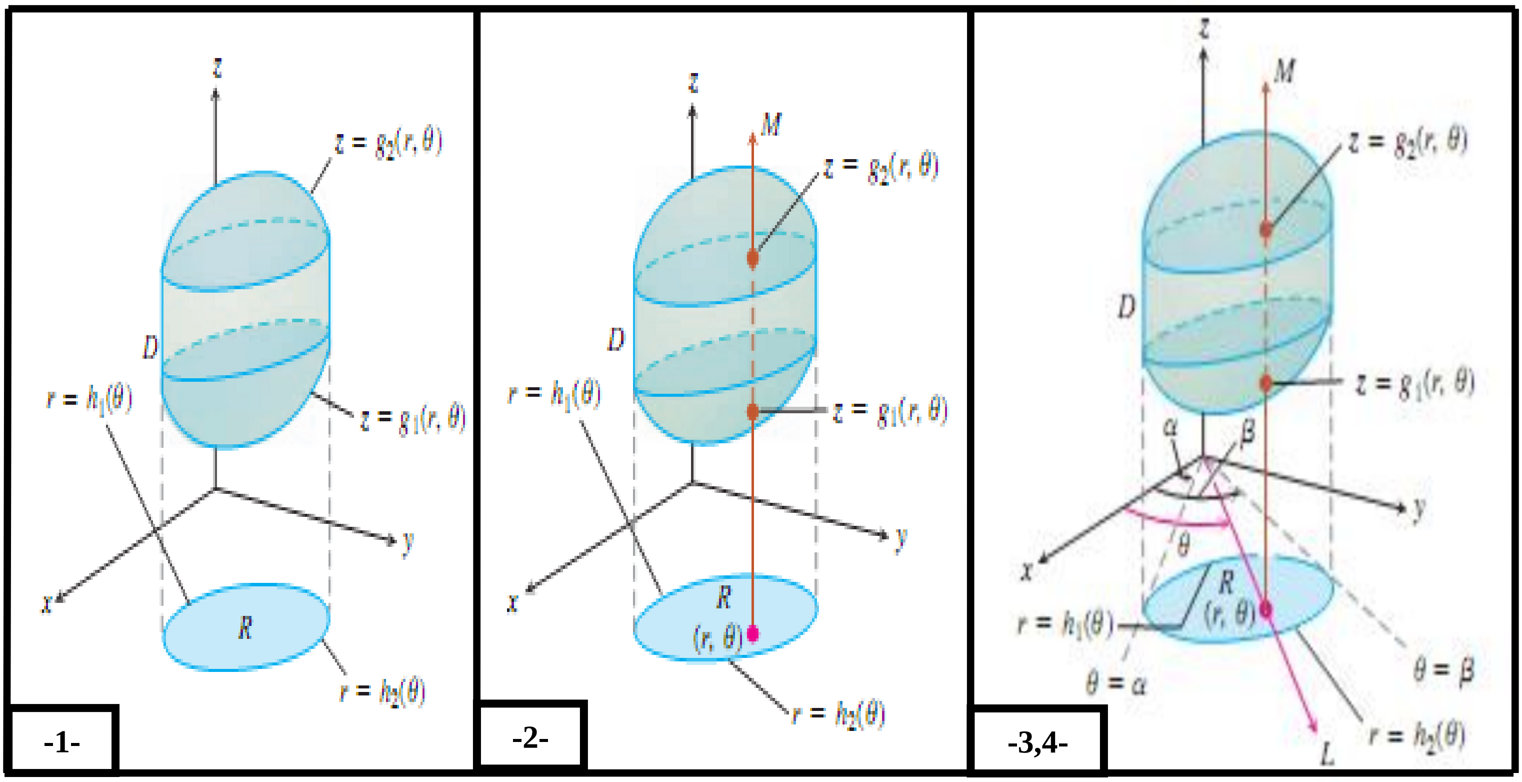
To evaluate

$$\iiint_D f(r, \theta, z)\,dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.
2. *Find the z-limits of integration*. Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.
3. *Find the r-limits of integration*. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.
4. *Find the θ-limits of integration*. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\,dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\,dz\,r\,dr\,d\theta.$$



EXAMPLE Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y-1)^2 = 1$. Its polar coordinate equation is

$$\begin{aligned} x^2 + (y-1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r\sin\theta &= 0 \\ r &= 2\sin\theta. \end{aligned}$$

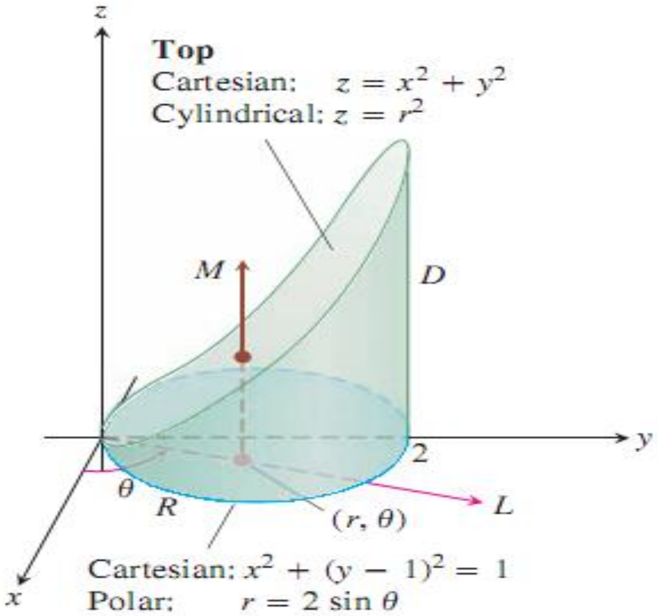


FIGURE Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).

We find the limits of integration, starting with the z-limits. A line M through a typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2\sin\theta$.

Finally we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Spherical Coordinates and Integration

Spherical coordinates locate points in space with two angles and one distance, as shown in Figure **below** The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin. Unlike *r, the variable ρ is never negative.* The second coordinate, ϕ, is the angle $\overrightarrow{OP}$ makes with the positive z-axis. It is required to lie in the interval $[0, \pi]$. The third coordinate is the angle θ as measured in cylindrical coordinates.

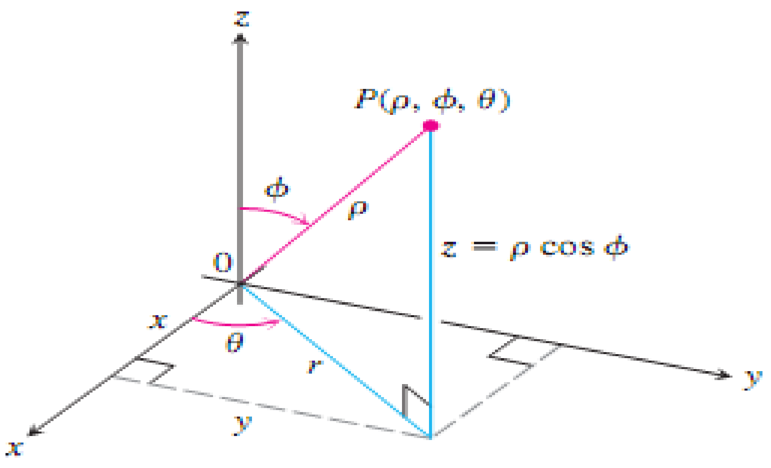


FIGURE The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \le \phi \le \pi)$.
3. θ is the angle from cylindrical coordinates $(0 \le \theta \le 2\pi)$.

$$\iiint_D f(\rho, \phi, \theta)\, dV = \iiint_D f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

In spherical coordinates, we have

$$dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is as follows. We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin \phi, \qquad x = r\cos\theta = \rho \sin\phi \cos\theta,$$
$$z = \rho \cos \phi, \qquad y = r\sin\theta = \rho \sin\phi \sin\theta, \tag{1}$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

EXAMPLE Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$.

Solution We use Equations (1) to substitute for x, y, and z:

$$x^2 + y^2 + (z-1)^2 = 1$$
$$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 = 1 \qquad \text{Eqs. (1)}$$
$$\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$
$$\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} = 2\rho\cos\phi$$
$$\rho^2 = 2\rho\cos\phi$$
$$\rho = 2\cos\phi. \qquad \rho > 0$$

The angle ϕ varies from 0 at the north pole of the sphere to $\pi/2$ at the south pole; the angle θ does not appear in the expression for ρ, reflecting the symmetry about the z-axis (see Figure).

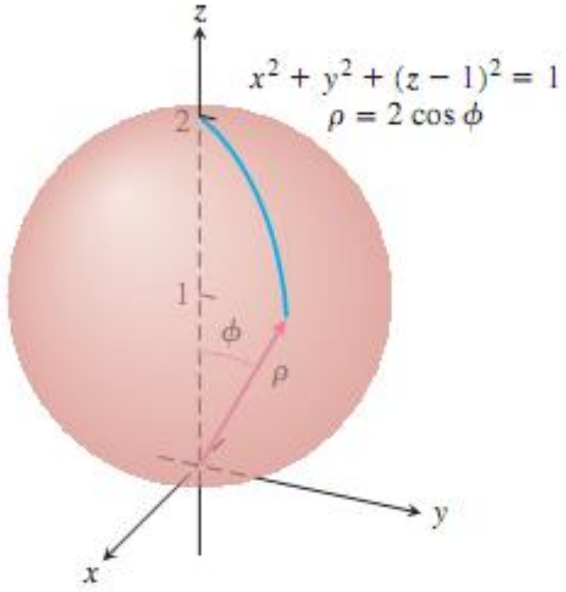


FIGURE The sphere in Example

How to Integrate in Spherical Coordinates

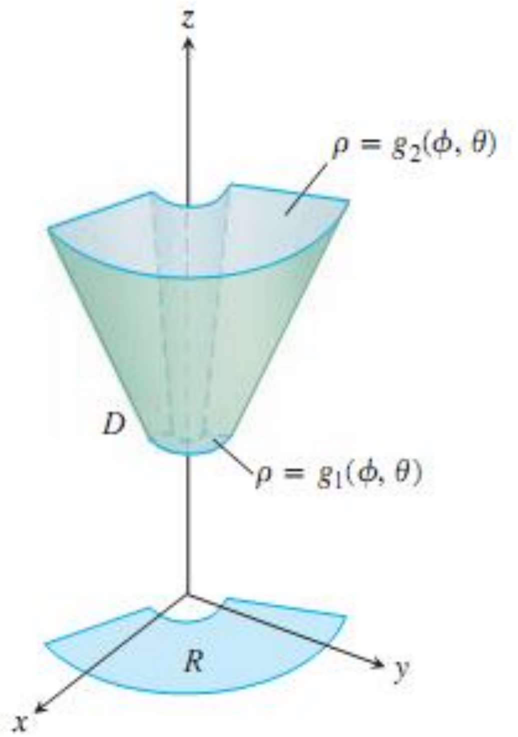
To evaluate

$$\iiint_D f(\rho, \phi, \theta)\, dV$$

over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *Sketch.* Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.
2. *Find the ρ-limits of integration.* Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.
3. *Find the ϕ-limits of integration.* For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$. These are the ϕ -limits of integration.
4. *Find the θ-limits of integration.* The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$



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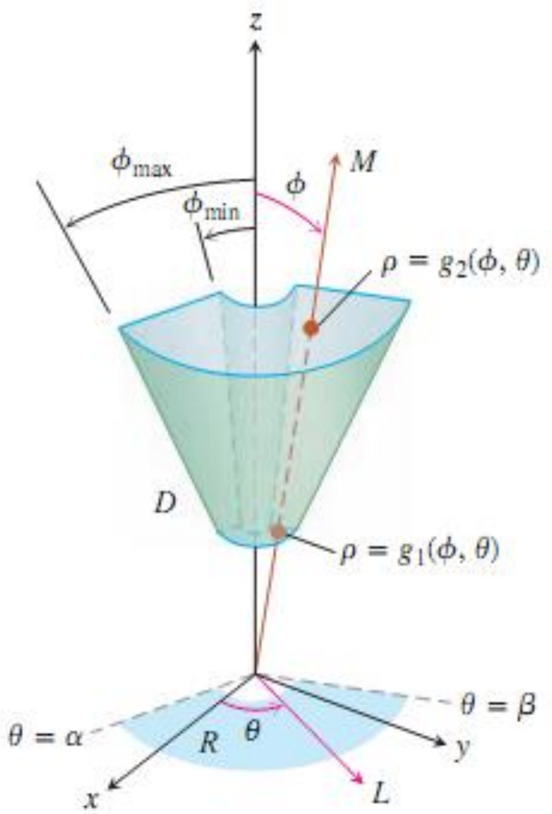


2, 3, 4

EXAMPLE Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

Solution The volume is $V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

1 To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy-plane (Figure).

2 *The ρ-limits of integration.* We draw a ray M from the origin through D making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

3 *The ϕ-limits of integration.* The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

4 *The θ-limits of integration.* The ray L sweeps over R as θ runs from 0 to 2π. The volume is

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$

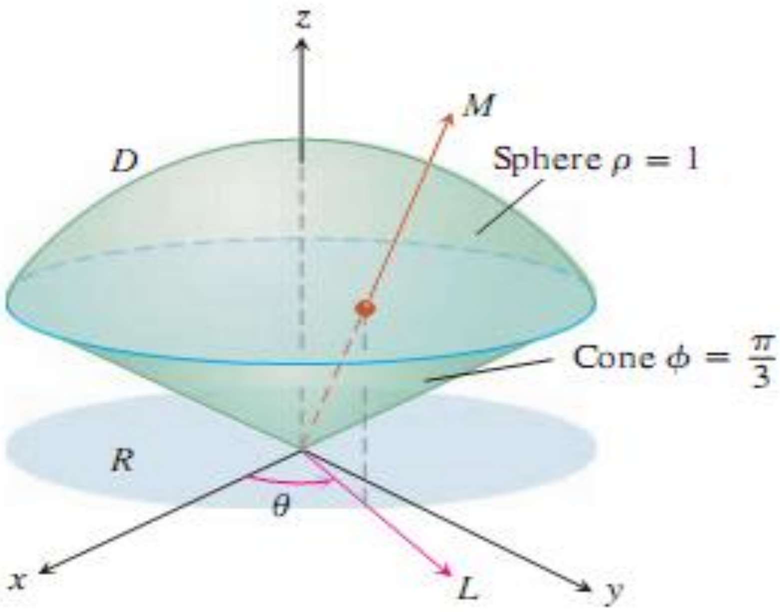


FIGURE The ice cream cone in Example .

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$dV = dx\,dy\,dz \quad \longrightarrow \quad \textbf{(Cartizain)}$$

$$= dz\,r\,dr\,d\theta \quad \longrightarrow \quad \textbf{(Polar)}$$

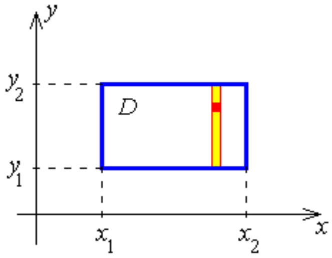
$$= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \quad \longrightarrow \quad \textbf{(Spher)}$$

Practices

Double integrals :

Example

Evaluate $\int_{-1}^{1}\int_{0}^{3} x^2 + y^2 \; dy \; dx$



Solution :

$$\int_{-1}^{1}\int_{0}^{3} x^2 + y^2 \; dy \; dx = \int_{-1}^{1}(\int_{0}^{3} x^2 + y^2 \; dy \;)\, dx = \int_{-1}^{1} \; x^2 y + \frac{y^3}{3}\Big|_0^3 dx = \int_{-1}^{1} 3x^2 + 9 \; dx$$

$$= \quad 3x^2 + 9x \Big|_{-1}^{1} \; = 20$$

Example 2

Evaluate $\int_{0}^{\pi/2}\int_{0}^{\cos\theta} r^2 \sin^2\theta \; dr \; d\theta$ { Note: $\cos^3\Theta = \cos^2\Theta\cos\Theta$}

Solution: {$\cos^3\Theta \sin^2\Theta d\Theta = (1-\sin^2\Theta)\sin^2\Theta d(\sin\Theta) = (1-u^2)u^2 du$}

$\int_{0}^{\pi/2}\int_{0}^{\cos\theta} r^2 \sin^2\theta \; dr \; d\theta =$ تبسيط (ملاحظة u=sinΘ)

$$\int_{0}^{\pi/2} \frac{r^3}{3}\Big|_0^{\cos\theta} \sin^2\theta \;\; d\theta = \frac{1}{3}\int_{0}^{\pi/2}\cos^3\theta \sin^2\theta \;\; d\theta$$

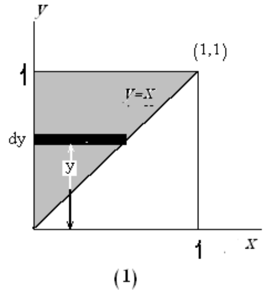
$$= \frac{1}{3}\int_{0}^{\pi/2}(1-\sin^2\theta)\sin^2\theta \;\; d(\sin\theta) = \frac{1}{3}\int_{0}^{\pi/2}(1-u^2)u^2 du = \frac{1}{3}(\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5})\Big|_0^{\pi/2} = \frac{2}{45}$$

Example

Evaluate $\iint_R x^3y^2\, dA$ where R is the region bounded by y = x; y = 1 and x=0, integrate in order (dxdy) and then in order of (dydx) .

Solution :

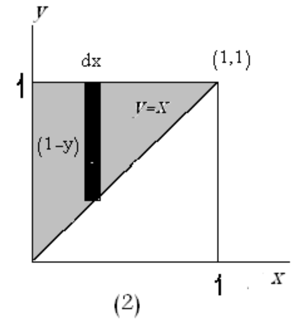
First integrated with respect to order (dxdy) as in(figure.1)



(1)

$$, \iint_R x^3y^2\, dA = \int_0^1 \int_{x=0}^{x=y} x^3y^2 dxdy = \int_0^1 y^2 \frac{x^4}{4}\Big|_0^y dy = \frac{1}{4}\int_0^1 y^6 dy = \frac{y^7}{28}\Big|_0^1 = \frac{1}{28}$$

Second integrated with respect to order (dydx) as in (figure.2)



(2)

$$\iint_R x^3y^2\, dA =$$

$$\int_0^1 \int_x^1 x^3y^2 dxdy = \int_0^1 x^3 \frac{y^3}{3}\Big|_x^1 dx = \frac{1}{3}\int_0^1 x^3(1-x^3)\,dx = \frac{1}{3}(\frac{x^4}{4} - \frac{x^7}{7})\Big|_0^1 = \frac{1}{3}(\frac{1}{4} - \frac{1}{7}) = \frac{1}{28}$$

Example

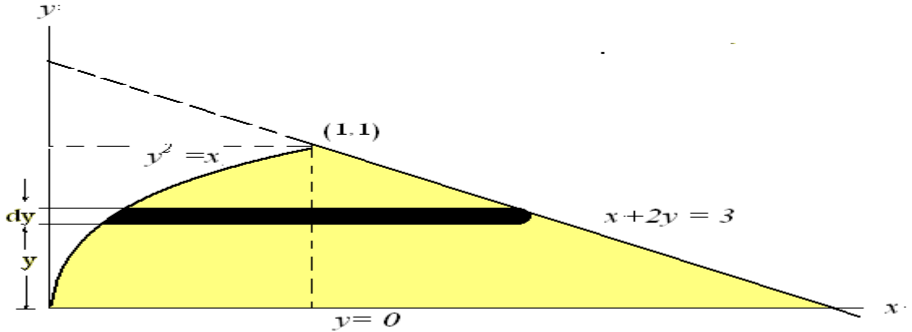
Evaluate $\iint_R x + y\; dxdy$ where R is the region bounded by $y^2 = x$; x+2y = 3 and y= 0 in the first quadrant.

Solution

$$\iint_R x + y\; dxdy =$$

$$\int_0^1 \int_{y^2}^{3-2y} (x+y) dxdy = \int_0^1 \frac{x^2}{2} + yx\Big|_{y^2}^{3-2y} dy = \int_0^1 \left\{ \frac{(3-2y)^2 - (y^2)^2}{2} + y\left[(3-2y) - y^2\right] \right\} dy$$

$$= \int (\frac{9}{2} - 3y - y^3 - y^4) dy = 2.55$$



Example

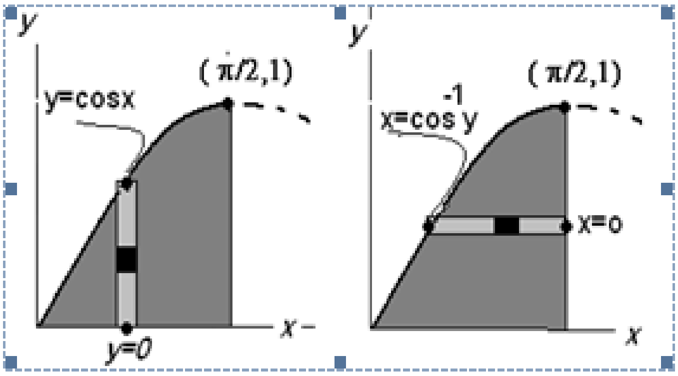
Evaluate

$$\int_0^1 \int_{\cos^{-1} y}^{\pi/2} \cos(a\sin x)\, dx dy = \int_0^{\pi/2} \int_0^{\cos x} \cos(a\sin x)\, dy dx$$

$$\int_0^{\pi/2} y\cos(a\sin x)\Big|_0^{\cos x} dx = \int_0^{\pi/2} \cos x \cos(a\sin x)\, dx$$

$$= \int_0^{\pi/2} \cos(a\sin x)\, d(\sin x) = \frac{1}{a}\sin(a\sin x)\Big|_0^{\pi/2}$$

$$= \frac{1}{a}\sin a$$

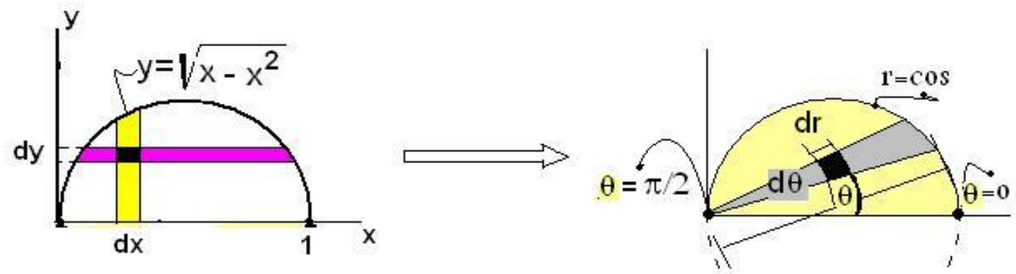


Example:

Evaluate $\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{y^2}{\sqrt{y^2+x^2}}\, dy\, dx$, change from Cartesian to polar coordinate then evaluate.

Solution4

$$\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{y^2}{\sqrt{y^2+x^2}}\, dy\, dx \equiv \int_0^{\pi/2} \int_0^{\cos\theta} r^2 \sin^2\theta\, dr\, d\theta$$



$$\int_0^{\pi/2} \int_0^{\cos\theta} r^2 \sin^2\theta\, dr\, d\theta$$

$$\int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{\cos\theta} \sin^2\theta\, dr\, d\theta = \frac{1}{3}\int_0^{\pi/2} \cos^3\theta \sin^2\theta \;\; d\theta =$$

$$= \frac{1}{3}\int_0^{\pi/2} (1-\sin^2 o)\sin^2\theta \;\; d(\sin\theta) \;= \frac{1}{3}\left(\frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5}\right)\Big|_0^{\pi/2} \;= \frac{2}{45}$$

Example

Find the area enclosed by one loop of the curve $r = \cos 2\theta$.

Boundaries:

$0 \le r \le \cos 2\theta \; ; \quad -\frac{\pi}{4} \le \theta \le +\frac{\pi}{4}$

Solution : the Area is :

$$A = \iint_D 1\, dA = \int_{-\pi/4}^{+\pi/4} \int_0^{\cos 2\theta} 1\, r\, dr\, d\theta$$

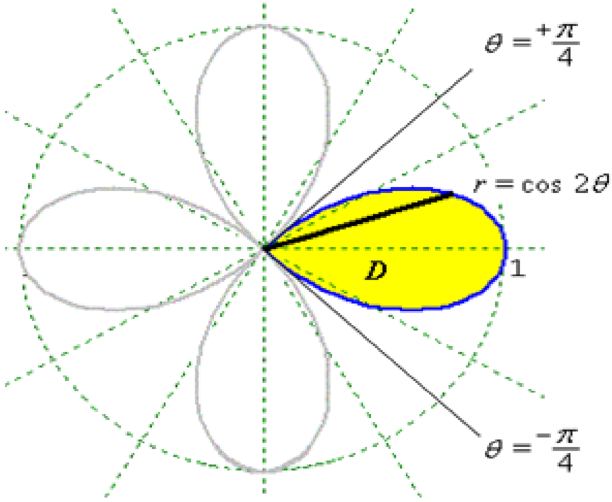
$$= \int_{-\pi/4}^{+\pi/4} \left[\frac{r^2}{2} \right]_0^{\cos 2\theta} d\theta$$

$$= \int_{-\pi/4}^{+\pi/4} \left(\frac{\cos^2 2\theta}{2} - 0 \right) d\theta$$

$$= \int_{-\pi/4}^{+\pi/4} \frac{\cos 4\theta + 1}{4}\, d\theta$$

$$= \left[\frac{\sin 4\theta}{16} + \frac{\theta}{4} \right]_{-\pi/4}^{+\pi/4} = \left(0 + \frac{\pi}{16} \right) - \left(0 - \frac{\pi}{16} \right)$$

Therefore $A = \frac{\pi}{8}$



Evaluate :

$\iint_R y\sqrt{1+x^3}\, dA$ where R is the triangle with vertices (0, 0), (3, 0) and (3, 2)

Solution:

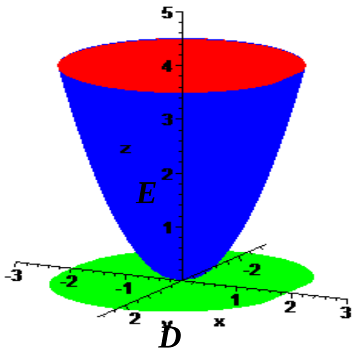
$$\iint_R y\sqrt{1+x^3}\, dA = \int_0^3 \int_0^{2x/3} y\sqrt{1+x^3}\, dy\, dx = \int_0^3 \frac{y^2}{2}\Big|_0^{2x/3} (1+x^3)^{1/2} dx = \frac{2}{9}\int_0^3 x^2 (1+x^3)^{1/2} dx$$

$$= \frac{2}{27} \cdot \frac{(1+x^3)^{3/2}}{3/2}\Big|_0^3 = \frac{4}{81}\left[28\sqrt{28} - 1\right] \approx 7.267262885$$

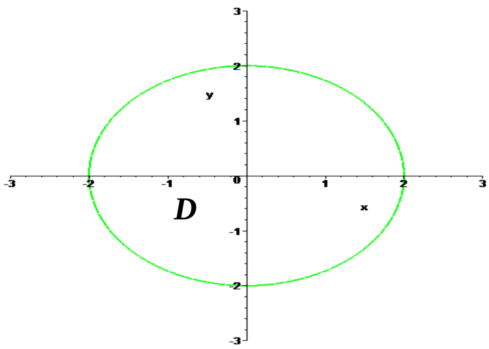
Triple integrals

Example : Use a triple integral to find the volume of the solid bounded by the graphs of $z = x^2 + y^2$ and the plane $z = 4$.

Solution: The following graph shows a plot of the paraboloid $z = x^2 + y^2$ (in blue), the plane $z = 4$ (in red), and its projection onto the *x*-*y* plane (in green).



The triple integral $\iiint_E dV$ will evaluate the volume of this surface. In the *z* direction, the surface *E* is bounded between the graphs of the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. This will make up the limits of integration in terms of *z*. The limits for *y* and *x* are determined by looking at the projection *D* given on the *x*-*y* plane, which is the graph of the circle $x^2 + y^2 = 4$ given as follows:



Taking the equation $x^2 + y^2 = 4$ and solving for *y* gives $y = \pm\sqrt{4 - x^2}$. Thus the limits of integration of *y* will range from $y = -\sqrt{4 - x^2}$ to $y = \sqrt{4 - x^2}$. The integration limits in terms of *x* hence range from *x* = -2 to *x* = 2. Thus the volume of the region *E* can be found by evaluating the following triple integral:

Volume of $E = \iiint_E dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dz\, dy\, dx$. If we evaluate the innermost integral we get the following:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]\Big|_{z=x^2+y^2}^{z=4}\ dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+y^2)\right]dy\,dx$$

Since the limits involving y involve two radicals, integrating the rest of this result in rectangular coordinates is a tedious task. However, since the region D on the x-y plane given by $x^2+y^2=4$ is circular, it is natural to represent this region in polar coordinates.
Using the fact that the radius r ranges from $r=0$ to $r=2$ and that θ ranges from $\theta=0$ to $\theta=2\pi$ and also that in polar coordinates, the conversion equation is $r^2=x^2+y^2$, the iterated integral becomes

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4-(x^2+y^2)\right]dy\,dx = \int_{0}^{2\pi}\int_{0}^{2} (4-r^2)r\,dr\,d\theta$$

Evaluating this integral in polar coordinates, we obtain

$$\int_{0}^{2\pi}\int_{0}^{2} (4-r^2)r\,dr\,d\theta = \int_{0}^{2\pi}\int_{0}^{2} (4r-r^3)\,dr\,d\theta \qquad \text{(Distribute } r\text{)}$$

$$= \int_{\theta=0}^{\theta=2\pi} (2r^2-\frac{1}{4}r^4)\Big|_{r=0}^{r=2} d\theta \qquad \text{(Integrate)}$$

$$= \int_{\theta=0}^{\theta=2\pi} [(2(2)^2 - \frac{1}{4}(2)^4) - 0]\,d\theta \qquad \text{(Sub in limits of integration)}$$

$$= \int_{\theta=0}^{\theta=2\pi} 4\,d\theta \qquad \text{(Simplify)}$$

$$= 4\theta\Big|_{\theta=0}^{\theta=2\pi} \qquad \text{(Integrate)}$$

$$= 4(2\pi) - 4(0) \qquad \text{(Sub in limits of integration)}$$

$$= 8\pi$$

Thus, the volume of E is 8π

Example :

Verify the formula $V=\frac{4}{3}\pi a^3$ for the volume of a sphere of radius a.

$$V = \iiint_V 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \left(\int_0^a r^2\,dr\right)\left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_0^{2\pi}1\,d\phi\right)$$

$$= \left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi} \left[\phi\right]_0^{2\pi} = \left(\frac{a^3}{3}-0\right)(+1+1)(2\pi-0)$$

Therefore : $V=\frac{4}{3}\pi a^3$

Evaluate the following integrals :

1- Find the volume of the solid bounded by y=x and $y=x^2$, z=0 and z=x+y

$$V=\int_0^1\int_{x^2}^{x}\int_0^{x+y} dzdydx = 0.15$$

2- $\displaystyle\int_0^2\int_0^{\frac{6-3x}{2}}\int_0^{6-3x-2y} x\ dzdydx = 3$

3- $\displaystyle 4\int_0^2\int_0^{\sqrt{2y-y^2}}\int_0^{\sqrt{x^2+y^2}} \sqrt{x^2+y^2}\,dzdxdy = 9.4248$

4- $\displaystyle\int_0^1\int_0^{x^2}\int_0^{y}(1)dzdydx$ $\displaystyle =\int_0^1\int_0^{x^2}\int_0^{y} dzdydx = \int_0^1\int_0^{x^2} ydydx = \int_0^1 \left.\frac{y^2}{2}\right|_0^{x^2} dx = \int_0^1 \frac{x^4}{2}dx = \left.\frac{x^5}{10}\right|_0^1 = \frac{1}{5}$

Homework

1-
Evaluate $\iint_R x\, e^{y^2}\, dA$ where R is the region bounded by y=x^2; x=0 and y= 4

Solution : (13.4)

2-
Evaluate in order of dxdy and dydx

$$\int_0^2 \int_{x^2}^4 x\, e^{y^2} dA = \int_0^2 \int_{x^2}^4 x\, e^{y^2} dydx$$

3-
Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{x-x^2}} \frac{y^2}{\sqrt{y^2+x^2}}\, dy\, dx$

4-
Evaluate $\displaystyle\int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2}\, dxdy$

5- Evaluate $I = \iint_R \left(6x+2y^2\right) dA$
where R is the region enclosed by the parabola $x = y^2$ and the line $x + y = 2$.

6-
Find $\displaystyle\iint_R \frac{1}{\left(x^2+y^2+1\right)^2} dA$ where R is the region in the first quadrant bounded by the circle $x^2+y^2=9 \quad x=0, \; and \;\; y=x$

7-

Evaluate the integral $\iint_R (x+y) dA$ where R is the region bounded by xy = 4 and x+y = 5

Differential Equations

(Differential Equations)

Differential Equations :

A differential equation is an equation that involves one or more derivatives, or differentials. Differential equations are classified by:

1. ***Type:*** Ordinary or partial.
2. ***Order:*** The order of differential equation is the highest order derivative that occurs in the equation.
3. ***Degree:*** The exponent of the highest power of the highest order derivative.

Ordinary D.Eqs : is a differential equation that the unknown function depends on only ***one*** independent variable.

Partial D.Eqs : is a differential equation that the unknown function depends on ***two or more*** independent variable

Example for a partial D.Eqs.. is :

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

Example for ordinary D.Eqs.. is :

Ex1:

$$\frac{dy}{dx} = 5x + 3$$ 1st order-1st degree

Ex2:

$$\left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{d^2 y}{dx}\right)^5$$ 3rd order-2nd degree

Ex3:

$$4\frac{d^3 y}{dx^3} + \sin x \frac{d^2 y}{dx^2} + 5xy = 0$$ 3rd order-1st degree

Exercise: Find the order and degree of these differential equations.

1. $\frac{dy}{dx} + \cos x = 0$ ans:1st order-1st degree
2. $3dx + 4y^2 dy = 0$ ans:1st order-1st degree
3. $\frac{d^2 y}{dx^2} + y = y^2$, **(H. W)**
4. $(y'')^2 + 2y' = x^2$ **(H. W)**
5. $y''' + 2(y'')^2 = xy$ **(H. W)**

Solution of the differential equation :

The solution of the differential equation in the unknown function y and the independent variable x is a function y(x) that satisfies the differential equation.

Ordinary Differential Equations:

Ordinary Differential Equations are equations involve derivatives.

Initial Condition(s) :

Initial Condition(s) are a condition, or set of conditions, on the solution that will allow us to determine which solution that we are after. Initial conditions are of the form,

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

So, in other words, initial conditions are values of the solution and/or its derivative(s) at specific points. *The number of initial conditions that are required for a given differential equation will depend upon the* ***order*** *of the differential equation as we will see.*

Initial Value Problem:

An **I**nitial **V**alue **P**roblem (or **IVP**) is a differential *equation* ***along with*** an appropriate number of *initial* conditions.

Example	The following is an IVP. $4x^2y'' + 12xy' + 3y = 0 \qquad y(4) = \frac{1}{8},\ y'(4) = -\frac{3}{64}$

Interval of Validity:

The interval of validity for an IVP with initial condition(s) is the largest possible interval on which the solution is **valid** and contains $\mathbf{t_0}$.

$$y(t_0) = y_0 \qquad \text{and/or} \qquad y^{(k)}(t_0) = y_k$$

General Solution :

The general solution to a differential equation is the most **general** form that the solution can take and ***doesn't take any initial conditions*** into account.

Actual Solution

The actual solution to a differential equation is the ***specific solution*** that not only satisfies the differential equation, but also satisfies the ***given initial condition(***s).

Explicit solution :

An explicit solution is any solution that is given in the form **y = y(t)** . In other words, the only place that "**y**" actually shows up is once on the left side and only raised to the first power.

implicit solution :

An implicit solution is any solution that isn't in explicit form. Note that it is possible to have either general implicit/explicit solutions and actual implicit/explicit solutions.

Methods to solve Ordinary *First* Order D.E.s.

1- Linear Differential Equations
2- Separable Equations.
3- Homogeneous.
4- Linear equation of first order differential equations.
5- Exact differential equations.
6- Bernoulli differential equation

1- Linear Differential Equations :

A linear differential equation is any differential equation that can be written in the following form :

$$a_n(t)y^{(n)}(t)+a_{n-1}(t)y^{(n-1)}(t)+\cdots+a_1(t)y'(t)+a_0(t)y(t)=g(t) \qquad \textbf{...... (1)}$$

Example :

$$4x^2y''+12xy'+3y=0$$

The important thing to note about linear differential equations is that ***there are no products of the function, y(t) , and its derivatives(y*y')*** and neither ***the function or its derivatives occur to any power other than the first power($(y)^2$, $(y')^2$)*** .
The coefficients $a_0(t)$ .... $a_n(t)$ and g(t) can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions. Only the function, y (t) and its derivatives are used in determining if a differential equation is linear.

If a differential equation cannot be written in the form, **(1)** then it is called a ***non-linear differential equation***. For example of non- linear differential equation is in equation **(2)** that given below :

Example of non- linear differential equation :

$$\sin(y)\frac{d^2y}{dx^2}=(1-y)\frac{dy}{dx}+y^2e^{-5y} \qquad \textbf{.................... (2)}$$

Example Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

Solution We'll need the first and second derivative to do this.

$$y'(x) = -\frac{3}{2}x^{-\frac{5}{2}} \qquad y''(x) = \frac{15}{4}x^{-\frac{7}{2}}$$

Plug these as well as the function into the differential equation.

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right) + 12x\left(-\frac{3}{2}x^{-\frac{5}{2}}\right) + 3\left(x^{-\frac{3}{2}}\right) = 0$$

$$15x^{-\frac{3}{2}} - 18x^{-\frac{3}{2}} + 3x^{-\frac{3}{2}} = 0$$

$$0 = 0$$

So, $y(x) = x^{-\frac{3}{2}}$ does satisfy the differential equation and hence is a solution. Why then did I include the condition that $x > 0$? I did not use this condition anywhere in the work showing that the function would satisfy the differential equation.

To see why recall that

$$y(x) = x^{-\frac{3}{2}} = \frac{1}{\sqrt{x^3}}$$

In this form it is clear that we'll need to avoid $x = 0$ at the least as this would give division by zero.

Also, there is a general rule of thumb that we're going to run with in this class. This rule of thumb is : Start with real numbers, end with real numbers. In other words, if our differential equation only contains real numbers then we don't want solutions that give complex numbers. So, in order to avoid complex numbers we will also need to avoid negative values of x.

Example 2 $y(x)=x^{-\frac{3}{2}}$ is a solution to $4x^2y''+12xy'+3y=0$, $y(4)=\frac{1}{8}$, and $y'(4)=-\frac{3}{64}$.

Solution As we saw in previous example the function is a solution and we can then note that

$$y(4)=4^{-\frac{3}{2}}=\frac{1}{\left(\sqrt{4}\right)^3}=\frac{1}{8}$$

$$y'(4)=-\frac{3}{2}4^{-\frac{5}{2}}=-\frac{3}{2}\frac{1}{\left(\sqrt{4}\right)^5}=-\frac{3}{64}$$

and so this solution also meets the initial conditions of $y(4)=\frac{1}{8}$ and $y'(4)=-\frac{3}{64}$. In fact, $y(x)=x^{-\frac{3}{2}}$ is the only solution to this differential equation that satisfies these two initial conditions.

2 - Separable Equations:

A separable differential equation is any differential equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x) \qquad \text{.................. } (1)$$

Note that in order for a differential equation to be separable all the y's in the differential equation ***must*** be multiplied by the derivative and all the ***x's*** in the differential equation must be on the other side of the equal sign.

Solving separable differential equation is fairly easy. We first rewrite the differential equation as the following

$$N(y)dy = M(x)dx$$

Then you integrate both sides.

$$\int N(y)dy = \int M(x)dx \qquad \text{............}(2)$$

So, after doing the integrations in (2) you will have an implicit solution that you can hopefully solve for the explicit solution, y(x). ***Note that it won't always be possible to solve for an explicit solution***.

Recall from the Definitions section that an implicit solution is a solution that is not in the form ***y = y(x)*** while an explicit solution has been written in that form.

Example 1 :

Solve: $\frac{dy}{dx} = e^{x+y}$

Sol.:

$$\frac{dy}{dx} = e^{x} \cdot e^{y} \Rightarrow \Rightarrow \frac{dy}{e^{y}} = e^{x}dx \Rightarrow \Rightarrow \int e^{-y}dy = \int e^{x} \cdot dx \Rightarrow \Rightarrow$$

$$\Rightarrow \Rightarrow -\int e^{-y} \cdot (-dy) = \int e^{x}dx \Rightarrow -e^{-y} = e^{x} + c$$

Ex.2:

Solve : $(1+x)\frac{dy}{dx} = x(y^{2}+1)$

Sol.:

$$\int \frac{dy}{(y^{2}+1)} = \int \frac{x}{x+1}dx$$

$$\tan^{-1} y = \int dx - \int \frac{1}{x+1}dx$$

$$\tan^{-1} y = x - \ln|x+1| + c$$

Ex.3: Solve $\frac{dy}{dx} = (y-x)^{2} \quad \cdots(1)$

Sol.: *put* $y - x = u$, $\frac{dy}{dx} - 1 = \frac{du}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 1 \quad \ldots\ldots (2)$

$$\frac{du}{dx} + 1 = u^{2} \Rightarrow \int \frac{du}{u^{2}-1} = \int dx$$

$$\therefore \int \left[\frac{1/2}{u-1} + \frac{-1/2}{u+1}\right]du = \int dx$$

$$\frac{1}{2}\left[\ln(u-1) - \ln(u+1)\right] = x + c$$

$$\frac{1}{2}\ln\frac{u-1}{u+1} = x + c$$

$$\frac{u-1}{u+1} = e^{2x+c}$$

H.W. :

Solve the differential equation :

1- $\frac{dy}{dx} = (1 + y^{2})e^{x}.$

2- $2\sqrt{xy}\,\frac{dy}{dx} = 1, \quad x, y > 0$

Example : Solve the following differential equation and determine the interval of validity for the solution.

$$\frac{dy}{dx} = 6y^2 x \qquad y(1) = \frac{1}{25}$$

Solution :

$$y^{-2} dy = 6x\,dx$$
$$\int y^{-2} dy = \int 6x\,dx$$
$$-\frac{1}{y} = 3x^2 + c$$

So, we now have an implicit solution. This solution is easy enough to get an explicit solution, however before getting that it is usually easier to find the value of the constant at this point. So apply the initial condition and find the value of **c**.

$$-\frac{1}{1/25} = 3(1)^2 + c \qquad c = -28$$

Plug this into the general solution and then solve to get an explicit solution.

$$-\frac{1}{y} = 3x^2 - 28$$
$$y(x) = \frac{1}{28 - 3x^2}$$

Now, as far as solutions go we've got the solution. We do need to start worrying about intervals of validity however.

Recall that there are ***two conditions*** that define an interval of validity. **First, it must be a continuous interval with no breaks or holes in it. Second it must contain the value of the independent variable in the initial condition**, x = 1 in this case.

So, for our case we've got to avoid two values of x. Namely, $x \neq \pm\sqrt{\frac{28}{3}} \approx \pm 3.05505$ since these will give us division by zero. This gives us three possible intervals of validity.

$$-\infty < x < -\sqrt{\frac{28}{3}} \qquad -\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}} \qquad \sqrt{\frac{28}{3}} < x < \infty$$

However, only one of these will contain the value of x from the initial condition and so we can see that

$$-\sqrt{\frac{28}{3}} < x < \sqrt{\frac{28}{3}}$$

must be the interval of validity for this solution.

Example 2 Solve the following IVP , and find explicit solution

$$y' = \frac{3x^2+4x-4}{2y-4} \qquad y(1)=3$$

Solution

This differential equation is clearly separable, so let's put it in the proper form and then integrate both sides.

$$(2y-4)dy = (3x^2+4x-4)dx$$

$$\int(2y-4)dy = \int(3x^2+4x-4)dx$$

$$y^2-4y = x^3+2x^2-4x+c$$

We now have our implicit solution, so as with the first example let's apply the initial condition at this point to determine the value of c.

$$(3)^2-4(3) = (1)^3+2(1)^2-4(1)+c \qquad c=-2$$

The implicit solution is then

$$y^2-4y = x^3+2x^2-4x-2$$

We now need to find the explicit solution. This is actually easier than it might look and you already know how to do it. First we need to rewrite the solution a little

$$y^2-4y-(x^3+2x^2-4x-2)=0$$

To solve this all we need to recognize is that this is quadratic in y and so we can use the quadratic formula to solve it. However, unlike quadratics you are used to, at least some of the "constants" will not actually be constant, but will in fact involve x's.

So, upon using the quadratic formula on this we get.

$$\mathbf{y} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \;\Rightarrow\; y(x) = \frac{4 \pm \sqrt{16-4(1)\left(-(x^3+2x^2-4x-2)\right)}}{2}$$

$$= \frac{4 \pm \sqrt{16+4(x^3+2x^2-4x-2)}}{2}$$

Next, notice that we can factor a 4 out from under the square root (it will come out as a 2…) and then simplify a little.

$$y(x) = \frac{4 \pm 2\sqrt{4+(x^3+2x^2-4x-2)}}{2}$$

$$= 2 \pm \sqrt{x^3+2x^2-4x+2}$$

We are almost there. Notice that we've actually got two solutions here (the "$\pm$") and we only want a single solution. In fact, only one of the signs can be correct. So, to figure out which one is correct we can reapply the initial condition to this. Only one of the signs will give the correct value so we can use this to figure out which one of the signs is correct. Plugging $x = 1$ into the solution gives.

$$3 = y(1) = 2 \pm \sqrt{1+2-4+2} = 2 \pm 1 = 3,1$$

In this case it looks like the "+" is the correct sign for our solution. Note that it is completely possible that the "–" could be the solution so don't always expect it to be one or the other.

The explicit solution for our differential equation is. $y(x) = 2+\sqrt{x^3+2x^2-4x+2}$

Example : Solve the following IVP :

$$y' = e^{-y}(2x-4) \qquad y(5) = 0$$

Solution

This differential equation is easy enough to separate, so let's do that and then integrate both sides.

$$e^{y}\,dy = (2x-4)\,dx$$

$$\int e^{y}\,dy = \int (2x-4)\,dx$$

$$e^{y} = x^2 - 4x + c$$

Applying the initial condition gives

$$1 = 25 - 20 + c \qquad c = -4$$

This then gives an implicit solution of.

$$e^{y} = x^2 - 4x - 4$$

We can easily find the explicit solution to this differential equation by simply taking the natural log of both sides.

$$y(x) = \ln\left(x^2 - 4x - 4\right)$$

H.W. : Solve the following IVP

1- $\dfrac{dr}{d\theta} = \dfrac{r^2}{\theta} \qquad r(1) = 2$

2- $\dfrac{dy}{dt} = e^{y-t}\sec(y)\left(1+t^2\right) \qquad y(0) = 0$

3-Homogeneous differential equations:

Sometimes a D.Eq. which variables can't be separated can be transformed by a change of variables into an equation which variables can be separated. This is the case with any equation that can be put into form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \ldots(1)$$

Such an equation is called ***homogenous***.

Put $\frac{y}{x} = u \Rightarrow y = ux$, $\frac{dy}{dx} = u + x \cdot \frac{du}{dx}$ and (1) becomes

$$x \cdot \frac{du}{dx} + u = f(u)$$

Example 1:

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

Sol.:

$$\frac{dy}{dx} = \frac{1 + \frac{y^2}{x^2}}{\frac{y}{x}} \Rightarrow \text{homo.} \quad \text{Put} \quad \frac{y}{x} = u \Rightarrow \frac{dy}{dx} = x \cdot \frac{du}{dx} + u$$

$$x \cdot \frac{du}{dx} + u = \frac{1 + u^2}{u} \Rightarrow x \cdot \frac{du}{dx} = \frac{1 + u^2 - u^2}{u}$$

$$x \cdot \frac{du}{dx} = \frac{1}{u} \quad , \quad \int u \cdot du = \int \frac{dx}{x}$$

$$\frac{u^2}{2} = \ln x + c \Rightarrow \frac{y^2}{2x^2} = \ln x + c$$

Ex.2: Solve the homogenous D.Eq $xdy - 2ydx = 0$

Sol.: $xdy = 2ydx \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$ put $\frac{y}{x} = u \Rightarrow \frac{dy}{dx} = x \cdot \frac{du}{dx} + u$

$$x \cdot \frac{du}{dx} + u = 2u \qquad \ln|x| - \ln|u| = c \Rightarrow \frac{x}{u} = c \Rightarrow \frac{x^2}{y} = c$$

Example : Solve the following IVP :

$$x\,y\,y' + 4x^2 + y^2 = 0 \qquad y(2) = -7, \qquad x > 0$$

Solution

Let's first divide both sides by x^2 to rewrite the differential equation as follows,

$$\frac{y}{x}y' = -4 - \frac{y^2}{x^2} = -4 - \left(\frac{y}{x}\right)^2$$, **v = (y / x) and (dy/dx) = v + x v'**

So, let's plug the substitution into this form of the differential equation to get,

$$v(v + xv') = -4 - v^2$$

Next, rewrite the differential equation to get everything separated out.

$$v\,x\,v' = -4 - 2v^2$$

$$x\,v' = -\frac{4+2v^2}{v}$$

$$\frac{v}{4+2v^2}\,dv = -\frac{1}{x}\,dx$$

Integrating both sides gives,

$$\tfrac{1}{4}\ln\left(4+2v^2\right) = -\ln(x) + c$$

We need to do a little rewriting using basic logarithm properties in order to be able to easily solve this for v.

$$\ln\left(4+2v^2\right)^{\frac{1}{4}} = \ln(x)^{-1} + c$$

Now exponentiate both sides and do a little rewriting

$$\left(4+2v^2\right)^{\frac{1}{4}} = \mathbf{e}^{\ln(x)^{-1}+c} = \mathbf{e}^{c}\mathbf{e}^{\ln(x)^{-1}} = \frac{c}{x}$$

Note that because c is an unknown constant then so is $\mathbf{e}^c$ and so we may as well just call this c as we did above.

Finally, let's solve for v and then plug the substitution back in and we'll play a little fast and loose with constants again.

$$4+2v^2 = \frac{c^4}{x^4} = \frac{c}{x^4}$$

$$v^2 = \tfrac{1}{2}\left(\frac{c}{x^4} - 4\right)$$

$$\frac{y^2}{x^2} = \tfrac{1}{2}\left(\frac{c-4x^4}{x^4}\right)$$

$$y^2 = \tfrac{1}{2}x^2\left(\frac{c-4x^4}{x^4}\right) = \frac{c-4x^4}{2x^2}$$

At this point it would probably be best to go ahead and apply the initial condition. Doing that gives,

$$49 = \frac{c-4(16)}{2(4)} \qquad \Rightarrow \qquad c = 456$$

Finally, plug in c and solve for y to get,

$$y^2 = \frac{228 - 2x^4}{x^2} \qquad \Rightarrow \qquad y(x) = \pm\sqrt{\frac{228 - 2x^4}{x^2}}$$

The initial condition tells us that the "–" must be the correct sign and so the actual solution is,

$$y(x) = -\sqrt{\frac{228 - 2x^4}{x^2}}$$

***H.W.*:** Solve the following IVP :

$$x\,y' = y(\ln x - \ln y) \qquad y(1) = 4, \qquad x > 0$$

Special case :

When we have the differential equation as the form below or its possible to put it like :

$$y' = G(ax + by)$$

In these cases, we'll use the substitution,

$$v = ax + by \qquad \Rightarrow \qquad v' = a + by'$$

Plugging this into the differential equation gives,

$$\tfrac{1}{b}(v' - a) = G(v)$$

$$v' = a + bG(v) \qquad \Rightarrow$$

So. as this form below we made the equation like separable :

$$\boxed{\frac{dv}{a + bG(v)} = dx}$$

So, with this substitution we'll be able to rewrite the original differential equation as a new separable differential equation that we can solve.

Example Solve the following IVP

$$y' - (4x - y + 1)^2 = 0 \qquad y(0) = 2$$

Solution

In this case we'll use the substitution.

$$v = 4x - y \qquad v' = 4 - y'$$

So, plugging this into the differential equation gives,

$$4 - v' - (v+1)^2 = 0$$

$$v' = 4 - (v+1)^2$$

$$\frac{dv}{(v+1)^2 - 4} = -dx$$

Using Partial fraction to find the integration :

$$\int \frac{dv}{v^2 + 2v - 3} = \int \frac{dv}{(v+3)(v-1)} = \int -dx$$

$$\frac{1}{4}\int \frac{1}{v-1} - \frac{1}{v+3}\,dv = \int -dx$$

$$\tfrac{1}{4}\left(\ln(v-1) - \ln(v+3)\right) = -x + c$$

$$\ln\left(\frac{v-1}{v+3}\right) = c - 4x$$

$$\frac{v-1}{v+3} = \mathbf{e}^{c-4x} = c\,\mathbf{e}^{-4x}$$

$$v - 1 = c\,\mathbf{e}^{-4x}(v+3)$$

$$v\left(1 - c\,\mathbf{e}^{-4x}\right) = 1 + 3c\,\mathbf{e}^{-4x}$$

So, let's solve for v and then go ahead and go back into terms of y.

$$v = \frac{1 + 3c\,\mathbf{e}^{-4x}}{1 - c\,\mathbf{e}^{-4x}}$$

$$4x - y = \frac{1 + 3c\,\mathbf{e}^{-4x}}{1 - c\,\mathbf{e}^{-4x}}$$

$$y(x) = 4x - \frac{1 + 3c\,\mathbf{e}^{-4x}}{1 - c\,\mathbf{e}^{-4x}}$$

The last step is to then apply the initial condition and solve for c.

$$2 = y(0) = -\frac{1+3c}{1-c} \qquad \Rightarrow \qquad c = -3$$

The solution is then,

$$y(x) = 4x - \frac{1 - 9\mathbf{e}^{-4x}}{1 + 3\mathbf{e}^{-4x}}$$

H.W: Solve the following IVP

$$y' = e^{9y-x} \qquad y(0) = 0$$

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4 – Linear equations

In order to solve a linear first order differential equation we MUST start with the differential equation in the form shown below. If the differential equation is not in this form then the process we're going to use will not work.

$$\frac{dy}{dx} + p(x)y = g(x)$$

The equation of the form $\frac{dy}{dx} + p(x) \cdot y = g(x)$, where p and g are functions of only "x" or constant is called linear in y and $\frac{dy}{dx}$.

Find integrating factor $(I.f.) = \mu(x) = e^{\int P dx}$, then the general solution is :

$$y \cdot \mu(x) = \int \mu(x) \; g(x) \, dx$$

Solution Process

The solution process for a first order linear differential equation is as follows.

1. Put the differential equation in the correct initial form,
2. Find the integrating factor, $\mu(x)$,

$$\mu(x) = e^{\int p(x)\, dx}$$

3. Multiply everything in the differential equation by $\mu(x)$ and verify that the left side becomes the product rule $(\mu(x)\, y(x))'$ and write it as such.
4. Integrate both sides, make sure you properly deal with the constant of integration.
5. Solve for the solution $y(x)$ ▪

Ex.1: Solve : $\frac{dy}{dx}-\frac{y}{x}=x\cdot e^{x}$

$$P(x)=-\frac{1}{x},\quad Q(x)=x.e^{x}$$

$$(I.f.)=\mu(t)=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=\frac{1}{x}$$

Solution is

$$y\cdot\frac{1}{x}=\int\frac{1}{x}\cdot xe^{x}\cdot dx$$

$$\frac{y}{x}=e^{x}+c$$

Ex.2:

Solve $\frac{dy}{dx}+x.y=x$

$P=x,\ \ Q=x$

$$(I.f.)=e^{\int xdx}=e^{\frac{x^{2}}{2}}$$

Solution is

$$y\cdot e^{\frac{x^{2}}{2}}=\int e^{\frac{x^{2}}{2}}\cdot x\cdot dx$$

$$y\cdot e^{\frac{x^{2}}{2}}=e^{\frac{x^{2}}{2}}+c\Rightarrow y=1+ce^{\frac{-x^{2}}{2}}\ \text{is the solution}$$

H.W:

Find the solution to the following differential equation.

1- $\frac{dv}{dt}=9.8-0.196v$ ans: $v(t)=50+ce^{-0.196t}$

2- $\frac{dy}{dx}+2y=e^{-x}$ ans: $y=e^{-x}+ce^{-2x}$

3- $x\frac{dy}{dx}+3y=\frac{\sin x}{x^{2}}$ ans: $x^{3}y=c-\cos x$

4- *Exact differential equations* :

The next type of first order differential equations that we'll be looking at is Exact differential equations.

The conditions for this method are :

1- The differential equation must be in the following form :

$$M(x,y)dx + N(x,y)dy = 0$$

The " **+** " should be between ***M*** and ***N***, and equal the equation to ***"zero"***

2- The equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if :

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then the general solution is :

$$c = \int Mdx + \int (terms\ \ in\ \ N\ \ \text{do not contains}\ \ x)\text{dy}$$

Where ***" c "*** is constant of the integration.

Ex.1:

Show that the following D.Eq. are exact D.Eq.

a) $(3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 + 2x \quad , \quad \frac{\partial N}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The D.Eq. is exact.

b) $[x\cos(x+y) + \sin(x+y)]dx + (x\cos(x+y)dy = 0$

$$\frac{\partial M}{\partial y} = -x\sin(x+y) + \cos(x+y)$$

$$\frac{\partial N}{\partial x} = -x\sin(x+y) + \cos(x+y)$$

$\therefore$ the D.Eq. is exact.

Ex.2: Is the D.Eq. $\dfrac{dy}{dx} = -\dfrac{(x^2+y^2)}{2xy}$ exact or not?

Sol.

$$2xydy = -(x^2 + y^2)dx$$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \therefore \text{the D.Eq. is exact}$$

Ex.3:

Solve the exact D.Eqs. in Ex.1(a) above $(3x^2y+2xy)dx+(x^3+x^2+2y)dy=0$

Sol.

$$c=\int(3x^2y+2xy)dx+\int 2y\,dy$$

$$=3y\cdot\frac{x^3}{3}+2y\cdot\frac{x^2}{2}+2\cdot\frac{y^2}{3}$$

the solution is $x^3y+x^2y+y^2=c$

Ex.4:

Solve $(x+y)dx+(x+y^2)dy=0$

Sol.

$$\frac{\partial M}{\partial y}=1 \quad , \quad \frac{\partial N}{\partial x}=1$$

$\therefore$ the D.Eq. is exact

$$c=\int Mdx+\int(\textit{terms in } N \text{ do not contains } x)dy$$

$$=(x+y)dx+\int y^2dy$$

$$=\frac{x^2}{2}+xy+\frac{y^3}{3}$$

the solution is $\frac{x^2}{2}+xy+\frac{y^3}{3}=c$

H.W:

1- Solve the following differential equation.

$$2xy-9x^2+\left(2y+x^2+1\right)\frac{dy}{dx}=0$$

2- Solve the following IVP and find the interval of validity for the solution.

$$2xy-9x^2+\left(2y+x^2+1\right)\frac{dy}{dx}=0, \qquad y(0)=-3$$

6- *Bernoulli differential equation*

In this section we are going to take a look at differential equations in the form,

$$y' + p(x)y = q(x)y^n$$

Where ***p(x)*** and ***q(x)*** are continuous functions on the interval we're working on and ***n*** is a real number. Differential equations in this form are called ***Bernoulli Equations***.

First notice that if ***n = 0*** or ***n = 1***, then the equation is linear and we already know how to solve it in these cases. Therefore, in this section we're going to be looking at solutions for values of ***n*** other than these ***two***.

In order to solve these we'll first divide the differential equation by y^n to get,

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

We are now going to use the substitution $v = y^{1-n}$ to convert this into a differential equation in terms of v. As we'll see this will lead to a differential equation that we can solve.

So, taking the derivative gives us,

$$v' = (1-n)y^{-n}y'$$

Now, plugging this as well as our substitution into the differential equation gives,

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

This is a linear differential equation that we can solve for v and once we have this in hand we can also get the solution to the original differential equation by plugging v back into our substitution and solving for y.

Example 1 Solve the following IVP

$$y' + \frac{4}{x}y = x^3 y^2 \qquad y(2) = -1, \qquad x > 0$$

Solution

So, the first thing that we need to do is get this into the "proper" form and that means dividing everything by y^2. Doing this gives,

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3$$

The substitution and derivative that we'll need here is,

$$v = y^{-1} \qquad\qquad v' = -y^{-2}y'$$

With this substitution the differential equation becomes,

$$-v' + \frac{4}{x}v = x^3$$

Here's the solution to this differential equation.

$$v' - \frac{4}{x}v = -x^3 \qquad \Rightarrow \qquad \mu(x) = \mathbf{e}^{\int -\frac{4}{x}dx} = \mathbf{e}^{-4\ln|x|} = x^{-4}$$

$$\int \left(x^{-4}v\right)' dx = \int -x^{-1}\,dx$$

$$x^{-4}v = -\ln|x| + c \qquad \Rightarrow \qquad v(x) = cx^4 - x^4\ln x$$

$$y^{-1} = x^4\left(c - \ln x\right)$$

At this point we can solve for y and then apply the initial condition or apply the initial condition and then solve for y. We'll generally do this with the later approach so let's apply the initial condition to get,

$$(-1)^{-1} = c2^4 - 2^4\ln 2 \qquad \Rightarrow \qquad c = \ln 2 - \frac{1}{16}$$

Plugging in for c and solving for y gives,

$$y(x) = \frac{1}{x^4\left(\ln 2 - \frac{1}{16} - \ln x\right)} = \frac{-16}{x^4\left(1 + 16\ln x - 16\ln 2\right)} = \frac{-16}{x^4\left(1 + 16\ln\frac{x}{2}\right)}$$

Example Solve the following IVP

$$6y' - 2y = x\,y^4 \qquad y(0) = -2$$

Solution

First get the differential equation in the proper form and then write down the substitution.

$$6y^{-4}y' - 2y^{-3} = x \qquad \Rightarrow \qquad v = y^{-3} \qquad v' = -3y^{-4}y'$$

Plugging the substitution into the differential equation gives,

$$-2v' - 2v = x \qquad \Rightarrow \qquad v' + v = -\tfrac{1}{2}x \qquad \mu(x) = \mathbf{e}^{x}$$

Again, we've rearranged a little and given the integrating factor needed to solve the linear differential equation. Upon solving the linear differential equation we have,

$$v(x) = -\tfrac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now back substitute to get back into y's.

$$y^{-3} = -\tfrac{1}{2}(x-1) + c\mathbf{e}^{-x}$$

Now we need to apply the initial condition and solve for c.

$$-\tfrac{1}{8} = \tfrac{1}{2} + c \qquad \Rightarrow \qquad c = -\tfrac{5}{8}$$

Plugging in c and solving for y gives,

$$y(x) = -\frac{2}{\left(4x - 4 + 5\mathbf{e}^{-x}\right)^{\frac{1}{3}}}$$

H.W. :

Solve the following IVP

$$y' + \frac{y}{x} - \sqrt{y} = 0 \qquad y(1) = 0$$

Second Order Differential Equations:

The most general linear second order differential equation is in the form.

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

In fact, we will rarely look at ***non-constant*** coefficient(***p(t), q(t) ,r(t)***) linear second order differential equations.

In the case where we assume constant coefficients we will use the following differential equation form:

$$ay'' + by' + cy = F(x) \qquad \ldots\ldots\ldots\ldots\ldots \textbf{(1)}$$

where **a**, **b** and **c** are constants coefficients.

If $F(x) = 0$ then (1) is called homogenous.

If $F(x) \neq 0$ then (1) is called ***non*** homogenous.

Ex:

1) $y''-x^2y' + \sin x\, y = 0$ is linear, 2^{nd} order, homo.
2) $y''-(y')^2 + y = \sin x$ is nonlinear, 2^{nd} order, non homo.

Then, we have two type of the second order linear D.E.s

1- The second order, constants coefficients, linear , Homogeneous D.E.s, is:

The method that solve the second order, constants coefficients, linear , Homogeneous D.E.s, is : ***characteristic equation***.

2- The second order, constants coefficients, linear , Non-Homogeneous D.E.s, is:

The method that solve the second order, constants coefficients, linear , Non-Homogeneous D.E.s, is :

a. Un determined coefficients.

b. Variation of parameters.

1) The Second order linear homogenous D.Eq. with constant coefficient:

The general form is

$$ay'' + by' + cy = 0 \qquad \ldots(2)$$

where a, b and c are constants.

The general solution

Put y'=Dy and y''=D^2y in eq. (2) (D is an operator)

$\Rightarrow$ a D^2y+bDy+cy=0

$\Rightarrow (aD^2 + bD + c)y = 0$ (using D-operator)

now substitute D by r and leave y then $ar^2 + br + c = 0$

is called **characteristic equation** of the differential equation and the solution of this equation (the roots r) give the solution of the differential equation where

$$r = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

There are three values of r :

1- real $r_1 \neq r_2$ (not equal root)
2- real $r_1 = r_2$ (equal root) or (repeated roots)
3- complex root ($\alpha \pm \beta i$)

Case 1: If ($b^2 - 4ac > 0$), then r_1 and r_2 are distinct ($r_1 \neq r_2$) and real roots, and the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

***Case 2**:* If ($b^2 - 4ac = 0$) , then $r_1 = r_2 = r$, and the general solution is:

$$y = (c_1 + c_2 x)e^{rx}$$

Case 3: If ($b^2 - 4ac < 0$) then the roots are two complex conjugate roots $r = \alpha \pm i\beta$, $i = \sqrt{-1}$, and the general solution is:

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Ex.1: Solve $y'' - 2y' - 3y = 0$

Solution:

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0 \quad , \quad y = 1 \;,\; y' = r \quad , \quad y'' = r^2$

$(r+1)(r-3) = 0$

(Not equal roots)

$r + 1 = 0 \quad \Rightarrow \quad r1 = -1$

$r - 3 = 0 \quad \Rightarrow \quad r2 = 3$

the general solution is

$y = c_1 e^{-x} + c_2 e^{3x}$

Ex.2: Solve $y'' - 6y' + 9y = 0$

Solution:

$y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

(Equal roots)

$(r-3)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 3$

$\therefore \quad y = (c_1 + c_2 x)e^{3x}$

Ex.3: Solve $y'' + y' + y = 0$

Solution:

$y'' + y' + y = 0$

$r^2 + r + 1 = 0 \quad \mathrm{a} = 1, \mathrm{b} = 1, \mathrm{c} = 1$

$$r = \frac{-b \pm \sqrt{1 - 4.1.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} \qquad = \frac{-1 \pm \sqrt{3}i}{2}$$

$$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i \qquad \alpha = \frac{-1}{2} \quad , \quad \beta = \frac{\sqrt{3}}{2}$$

, (Complex roots)

$$\therefore \quad y = e^{\frac{-1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

2) The Second order linear non homogenous D.Eq. with constant coefficient:

The general form is: $ay'' + by' + cy = F(x)$ ………………(3)

where a, b and c are constants coefficient.

The general solution

If y_h is the solution of the homo. D.Eq. $ay'' + by' + cy = 0$, then the general solution of eq. (3) is:

$y = y_h + y_p$ y_h (homogeneous function)

y_p (particular integral)

Where :

i) y_h is y ***h***omogenous.

ii) y_p (use the table)

In this section we will take a look at the method that can be used to find a particular solution($\boldsymbol{y_p}$) to a nonhomogeneous differential equation.

Methods of finding y_p:

There are two methods:

1- ***Undetermined coefficients.***

2- ***Variation of parameters.***

1- *Undetermined coefficients:*

In this method y_p depends on the roots $\boldsymbol{r_1}$, and $\boldsymbol{r_2}$ of characteristic equation and on the form of ***F(x)*** in eq. (3) as follows:

$F(x)$	Choice of y_p	M.R.
kx^n nth degree polynomial	$k_n x^n + k_{n-1}x^{n-1} + k_{n-2}x^{n-2} + \cdots + k_0$	0
ke^{px}	ce^{px}	p
$(k\sin\beta x)$ *or* $(k\cos\beta x)$	$c_1\cos\beta x + c_2\sin\beta x$	$\mp i\beta$

Note*: For repeated term (root), multiply by* x.

Ex.1: Use the table to write y_p

1) $F(x) = 3x^2$, $k = 3$, $n = 2$

$$y_p = k_2x^2 + k_1x + k_0$$

2) $F(x) = \frac{-1}{2}e^{-3x}$, $k = \frac{-1}{2}$ $\Rightarrow$ and p = -3

$$y_p = ce^{-3x}$$

3) $F(x) = 2\cos 3x$, $k = 2$, $\beta = 3$, k × $c_1 = c_1$, k × $c_2 = c_2$

$$y_p = c_1\cos 3x + c_2\sin 3x$$

4) $F(x) = 3x^2 - 3x + 5 - 2e^{3x}$,

$$y_p = k_2x^2 + k_1x + k_0 + ce^{3x}$$

5) $F(x) = 2\cos x - \frac{1}{2}\sin x$, note , for angle(ɵ) of ***sin*** and ***cos*** are equal then :

$$y_p = c_1\cos x + c_2\sin x$$

6) $F(x) = \sin x - \cos 2x$ note , for angle (ɵ) of ***sin*** and ***cos*** are ***not*** equal then :

$$y_p = c_1\cos x + c_2\sin x + A\cos 2x + B\sin 2x$$

Ex.2: Solve $y'' - y' - 2y = 4x^2$ ....(1)

Solution:

$$y = y_h + y_p$$

First we will find y_h :

$$y'' - y - 2y = 0$$

the char. Eq. $r^2 - r - 2 = 0$

$$(r+1)(r-2) = 0$$

$$r_1 = -1, \; r_2 = 2$$

r1 and r2 are not equal roots. then :

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

Second we will find y_p :

$F(x) = 4x^2$, is polynomial of second degree then

$$y_p = k_2 x^2 + k_1 x + k_0$$

Now, we are going to find k_2, k_1, k_0.

$$y_p = k_2 x^2 + k_1 x + k_0$$
$$\Rightarrow \quad y'_p = 2k_2 x + k_1 \quad , \quad y''_p = 2k_2$$

differentiate the $\mathbf{y_p}$ first and second derivative.

Substitution y_p, y_p' , y_p'' in (1)

$$2k_2 - (2k_2 x + k_1) - 2(k_2 x^2 + k_1 x + k_0) = 4x^2$$

Then, find k_2, k_1, k_0.

$$coeff.of\ x^2 : -2k_2 = 4 \quad \Rightarrow \quad k_2 = -2$$
$$coeff.of\ x : -2k_2 - 2k_1 = 0 \quad \Rightarrow \quad k_1 = 2$$
$$const : 2k_2 - k_1 - 2k_0 = 0 \quad \Rightarrow \quad k_0 = -3$$

Now, the y_p is,

$$y_p = -2x^2 + 2x - 3$$

Then the solution of the equation **(1)** is :

$$y = y_h + y_p = (c_1 e^{-x} + c_2 e^{2x}) - 2x^2 + 2x - 3$$

Ex.3: $y'' - y' - 2y = e^{3x}$

Solution:

$y'' - y' - 2y = e^{3x} \quad \dots(1)$

$y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r_1 = 2, r_2 = -1$

$y_h = (c_1 e^{2x} + c_2 e^{-x})$, Put

$y_p = ce^{3x} \quad \dots(2)$

$y'_p = 3ce^{3x} \quad , \quad y''_p = 9ce^{3x}$

Substitute In (1)

$9ce^{3x} - 3ce^{3x} - 2ce^{3x} = e^{3x}$

$9c - 3c - 2c = 1 \Rightarrow 4c = 1 \Rightarrow c = \frac{1}{4}$

In (2) $\Rightarrow y_p = \frac{1}{4} e^{3x}$

$y = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} + \frac{1}{4} e^{3x}$

Modification rule قاعدة التعديل

1) اذا كان $F(x) = kx^n$ وكان احد جذري المعادلة القياسية = 0 ← يضرب y_p السابق في x.

2)
- a اذا كان $F(x) = ke^{px}$ وكان احد جذري/ المعادلة القياسية =p ← يضرب y_p السابق في x.
- b اذا كان $F(x) = ke^{px}$ وكان جذري المعادلة القياسية =p ← يضرب y_p السابق في x^2.

3) اذا كان $F(x) = \begin{cases} k\cos\beta x \\ k\sin\beta x \end{cases}$ وكان $\alpha = 0$, $r = \mp i\beta$ ← يضرب y_p السابق في x.

Ex.4: Solve $y'' + y = \sin x$

Solution:

$y'' + y = 0$

$r^2+1=0,\ r^2=-1 \Rightarrow r = \pm i,\ \boldsymbol{\alpha=0},\ \beta=1$

$y_h = c_1 \cos x + c_2 \sin x$

$y_p = x(c_3\cos x+c_4\sin x),$

$y'_p=x(-c_3\sin x+c_4\cos x)+(c_3\cos x+c_4\sin x)$

$y''_p=x(-c_3\cos x-c_4\sin x)+(-c_3\sin x+c_4\cos x)+(-c_3\sin x+c_4\cos x)$

Substitution $y_p,\ y'_p,\ y''_p$.

$-2c_3\sin x+2c_4\cos x=\sin x$

$-2c_3=1\Rightarrow c_3=-1/2,$

$2c_4=0\Rightarrow c_4=0$

$$y_g = c_1 \cos x + c_2 \sin x - \frac{x}{2}\cos x$$

2 - Variation of parameters.

To solve $ay'' + by' + cy = F(x)$ using Variation of parameters method,

Let $y_h = c_1u_1 + c_2u_2$, be the homogenous solution of $ay'' + by' + cy = F(x)$

and

The particular solution has the form:

$y_p = u_1v_1 + u_2v_2$

where $\mathbf{v_1}$ and $\mathbf{v_2}$ are unknown functions of **x** which must be determined.

First solve the following linear equations for $\mathbf{v'_1}$ and $\mathbf{v'_2}$:

$$v'_1u_1 + v'_2u_2 = 0$$

$$v'_1u'_1 + v'_2u'_2 = F(x)$$

which can be solved with respect to $\mathbf{v'_1}$ and $\mathbf{v'_2}$ by Grammar rule as follows

$$D = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}, \quad D_1 = \begin{vmatrix} 0 & u_2 \\ F(x) & u_2' \end{vmatrix}, \quad D_2 = \begin{vmatrix} u_1 & 0 \\ u_1' & F(x) \end{vmatrix}$$

and $v_1' = \dfrac{D_1}{D}, \quad v_2' = \dfrac{D_2}{D}$

by ***integration*** of $\mathbf{v'_1}$ and $\mathbf{v'_2}$ with respect to **x** we can find $\mathbf{v_1}$ and $\mathbf{v_2}$.

Ex.1:

Solve $y'' - y' - 2y = e^{3x}$ ……. (1)

To find $\mathbf{y_h}$,

$$y'' - y' - 2y = e^{3x}$$

$$r^2 - r - 2 = 0$$

$$(r + 1)(r - 2) = 0$$

Then, $r_1 = -1$, $r_2 = 2$.

The $\mathbf{y_h}$ is then :

$$y_h = c_1 e^{-x} + c_2 e^{2x}$$

Then, $u_1 = e^{-x}$ and , $u_2 = e^{2x}$

Now, to solve with variation of parameters method,

$$u_1 = e^{-x} \quad \Rightarrow \quad u_1' = -e^{-x}$$
$$u_2 = e^{2x} \quad \Rightarrow \quad u_2' = 2e^{2x}$$

Suppose, $y_p = v_1 u_1 + v_2 u_2$

$$v_1' u_1 + v_2' u_2 = 0$$
$$v_1' u_1' + v_2' u_2' = F(x)$$

$$v_1'(e^{-x}) + v_2'(e^{2x}) = 0$$
$$v_1'(-e^{-x}) + v_2'(2e^{2x}) = e^{3x}$$

Solving this system by Cramer rule gives

$$D = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^{x}, \quad D_1 = \begin{vmatrix} 0 & e^{2x} \\ e^{3x} & 2e^{2x} \end{vmatrix} = -e^{5x}, \quad D_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{3x} \end{vmatrix} = e^{2x}$$

$$v'_1 = \frac{-e^{5x}}{3e^{x}} = \frac{-1}{3}e^{4x} \Rightarrow v_1 = \int \frac{-1}{3}e^{4x} = -\frac{1}{12}e^{4x},$$

$$v'_2 = \frac{e^{2x}}{3e^{x}} = \frac{1}{3}e^{x} \Rightarrow v_2 = \int \frac{1}{3}e^{x} = \frac{1}{3}e^{x}$$

$$\therefore y_p = -\frac{1}{2}e^{4x}e^{-x} + \frac{1}{3}e^{x}e^{2x} = \frac{1}{4}e^{3x}$$

The general solution is : $y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4}e^{3x}$

$$y = \quad y_p \quad + y_h$$

Ex.2: solve

$$y'' + y = \sec x$$

Solution:

$y'' + y = 0$

$r^2+1=0 \Rightarrow r^2=-1 \Rightarrow r = \pm i \qquad \alpha=0,\ \beta=1$

$y_h = c_1\cos x + c_2\sin x,\quad u_1 = \cos x,\ u_2 = \sin x,\quad f(x)=\sec x$

$y_p = v_1u_1+v_2u_2$

$\quad = v_1\cos x + v_2\sin x,\quad$ then

$$v_1'(\cos x) + v_2'(\sin x) = 0$$
$$v_1'(-\sin x) + v_2'(\cos x) = \sec x$$

$$D = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1,$$

$$D_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\sin x \frac{1}{\cos x} = -\tan x,$$

$$D_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$v_1' = \frac{-\tan x}{1} = -\tan x \Rightarrow v_1 = \int \frac{-\sin x}{\cos x}dx = \ln|\cos x|$$

$$v'_2 = 1 \Rightarrow v_2 = \int dx = x$$

$y_p = \ln|\cos x| \cos x + x \sin x$

$y = c_1\cos x + c_2\sin x + \ln|\cos x| \cos x + x \sin x$

C. Higher order Differential Equations:

Higher order linear Differential Equations:

The general form with constant coefficient is:

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y + a_n y = F(x) \qquad \ldots\ldots(1)$$

If $F(x) = 0$ then (1) is called ***homogenous***, *otherwise* (1) is called ***nonhomogenous***.

The general solution

The methods of solving second order homogenous D.Eqs. with constant coefficients can be extended to solve higher order homogenous and nonhomogenous D.Eq. with constant coefficients.

a) Homogenous: the characteristic equation of nth order homogenous D. Eq.:
$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y + a_n y = 0$ is:
$r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0$

Let r_1, r_1 , r_2 , ... , r_n be the roots of characteristic equation then:

1) If r_1 , r_2 , ... , r_n are all distinct then the solution is:
$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}$

2) If r_1 repeated m times, then y_h will contain the terms:
$c_1 e^{r_1 x} + c_2 x e^{r_1 x} + \ldots + c_m x^{m-1} e^{r_1 x}$

3) If some of roots are complex ($r = \alpha \mp i\beta$) then y_h will contain
$(c_1 \cos \beta x + c_2 \sin \beta x) e^{\alpha x}$

Note : the general solution for ordinary differential equations is in the form of :

$$\boldsymbol{y = y_h + y_p}$$

Then, finding the roots just to find $\boldsymbol{y_h}$.

Now, we are going to find the roots and solve D.Eq. of higher order :

There are two methods to factorize f(x): long division & fast division.

1- *First method: **Fast** division*
2- *Second method: **long** division*

Ex.: Find all roots of the given differential equation and solve it.

$y''' + 4y'' - 3y' - 18y = 0$, using **fast** and long **division**.

Solution:

$$y^3 + 4y^2 - 3y - 18 = 0$$

$$r^3 + 4r^2 - 3r - 18 = 0$$

First method: Fast division

Find all roots of $r^3 + 4r^2 - 3r - 18 = 0$,

$r: \mp 1, \mp 2, \mp 3, \mp 6, \mp 9, \mp 18$

$f(2) = 8 + 16 - 6 - 18 = 0,$

" **r = 2** " is the root that make the equation above is zero, then :

	1	4	-3	-18
2	↓	2	12	18
	1	6	9	0

$$r^2 + 6r + 9 = 0$$

$$(r - 2)(r^2 + 6r + 9) = 0$$

$$(r - 2)(r ++ 3)(r + 3) = 0$$

Then the roots that we got it using fast division are :

$r_1 = 2, r_2 = -3, r_3 = -3$

then the solution of the given Differential Equation is :

$y = C_1 e^{2x} + C_2 e^{-3x} + C_3 x e^{-3x}$

Second method: long division

$r^3 + 4r^2 - 3r - 18 = 0$, " **r = 2** " is the root that make the equation is zero, then :

$(r - 2)(r^2 + 6r + 9) = 0$

$(r - 2)(r + 3)(r + 3) = 0$

Then the roots $r_1 = 2, r_{2,3} = -3$, using long division.

$$\begin{array}{r} r^2 + 6r + 9 \\ (r-2)\overline{)\,r^3 + 4r^2 - 3r - 18} \\ \underline{\mp r^3 \pm 2r^2} \\ 6r^2 - 3r \\ \underline{\mp 6r^2 \pm 12r} \\ 9r - 18 \\ \underline{\mp 9r \pm 18} \\ \mathbf{0} \end{array}$$

then the solution of the given Differential Equation is :

$y = C_1 e^{2x} + C_2 e^{-3x} + C_3 x e^{-3x}$

Ex.: $y'''' - 3y''' - 2y'' + 2y' + 12y = 0$

$r^4 - 3r^3 - 2r^2 + 2r + 12 = 0$

$r = 2$ is a root $\Rightarrow$ $(r - 2)$ is a factor

$\Rightarrow r^3 - r^2 - 4r - 6 = 0$

$\Rightarrow (r-2)(r^3 - r^2 - 4r - 6) = 0$, $r = 3$ root $\Rightarrow$ $(r - 3)$ is a factor

	1	-3	-2	2	12
2	↓	2	-2	-8	-12
	1	-1	-4	-6	0

$\Rightarrow r^2 + 2r + 2 = 0$

$(r-2)(r-3)(r^2 + 2r + 2) = 0$

$r_1 = 2, \quad r_2 = 3, \quad r = -1 \mp i \quad \alpha = -1 \ , \beta = 1$

$\Rightarrow \quad y_h = c_1 e^{2x} + c_2 e^{3x} + (c_3 \cos x + c_4 \sin x) e^{-x}$

	1	-1	-4	-6
3	↓	3	6	6
	1	2	2	0

<u>**H.W**</u> : Solve the above Example using long division.

b) Nonhomogeneous: the general form of *n*th order nonhomogeneous differential equation is:

$$y^{(n)}+a_1y^{(n-1)}+\ldots+a_{n-1}y+a_ny=F(x) \qquad ......(1)$$

The general solution is $y_g = y_h + y_p$

Methods of finding y_p:

1) Undetermined coefficients

We can extend the methods of solving *second order* non homogenous D.Eqs. with constant coefficients to solve higher order non-homogenous D.Eq. with constant coefficients.

Ex.1: $y^{(4)}-8y''+16y=-18\sin x$

Solution:

$y_g = y_h + y_p$

$y^{(4)}-8y''+16y=0$

$r^4-8r^2+16=0 \Rightarrow (r^2-4)^2=0 \Rightarrow r^2=4 \Rightarrow r=\pm 2$

$\mathbf{y_h=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}+c_4xe^{-2x}}$

Now, we will find y_p,

let $y_p= A\cos x + B\sin x$, $y'_p= -A\sin x + B\cos x$, $y''_p= -A\cos x - B\sin x$

$y'''_p= A\sin x - B\cos x$, $y^{(4)}_p= A\cos x + B\sin x$

$A\cos x + B\sin x + 8A\cos x + 8B\sin x + 16A\cos x+16B\sin x = -18\sin x$

$25A\cos x+25B\sin x=-18\sin x$

$25A = 0 \quad \Rightarrow A = 0$

$25B = -18 \Rightarrow B = -18/25$

$y \quad = \qquad y_h \qquad + \quad y_p$

$$y = c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}+c_4xe^{-2x} - \frac{18}{25}\sin x$$

2) Variation of parameters

In this method, the particular solution yp has the form $y_p=v_1u_1+v_2u_2+\ldots+v_nu_n$

Where $u_1, u_2, \ldots, u_n$ are taken from $y_h=c_1u_1+c_2u_2+\ldots+c_nu_n$.

To find $v_1, v_2, \ldots, v_n$, we must solve the following linear eqs. For $v'_1, v'_2, \ldots, v'_n$:

$$\begin{aligned}
v'_1u_1+v'_2u_2+\ldots+v'_nu_n &= 0\\
v'_1u'_1+v'_2u'_2+\ldots+v'_nu'_n &= 0\\
&\vdots\\
v'_1u_1^{(n-2)}+v'_2u_2^{(n-2)}+\ldots+v'_nu_n^{(n-2)} &= 0\\
v'_1u_1^{(n-1)}+v'_2u_2^{(n-1)}+\ldots+v'_nu_n^{(n-1)} &= f(x)
\end{aligned}$$

Ex2: solve $y''' + y' = \sec x$

<u>Solution</u>:

Let $y'''+y'=0$

$r^3+r=0 \Rightarrow r(r^2+1)=0 \Rightarrow r=0,\ r^2=-1 \Rightarrow r1=0,\ r=\pm i$

$y_h=c_1+c_2\cos x+c_3\sin x$

$u_1=1,\ u_2=\cos x,\ u_3=\sin x,\ f(x)=\sec x$

$$v_1' + v_2'\cos x + v_3'\sin x = 0$$
$$v_1'(0) + v_2'(-\sin x) + v_3(\cos x) = 0$$
$$v_1'(0) + v_2'(-\cos x) - v_3'(\sin x) = \sec x$$

$$D = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

$$D_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \sec x & -\cos x & -\sin x \end{vmatrix} = \sec x(\sin^2 x + \cos^2 x) = \sec x$$

$$D_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \sec x & -\sin x \end{vmatrix} = \begin{vmatrix} 0 & \cos x \\ \sec x & -\sin x \end{vmatrix} = -\cos x \sec x = -1$$

$$D_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \sec x \end{vmatrix} = \begin{vmatrix} -\sin x & 0 \\ -\cos x & \sec x \end{vmatrix} = -\sin x \sec x = -\tan x$$

$$v_1' = \frac{D_1}{D} = \sec x \Rightarrow v_1 = \int \sec x dx = \ln(\sec x + \tan x)$$

$$v_2' = \frac{D_2}{D} = -1 \Rightarrow v_2 = \int -1\, dx = -x$$

$$v_3' = \frac{D_3}{D} = -\tan x \Rightarrow v_3 = -\int \tan x\, dx = \ln\cos x$$

$y_p= \ln(\sec x+\tan x)-x\cos x-\ln\cos x\sin x$

$y_g= c_1+c_2\cos x+c_3\sin x+ \ln(\sec x+\tan x)-x\cos x - \ln\cos x\sin x$

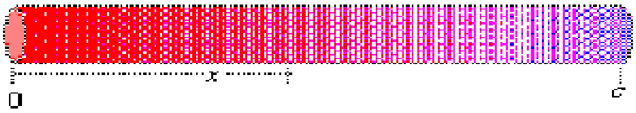
Partial Differential Equations

(Partial Differential Equations)

Introduction :

Much of modern science, engineering, and mathematics is based on the study of *partial differential equations*, where a partial differential equation is an equation involving partial derivatives which implicitly defines a *function of two or more variables*.

For example, if ***u (x, t)*** is the temperature of a metal bar at a distance ***x*** from the initial end of the bar,



then under suitable conditions **u (x, t)** is a solution to the ***heat equation***, where " **k** " is a constant. :

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

As another example, consider that if ***u (x; t)*** is the displacement of a string a time t ; then the **vibration** of the string is likely to satisfy the one dimensional wave equation for a constant, which is :

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

When a partial differential equation occurs in an application, ***our goal is usually that of solving the equation***, where a ***given function is a solution of a partial differential equation if it is implicitly defined by that equation***. That is, a solution is a function that satisfies the equation.

EXAMPLE Show that if a is a constant, then $u(x, y) = \sin(at)\cos(x)$ is a solution to

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \qquad (2)$$

Solution: Since a is constant, the partials with respect to t are

$$\frac{\partial u}{\partial t} = a\cos(at)\cos(x), \qquad \frac{\partial^2 u}{\partial t^2} = -a^2\sin(at)\sin(x) \qquad (3)$$

Moreover, $u_x = -\sin(at)\sin(x)$ and $u_{xx} = -\sin(at)\cos(x)$, so that

$$a^2\frac{\partial^2 u}{\partial x^2} = -a^2\sin(at)\cos(x) \qquad (4)$$

Since (3) and (4) are the same, $u(x, t) = \sin(at)\cos(x)$ is a solution to (2).

Home Work : Show that $u(x, t) = e^y\sin(x)$ is a solution to *Laplace's Equation*,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1) Direct Partial integration:

As explained in the previous class, the integration is the reverse process of differentiation. The partial integration is the same as ordinary integration when its required the integrate of the function.

1- a) Direct Partial integration without boundary conditions:

Example: Integrate the partial differential equation given below with respect to t.

$$\partial u/\partial t = 5\cos x \sin t$$

Solution **:**

The (**5 cosx**) term is considered as a ***constant***.

$$\text{and} \quad u = \int 5\cos x \sin t \, dt = (5\cos x)\int \sin t \, dt$$

$$= (5\cos x)(-\cos t) + c$$

$$= -5\cos x \cos t + f(x)$$

Example: Integrate the partial differential equation given below with respect to **y,** then with respect **x**. *(***Integrate the partial differential equation***).

$$\frac{\partial^2 u}{\partial x \partial y} = 6x^2 \cos 2y$$

Solution **:** Integrate with respect to **y :**

$$\frac{\partial u}{\partial x} = \int 6x^2 \cos 2y \, dy = (6x^2)\int \cos 2y \, dy$$

$$= (6x^2)\left(\frac{1}{2}\sin 2y\right) + f(x)$$

$$= 3x^2 \sin 2y + f(x)$$

integrating $\frac{\partial u}{\partial x}$ partially with respect to x gives:

$$u = \int [3x^2 \sin 2y + f(x)] \, dx$$

$$= x^3 \sin 2y + \int f(x) \, dx + g(y)$$

$f(x)$ and $g(y)$ are functions that may be determined, if extra information, called **boundary conditions** or **initial conditions**, are known.

1-b) Solution of partial differential equations by direct partial integration *with boundary conditions*:

The simplest form of partial differential equations occurs when a solution can be determined by direct partial integration. This is demonstrated in the following worked problems.

Example : Solve the differential equation $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ given the boundary conditions that at $x=0$, $\frac{\partial u}{\partial x} = \sin 2y$ and $u = \cos y$.

***Solution* :**

Since $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ then integrating partially with respect to x gives:

$$\frac{\partial u}{\partial x} = \int 6x^2(2y-1)\mathrm{d}x = (2y-1)\int 6x^2\mathrm{d}x$$

$$= (2y-1)\frac{6x^3}{3} + f(y)$$

$$= 2x^3(2y-1) + f(y)$$

where $f(y)$ is an arbitrary function. From the boundary conditions, when $x=0$,

$$\frac{\partial u}{\partial x} = \sin 2y.$$

Hence, $\sin 2y = 2(0)^3(2y-1) + f(y)$, from which, $f(y) = \sin 2y$

Now $\frac{\partial u}{\partial x} = 2x^3(2y-1) + \sin 2y$

Integrating partially with respect to x gives:

$$u = \int [2x^3(2y-1) + \sin 2y]\mathrm{d}x$$

$$= \frac{2x^4}{4}(2y-1) + x(\sin 2y) + g(y)$$

From the boundary conditions, when $x=0$, $u=\cos y$, hence

$$\cos y = \frac{(0)^4}{2}(2y-1) + (0)\sin 2y + g(y)$$

from which, $g(y) = \cos y$

Hence, the solution of $\frac{\partial^2 u}{\partial x^2} = 6x^2(2y-1)$ for the given boundary conditions is:

$$u = \frac{x^4}{2}(2y-1) + x\sin 2y + \cos y$$

(H. W.) Solve the differential equation:

$\frac{\partial^2 u}{\partial x \partial y} = \cos(x+y)$ given that $\frac{\partial u}{\partial x} = 2$ when $y=0$, and $u = y^2$ when $x=0$.

Some important engineering partial differential equations :

There are many types of partial differential equations. Some typically found in engineering and science include:

(a) The wave equation, where the equation of motion is given by:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where $c^2 = \frac{T}{\rho}$, with T being the tension in a string and ρ being the mass/unit length of the string.

(b) The heat conduction equation is of the form:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$$

where $c^2 = \frac{h}{\sigma\rho}$, with h being the thermal conductivity of the material, σ the specific heat of the material, and ρ the mass/unit length of material.

(c) Laplace's equation, used extensively with electrostatic fields is of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(d) The transmission equation, where the potential u in a transmission cable is of the form:

$\frac{\partial^2 u}{\partial x^2} = A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial u}{\partial t} + Cu$ where A, B and C are constants.

Separating the variables :

Solutions to many (but not all!) partial differential equations can be obtained using the technique known as ***separation of variables***.

In separation of variables, we first assume that the solution is of the separated form :

$$u(x,t) = X(x)T(t)$$

We then substitute the separated form into the equation, and if possible, move the **x**-terms to one side and the **t**-terms to the other. If not possible, then ***this method will not work***; and correspondingly, we say that the partial differential equation ***is not separable***.

Once separated, the two sides of the equation must be constant, thus requiring the solutions to ordinary deferential equations. A ***table of solutions*** to common deferential equations is given below :

Equation	General Solution
$y'' + \omega^2 y = 0$	$y(x) = A\cos(\omega x) + B\sin(\omega x)$
$y' = ky$	$y(t) = Pe^{kt}$

The product of **X (x)** and **T (t)** is the separated solution of the partial deferential equation.

Example :

Use separation of variables to find the product solution of :

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

Solution :

Let : $U = X\,Y$, $\frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = Y'X$,

Then $X'Y + Y'X = XY$, ... $(1/XY)$

$$\frac{X'}{X} + \frac{Y'}{Y} = 1$$

$$\frac{X'}{X} = 1 - \frac{Y'}{Y} = c_1$$

$$\frac{X'}{X} = c_1 \quad \text{and} \quad \frac{Y'}{Y} = 1 - c_1$$

For X : $\frac{X'}{X} = c_1$, $X' = C_1 X$, $r = c_1$, $X = c_2 e^{rX}$ ➡ $X = c_2 e^{C_1 X}$

For Y : $\frac{Y'}{Y} = 1 - c_1$, $Y' = (1 - c_1)Y$ and $r = (1-c_1)$, $Y = c_3 e^{rY}$

Then $Y = c_3 e^{(1-c1)Y}$,

Now : $U = X\,Y$

$U = (c_2 e^{C1\,X})*(c_3 e^{(1-C1)Y})$

Example .

Find the product solution of $\frac{\partial^2 u}{\partial x^2} - u = 0$

Solution :

$$U = X\ Y \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''Y$$

$$\frac{\partial^2 u}{\partial x^2} - u = 0 \quad , \quad \frac{\partial^2 u}{\partial x^2} = u$$

$$X''Y = XY \quad \dots\dots\dots\dots \quad (1/XY)$$

$$\frac{X''}{X} = 1$$

$$X''-X=0, \qquad r^2 - 1 = 0, \qquad r = \mp 1$$

$$U = A e^{x} + Be^{-x}$$

$$U(x,y) = X * Y$$

$$U(x,y) = (Ae^{x} + Be^{-x}) * Y$$

Example : Use separation of variables to find the product solution of :

$$\frac{\partial U}{\partial x} + 3\frac{\partial U}{\partial y} = 0$$

Solution. Here the dependent variables are x and y, so we substitute $U = XY$ in the given differential equation where X depends on only x and Y depends on only y.

$$\frac{\partial U}{\partial x} + 3\frac{\partial U}{\partial y} = 0 \quad \text{or} \quad X'Y = -3XY'$$

$$\frac{X'}{3X} = -\frac{Y'}{Y}$$

$$\frac{X'}{3X} = -\frac{Y'}{Y} = c$$

$$X' - 3cX = 0 \qquad Y' + cY = 0$$

From our knowledge of ordinary differential equations, have solutions

$$X = b_1 e^{3cx}, \; Y = b_2 e^{-cy}$$

Thus

$$U = XY = b_1 b_2 e^{c(3x-y)} = Be^{c(3x-y)}, \qquad \text{where} \quad B = b_1 b_2$$

Solving wave equation using separating the variables :

Let $u(x,t) = X(x)T(t)$, where $X(x)$ is a function of x only and $T(t)$ is a function of t only, be a trial solution to the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$. If the trial solution is simplified to $u = XT$, then $\frac{\partial u}{\partial x} = X'T$ and $\frac{\partial^2 u}{\partial x^2} = X''T$. Also $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial t^2} = XT''$.

Substituting into the partial differential equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ gives:

$$X''T = \frac{1}{c^2}XT''$$

Separating the variables gives:

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

Let $\mu = \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$ where μ is a constant.

Thus, since $\mu = \frac{X''}{X}$ (a function of x only), it must be independent of t; and, since $\mu = \frac{1}{c^2}\frac{T''}{T}$ (a function of t only), it must be independent of x.

If μ is independent of x *and* t, it can only be a constant. If $\mu = \frac{X''}{X}$ then $X'' = \mu X$ or $X'' - \mu X = 0$ and if $\mu = \frac{1}{c^2}\frac{T''}{T}$ then $T'' = c^2\mu T$ or $T'' - c^2\mu T = 0$.

Such ordinary differential equations are of the form found , and their solutions will depend on whether $\mu > 0$, $\mu = 0$ or $\mu < 0$.

The next example will be a reminder of solving ordinary differential equations of this type.

Example : Application to Wave Equation

$$\frac{\partial^2 u}{dt^2} = c^2 \frac{\partial^2 u}{dx^2}$$

Let U(x,t) = X(x) T(t), then

$$\frac{\partial u}{\partial t} = X(x)\ T'(t)\ ,\ \frac{\partial^2 u}{\partial t^2} = X(x)\ T''(t)$$

$$\frac{\partial u}{\partial x} = X'(x)\ T(t)\ ,\ \frac{\partial^2 u}{\partial x^2} = X''(x)\ T(t)$$

Putting these values in the equation we get

$X(x)\ T''(t) = c^2 X''(x)\ T(t)$

or $\dfrac{1}{c^2}\dfrac{T''(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = k$

***Case One:* k = 0**

$T''(t) = 0 \quad , X''(x) = 0$

$T(t) = at + b$ and $X(x) = px + r$

***Case Two:* k > 0**

$$T''(t) = kc^2 T(t)$$

and

$$X''(x) = kX(x)$$

$$X(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x} \qquad T(t) = Ee^{\sqrt{k}ct} + Fe^{-\sqrt{k}ct}$$

***Case Three:* k < 0**

$$T''(t) = -kc^2 T(t)$$

and

$$X''(x) = -kX(x)$$

$$X(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x) \quad , \quad T(t) = E\cos(\sqrt{k}ct) + F\sin(\sqrt{k}ct)$$

Example : Find the general solution of the following differential equations:

(a) $X'' - 4X = 0$ (b) $T'' + 4T = 0.$

Solution

(a) If $X'' - 4X = 0$ then the auxiliary equation is:

$m^2 - 4 = 0$ i.e. $m^2 = 4$ from which,
$m = +2$ or $m = -2$

Thus, the general solution is:

$$\boldsymbol{X = Ae^{2x} + Be^{-2x}}$$

(b) If $T'' + 4T = 0$ then the auxiliary equation is:

$m^2 + 4 = 0$ i.e. $m^2 = -4$ from which,
$m = \sqrt{-4} = \pm j2$

Thus, the general solution is:

$$T = e^0\{A\cos 2t + B\sin 2t\} = \boldsymbol{A\cos 2t + B\sin 2t}$$

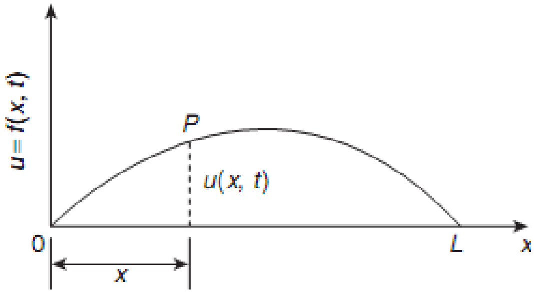
(<u>H.W.</u>)

1 - Solve $T'' - c^2\mu T = 0$ given $c = 3$ and $\mu = -1$

2 - Solve $X'' = \mu X$ given $\mu = 1$

The wave equation :

An elastic string is a string with elastic properties, i.e. the string satisfies Hooke's law. Figure below shows a flexible elastic string stretched between two points at x =0 and x =L with uniform tension T. The string will vibrate if the string is displace slightly from its initial position of rest and released, the end points remaining fixed. The position of any point P on the string depends on its distance from one end, and on the instant in time. Its displacement u at any time t can be expressed as u = f (x, t), where x is its distance from 0.



The equation of motion is :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The boundary and initial conditions are:

(i) The string is fixed at both ends, i.e. $x = 0$ and $x = L$ for all values of time t.

Hence, $u(x, t)$ becomes:

$$\left.\begin{array}{l} u(0, t) = 0 \\ u(L, t) = 0 \end{array}\right\} \text{ for all values of } t \geq 0$$

(ii) If the initial deflection of P at $t = 0$ is denoted by $f(x)$ then $u(x, 0) = f(x)$

(iii) Let the initial velocity of P be $g(x)$, then

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x)$$

Initially a **trial solution** of the form $u(x, t) = X(x)T(t)$ is assumed, where $X(x)$ is a function of x only and $T(t)$ is a function of t only. The trial solution may be simplified to $u = XT$ and the variables separated as explained in the previous section to give:

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$$

When both sides are equated to a constant μ this results in two ordinary differential equations:

$$T'' - c^2\mu T = 0 \quad \text{and} \quad X'' - \mu X = 0$$

Three cases are possible, depending on the value of μ.

Case 1: $\mu > 0$

For convenience, let $\mu = p^2$, where p is a real constant. Then the equations

$$X'' - p^2 X = 0 \quad \text{and} \quad T'' - c^2 p^2 T = 0$$

have solutions: $X = Ae^{px} + Be^{-px}$ and $T = Ce^{cpt} + De^{-cpt}$ where A, B, C and D are constants.

But $X=0$ at $x=0$, hence $0=A+B$ i.e. $B=-A$ and $X=0$ at $x=L$, hence
$0=Ae^{pL}+Be^{-pL}=A(e^{pL}-e^{-pL})$.
Assuming $(e^{pL}-e^{-pL})$ is not zero, then $A=0$ and since $B=-A$, then $B=0$ also.
This corresponds to the string being stationary; since it is non-oscillatory, this solution will be disregarded.

Case 2: $\mu=0$

In this case, since $\mu=p^2=0$, $T''=0$ and $X''=0$. We will assume that $T(t)\neq 0$. Since $X''=0$, $X'=a$ and $X=ax+b$ where a and b are constants. But $X=0$ at $x=0$, hence $b=0$ and $X=ax$ and $X=0$ at $x=L$, hence $a=0$. Thus, again, the solution is non-oscillatory and is also disregarded.

Case 3: $\mu<0$

For convenience,
let $\mu=-p^2$ then $X''+p^2X=0$ from which,

$$X=A\cos px+B\sin px \qquad (1)$$

and $T''+c^2p^2T=0$ from which,

$$T=C\cos cpt+D\sin cpt \qquad (2)$$

Thus, the suggested solution $u=XT$ now becomes:

$$u=\{A\cos px+B\sin px\}\{C\cos cpt+D\sin cpt\} \qquad (3)$$

Applying the boundary conditions:

(i) $u=0$ when $x=0$ for all values of t,
thus $0=\{A\cos 0+B\sin 0\}\{C\cos cpt+D\sin cpt\}$

i.e. $0=A\{C\cos cpt+D\sin cpt\}$

from which, $A=0$, (since $\{C\cos cpt+D\sin cpt\}\neq 0$)

Hence, $u=\{B\sin px\}\{C\cos cpt+D\sin cpt\}$ (4)

(ii) $u=0$ when $x=L$ for all values of t

Hence, $0=\{B\sin pL\}\{C\cos cpt+D\sin cpt\}$
Now $B\neq 0$ or $u(x,t)$ would be identically zero.
Thus $\sin pL=0$ i.e. $pL=n\pi$ or $p=\dfrac{n\pi}{L}$ for integer values of n.

Substituting in equation (4) gives:

$$u=\left\{B\sin\frac{n\pi x}{L}\right\}\left\{C\cos\frac{cn\pi t}{L}+D\sin\frac{cn\pi t}{L}\right\}$$

i.e. $$u=\sin\frac{n\pi x}{L}\left\{A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}\right\}$$

(where constant $A_n=BC$ and $B_n=BD$). There will be many solutions, depending on the value of n. Thus, more generally,

$$u_n(x,t)=\sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L}+B_n\sin\frac{cn\pi t}{L}\right)\right\} \qquad (5)$$

To find A_n and B_n we put in the initial conditions not yet taken into account.

(i) At $t=0$, $u(x,0)=f(x)$ for $0\le x\le L$

Hence, from equation (5),

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}\left\{A_n\sin\frac{n\pi x}{L}\right\} \qquad (6)$$

(ii) Also at $t=0$, $\left[\dfrac{\partial u}{\partial t}\right]_{t=0}=g(x)$ for $0\le x\le L$
Differentiating equation (5) with respect to t gives:

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}\left(A_n\left(-\frac{cn\pi}{L}\sin\frac{cn\pi t}{L}\right)+B_n\left(\frac{cn\pi}{L}\cos\frac{cn\pi t}{L}\right)\right)\right\}$$

and when $t=0$,

$$g(x)=\sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}B_n\frac{cn\pi}{L}\right\}$$

i.e. $$g(x)=\frac{c\pi}{L}\sum_{n=1}^{\infty}\left\{B_n n\sin\frac{n\pi x}{L}\right\} \qquad (7)$$

From Fourier series it may be shown that:
A_n is twice the mean value of $f(x)\sin\dfrac{n\pi x}{L}$ between $x=0$ and $x=L$

i.e. $$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

for $n=1,2,3,\ldots$ (8)

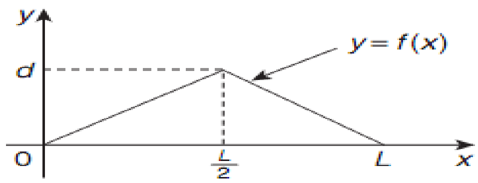
and $B_n\left(\frac{cn\pi}{L}\right)$ is twice the mean value of $g(x)\sin\frac{n\pi x}{L}$ between $x=0$ and $x=L$

i.e. $$B_n = \frac{L}{cn\pi}\left(\frac{2}{L}\right)\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

or $$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

For stretched string problems as in next example below, the main parts of the procedure are:

1. Determine A_n from equation (8).
   Note that $\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$ is **always** equal to $\frac{8d}{n^2\pi^2}\sin\frac{n\pi}{2}$
2. Determine B_n from equation (9)
3. Substitute in equation (5) to determine $u(x,t)$



1- $$u_n(x,t) = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)\right\}$$

2- $$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{8d}{n^2\pi^2}\sin\frac{n\pi}{2}$$

3- $$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

4- $$g(x) = \frac{c\pi}{L}\sum_{n=1}^{\infty}\left\{B_n\, n\sin\frac{n\pi x}{L}\right\}$$

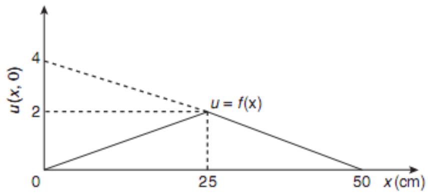
$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = 0 = g(x) \text{ thus, } B_n = 0$$

$B_n = 0$

$$\boxed{u_n(x,t) = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)\right\}}$$

Example : Figure below shows a stretched string of length 50 cm which is set oscillating by displacing its mid-point a distance of 2 cm from its rest position and releasing it with zero velocity where $c^2 = 1$, to determine the resulting motion u(x, t). . Solve the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$



Solution :

$$u_n(x,t) = \sum_{n=1}^{\infty}\left\{\sin\frac{n\pi x}{L}\left(A_n\cos\frac{cn\pi t}{L} + B_n\sin\frac{cn\pi t}{L}\right)\right\}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

The boundary and initial conditions given are:

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(50,t) &= 0 \end{aligned}\right\} \quad \text{i.e. fixed end points}$$

$$u(x,0) = f(x) = \frac{2}{25}x \quad 0 \le x \le 25$$

$$u(50,t) = f(x) = -\frac{2}{25}x + 4 = \frac{100-2x}{25} \qquad 25 \le x \le 50$$

$$A_n = \frac{16}{n^2\pi^2}\sin\frac{n\pi}{2} \quad \textbf{(Always)}$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$g(x) = \frac{c\pi}{L}\sum_{n=1}^{\infty}\left\{B_n\, n\,\sin\frac{n\pi x}{L}\right\}$$

$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = 0 = g(x) \text{ thus, } B_n = 0$$

$$B_n = 0$$

$$u_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{50}\left\{A_n\cos\frac{n\pi t}{50} + B_n\sin\frac{n\pi t}{50}\right\}$$

$$u_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{50}\left\{\frac{16}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{50} + (0)\sin\frac{n\pi t}{50}\right\}$$

Hence,

$$u(x,t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi x}{50}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{50}$$

(H.W.)

1. An elastic string is stretched between two points 40 cm apart. Its centre point is displaced 1.5 cm from its position of rest at right angles to the original direction of the string and then released with zero velocity. Determine the subsequent motion u(x, t) by applying the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ with $c^2 = 9$.

2. The centre point of an elastic string between two points P and Q, 80 cm apart, is deflected a distance of 1 cm from its position of rest perpendicular to PQ and released initially with zero velocity. Apply the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ where c = 8, to determine the motion of a point distance x from P at time t.

Classifications of PDEs :

A general second order linear partial differential equation in two Cartesian variables can be written as :

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

Three main types arise, based on the value of $D = B^2 - 4AC$ (a discriminant):

1. **Hyperbolic**, wherever (x, y) is such that $D > 0$;
2. **Parabolic**, wherever (x, y) is such that $D = 0$;
3. **Elliptic**, wherever (x, y) is such that $D < 0$.

Example : Classify the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\, \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x, 0) = -x^2$$

$$u(x, 0) = 0$$

Solution :

Compare this PDE to the standard form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\, \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1, \quad B = -3, \quad C = 2 \quad \Rightarrow \quad D = 9 - 4\times 2 = 1 > 0$

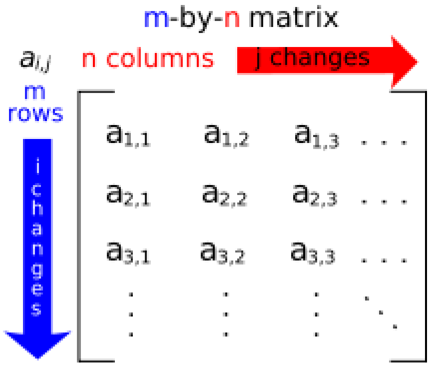
Therefore the PDE is **hyperbolic** everywhere.

SYSTEM OF LINEAR EQUATIONS & MATRICES

Revision of the matrices

In mathematics, a matrix is a rectangular table of elements (or entries), which may be numbers or, more generally, any abstract quantities that can be added and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformations and to record data that depend on multiple parameters. Matrices are described by the field of matrix theory. Matrices can be added, multiplied, and decomposed in various ways, which also makes them a key concept in the field of linear algebra.

In this material, the entries of a matrix are real or complex numbers unless otherwise noted.



Basic operations :

Sum :

Two or more matrices of identical dimensions *m* and *n* can be added. Given *m*-by-*n* matrices A and B, their sum A+B is the *m*-by-*n* matrix computed by adding corresponding elements:

$$\mathbf{A}+\mathbf{B} = (a_{i,j})_{1\le i\le m;\,1\le j\le n} + (b_{i,j})_{1\le i\le m;\,1\le j\le n}$$
$$= (a_{i,j}+b_{i,j})_{1\le i\le m;1\le j\le n}.$$

$$\begin{bmatrix}1&3&1\\1&0&0\\1&2&2\end{bmatrix}+\begin{bmatrix}0&0&5\\7&5&0\\2&1&1\end{bmatrix}=\begin{bmatrix}1+0&3+0&1+5\\1+7&0+5&0+0\\1+2&2+1&2+1\end{bmatrix}=\begin{bmatrix}1&3&6\\8&5&0\\3&3&3\end{bmatrix}.$$

Scalar multiplication :

Given a matrix **A** and a scalar number ***c***, the scalar multiplication ***c*A** is computed by multiplying every element of **A** by the scalar ***c*** (i.e. $(c\mathbf{A})_{i,j} = c \cdot a_{i,j}$). For example:

$$2\cdot\begin{bmatrix}1&8&-3\\4&-2&5\end{bmatrix}=\begin{bmatrix}2\cdot1&2\cdot8&2\cdot-3\\2\cdot4&2\cdot-2&2\cdot5\end{bmatrix}=\begin{bmatrix}2&16&-6\\8&-4&10\end{bmatrix}.$$

Matrix multiplication :

Multiplication of two matrices is well-defined only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If **A** is an *m*-by-*n* matrix and **B** is an *n*-by-*p* matrix, then their **matrix product AB** is the *m*-by-*p* matrix given by:

$$(\mathbf{AB})_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,n}b_{n,j}$$

for each pair (*i*,*j*). For example:

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (1\times3+0\times2+2\times1) & (1\times1+0\times1+2\times0) \\ (-1\times3+3\times2+1\times1) & (-1\times1+3\times1+1\times0) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}.$$

Matrix multiplication has the following properties:

- (**AB**)**C** = **A**(**BC**) for all *k*-by-*m* matrices **A**, *m*-by-*n* matrices **B** and *n*-by-*p* matrices **C** ("associatively").
- (**A**+**B**)**C** = **AC**+**BC** for all *m*-by-*n* matrices **A** and **B** and *n*-by-*k* matrices **C** ("right distributive").
- **C**(**A**+**B**) = **CA**+**CB** for all *m*-by-*n* matrices **A** and **B** and *k*-by-*m* matrices **C** ("left distributive").

Matrix multiplication is not commutative; that is, given matrices **A** and **B** and their product defined, then generally **AB** $\neq$ **BA**. It may also happen that **AB** is defined but **BA** is not defined.

Transposition :

Transposing a matrix means converting an *m by n* matrix into an *n by m* matrix, by "flipping" the rows and columns.

$$x_{i,j} = a_{j,i}$$

It is denoted by a superscript ***T***, i.e:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

As an aside, there is an interesting relationship between transposition and multiplication:

$(A \times B)^T = B^T \times A^T$

$A = (A^T)^T = A$

$(A + B)^T = A^T + B^T$

$(kA)^T = k(A)^T$

<u>H.W</u>: choose Matrix A and Matrix B then apply all the above process to them.

If you are interested, you can prove this for yourself fairly easily. Hint – look at the definition of matrix multiply, and try swapping the subscripts.

Equality :

two matrices are considered to be ***equal*** if they have the same order, and if all their corresponding elements are equal.

Square Matrix :

A square matrix is a matrix where the number of **rows** and **columns** are equal ,i.e. a *2 by 2* matrix, a *3 by 3* matrix etc.

Unit (Identity) Matrix :

A unit matrix is ***a square matrix*** in which all the elements on the leading diagonal are 1, and all the other elements are 0, i.e.:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zero (Null) Matrix:

A zero, or null, matrix is one where every element is zero, i.e.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Diagonal Matrix :

A diagonal matrix is a **square** matrix in which all the elements are zero except for the elements on the leading diagonal, i.e.:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Symmetric Matrix

A symmetric matrix is a **square** matrix where

$$a_{i,j} = a_{j,i}$$

for all elements, i.e., the matrix is symmetrical about the leading diagonal. For example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

Skew Symmetric Matrix(anti-symmetrical Matrix) :

A skew symmetric matrix is a ***square*** matrix where :

$$a_{i,j} = -a_{j,i}$$

for all elements. i.e., the matrix is ***anti-symmetrical*** about the leading diagonal. This of course requires that elements along the **diagonal must be zero**. For example :

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

Orthogonal Matrix:

An orthogonal matrix is a ***square*** matrix which produces an identity matrix if it is multiplied by its own transpose, i.e.:

$$A \times A^T = I$$

a - $A \times A^T = I = A^T \times A$

Or

b- $A^{-1} = A^T$

RANK :

The ***RANK*** of a matrix is equal to the highest order non-zero determinant that can be formed from its sub-matrices

$$A = \begin{bmatrix} 4 & 5 & 2 & 14 \\ 3 & 9 & 6 & 21 \\ 8 & 10 & 7 & 28 \\ 1 & 2 & 9 & 5 \end{bmatrix}$$

$$\det A = 0$$

$$\begin{vmatrix} 4 & 5 & 2 \\ 3 & 9 & 6 \\ 8 & 10 & 7 \end{vmatrix} = 63$$

Rank of A = 3

The rank of a matrix can also be measured by the maximum number of linearly independent columns of A.

This also equals the maximum number of linearly independent rows

$$1\begin{bmatrix}4\\3\\8\\1\end{bmatrix}+2\begin{bmatrix}5\\9\\10\\2\end{bmatrix}+0\begin{bmatrix}2\\6\\7\\9\end{bmatrix}+(-1)\begin{bmatrix}14\\21\\28\\5\end{bmatrix}=0$$

$c_1\underline{a}_1 + c_2\underline{a}_2 + c_3\underline{a}_3 + c_4\underline{a}_4 = 0$

A FULL COLUMN RANK matrix has the same number of linearly independent columns (rows) equal to the number of columns

A FULL ROW RANK matrix has the same number of linearly independent rows (columns) equal to the number of rows

If A does not have full row and column rank it is <u>SINGULAR</u> : det(A)=0
If A does have full row and column rank it is <u>NON-SINGULAR</u> : det(A)≠0

rank $(I_n) = n$
rank (kA) = rank (A)
rank (A^T) = rank (A)
If A is (m x n) then rank (A) is $\le$ min {m, n}
rank $AB \le$ min{rank (A), rank (B)}

Inverse Matrix :

The inverse of a square matrix A, sometimes called a reciprocal matrix, is a matrix A^{-1} such that :

$$AA^{-1} = I,$$

Where " **I** "is the identity matrix. A square matrix " ***A*** " has an inverse if the determinant $|A| \neq 0$ A matrix possessing an inverse is called nonsingular, or invertible.

The matrix inverse of a square matrix m may be taken in Mathematic using the function Inverse[m].

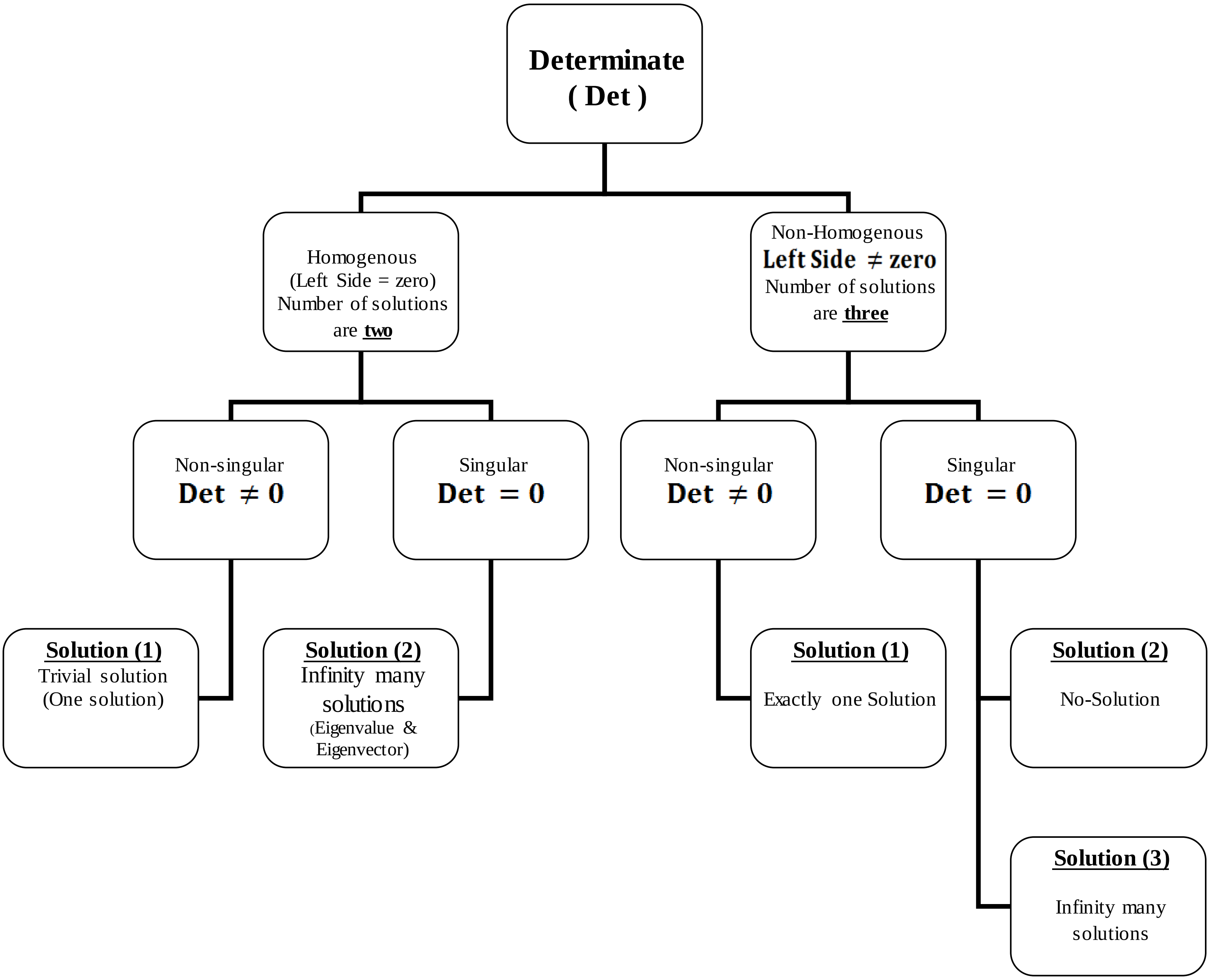
For a 2 X 2 Matrix :

$$A \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

The Inverse of " **A** " is then $\mathbf{A^{-1}}$:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(Solution systems of linear equations using Gaussian Elimination)



Definitions:
Singular : if the determinant of a matrix is zero we call that matrix singular.
Non-singular : if the determinant of a matrix isn't zero we call the matrix non-singular.

Gaussian Elimination (elementary row operations) :

Before working an example let's first define the elementary row operations. There are three of them.

1. Interchange two rows. This is exactly what it says. We will interchange row i with row j. The notation that we'll use to denote this operation is : $R_i \leftrightarrow R_j$
2. Multiply row i by a constant, c. This means that every entry in row i will get multiplied by the constant c. The notation for this operation is : cR_i
3. Add a multiple of row i to row j. In our heads we will multiply row i by an appropriate constant and then add the results to row j and put the new row back into row j leaving row i in the matrix unchanged. The notation for this operation is : $cR_i + R_j$

Example 1 Solve the following system of equations.

$$-2x_1 + x_2 - x_3 = 4$$
$$x_1 + 2x_2 + 3x_3 = 13$$
$$3x_1 + x_3 = -1$$

$$\begin{pmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{pmatrix} \xrightarrow[-3R_1 + R_3]{2R_1 + R_2} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{pmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{pmatrix} \xrightarrow{6R_2+R_3} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

We can now convert back to equations.

$$\begin{pmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{pmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 + 3x_3 &= 13 \\ x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

The solution to this system of equation is

$$x_1 = -1 \qquad x_2 = 4 \qquad x_3 = 2$$

Example 2 Solve the following system of equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= 1 \end{aligned}$$

Solution

First write down the augmented matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{pmatrix} \begin{matrix} R_1 + R_2 \\ -2R_1 + R_3 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{pmatrix}$$

$$\begin{matrix} -R_2 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{pmatrix} \begin{matrix} -3R_2 + R_3 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{pmatrix} \Rightarrow \begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ x_2 - x_3 &= -1 \\ 0 &= 8 \end{aligned}$$

When we get something like the third equation that simply doesn't make sense we immediately know that there is ***no solution***. In other words, there is no set of three numbers that will make all three of the equations true at the same time.

Example 3 Solve the following system of equations.

$$x_1 - 2x_2 + 3x_3 = -2$$
$$-x_1 + x_2 - 2x_3 = 3$$
$$2x_1 - x_2 + 3x_3 = -7$$

Now write down the augmented matrix for this system.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & -7 \end{pmatrix}$$

The steps for this problem are identical to the steps for the second problem so we won't write them all down. Upon performing the same steps we arrive at the following matrix.

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This time the last equation reduces to

$$0 = 0$$

and unlike the second example this is not a problem. Zero does in fact equal zero!

We could stop here and go back to equations to get a solution and there is a solution in this case. However, if we go one more step and get a zero above the one in the second column as well as below it our life will be a little simpler. Doing this gives,

$$\begin{pmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{2R_2 + R_1}{\Rightarrow} \begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If we now go back to equation we get the following two equations.

$$\begin{pmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{aligned} x_1 + x_3 &= -4 \\ x_2 - x_3 &= -1 \end{aligned}$$

We have two equations and three unknowns. This means that we can solve for two of the variables in terms of the remaining variable. Since x_3 is in both equations we will solve in terms of that.

$$x_1 = -x_3 - 4$$
$$x_2 = x_3 - 1$$

What this solution means is that we can pick the value of x_3 to be anything that we'd like and then find values of x_1 and x_2. In these cases we typically write the solution as follows,

$$x_1 = -t - 4$$
$$x_2 = t - 1 \qquad t = \text{ any real number}$$
$$x_3 = t$$

In this way we get an infinite number of solutions, one for each and every value of t.

Eigenvalues and Eigenvectors :

If we multiply an n x n matrix by an n x 1 vector we will get a new n x 1 vector back. In other words,

$$A\vec{\eta} = \vec{y}$$

What we want to know is if it is possible for the following to happen. Instead of just getting a brand new vector out of the multiplication is it possible instead to get the following,

$$A\vec{\eta} = \lambda\vec{\eta} \qquad \textbf{(1)}$$

In other words is it possible, at least for certain λ and $\vec{\eta}$, to have matrix multiplication be the same as just multiplying the vector by a constant? Of course, we probably wouldn't be talking about this if the answer was no. So, it is possible for this to happen, however, it won't happen for just any value of λ or $\vec{\eta}$. If we do happen to have a λ and $\vec{\eta}$ for which this works (and they will always come in pairs) then we call λ an **eigenvalue** of A and $\vec{\eta}$ an **eigenvector** of A.

So, how do we go about find the eigenvalues and eigenvectors for a matrix? Well first notice that if $\vec{\eta} = \vec{0}$ then (1) is going to be true for any value of λ and so we are going to make the assumption that $\vec{\eta} \neq \vec{0}$. With that out of the way let's rewrite (1) a little.

$$A\vec{\eta} - \lambda\vec{\eta} = \vec{0}$$

$$A\vec{\eta} - \lambda I_n\vec{\eta} = \vec{0}$$

$$\left(A - \lambda I_n\right)\vec{\eta} = \vec{0}$$

Notice that before we factored out the $\vec{\eta}$ we added in the appropriately sized identity matrix. This is equivalent to multiplying things by a one and so doesn't change the value of anything. We needed to do this because without it we would have had the difference of a matrix, A, and a constant, λ, and this can't be done. We now have the difference of two matrices of the same size which can be done.

So, with this rewrite we see that

$$\left(A - \lambda I_n\right)\vec{\eta} = \vec{0} \qquad \textbf{(2)}$$

is equivalent to (1). In order to find the eigenvectors for a matrix we will need to solve a homogeneous system. Recall the fact from the previous section that we know that we will either have exactly one solution ($\vec{\eta} = \vec{0}$) or we will have infinitely many nonzero solutions. Since we've already said that we don't want $\vec{\eta} = \vec{0}$ this means that we want the second case.

Knowing this will allow us to find the eigenvalues for a matrix. Recall from this fact that we will get the second case only if the matrix in the system is singular. Therefore we will need to determine the values of λ for which we get,

$$\det\left(A - \lambda I\right) = 0$$

Once we have the eigenvalues we can then go back and determine the eigenvectors for each eigenvalue. Let's take a look at a couple of quick facts about eigenvalues and eigenvectors.

Fact

If A is an n x n matrix then $\det\left(A-\lambda I\right)=0$ is an n^{th} degree polynomial. This polynomial is called the **characteristic polynomial**.

To find eigenvalues of a matrix all we need to do is solve a polynomial. That's generally not too bad provided we keep n small. Likewise this fact also tells us that for an n x n matrix, A, we will have n eigenvalues if we include all repeated eigenvalues.

Fact

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the complete list of eigenvalues for A (including all repeated eigenvalues) then,

1. If λ occurs only once in the list then we call λ **simple**.
2. If λ occurs $k>1$ times in the list then we say that λ has **multiplicity** $\boldsymbol{k}$.
3. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ $(k \le n)$ are the simple eigenvalues in the list with corresponding eigenvectors $\vec{\eta}^{(1)}$, $\vec{\eta}^{(2)}$, …, $\vec{\eta}^{(k)}$ then the eigenvectors are all linearly independent.
4. If λ is an eigenvalue of multiplicity $k > 1$ then λ will have anywhere from 1 to k linearly independent eigenvectors.

The usefulness of these facts will become apparent when we get back into differential equations since in that work we will want linearly independent solutions.

Let's work a couple of examples now to see how we actually go about finding eigenvalues and eigenvectors.

Example 1 Find the eigenvalues and eigenvectors of the following matrix.

$$A=\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$$

Solution

The first thing that we need to do is find the eigenvalues. That means we need the following matrix,

$$A-\lambda I=\begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix}$$

In particular we need to determine where the determinant of this matrix is zero.

$$\det\left(A-\lambda I\right)=\left(2-\lambda\right)\left(-6-\lambda\right)+7=\lambda^2+4\lambda-5=\left(\lambda+5\right)\left(\lambda-1\right)$$

So, it looks like we will have two simple eigenvalues for this matrix, $\lambda_1=-5$ and $\lambda_2=1$. We will now need to find the eigenvectors for each of these. Also note that according to the fact above, the two eigenvectors should be linearly independent.

To find the eigenvectors we simply plug in each eigenvalue into (2) and solve. So, let's do that.

$\lambda_1 = -5$:

In this case we need to solve the following system.

$$\begin{pmatrix} 7 & 7 \\ -1 & -1 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Recall that officially to solve this system we use the following augmented matrix.

$$\begin{pmatrix} 7 & 7 & 0 \\ -1 & -1 & 0 \end{pmatrix} \begin{matrix} \frac{1}{7}R_1 + R_2 \\ \Rightarrow \end{matrix} \begin{pmatrix} 7 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Upon reducing down we see that we get a single equation

$$7\eta_1 + 7\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

that will yield an infinite number of solutions. This is expected behavior. Recall that we picked the eigenvalues so that the matrix would be singular and so we would get infinitely many solutions.

Now, let's get back to the eigenvector, since that is what we were after. In general then the eigenvector will be any vector that satisfies the following,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \qquad , \eta_2 \neq 0$$

Here's the eigenvector for this eigenvalue.

$$\vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \text{using } \eta_2 = 1$$

Now we get to do this all over again for the second eigenvalue.

$\lambda_2 = 1$:

We'll do much less work with this part than we did with the previous part. We will need to solve the following system.

$$\begin{pmatrix} 1 & 7 \\ -1 & -7 \end{pmatrix} \vec{\eta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\eta_1 + 7\eta_2 = 0 \qquad \eta_1 = -7\eta_2$$

The eigenvector is the

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -7\eta_2 \\ \eta_2 \end{pmatrix} \qquad , \eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}, \qquad \text{using } \eta_2 = 1$$

Summarizing we have,

$$\lambda_1 = -5 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -7 \\ 1 \end{pmatrix}$$

Example 2 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 1 & -1 \\ \frac{4}{9} & -\frac{1}{3} \end{pmatrix}$$

Solution

This matrix has fractions in it. That's life so don't get excited about it. First we need the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ \frac{4}{9} & -\frac{1}{3}-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(-\frac{1}{3}-\lambda\right) + \frac{4}{9}$$

$$= \lambda^2 - \frac{2}{3}\lambda + \frac{1}{9}$$

$$= \left(\lambda - \frac{1}{3}\right)^2 \qquad \Rightarrow \qquad \lambda_{1,2} = \frac{1}{3}$$

So, it looks like we've got an eigenvalue of multiplicity 2 here. Remember that the power on the term will be the multiplicity.

Now, let's find the eigenvector(s). This one is going to be a little different from the first example. There is only one eigenvalue so let's do the work for that one. We will need to solve the following system.

$$\begin{pmatrix} \frac{2}{3} & -1 \\ \frac{4}{9} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad R_1 = \frac{3}{2} R_2$$

So, the rows are multiples of each other. We'll work with the first equation in this example to find the eigenvector.

$$\frac{2}{3}\eta_1 - \eta_2 = 0 \qquad \eta_2 = \frac{2}{3}\eta_1$$

In this case the eigenvector will be,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \frac{2}{3}\eta_1 \end{pmatrix}, \qquad \eta_1 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad \eta_1 = 3$$

Also in this case we are only going to get a single (linearly independent) eigenvector. We can get other eigenvectors, by choosing different values of η_1. However, each of these will be linearly dependent with the first eigenvector. If you're not convinced of this try it. Pick some values for η_1 and get a different vector and check to see if the two are linearly dependent.

Example 3 Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} -4 & -17 \\ 2 & 2 \end{pmatrix}$$

Solution

So, we'll start with the eigenvalues.

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & -17 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (-4-\lambda)(2-\lambda) + 34$$

$$= \lambda^2 + 2\lambda + 26$$

This doesn't factor, so upon using the quadratic formula we arrive at,

$$\lambda_{1,2} = -1 \pm 5i$$

In this case we get complex eigenvalues which are definitely a fact of life with eigenvalue/eigenvector problems so get used to them.

Finding eigenvectors for complex eigenvalues is identical to the previous two examples, but it will be somewhat messier. So, let's do that.

$\lambda_1 = -1 + 5i$:

The system that we need to solve this time is

$$\begin{pmatrix} -4-(-1+5i) & -17 \\ 2 & 2-(-1+5i) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3-5i & -17 \\ 2 & 3-5i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now, it's not super clear that the rows are multiples of each other, but they are. In this case we have,

$$R_1 = -\frac{1}{2}(3+5i) R_2$$

We'll work with the second row this time.

$$2\eta_1 + (3-5i)\eta_2 = 0$$

$$2\eta_1 = -(3-5i)\eta_2$$

$$\eta_1 = -\frac{1}{2}(3-5i)\eta_2$$

So, the eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(3-5i)\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} -3+5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

As with the previous example we choose the value of the variable to clear out the fraction.

Now, the work for the second eigenvector is almost identical and so we'll not dwell on that too much.

$\lambda_2 = -1-5i$:

The system that we need to solve here is

$$\begin{pmatrix} -4-(-1-5i) & -17 \\ 2 & 2-(-1-5i) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+5i & -17 \\ 2 & 3+5i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Working with the second row again gives,

$$2\eta_1 + (3+5i)\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\frac{1}{2}(3+5i)\eta_2$$

The eigenvector in this case is

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(3+5i)\eta_2 \\ \eta_2 \end{pmatrix}, \qquad \eta_2 \neq 0$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}, \qquad \eta_2 = 2$$

Summarizing,

$$\lambda_1 = -1+5i \qquad \vec{\eta}^{(1)} = \begin{pmatrix} -3+5i \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1-5i \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -3-5i \\ 2 \end{pmatrix}$$

There is a nice fact that we can use to simplify the work when we get complex eigenvalues. We need a bit of terminology first however.

If we start with a complex number,

$$z = a + bi$$

then the **complex conjugate** of z is

$$\bar{z} = a - bi$$

To compute the complex conjugate of a complex number we simply change the sign on the term that contains the "i". The complex conjugate of a vector is just the conjugate of each of the vector's components.

We now have the following fact about complex eigenvalues and eigenvectors.

Fact

If A is an n x n matrix with only real numbers and if $\lambda_1 = a + bi$ is an eigenvalue with eigenvector $\vec{\eta}^{(1)}$. Then $\lambda_2 = \overline{\lambda_1} = a - bi$ is also an eigenvalue and its eigenvector is the conjugate of $\vec{\eta}^{(1)}$.

(H.W.1) Find the eigenvalues and eigenvectors of the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution :

the eigenvalues and eigenvectors for this matrix

$$\lambda_1 = 2 \qquad \vec{\eta}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad \vec{\eta}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -1 \qquad \vec{\eta}^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

H.W.(2) : Determined the eigenvalues of the matrix **A** given below :

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 4 & 0 \\ 3 & 5 & -3 \end{bmatrix}$$

Solution :

$\lambda_1 = 4$, $\lambda_2 = -5$, $\lambda_3 = 3$,

H.W.(3) : Determined the eigenvector of the H.W.2 at $\lambda_3 = 3$:

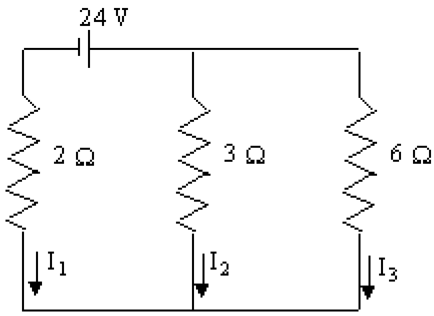
Solution :

The eigenvector $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ where $s \neq 0$ and corresponds to $\lambda_3 = 3$.

Application of matrices to electric circuits :

Example (1) :

Find the electric currents shown by solving the matrix equation (obtained ***using*** ***Kirchhoff's Law***) arising from this circuit (Use inverse matrix):



Solution :

$$\begin{pmatrix} I_1 + I_2 + I_3 \\ -2I_1 + 3I_2 \\ -3I_2 + 6I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \\ 0 \end{pmatrix}$$

We can write this as:

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & 3 & 0 \\ 0 & -3 & 6 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \\ 0 \end{pmatrix}$$

So we have:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 3 & 0 \\ 0 & -3 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 24 \\ 0 \end{pmatrix}$$

Using a computer algebra system to perform the inverse and multiply by the constant matrix, we get:

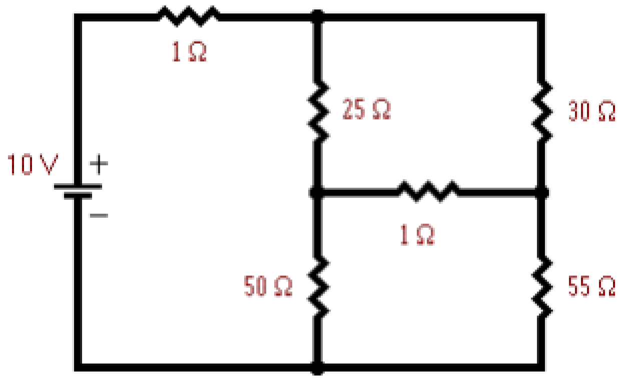
$$I_1 = -6 \text{ A}$$

$$I_2 = 4 \text{ A}$$

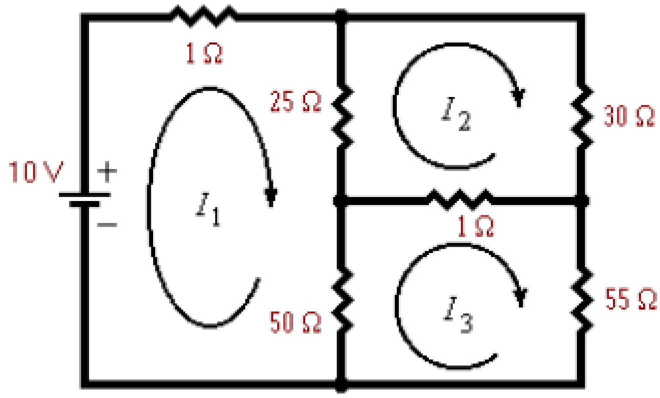
$$I_3 = 2 \text{ A}$$

We observe that I_1 is negative, as expected from the circuit diagram.

Example (2) :
Find the electric currents shown by solving the matrix equation (obtained ***using Kirchhoff's Law***) arising from this circuit (Use ***Gaussian Elimination***):



Solution :



$$\begin{cases} 1i_1 + 25(i_1 - i_2) + 50(i_1 - i_3) = 10 \\ 25(i_2 - i_1) + 30i_2 + 1(i_2 - i_3) = 0 \\ 50(i_3 - i_1) + 1(i_3 - i_2) + 55i_3 = 0 \end{cases} \Rightarrow \begin{cases} 76i_1 - 25i_2 - 50i_3 = 10 \\ -25i_1 + 56i_2 - 1i_3 = 0 \\ -50i_1 - 1i_2 + 106i_3 = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{bmatrix} 76 & -25 & -50 & 10 \\ -25 & 56 & -1 & 0 \\ -50 & -1 & 106 & 0 \end{bmatrix}$$

$$\Downarrow$$

Then by using the value of currents are :

$$\begin{bmatrix} 1 & 0 & 0 & 0.245 \\ 0 & 1 & 0 & 0.111 \\ 0 & 0 & 1 & 0.117 \end{bmatrix}$$

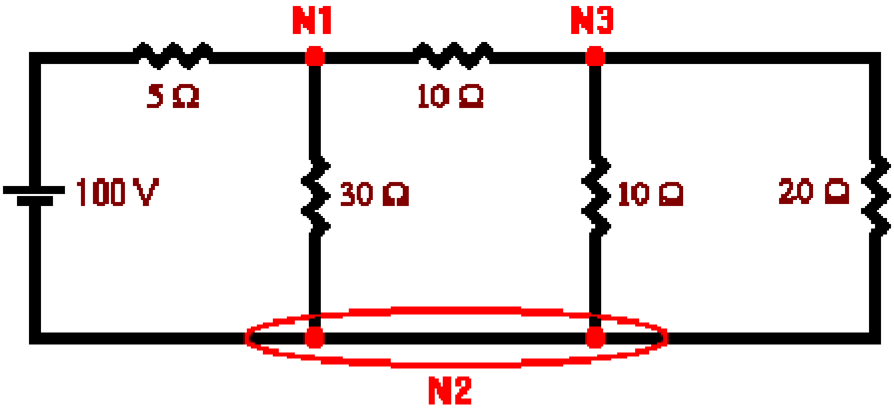
$i_3 = 0.117$, $i_2 = 0.111$, $i_1 = 0.245$

(H.W) :

Find the electric currents shown by solving the matrix equation (obtained ***using Nodal Voltage Analysis)*** arising from this circuit (Use ***Gaussian Elimination***):

Solution :

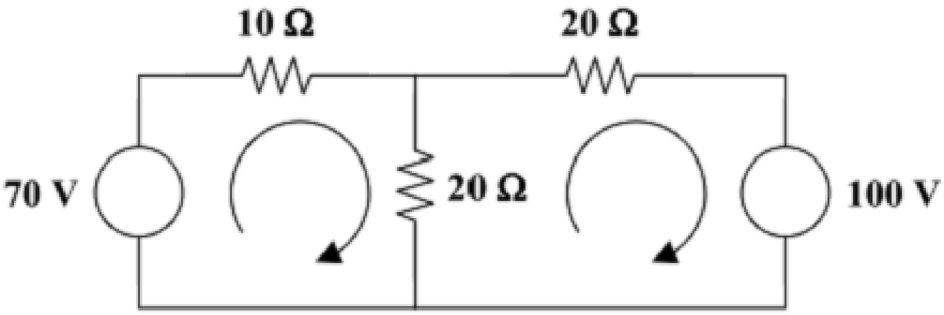
$V_1 = 75$ V, $V_3 = 50$ V



N2 is the reference node as so the voltage is 0V.

(H.W) : Using Matrix to find the currents in the circuit below :

Given the circuit below, solve for the loop currents i_1 and i_2 indicated using mesh analysis.



Solution : $i_1 = \mathbf{1\ A}$ and $i_2 = \mathbf{-2\ A}$

Rank of matrix ①

Determinate ↔ Gaussian elemination

1- Determinate method

→ Find determinate of (n×n) matrix

- dete ≠ 0 ∴ $\rho(A) = n$ (↑ Rank)
- dete = 0: take another (n×n) and find dete

Example: Find the rank of matrix using determinat method.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{bmatrix}$$

Solu: n×n

$$|2*2| = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} = -12 \neq 0$$

$$\therefore \rho(A) = 2 = n$$

②

Example: Find the rank of matrix using determinate method.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

Solution:
$$|3\times3| = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

all (3×3) is zero, then take (2×2).

$$|2\times2| = \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 4 \neq 0$$

$$\therefore \rho(A) = 2$$

H.W: Find the rank of matrix using determinate method.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & & -6 \\ 0 & 1 & 2 \end{bmatrix}$$

Soln: $\rho(A) = 2$

2- Gaussian Elemination method

③

After solve using row elemination, the rank of matrix is the ~~...~~ number of rows that are not all elements is zero.

Example:- Find the rank of matrix using Gaussian elemination method.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{bmatrix} \rightarrow -2R_1 + R_2 \rightarrow R_2$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix}$$

$\therefore \rho(A) = 2$

Example:- Find the rank of matrix using Gaussian Elemination method.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

Solution:-

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} \rightarrow -R_1 + R_2 \rightarrow R_2 \\ \\ \rightarrow -3R_1 + R_3 \rightarrow R_3 \end{matrix}$$

$\Downarrow$

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \rightarrow -R_2 + R_3 \rightarrow R_3$$

$\Downarrow$

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\rho(A) = 2}$

H.W: use Gaussian elemination method to find the Rank of the following matrix ④

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 3 & -6 \\ 0 & 1 & 2 \end{bmatrix}$$

solution:- $\rho(A) = 2$

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution

We use elementary row operations:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix} \dashrightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

A *has rank* $= 3$.

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$$

Solution A *has rank* $= 3$

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

$\mathcal{L}$

The Laplace Transform

(Laplace Transforms)

Laplace Transforms :

The solution of most electrical circuit problems can be reduced ultimately to the solution of differential equations. The use of ***Laplace transforms*** provides an alternative method to those discussed in the previous subjects for solving linear differential equations.

Definition

Suppose that *f(t)* is a piecewise continuous function. The Laplace transform of *f(t)* is denoted $\mathcal{L}\{f(t)\}$ and defined as

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} \mathbf{e}^{-st} f(t)\,dt \qquad \textbf{(1)}$$

There is an alternate notation for Laplace transforms. For the sake of convenience we will often denote Laplace transforms as,

$$\mathcal{L}\{f(t)\} = F(s)$$

With this alternate notation, note that the transform is really a function of a new variable, s, and that all the t's will drop out in the integration process.

Now, the integral in the definition of the transform is called an ***improper integral*** and it would probably be best to know how these kinds of integrals work before we actually jump into computing some transforms.

Improper Integrals :

In this section we need to take a look at integrals that are called Improper Integrals.

Infinite Interval :

In this kind of integral one or both of the limits of integration are infinity. In these cases the interval of integration is said to be over an infinite interval.

Let's now formalize up the method for dealing with infinite intervals. There are essentially three cases that we'll need to look at.

1. If $\int_a^t f(x)\,dx$ exists for every $t > a$ then,

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the limit exists and is finite.

2. If $\int_t^b f(x)\,dx$ exists for every $t < b$ then,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided the limits exists and is finite.

3. If $\int_{-\infty}^{c} f(x)\,dx$ and $\int_c^{\infty} f(x)\,dx$ are both convergent then,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

Where c is any number. Note as well that this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

Example Determine if the following integral is convergent or divergent and if it's convergent find its value.

$$\int_1^\infty \frac{1}{x}\,dx$$

Solution
So, the first thing we do is convert the integral to a limit.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx$$

Now, do the integral and the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty} \ln(x)\Big|_1^t \\ &= \lim_{t\to\infty}\left(\ln(t)-\ln 1\right) \\ &= \infty\end{aligned}$$

So, the limit is infinite and so the integral is divergent.

Example If $c \neq 0$, evaluate the following integral.

$$\int_0^\infty \mathbf{e}^{ct}\,dt$$

Solution
Remember that you need to convert improper integrals to limits as follows,

$$\int_0^\infty \mathbf{e}^{ct}\,dt = \lim_{n\to\infty}\int_0^n \mathbf{e}^{ct}\,dt$$

Now, do the integral, then evaluate the limit.

$$\begin{aligned}\int_0^\infty \mathbf{e}^{ct}\,dt &= \lim_{n\to\infty}\int_0^n \mathbf{e}^{ct}\,dt \\ &= \lim_{n\to\infty}\left(\frac{1}{c}\mathbf{e}^{ct}\right)\Bigg|_0^n \\ &= \lim_{n\to\infty}\left(\frac{1}{c}\mathbf{e}^{cn}-\frac{1}{c}\right)\end{aligned}$$

Now, at this point, we've got to be careful. The value of c will affect our answer. We've already assumed that c was non-zero, now we need to worry about the sign of c. If c is positive the exponential will go to infinity. On the other hand, if c is negative the exponential will go to zero.

So, the integral is only convergent (*i.e.* the limit exists and is finite) provided $c<0$. In this case we get,

$$\int_0^\infty \mathbf{e}^{ct}\,dt = -\frac{1}{c} \qquad \text{provided } c<0 \qquad \textbf{(2)}$$

Now that we remember how to do these (***improper integrals***), let's compute some Laplace transforms. We'll start off with probably the simplest Laplace transform to compute.

Example Compute $\mathcal{L}\{1\}$.

Solution
There's not really a whole lot do here other than plug the function *f(t)* = 1 into (1)

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathbf{e}^{-st} f(t)\,dt \qquad \textbf{(1)}$$

$$\mathcal{L}\{1\} = \int_0^\infty \mathbf{e}^{-st}\,dt$$

Now, at this point notice that this is nothing more than the integral in the previous example with $c = -s$. Therefore, all we need to do is reuse (2) with the appropriate substitution. Doing this gives,

$$\mathcal{L}\{1\} = \int_0^\infty \mathbf{e}^{-st}\,dt = -\frac{1}{-s} \qquad \text{provided } -s < 0$$

Or, with some simplification we have,

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \text{provided } s > 0$$

Example Compute $\mathcal{L}\{\mathbf{e}^{at}\}$

Solution
Plug the function into the definition of the transform and do a little simplification.

$$\mathcal{L}\{\mathbf{e}^{at}\} = \int_0^\infty \mathbf{e}^{-st}\mathbf{e}^{at}\,dt = \int_0^\infty \mathbf{e}^{(a-s)t}\,dt$$

Once again, notice that we can use (2) provided $c = a - s$. So let's do this.

$$\begin{aligned}\mathcal{L}\{\mathbf{e}^{at}\} &= \int_0^\infty \mathbf{e}^{(a-s)t}\,dt \\ &= -\frac{1}{a-s} \qquad \text{provided } a - s < 0 \\ &= \frac{1}{s-a} \qquad \text{provided } s > a\end{aligned}$$

Example Compute $\mathcal{L}\{\sin(at)\}$.

Solution

Note that we're going to leave it to you to check most of the integration here. Plug the function into the definition. This time let's also use the alternate notation.

$$\begin{aligned}\mathcal{L}\{\sin(at)\} &= F(s)\\ &= \int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\\ &= \lim_{n\to\infty}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\end{aligned}$$

Now, if we integrate by parts we will arrive at,

$$F(s) = \lim_{n\to\infty}\left(-\left(\frac{1}{a}\mathbf{e}^{-st}\cos(at)\right)\Bigg|_0^n - \frac{s}{a}\int_0^n \mathbf{e}^{-st}\cos(at)\,dt\right)$$

Now, evaluate the first term to simplify it a little and integrate by parts again on the integral. Doing this arrives at,

$$F(s) = \lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right) - \frac{s}{a}\left(\left(\frac{1}{a}\mathbf{e}^{-st}\sin(at)\right)\Bigg|_0^n + \frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right)$$

Now, evaluate the second term, take the limit and simplify.

$$\begin{aligned}F(s) &= \lim_{n\to\infty}\left(\frac{1}{a}\left(1-\mathbf{e}^{-sn}\cos(an)\right) - \frac{s}{a}\left(\frac{1}{a}\mathbf{e}^{-sn}\sin(an) + \frac{s}{a}\int_0^n \mathbf{e}^{-st}\sin(at)\,dt\right)\right)\\ &= \frac{1}{a} - \frac{s}{a}\left(\frac{s}{a}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\right)\\ &= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty \mathbf{e}^{-st}\sin(at)\,dt\end{aligned}$$

Now, notice that in the limits we had to assume that $s>0$ in order to do the following two limits.

$$\lim_{n\to\infty}\mathbf{e}^{-sn}\cos(an) = 0$$
$$\lim_{n\to\infty}\mathbf{e}^{-sn}\sin(an) = 0$$

Without this assumption, we get a divergent integral again. Also, note that when we got back to the integral we just converted the upper limit back to infinity. The reason for this is that, if you think about it, this integral is nothing more than the integral that we started with. Therefore, we now get,

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}F(s)$$

Now, simply solve for $F(s)$ to get,

$$\mathcal{L}\{\sin(at)\} = F(s) = \frac{a}{s^2+a^2} \qquad \text{provided } s>0$$

Before moving on to the next section, we need to do a little side note. On occasion you will see the following as the definition of the Laplace transform.

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

Note the change in the lower limit from zero to negative infinity. In these cases there is almost always the assumption that the function *f(t)* is in fact defined as follows,

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ f(t) & \text{if } t \ge 0 \end{cases}$$

In other words, it is assumed that the function is zero if $t<0$. In this case the first half of the integral will drop out since the function is zero and we will get back to the definition given in (1). A Heaviside function is usually used to make the function zero for $t<0$. We will be looking at these in a later section.

Table of the common functions used by Laplace transform

$f(t)$	$F(s) = L\{f(t)\}$
a	$\frac{a}{s}$
$t^n,\ n = 1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

Example Find the Laplace transforms of the given functions.

(a) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

(b) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

(c) $h(t) = 3\sinh(2t) + 3\sin(2t)$ **(H.W.)**

Solution

Okay, there's not really a whole lot to do here other than go to the table, transform the individual functions up, put any constants back in and then add or subtract the results.

(a) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

(b) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$G(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

Properties of Laplace transform:

1- Linearity: $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$.

2- *Shifting property.* If $\mathscr{L}\{f\} = F(s)$ then $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$.

3- *Time scaling.* Let $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$,

4- *Differentiation of the frequency.* Let $\mathscr{L}\{f(t)\} = F(s)$. Then $\mathscr{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$

5- *Differentiation.* Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$.

$\mathscr{L}\{f''(t)\} = s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$.

1-Linearity :

Example Using this property we can easily find, using the information above, the Laplace transform of, e.g., $5 - 3t + \pi\cos t$:

$$\mathscr{L}\{5 - 3t + \pi\cos t\} = 5\,\mathscr{L}\{1\} - 3\,\mathscr{L}\{t\} + \pi\,\mathscr{L}\{\cos t\} = \frac{5}{s} - \frac{3}{s^2} + \frac{\pi s}{s^2+1}.$$

2-Shifting property:

Example Now, to find, e.g., $\mathscr{L}\{e^{3t}\sin t\}$ we do not need to evaluate the integral:

since $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$.

$$\mathscr{L}\{e^{3t}\sin t\} = \frac{1}{(s-3)^2+1},$$

3-Time scaling:

Example Find $\mathscr{L}\{\cos 3t\}$.

By the previous property and the fact that $\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$ we find

$$\mathscr{L}\{\cos 3t\} = \frac{1}{3}\frac{s/3}{(s/3)^2+1} = \frac{s}{s^2+3^2}.$$

4-Differentiation of the frequency:

Example What is $\mathscr{L}\{t^3\}$?

the fact that $\mathscr{L}\{1\} = 1/s$,

$$\mathscr{L}\{t^3 \times \mathbf{1}\} = (-1)^3\left(\frac{1}{s}\right)''' = \frac{3\cdot 2}{s^4}$$

5- Differentiation used in Solution of initial value problems:

Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$.

$\mathscr{L}\{f''(t)\} = s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$.

More Examples about Laplace Transform

Laplace transform

Example 1. $\mathcal{L}\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}$. From $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$,

Example 2. $\mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$. From $\mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}$,

Example 3. $\mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$. From $\mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}$,

Example 4. $\mathcal{L}\{t^3+5t-2\} = \mathcal{L}\{t^3\}+5\mathcal{L}\{t\}-2\mathcal{L}\{1\} = \dfrac{3!}{s^4}+5\dfrac{1}{s^2}-2\dfrac{1}{s}$.

Example 5. $\mathcal{L}\{e^{2t}(t^3+5t-2)\} = \dfrac{3!}{(s-2)^4}+5\dfrac{1}{(s-2)^2}-2\dfrac{1}{s-2}$.

Example 6. $\mathcal{L}\{(t^2+4)e^{2t}-e^{-t}\cos t\} = \dfrac{2}{(s-2)^3}+\dfrac{4}{s-2}-\dfrac{s+1}{(s+1)^2+1}$,

because $\mathcal{L}\{t^2+4\} = \dfrac{2}{s^3}+\dfrac{4}{s}$, $\Rightarrow \mathcal{L}\{(t^2+4)e^{2t}\} = \dfrac{2}{(s-2)^3}+\dfrac{4}{s-2}$.

Example Find the transform of each of the following functions.

(a) $f(t) = t\cosh(3t)$

(b) $h(t) = t^2\sin(2t)$

(c) $g(t) = t^{\frac{3}{2}}$

(d) $f(t) = (10t)^{\frac{3}{2}}$

(e) $f(t) = tg'(t)$

(c)–(e) → **(H.W.)**

Solution

(a) $f(t) = t\cosh(3t)$

This function is not in the table of Laplace transforms. However we can use #30 in the table to compute its transform. This will correspond to #30 if we take $n=1$.

$$F(s) = \mathcal{L}\{tg(t)\} = -G'(s), \quad \text{where } g(t) = \cosh(3t)$$

So, we then have,

$$G(s) = \frac{s}{s^2-9} \qquad G'(s) = -\frac{s^2+9}{(s^2-9)^2}$$

Using #30 we then have,

$$F(s) = \frac{s^2+9}{(s^2-9)^2}$$

(b) $h(t) = t^2 \sin(2t)$

This part will also use #30 in the table. In fact we could use #30 in one of two ways. We could use it with $n = 1$.

$$H(s) = \mathcal{L}\{tf(t)\} = -F'(s), \qquad \text{where } f(t) = t\sin(2t)$$

Or we could use it with $n = 2$.

$$H(s) = \mathcal{L}\{t^2 f(t)\} = F''(s), \qquad \text{where } f(t) = \sin(2t)$$

Since it's less work to do one derivative, let's do it the first way. So using #9 we have,

$$F(s) = \frac{4s}{(s^2+4)^2} \qquad F'(s) = -\frac{12s^2-16}{(s^2+4)^3}$$

The transform is then,

$$H(s) = \frac{12s^2-16}{(s^2+4)^3}$$

H . W.

1 - $\mathcal{L}\{te^{at}\}$ **2 -** $\mathcal{L}\{t\sin bt\}$ **3 -** $\mathcal{L}\{t\cos bt\}$ **4-** $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$

Table of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n e^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	e^{-cs}
27.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	28.	$u_c(t)g(t)$	$e^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$e^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^{\infty}F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Table Notes

1. This list is not a complete listing of Laplace transforms and only contains some of the more commonly used Laplace transforms and formulas.

2. Recall the definition of hyperbolic functions.

$$\cosh(t) = \frac{e^{t} + e^{-t}}{2} \qquad \sinh(t) = \frac{e^{t} - e^{-t}}{2}$$

3. Be careful when using "normal" trig function vs. hyperbolic functions. The only difference in the formulas is the "+ a^2" for the "normal" trig functions becomes a "- a^2" for the hyperbolic functions!

4. Formula #4 uses the Gamma function which is defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx$$

If n is a positive integer then,

$$\Gamma(n+1) = n!$$

The Gamma function is an extension of the normal factorial function. Here are a couple of quick facts for the Gamma function

$$\Gamma(p+1) = p\Gamma(p)$$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Inverse Laplace Transforms :

In these cases we say that we are finding the Inverse Laplace Transform of ***F(s)*** and use the following notation.

$$f(t)=\mathcal{L}^{-1}\{F(s)\}$$

As with Laplace transforms, we've got the following fact to help us take the inverse transform.

Fact

Given the two Laplace transforms *F(s)* and *G(s)* then

$$\mathcal{L}^{-1}\{aF(s)+bG(s)\}=a\mathcal{L}^{-1}\{F(s)\}+b\mathcal{L}^{-1}\{G(s)\}$$

for any constants a and b.

So, we take the inverse transform of the individual transforms, put any constants back in and then add or subtract the results back up.

Example Find the inverse transform of each of the following.

(a) $F(s)=\frac{6}{s}-\frac{1}{s-8}+\frac{4}{s-3}$

(b) $H(s)=\frac{19}{s+2}-\frac{1}{3s-5}+\frac{7}{s^5}$

(c) $F(s)=\frac{6s}{s^2+25}+\frac{3}{s^2+25}$

(d) $G(s)=\frac{8}{3s^2+12}+\frac{3}{s^2-49}$

(H.W.) (c), (d)

Solution

(a) $F(s)=\frac{6}{s}-\frac{1}{s-8}+\frac{4}{s-3}$

$$F(s)=6\,\frac{1}{s}-\frac{1}{s-8}+4\frac{1}{s-3}$$

$$f(t)=6(1)-\mathbf{e}^{8t}+4\left(\mathbf{e}^{3t}\right)$$

$$=6-\mathbf{e}^{8t}+4\mathbf{e}^{3t}$$

(b) $H(s)=\dfrac{19}{s+2}-\dfrac{1}{3s-5}+\dfrac{7}{s^5}$

The first term in this case looks like an exponential with $a=-2$ and we'll need to factor out the 19. Be careful with negative signs in these problems, it's very easy to lose track of them.

The second term almost looks like an exponential, except that it's got a $3s$ instead of just an s in the denominator. It is an exponential, but in this case we'll need to factor a 3 out of the denominator before taking the inverse transform.

The denominator of the third term appears to be #3 in the table with $n=4$. The numerator however, is not correct for this. There is currently a 7 in the numerator and we need a 4! = 24 in the numerator. This is very easy to fix. Whenever a numerator is off by a multiplicative constant, as in this case, all we need to do is put the constant that we need in the numerator. We will just need to remember to take it back out by dividing by the same constant.

So, let's first rewrite the transform.

$$H(s)=\frac{19}{s-(-2)}-\frac{1}{3\left(s-\frac{5}{3}\right)}+\frac{7\frac{4!}{4!}}{s^{4+1}}$$

$$=19\frac{1}{s-(-2)}-\frac{1}{3}\frac{1}{s-\frac{5}{3}}+\frac{7}{4!}\frac{4!}{s^{4+1}}$$

Let's now take the inverse transform.

$$h(t)=19\mathbf{e}^{-2t}-\frac{1}{3}\mathbf{e}^{\frac{5t}{3}}+\frac{7}{24}t^4$$

Example $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+4}\right\}=\mathcal{L}^{-1}\left\{\dfrac{3}{2}\cdot\dfrac{2}{s^2+2^2}\right\}=\dfrac{3}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+2^2}\right\}=\dfrac{3}{2}\sin 2t.$

Example $\mathcal{L}^{-1}\left\{\dfrac{2}{(s+5)^4}\right\}=\mathcal{L}^{-1}\left\{\dfrac{1}{3}\cdot\dfrac{6}{(s+5)^4}\right\}=\dfrac{1}{3}\mathcal{L}^{-1}\left\{\dfrac{3!}{(s+5)^4}\right\}=\dfrac{1}{3}e^{-5t}\mathcal{L}^{-1}\left\{\dfrac{3!}{s^4}\right\}=\dfrac{1}{3}e^{-5t}t^3.$

H. W.

$$\mathcal{L}^{-1}\left\{\frac{V}{R}\left(\frac{1}{s}\right)-\frac{V}{R}\left(\frac{1}{\frac{R}{L}+s}\right)\right\}$$, Where V, R, L are constants

H.W. Find the inverse transform of each of the following.

(a) $F(s)=\dfrac{6s-5}{s^2+7}$

(b) $F(s)=\dfrac{1-3s}{s^2+8s+21}$

(c) $G(s)=\dfrac{3s-2}{2s^2-6s-2}$

(d) $H(s)=\dfrac{s+7}{s^2-3s-10}$

So, let's remind you how to get the correct partial fraction decomposition. The first step is to factor the denominator as much as possible. Then for each term in the denominator we will use the following table to get a term or terms for our partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Example Find the inverse transform of each of the following.

(a) $G(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

(b) $F(s)=\dfrac{2-5s}{(s-6)(s^2+11)}$

(c) $G(s)=\dfrac{25}{s^3(s^2+4s+5)}$ **(H. W.)**

Solution

(a) $G(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

Here's the partial fraction decomposition for this part.

$$G(s)=\frac{A}{s+3}+\frac{B}{s-4}+\frac{C}{5s-1}$$

$$86s-78=A(s-4)(5s-1)+B(s+3)(5s-1)+C(s+3)(s-4)$$

As with the last example, we can easily get the constants by correctly picking values of s.

$$s=-3 \qquad -336=A(-7)(-16) \qquad \Rightarrow \qquad A=-3$$

$$s=\frac{1}{5} \qquad -\frac{304}{5}=C\left(\frac{16}{5}\right)\left(-\frac{19}{5}\right) \qquad \Rightarrow \qquad C=5$$

$$s=4 \qquad 266=B(7)(19) \qquad \Rightarrow \qquad B=2$$

So, the partial fraction decomposition for this transform is,

$$G(s) = -\frac{3}{s+3} + \frac{2}{s-4} + \frac{5}{5s-1}$$

Now, in order to actually take the inverse transform we will need to factor a 5 out of the denominator of the last term. The corrected transform as well as its inverse transform is.

$$G(s) = -\frac{3}{s+3} + \frac{2}{s-4} + \frac{1}{s-\frac{1}{5}}$$

$$g(t) = -3\mathbf{e}^{-3t} + 2\mathbf{e}^{4t} + \mathbf{e}^{\frac{t}{5}}$$

(b) $F(s) = \dfrac{2-5s}{(s-6)(s^2+11)}$

So, for the first time we've got a quadratic in the denominator. Here's the decomposition for this part.

$$F(s) = \frac{A}{s-6} + \frac{Bs+C}{s^2+11}$$

Setting numerators equal gives,

$$2-5s = A(s^2+11) + (Bs+C)(s-6)$$

$$2-5s = A(s^2+11) + (Bs+C)(s-6)$$

$$= S^2 A + 11A + BS^2 - 6BS + CS - 6C$$

$$= (A+B)s^2 + (-6B+C)s + 11A - 6C$$

So, setting coefficients equal gives the following system of equations that can be solved.

$$\left.\begin{aligned} s^2 &: \quad A+B=0 \\ s^1 &: \ -6B+C=-5 \\ s^0 &: \ 11A-6C=2 \end{aligned}\right\} \quad \Rightarrow \quad A=-\frac{28}{47},\quad B=\frac{28}{47},\quad C=-\frac{67}{47}$$

$$F(s) = \frac{1}{47}\left(-\frac{28}{s-6} + \frac{28s-67}{s^2+11}\right)$$

$$= \frac{1}{47}\left(-\frac{28}{s-6} + \frac{28s}{s^2+11} - \frac{67\frac{\sqrt{11}}{\sqrt{11}}}{s^2+11}\right)$$

The inverse transform is then.

$$f(t) = \frac{1}{47}\left(-28\mathbf{e}^{6t} + 28\cos\left(\sqrt{11}t\right) - \frac{67}{\sqrt{11}}\sin\left(\sqrt{11}t\right)\right)$$

Solution Differential equations of initial value problems using Laplace

Structure of solutions:

- Take Laplace transform on both sides. You will get an algebraic equation for Y.
- Solve this equation to get $Y(s)$.
- Take inverse transform to get $y(t) = \mathcal{L}^{-1}\{y\}$.

Example Solve the initial value problem by Laplace transform,

$$y'' - 3y' - 10y = 2, \qquad y(0) = 1, y'(0) = 2.$$

Solution :

Step 1. Take Laplace transform on both sides: Let $\mathcal{L}\{y(t)\} = Y(s)$, and then

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY - 1, \qquad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y - s - 2.$$

Note the initial conditions are the first thing to go in!

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 10\mathcal{L}\{y(t)\} = \mathcal{L}\{2\}, \qquad \Rightarrow \qquad s^2Y - s - 2 - 3(sY - 1) - 10Y = \frac{2}{s}.$$

Now we get an algebraic equation for $Y(s)$.

Step 2: Solve it for $Y(s)$:

$$(s^2 - 3s - 10)Y(s) = \frac{2}{s} + s + 2 - 3 = \frac{s^2 - s + 2}{s}, \qquad \Rightarrow \qquad Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)}.$$

Step 3: Take inverse Laplace transform to get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. The main technique here is **partial fraction**.

$$Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+2} = \frac{A(s-5)(s+2) + Bs(s+2) + Cs(s-5)}{s(s-5)(s+2)}.$$

Compare the numerators:

$$s^2 - s + 2 = A(s-5)(s+2) + Bs(s+2) + Cs(s-5).$$

The previous equation holds for all values of s.
Set $s = 0$: we get $-10A = 2$, so $A = -\frac{1}{5}$.
Set $s = 5$: we get $35B = 22$, so $B = \frac{22}{35}$.
Set $s = -2$: we get $14C = 8$, so $C = \frac{4}{7}$.
Now, $Y(s)$ is written into sum of terms which we can find the inverse transform:

$$y(t) = A\mathcal{L}^{-1}\{\frac{1}{s}\} + B\mathcal{L}^{-1}\{\frac{1}{s-5}\} + C\mathcal{L}^{-1}\{\frac{1}{s+2}\} = -\frac{1}{5} + \frac{22}{35}e^{5t} + \frac{4}{7}e^{-2t}.$$

Example Solve

$$y'' - 10y' + 9y = 5t, \quad y(0) = -1,\ y'(0) = 2.$$

Applying the Laplace transform to both side, we find

$$(s^2 - 10s + 9)Y + s - 2 - 10 = \frac{5}{s^2} \implies Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}.$$

To find the inverse Laplace transform we will need first simplify the expression for $Y(s)$ using the partial fraction decomposition:

$$\frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}.$$

We find

$$B = \frac{5}{9},\ D = -2,\ C = \frac{31}{81},\ A = \frac{50}{81}.$$

Therefore, using the linearity of the inverse Laplace transform,

$$y(t) = \frac{50}{81} + \frac{5t}{9} + \frac{31}{81}e^{9t} - 2e^t.$$

H. W. : Solve $y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1,\ y'(0) = 0.$

$$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

$$\cos a \sin b = \frac{1}{2}[\sin(a+b) - \sin(a-b)]$$

$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

$$\sin a \sin b = -\frac{1}{2}[\cos(a+b) - \cos(a-b)]$$

$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$	$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$	$\cos 2\theta = 2\cos^2\theta - 1$ $\cos 2\theta = 1 - 2\sin^2\theta$
$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$, $\cos\theta \neq -1$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Probability and Statistics

(Probabilities and Statistics)

Introduction :

Probability and statistics are concerned with events which occur by chance. Examples include occurrence of accidents, errors of measurements, production of defective and non-defective items from a production line, and various games of chance, such as drawing a card from a well-mixed deck, flipping a coin, or throwing a symmetrical six-sided die. In each case we may have some knowledge of the likelihood of various possible results, but we cannot predict with any certainty the outcome of any particular trial. Probability and statistics are used throughout engineering. In electrical engineering, signals and noise are analyzed by means of probability theory.

Probability : The chance that something will happen.

Some Important Terms :

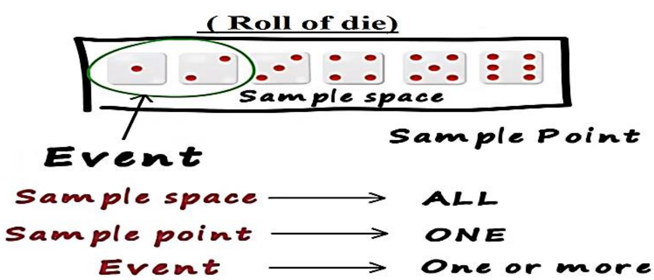
(a) Probability as a specific term is a measure of the likelihood(Probability) that a particular event will occur.

(b) Statistics is a word with a variety of meanings. To the man in the street it most often means simply a collection of numbers, such as the number of people living in a country or city.

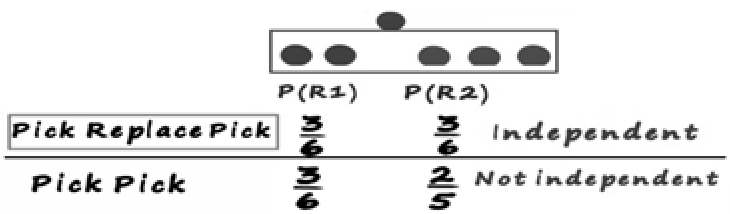
Events and the Sample Space

Data are obtained by observing either uncontrolled events in nature or by observing events in controlled situations. We use the term ***experiment*** to describe either method of data collection.

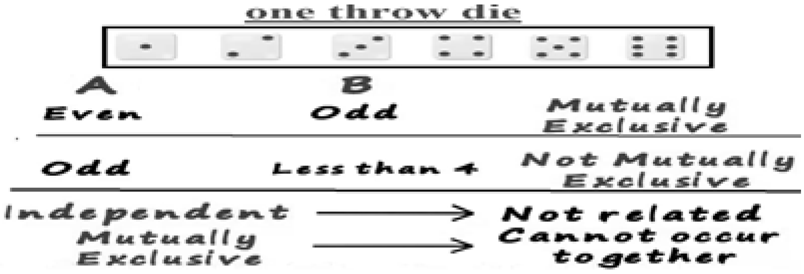
Experiment : is the process by which an observation (or measurement) is obtained.
Outcome : A possible result of one trial of a probability experiment.
Sample Point : is the **one of each** outcome.
Event : is the outcome that is observed on a single repetition of the experiment.
Sample space: is a collection of events. Or, The set of all events .



Dependent events : The outcome of one event is affected by another event.. Or, The probability of occurrence of one event depends on the occurrence of the other.
Independent event : Two or more events whose outcomes do not affect each other.



Mutually Exclusive Events (disjoint): Two outcomes or events are mutually exclusive when they cannot both occur simultaneously .Or , Two events are mutually exclusive if, when one event occurs, the other cannot, and vice versa.



Venn Diagram : A diagram of overlapping circles that shows the relationships among members of different sets.

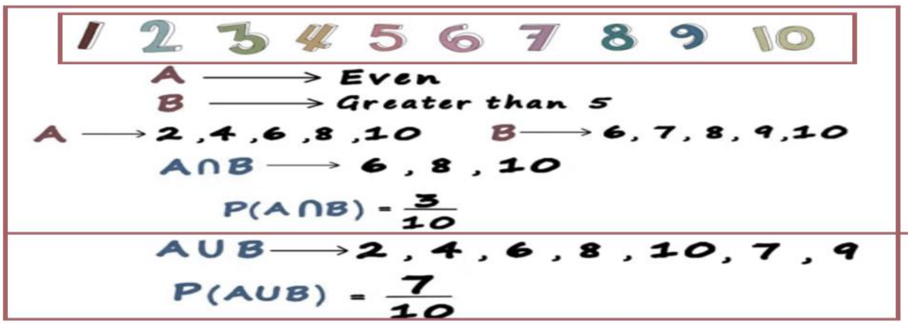
Event Relations and Probability rules :

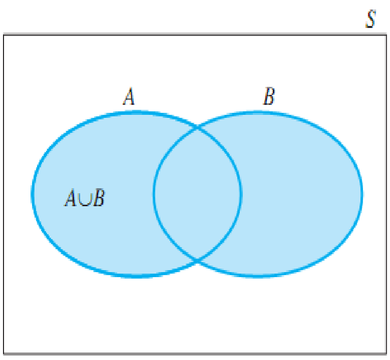
Sometimes the event of interest can be formed as a ***combination*** of several other events. Let A and B be two events defined on the ***sample space S***. Here are three important relationships between events.

Union : The union of events A and B, denoted by " **A ∪ B** ", is the event that either A or B or both occur.

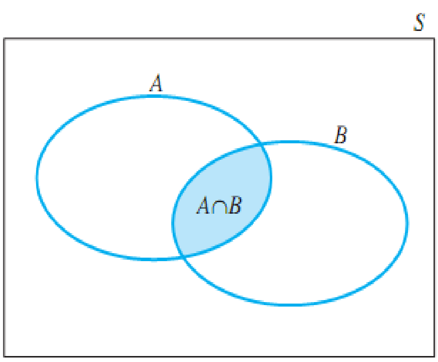
Intersection : The intersection of events A and B, denoted by " **A ∩ B** ", is the event that both A and B occur.

Complement : The complement of an event A, denoted by " $\mathbf{A^c}$ ", is the event that A does not occur.

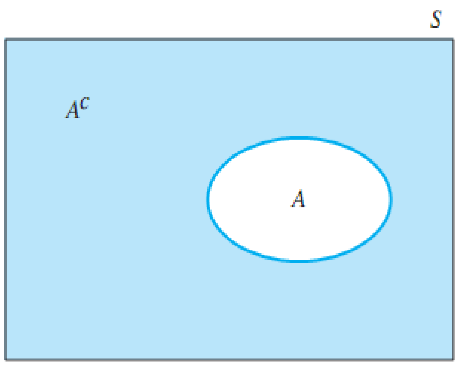




Venn diagram of A ∪ B



Venn diagram A ∩ B



The complement of an event

EXAMPLE

Two fair coins are tossed, and the outcome is recorded. These are the events of interest:

A: Observe at least one head

B: Observe at least one tail

Define the events A, B, $A \cap B$, $A \cup B$, and A^c as collections of simple events, and find their probabilities.

Solution

E_1: HH (head on first coin, head on second)

E_2: HT

E_3: TH

E_4: TT

and that each simple event has probability 1/4.

Event A, at least one head, occurs if E_1, E_2, or E_3 occurs, so that

$$A = \{E_1, E_2, E_3\}$$

$$P(A) = \frac{3}{4}$$

Similarly,

Event B at least one tail, occurs if

$$B = \{E_2, E_3, E_4\}$$

$$P(B) = \frac{3}{4}$$

$$A \cap B = \{E_2, E_3\}$$

$$P(A \cap B) = \frac{1}{2}$$

$$A \cup B = \{E_1, E_2, E_3, E_4\}$$

$$P(A \cup B) = \frac{4}{4} = 1$$

and $A^c = \{E_4\}$ $\quad P(A^c) = \frac{1}{4}$

The concept of unions and intersections can be extended to more than two events. For example, the union of three events A, B, and C, which is written as $A \cup B \cup C$, is the set of simple events that are in A or B or C or in any combination of those events. Similarly, the intersection of three events A, B, and C, which is written as $A \cap B \cap C$, is the collection of simple events that are common to the three events A, B, and C.

Calculating Probabilities for Unions and Complements :

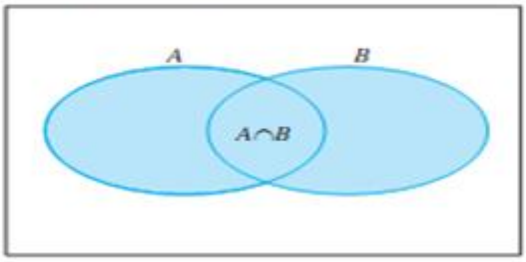
When we can write the event of interest in the form of a union, a complement, or an intersection, there are special probability rules that can simplify our calculations. The first rule deals with ***unions of events***.

The Addition Rule :

General addition rule

Given two events, A and B, the probability of their union, $A \cup B$, is equal to

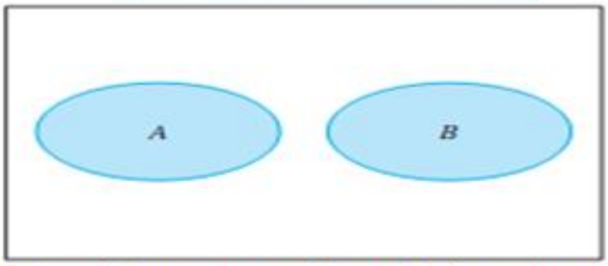
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Special case of additon rule (mutually exclusive)

When two events A and B are **mutually exclusive** or **disjoint,** it means that when A occurs, B cannot, and vice versa. This means that the probability that they both occur, $P(A \cap B)$, must be zero. Figure is a Venn diagram representation of two such events with no simple events in common.

$$P(A \cup B) = P(A) + P(B)$$



When two events A and B are **mutually exclusive,** then $P(A \cap B) = 0$ and the Addition Rule simplifies to

---

Example :

1	2	3	4	5	6	7	8	9	10

A ⟶ Even B ⟶ Greater than 5

P (A ∪ B) = P (A happening) + P (B happening) − P (A & B happening together)

$= \frac{5}{10} + \frac{5}{10} - \frac{3}{10}$

$= 0.7$

A --> 2 , 4 , 6 , 8 , 10

B --> 6 , 7 , 8 , 9 , 10

2 , 4 , 6 , 7 , 8 , 9 , 10

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1	2	3	4	5	6

A ⟶ Odd

B ⟶ Even

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A & B ---> Mutually Exclusive

$P(A \cup B) = P(A) + P(B)$

$= \frac{3}{6} + \frac{3}{6} = 1$

$P(A \cup B) = P(A) + P(B)$ Mutually Exclusive

Rule For Complements :

The second rule deals with *complements* of events. You can see from the Venn diagram in Figure that A and A^c are mutually exclusive and that $A \cup A^c = S$, the entire sample space. It follows that

$$P(A) + P(A^c) = 1 \text{ and } P(A^c) = 1 - P(A)$$

$$\boxed{P(A^c) = 1 - P(A)}$$

EXAMPLE An oil-prospecting firm plans to drill two exploratory wells. Past evidence is used to assess the possible outcomes listed in Table

TABLE **Outcomes for Oil-Drilling Experiment**

Event	Description	Probability
A	Neither well produces oil or gas	.80
B	Exactly one well produces oil or gas	.18
C	Both wells produce oil or gas	.02

Find $P(A \cup B)$ and $P(B \cup C)$.

Solution By their definition, events A, B, and C are jointly mutually exclusive because the occurrence of one event precludes the occurrence of either of the other two. Therefore,

$$P(A \cup B) = P(A) + P(B) = .80 + .18 = .98$$

and

$$P(B \cup C) = P(B) + P(C) = .18 + .02 = .20$$

A little note about a deck of cards

A deck of cards = 52 cards

Each deck has four parts (suits) with 13 cards in them.

Each suit has 3 face cards.

52 cards = 1 deck

13 spades	13 hearts	13 clubs	13 diamonds
♠	♥	♣	♦
Black	**Red**	**Black**	**Red**

Independence, Conditional Probability, And The Multiplication Rule :

Independent : Two events, A and B, are said to be independent if and only if the probability of event B is not influenced or changed by the occurrence of event A, or vice versa

Example : consider tossing a single die two times, and define two events:

A: Observe a 2 on the first toss

B: Observe a 2 on the second toss

If the die is fair, the probability of event *A* is $P(A) = 1/6$. Consider the probability of event *B*. Regardless of whether event *A* has or has not occurred, the probability of observing a 2 on the second toss is still 1/6. We could write:

$P(B \text{ given that } A \text{ occurred}) = 1/6$

$P(B \text{ given that } A \text{ did not occur}) = 1/6$

Since the probability of event *B* is not changed by the occurrence of event *A*, we say that *A* and *B* are *independent events*.

CONDITIONAL PROBABILITY.

The probability of an event *A*, given that the event *B* has occurred, is called the **conditional probability of *A*, given that *B* has occurred,** denoted by $P(A|B)$. The vertical bar is read "given" and the events appearing to the right of the bar are those that you know have occurred. We will use these probabilities to calculate the probability that *both A and B* occur when the experiment is performed.

Conditional Probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A) \times P(B|A)$$

Another way to look at the conditional probability formula is:

$$P(second|first) = \frac{P(first\ choice\ and\ second\ choice)}{P(first\ choice)}$$

CHECKING FOR INDEPENDENCE

Two events A and B are said to be **independent** if and only if either

$$P(A \cap B) = P(A)P(B)$$

or

$$P(B|A) = P(B) \text{ or equivalently, } P(A|B) = P(A)$$

Otherwise, the events are said to be **dependent.**

EXAMPLE

Toss two coins and observe the outcome. Define these events:

A: Head on the first coin

B: Tail on the second coin

Are events A and B independent?

Solution From previous examples, you know that $S = \{HH, HT, TH, TT\}$. Use these four simple events to find

Remember, independence ⇔ $P(A \cap B) = P(A)P(B)$.

$$P(A) = \frac{1}{2}, P(B) = \frac{1}{2}, \text{ and } P(A \cap B) = \frac{1}{4}.$$

Since $P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$ and $P(A \cap B) = \frac{1}{4}$, we have $P(A)P(B) = P(A \cap B)$

and the two events must be independent.

The Difference between Mutually Exclusive and Independent Events

Many students find it hard to tell the difference between *mutually exclusive* and *independent* events.

- When two events are *mutually exclusive* or *disjoint*, they cannot both happen together when the experiment is performed. Once the event B has occurred, event A cannot occur, so that $P(A|B) = 0$, or vice versa. The occurrence of event B certainly affects the probability that event A can occur.
- Therefore, mutually exclusive events must be *dependent*.
- When two events are *mutually exclusive* or *disjoint*, $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$.
- When two events are *independent*, $P(A \cap B) = P(A)P(B)$, and $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.

EXAMPLE

Two cards are drawn from a deck of 52 cards. Calculate the probability that the draw includes an ace and a ten.

Solution Consider the event of interest:

A: Draw an ace and a ten

Then $A = B \cup C$, where

B: Draw the ace on the first draw and the ten on the second

C: Draw the ten on the first draw and the ace on the second

Events B and C were chosen to be mutually exclusive and also to be intersections of events with known probabilities; that is,

$$B = B_1 \cap B_2 \text{ and } C = C_1 \cap C_2$$

where

B_1: Draw an ace on the first draw

B_2: Draw a ten on the second draw

C_1: Draw a ten on the first draw

C_2: Draw an ace on the second draw

Applying the Multiplication Rule, you get

$$P(B_1 \cap B_2) = P(B_1)P(B_2|B_1) = \left(\frac{4}{52}\right)\left(\frac{4}{51}\right)$$

and

$$P(C_1 \cap C_2) = \left(\frac{4}{52}\right)\left(\frac{4}{51}\right)$$

Then, applying the Addition Rule,

$$P(A) = P(B) + P(C) = \left(\frac{4}{52}\right)\left(\frac{4}{51}\right) + \left(\frac{4}{52}\right)\left(\frac{4}{51}\right) = \frac{8}{663}$$

Check each composition carefully to be certain that it is actually equal to the event of interest.

Example ': A bag contains green balls and yellow balls. You are going to choose two balls without replacement. If the probability of selecting a green ball and a yellow ball is $\frac{14}{39}$, what is the probability of selecting a yellow ball on the second draw, if you know that the probability of selecting a green ball on the first draw is $\frac{4}{9}$.

Solution:

Step 1: List what you know

$$P(Green) = \frac{4}{9}$$

$$P(Green\ AND\ Yellow) = \frac{14}{39}$$

Step 2: Calculate the probability of selecting a yellow ball on the second draw with a green ball on the first draw

$$P(Y|G) = \frac{P(Green\ AND\ Yellow)}{P(Green)}$$

$$P(Y|G) = \frac{14/39}{4/9}$$

$$P(Y|G) = \frac{14}{39} \times \frac{9}{4}$$

$$P(Y|G) = \frac{126}{156}$$

$$P(Y|G) = \frac{21}{26}$$

Step 3: Write your conclusion: Therefore the probability of selecting a yellow ball on the second draw after drawing a green ball on the first draw is $\frac{21}{26}$.

H.W : Refer to the probability table in Example , which is reproduced below.

	Too High (A)	Right Amount (B)	Too Little (C)
Child in College (D)	.35	.08	.01
No Child in College (E)	.25	.20	.11

Are events D and A independent? Explain.

Example .!: Two cards are chosen from a deck of cards. What is the probability that they both will be face cards?

Solution

Let A = 1^{st} Face card chosen

Let B = 2^{nd} Face card chosen

4 suits → 3 face cards per suit

Therefore, the total number of face cards in the deck = $4 \times 3 = 12$

$$P(A) = \frac{12}{52}$$

$$P(B) = \frac{11}{51}$$

$$P(A\ AND\ B) = \frac{12}{52} \times \frac{11}{51} \quad \text{or} \quad P(A \cap B) = \frac{12}{52} \times \frac{11}{51} = \frac{33}{663}$$

$$P(A \cap B) = \frac{11}{221}$$

H.W : : You have different pairs of gloves of the following colors: blue, brown, red, white and black. Each pair is folded together in matching pairs and put away in your closet. You reach into the closet and choose a pair of gloves. The first pair you pull out is blue. You replace this pair and choose another pair. What is the probability that you will choose the blue pair of gloves twice?

H.W : : Two cards are drawn from a deck of cards.

A: 1^{st} card is a club

B: 1^{st} card is a 7

C: 2^{nd} card is a heart

Describing a set of Data with numerical measures :

Graphs can help you describe the basic shape of a data distribution; "a picture is worth a thousand words." But there are limitations of using graph. Therefore, we need to find another way to convey a mental picture of the data.

One way to overcome these problems is to use ***numerical measures***, which can be calculated for either a **sample** or a **population** of measurements. You can use the data to calculate a set of numbers that will convey a good mental picture of the frequency distribution. These measures are called **parameters** when associated with the **population**, and they are called ***statistics*** when calculated from ***sample measurements***.

Definition :

Numerical descriptive measures associated with a **population** of measurements are called **parameters**; those computed from **sample measurements** are called **statistics**.

Measures Of Center :

Let's consider some rules for locating the ***center of a distribution of measurements***.

The arithmetic mean or average :

The arithmetic average of a set of measurements is a very common and useful measure of ***center***. The **definition of the arithmetic mean or average** of a set of ***n*** measurements is equal to the sum of the measurements divided by ***n***.

$$\text{Sample mean: } \quad \bar{x} = \frac{\Sigma x_i}{n}$$

Where :

Population mean: μ

$\bar{x}$ (x-bar) is a sample mean.

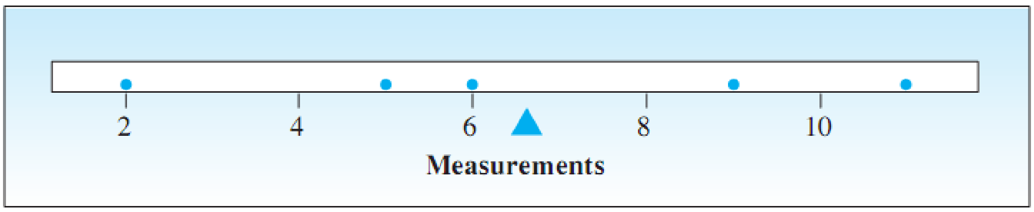
μ symbol (Greek lowercase mu) for the mean of a population.

EXAMPLE raw a dotplot for the $n = 5$ measurements 2, 9, 11, 5, 6. Find the sample mean and ımpare its value with what you might consider the "center" of these observations on e dotplot.

olution The dotplot in Figure seems to be centered between 6 and 8. To find e sample mean, calculate

$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{2 + 9 + 11 + 5 + 6}{5} = 6.6$$

FIGURE
Dotplot for Example



The statistic $\bar{x} = 6.6$ is the balancing point or fulcrum shown on the dotplot. It does seem to mark the center of the data.

Median:

A second measure of ***central tendency*** is the ***median***, which is the value in the ***middle position*** in the set of measurements ***ordered from smallest to largest***.

Definition : The median ***m*** of a set of n measurements is the value of ***x*** that falls in the ***middle position*** when the measurements are ordered ***from smallest to largest***.

We can know the order and the value of median by using the following :

The value " **.5(n + 1)** " indicates the **position** of the median in the ordered data set. If the position of the median is a number that ends in the value .5, you need to average the two adjacent values.

EXAMPLE Find the median for the set of measurements 2, 9, 11, 5, 6.

Solution Rank the $n = 5$ measurements from smallest to largest:

2 5 6 9 11
↑

The middle observation, marked with an arrow, is in the center of the set, or $m = 6$.

EXAMPLE Find the median for the set of measurements 2, 9, 11, 5, 6, 27.

Solution Rank the measurements from smallest to largest:

2 5 [6 9] 11 27
↑

Now there are two "middle" observations, shown in the box. To find the median, choose a value halfway between the two middle observations:

$$m = \frac{6 + 9}{2} = 7.5$$

Now if we use the value " **.5(n + 1)** :

For the $n = 5$ ordered measurements from Example , the position of the median is $.5(n + 1) = .5(6) = 3$, and the median is the *3rd ordered observation*, or $m = 6$. For the $n = 6$ ordered measurements from Example , the position of the median is $.5(n + 1) = .5(7) = 3.5$, and the median is the *average of the 3rd and 4th ordered observations*, or $m = (6 + 9)/2 = 7.5$.

If a distribution is ***tilt*** to the right, the ***mean*** shifts to the right; if a distribution is skewed to the left, the ***mean*** shifts to the left. The median is not affected by these extreme values because the numerical values of the measurements are not used in its calculation. **When a distribution is symmetric**, the mean and the median are ***equal***. If a distribution is strongly skewed by one or more extreme values, you should use the median rather than the mean as a measure of center.

The Mode :

Another way to ***locate the center of a distribution*** is to look for the value of ***x*** that occurs with the ***highest frequency***. This measure of the center is called the **mode**.

Definition : The mode is the category that occurs ***most frequently***, or the ***most frequently occurring*** value of ***x***. ***When measurements on a continuous variable*** have been ***grouped as a frequency*** or relative frequency ***histogram***, the class with the highest peak or frequency is called the ***modal class***, and the midpoint of that class is taken to be the mode.

Note : The mode is generally used to describe ***large data sets***, whereas the mean and median are used for both ***large*** and ***small*** data sets.

EXAMPLE

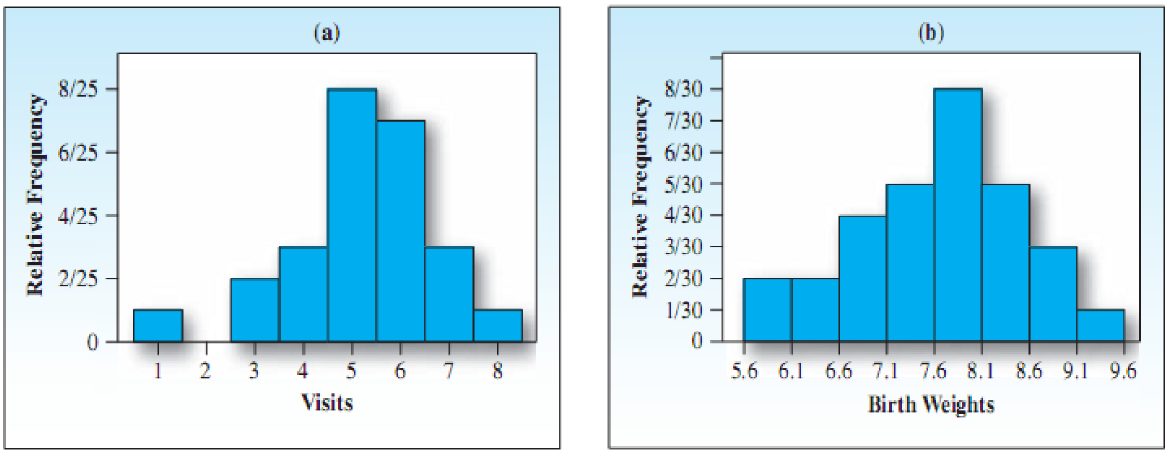
Starbucks and birth weight data

(a) Starbucks data

6	7	1	5	6
4	6	4	6	8
6	5	6	3	4
5	5	5	7	6
3	5	7	5	5

(b) Birth weight data

7.2	7.8	6.8	6.2	8.2
8.0	8.2	5.6	8.6	7.1
8.2	7.7	7.5	7.2	7.7
5.8	6.8	6.8	8.5	7.5
6.1	7.9	9.4	9.0	7.8
8.5	9.0	7.7	6.7	7.7



Solution :

For The visits :

Table: From the data in Example reproduced in Table (a), the **mode** of the distribution of the number of reported weekly visits to Starbucks for 30 Starbucks customers is 5.

Using the histogram: The ***modal class*** and the value of x occurring with the highest frequency are the same, as shown in Figure (a).

For the birth weight :

Table: For the birth weight data in Table (b), a birth weight of 7.7 occurs four times, and therefore the **mode** for the distribution of birth weights is 7.7

Using the histogram : Using the histogram to find the **modal class**, you find that the class with the highest peak is the fifth class, from 7.6 to 8.1. Our choice for the mode would be the midpoint of this class, or (**7.6 + 8.1**) 7.85. See Figure (b).

Measures Of Variability :

Data sets may have the same center but look different because of the way the numbers spread out from the center.

Measures of variability can help you create a mental picture of the spread of the data. We will present some of the more important ones. The simplest measure of variation is the ***range***.

Definition : The range, ***R*** , of a set of ***n*** measurements is defined as the difference between the **largest** and **smallest** measurements.

For example, the measurements " 5, 7, 1, 2, 4 " vary from 1 to 7. Hence, the range is (7 - 1 = 6) . The range is easy to calculate, easy to interpret, and is an adequate measure of variation for ***small sets of data***. for large data sets, the range is not an adequate measure of variability.

Definition : The ***variance of a population*** of ***N*** measurements is the average of the squares of the deviations of the measurements about their mean m. The population variance is denoted by " δ^2 " and is given by the formula.

$$\sigma^2 = \frac{\Sigma(x_i - \mu)^2}{N}$$

Most often, you will not have all the population measurements available but will need to calculate the *variance of a sample* of *n* measurements.

Definition: The ***variance of a sample*** of **n** measurements is the sum of the squared deviations of the measurements about their mean " $\bar{x}$ " divided by (n - 1). The sample variance is denoted by s^2 and is given by the formula.

$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n - 1}$$

For the set of $n = 5$ sample measurements presented in Table , the square of the deviation of each measurement is recorded in the third column. Adding, we obtain

$$\Sigma(x_i - \bar{x})^2 = 22.80$$

and the sample variance is

$$s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n - 1} = \frac{22.80}{4} = 5.70$$

TABLE Computation of $\Sigma(x_i - \bar{x})^2$

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
5	1.2	1.44
7	3.2	10.24
1	−2.8	7.84
2	−1.8	3.24
4	.2	.04
19	0.0	22.80

The **variance** (s^2) is measured in terms of the square of the original units of measurement. If the original measurements are in inches, the variance is expressed in square inches. Taking the square root of the variance, we obtain the ***standard deviation***, which returns the measure of variability to the original units of measurement.

Definition : The ***standard deviation*** of a set of measurements is equal to the positive square root of the variance.

NOTATION

n: number of measurements in the sample

s^2: sample variance

$s = \sqrt{s^2}$: sample standard deviation

N: number of measurements in the population

σ^2: population variance

$\sigma = \sqrt{\sigma^2}$: population standard deviation

For the set of $n = 5$ sample measurements in Table , the sample variance is $s^2 = 5.70$, so the sample standard deviation is $s = \sqrt{s^2} = \sqrt{5.70} = 2.39$. The more variable the data set is, the larger the value of s.

Shortcut method for calculating s^2.

THE COMPUTING FORMULA FOR CALCULATING s^2

$$s^2 = \frac{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}}{n - 1}$$

Σx_i^2 = Sum of the squares of the individual measurements

$(\Sigma x_i)^2$ = Square of the sum of the individual measurements

EXAMPLE Calculate the variance and standard deviation for the five measurements in Table , which are 5, 7, 1, 2, 4. Use the computing formula for s^2 and compare your results with those obtained using the original definition of s^2.

Table for Simplified Calculation of s^2 and s

x_i	x_i^2
5	25
7	49
1	1
2	4
4	16
19	95

Solution The entries in Table are the individual measurements, x_i, and their squares, x_i^2, together with their sums. Using the computing formula for s^2, you have

$$s^2 = \frac{\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}}{n-1} = \frac{95 - \frac{(19)^2}{5}}{4} = \frac{22.80}{4} = 5.70$$

and $s = \sqrt{s^2} = \sqrt{5.70} = 2.39$, as before.

Now that you have learned how to compute the variance and standard deviation, remember these points:

- The value of s is always greater than or equal to zero.
- The larger the value of s^2 or s, the greater the variability of the data set.
- If s^2 or s is equal to zero, all the measurements must have the same value.
- In order to measure the variability in the same units as the original observations, we compute the standard deviation $s = \sqrt{s^2}$.

Random Variables and their distributions

Definition and notation

Recall:

Dataset 1: number of errors X, $S = \{0, 1, 2, ...\}$.

Dataset 2: time Y, $S = \{x : x \geq 0\}$.

Other example: Toss of a coin, outcome Z: $S = \{H, T\}$.

In the above X, Y and Z are examples of *random variables*.

Important note: capital letters will denote random variables, lower case letters will denote particular values (realizations).

When the outcomes can be listed we have a *discrete random variable*, otherwise we have a *continuous random variable*.

Let $p_i = \mathrm{P}(X = x_i)$, $i = 1, 2, 3, ...$ Then any set of p_i's such that

i) $p_i \geq 0$, and

ii) $\sum_{i=1}^{\infty} p_i = \mathrm{P}(X \in S) = 1$

forms a *probability distribution* over $x_1, x_2, x_3, ...$

The *distribution function* $F(x)$ of a *discrete random variable* is given by

$$F(x_j) = \mathrm{P}(X \leq x_j) = \sum_{i=1}^{j} p_i = p_1 + p_2 + ... + p_j.$$

We now give some examples of distributions which can be used as models for discrete random variables.

Random variable

Random variable: a numerical characteristic that takes on different values due to chance.

Example :

Coin Flips

The number of heads in four flips of a coin (a numerical property of each different sequence of flips) is a **random variable** because the results will vary between trials.

Discrete and Continuous Random variables :

Random variables are classified into two broad types: discrete and continuous.

A **discrete random variable** has a countable set of distinct possible values.

A **continuous random variable** is such that any value (to any number of decimal places) within some interval is a possible value.

Examples for Discrete and Continuous Random variables

Discrete Random Variables:

- Number of heads in 4 flips of a coin (possible outcomes are 0, 1, 2, 3, 4)
- Number of classes missed last week (possible outcomes are 0, 1, 2, 3, ..., up to the maximum number of classes)
- Amount won or lost when betting $1 on the Pennsylvania Daily number lottery

Examples for Discrete and Continuous Random variables

Continuous Random Variables:

- Heights of individuals
- Time to finish a test
- Hours spent exercising last week

Discrete Random Variables

Probability distribution: A table, graph, or formula that gives the probability of a given outcome's occurrence

Probability Distribution for a discrete random variable :

For a discrete random variable, its **probability distribution** (also called the **probability distribution function**) is any table, graph, or formula that gives each ***possible value*** and the ***probability*** of that value.

Let $p_i = P(X = x_i)$, $i = 1, 2, 3, \ldots$. Then any set of p_i's such that

i) $p_i \geq 0$, and

ii) $\sum_{i=1}^{\infty} p_i = P(X \in S) = 1$

Note: The total of all probabilities across the distribution must be 1, and each individual probability must be between 0 and 1, inclusive.

Example

What if we flipped a fair coin four times? What are the possible outcomes and what is the probability of each?

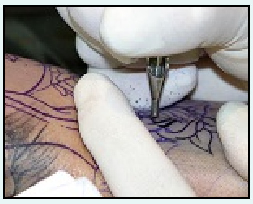
Figure 1 below is a probability distribution for the number of heads in 4 flips of a coin. Given that P(Heads)=.50, the probability of not flipping heads at all is 1/16, or .0625. In 6.25% of all trials, we can expect that there will be no heads. This may be written as P(X=0)=.0625. Similarly, the probability of flipping heads once in four trials is 4/16, or .25. In 25% of all trials, we can expect that heads will be flipped exactly once. This may be written as P(X=1)=.25.

This probability distribution could be constructed by listing all 16 possible sequences of heads and tails for four flips (i.e., HHHH, HTHH, HTTH, HTTT, etc.), and then counting how many sequences there are for each possible number of heads. Or, in section 5.4 you will see how these could be computed using binomial random variable techniques.

Figure 1. Probability Distribution for Number of Heads in 4 Flips of a Coin

Heads	0	1	2	3	4
Probability	1/16	4/16	6/16	4/16	1/16

Example



A census was conducted at a university. All students were asked how many tattoos they had.

Figure 2 presents a probability distribution for the discrete variable of number of tattoos for each student. From this table we can find that 85% of students in the population do not have a tattoo, 12% of students in the population have one tattoo, 1.5% of students in the population have two tattoos, and so on. This could be written as P(X=0)=.85, P(X=1)=.12, P(X=2)=.015, etc.

Figure 2. Probabilty Distribution for Number of Tattoos Each Student Has in a Population of Students

Tattoos	0	1	2	3	4
Probability	.850	.120	.015	.010	.005

Cumulative Probabilities

Cumulative probability: Likelihood (probability) of an outcome less than or equal to a given value occurring.

To find a **cumulative probability** we add the probabilities for all values qualifying as "less than or equal" to the specified value.

> Example
>
> Suppose we want to know the probability that the number of heads in four flips is less than two. If we let X represent number of heads we get on four flips of a coin, then:
>
> Because this is a discrete distribution, the probability of flipping less than two heads is equal to flipping one or zero heads:
>
> $$P(X < 2) = P(X = 0 \cup 1)$$
>
> The probability of flipping 1 head and the probability of flipping 0 heads are mutually exclusive events. Thus,
> $P(0 \cup 1) = P(X = 0) + P(X = 1)$
>
> We can use the values from Figure 1 above to solve this equation.
>
> $$P(X = 0) + P(X = 1) = (1/16) + (4/16) = 5/16$$

Cumulative distribution:

Cumulative distribution: A listing of all possible values along with the probability of that value and all lower values occurring (i.e., the **cumulative probability**).

> Example
>
> Cumulative probabilities are found by adding the probability up to each column of the table. In Figure 3 we find the cumulative probability for one head by adding the probabilities for zero and one. The cumulative probability for two heads is found by adding the probabilities for zero, one, and two. We continue with this procedure until we reach the maximum number of heads, in this case four, which should have a cumulative probability of 1.00 because 100% of trials must have four or fewer heads.

Figure 3. Probability Distribution and Cumulative Distribution for Number of Heads in 4 Flips.

Heads	0	1	2	3	4
Probability	1/16	4/16	6/16	4/16	1/16
Cumulative Probability	1/16	5/16	11/16	15/16	1

> Example
>
> Let's construct a cumulative distribution for the data concerning number of tattoos.

Figure 4. Probability Distribution and Cumulative Distribution for Number of Tattoos Each Student Has in a Population of Students.

Tattoos	0	1	2	3	4
Probability	.850	.120	.015	.010	.005
Cumulative Probability	.850	.970	.985	.995	1

Note that the cumulative probability for the last column is always 1. That is, 100% of trials will be less than or equal to the maximum value.

Expected Value of a Discrete Random Variable :

Law of Large Numbers: Given a large number of repeated trials, the average of the results will be approximately equal to the expected value.

Expected value: The mean value in the long run for many repeated samples, symbolized as ***E(X).***

Expected Value for a Discrete Random Variable

$$E(X) = \sum x_i p_i$$

x_i = value of the i[th] outcome

p_i = probability of the i[th] outcome

According to this formula, we take each observed X value and multiply it by its respective probability. We then add these products to reach our expected value.

Example

A fair six-sided die is tossed. You win \$2 if the result is a "1," you win \$1 if the result is a "6," but otherwise you lose \$1.

The Probability Distribution for X = Amount Won or Lost

X	+\$2	+\$1	-\$1
Probability	1/6	1/6	4/6

$$E(X) = \$2(\tfrac{1}{6}) + \$1(\tfrac{1}{6}) + (-\$1)(\tfrac{4}{6}) = \$\tfrac{-1}{6} = -\$0.17$$

The interpretation is that if you play many times, the average outcome is losing 17 cents per play. Thus, over time you should expect to lose money.

Example

Using the probability distribution for number of tattoos, let's find the mean number of tattoos per student.

Probabilty Distribution for Number of Tattoos Each Student Has in a Population of Students

Tattoos	0	1	2	3	4
Probability	.850	.120	.015	.010	.005

$$E(X) = 0(.85) + 1(.12) + 2(.015) + 3(.010) + 4(.005) = .20$$

The mean number of tattoos per student is .20.

Symbols for Population Parameters

	Sample Statistic	**Population Parameter**
Mean	$\bar{x}$	μ
Variance	s^2	σ^2
Standard Deviation	s	σ

Also recall that the standard deviation is equal to the square root of the variance. Thus, $\sigma = \sqrt{(\sigma^2)}$

Standard Deviation of a Discrete Random Variable :

To calculate the standard deviation we first must calculate the variance. From the variance, we take the square root and this provides us the standard deviation. Conceptually, the variance of a discrete random variable is the sum of the difference between each value and the mean times the probability of obtaining that value, as seen in the conceptual formulas below:

Conceptual Formulas

Variance for a Discrete Random Variable

$$\sigma^2 = \sum[(x_i - \mu)^2 p_i]$$

Standard Deviation for a Discrete Random Variable

$$\sigma = \sqrt{\sum[(x_i - \mu)^2 p_i]}$$

x_i = value of the i^{th} outcome
$\mu = E(X) = \sum x_i p_i$
p_i = probability of the i^{th} outcome

In these expressions we substitute our result for E(X) into **μ** because **μ** is the symbol used to represent the mean of a population .

However, there is an **easier** computational formula. The computational formula will give you the same result as the conceptual formula above, but the calculations are simpler.

Computational Formulas

Variance for a Discrete Random Variable

$$\sigma^2 = [\sum(x_i^2 p_i)] - \mu^2$$

Standard Deviation for a Discrete Random Variable

$$\sigma = \sqrt{[\sum(x_i^2 p_i)] - \mu^2}$$

x_i = value of the i^{th} outcome
$\mu = E(X) = \sum x_i p_i$
p_i = probability of the i^{th} outcome

Notice in the summation part of this equation that we only square each observed X value and not the respective probability. Also note that the **μ** is outside of the summation.

Example

Going back to the first example used above for expectation involving the dice game, we would calculate the standard deviation for this discrete distribution by first calculating the variance:

The Probability Distribution for X = Amount Won or Lost

X	+$2	+$1	-$1
Probability	1/6	1/6	4/6

$\sigma^2 = [\sum x_i^2 p_i] - \mu^2 = [2^2(\frac{1}{6}) + 1^2(\frac{1}{6}) + (-1)^2(\frac{4}{6})] - (-\frac{1}{6})^2$

$= [\frac{4}{6} + \frac{1}{6} + \frac{4}{6}] - \frac{1}{36} = \frac{53}{36} = 1.472$

The variance of this discrete random variable is 1.472.

$\sigma = \sqrt{(\sigma^2)}$

$\sigma = \sqrt{1.472} = 1.213$

The standard deviation of this discrete random vairable is 1.213.

Binomial Random Variable :

A specific type of discrete random variable that counts how often a particular event occurs in a fixed number of tries or trials.

For a variable to be a **binomial random variable**, **ALL** of the following conditions must be met:

1. There are a fixed number of trials (a fixed sample size)
2. On each trial, the event of interest either occurs or does not
3. The probability of occurrence (or not) is the same on each trial
4. Trials are independent of one another.

Examples of Binomial Random Variables

- Number of correct guesses at 30 true-false questions when you randomly guess all answers
- Number of winning lottery tickets when you buy 10 tickets of the same kind
- Number of tails when flipping a coin 10 times

Notation

n = number of trials

p = probability event of interest occurs on any one trial

Example

Number of correct guesses at 30 true-false questions when you randomly guess all answers

There are 30 trials, therefore n = 30

There are two possible outcomes (true and false) that are equally probable, therefore p = 1/2 = .5

Probabilities for Binomial Random Variables :

The conditions for being a binomial variable lead to a somewhat complicated formula for finding the probability any specific value occurs (such as the probability you get 20 right when you guess as 30 True-False questions.)

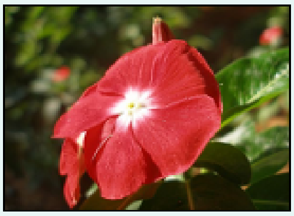
Binomial Random Variable Probability

$$P(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

n = number of trials
x = number of successes
p = probability event of interest occurs on any one trial

Example

Red Flowers



Cross-fertilizing a red and a white flower produces red flowers 25% of the time. Now we cross-fertilize five pairs of red and white flowers and produce five offspring. Find the probability that there will be no red flowered plants in the five offspring.

X = # of red flowered plants in the five offspring.

The number of red flowered plants has a binomial distribution with n = 5, p = .25

$$P(X=0) = \frac{5!}{0!(5-0)!}.25^0(1-.25)^5 = 1 \times .25^0 \times .75^5 = .237$$

There is a 23.7% chance that none of the five plants will be red flowered.

Cumulative probability: Likelihood that a certain number of successes or fewer will occur.

Binomial random variable probabilities are ***mutually exclusive***, therefore we can use the addition rule that we learned before.

Example

Red Flowers, cont.

Continuing with the red flowers example, what if we wanted to know the probability that there would be one or fewer red flowered plants?

$$P(X \text{ is } 1 \text{ or less}) = P(X=0) + P(X=1)$$
$$= \frac{5!}{0!(5-0)!}.25^0(1-.25)^5 + \frac{5!}{1!(5-1)!}.25^1(1-.25)^4$$
$$= .237 + .395 = .632$$

There is a 63.2% chance that one or fewer of the five plants will be red flowered.

In the red flowers example, we first computed P(X = x) and then P(X ≤ x). This latter expression is called finding a **cumulative probability** (***CDF***)because you are finding the probability that has accumulated from the minimum to some point, i.e. from 0 to 1 in this example.

Expected Value and Standard Deviation for Binomial Random Variable

The formula given earlier for discrete random variables could be used, but the good news is that for binomial random variables a shortcut formula for expected value (the mean) and standard deviation can also be used.

Binomial Random Variable Formulas

$$\mu = np$$

$$\sigma = \sqrt{np(1-p)}$$

n = number of trials
p = probability event of interest occurs on any one trial

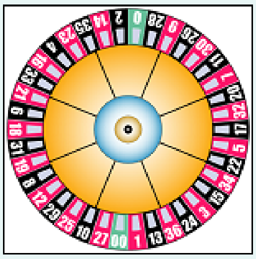
After you use this formula a couple of times, you'll realize this formula matches your intuition. For instance, the "expected" number of correct (random) guesses at 30 True-False questions is $np = (30)(.5) = 15$ (half of the questions). For a fair six-sided die rolled 60 times, the expected value of the number of times a "1" is tossed is $np = (60)(1/6) = 10$.

The standard deviations for these would be, for the True-False test, $\sigma = \sqrt{30(0.5)(1-0.5)} = \sqrt{7.5} = 2.74$, and for the die, $\sigma = \sqrt{60\left(\frac{1}{6}\right)\left(1-\frac{1}{6}\right)} = \sqrt{\frac{50}{6}} = 2.89$.

Example

Roulette

A roulette wheel has 38 slots, 18 are red, 18 are black, and 2 are green. You play five games and always bet on red.



How many games can you expect to win?

Recall, you play five games and always bet on red. $n = 5$ and $p = \frac{red\ slots}{total\ slots} = \frac{18}{38}$

$\mu = np = 5\left(\frac{18}{38}\right) = 2.3684$

$\sigma = \sqrt{np(1-p)} = \sqrt{5\left(\frac{18}{38}\right)\left(1-\frac{18}{38}\right)} = 1.1165$

Out of 5 games, you can expect to win 2.3684 (with a standard deviation of 1.1165).

What is the probability that you will win all five games?

$P(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$

$P(X=5) = \frac{5!}{5!(5-5)!}\left(\frac{18}{38}\right)^5\left(1-\frac{18}{38}\right)^{5-5}$

$P(X=5) = \frac{5!}{5!0!}\left(.4737^5\right).5263^0 = 1(.0238)(1) = .0238$

There is a 2.38% chance that you will win all five out of five games.

If you win three or more games, you make a profit. If you win two or fewer games, you lose money. **What is the probability that you will win no more than two games?**

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$P(X=0) = \frac{5!}{0!(5-0)!}\left(\frac{18}{38}\right)^0\left(1-\frac{18}{38}\right)^{5-0} = .0404$

$P(X=1) = \frac{5!}{1!(5-1)!}\left(\frac{18}{38}\right)^1\left(1-\frac{18}{38}\right)^{5-1} = .1817$

$P(X=2) = \frac{5!}{2!(5-2)!}\left(\frac{18}{38}\right)^2\left(1-\frac{18}{38}\right)^{5-2} = .3271$

$P(X \le 2) = .0404 + .1817 + .3271 = .5493$

There is a 54.93% chance that you will win no more than two games. In other words, there is a 54.93% chance that you will lose money.

The Poisson Probability Distribution :

Another discrete random variable that has numerous practical applications is the Poisson random variable. Its probability distribution provides a good model for data that represent the number of occurrences of a specified event in a given ***unit of time or space***.

Here are some examples of experiments for which the random variable x can be modeled by the Poisson random variable:

- The number of **calls received** by a technical support specialist during a given period of **time (time).**
- The number of **bacteria** per small volume of fluid(**Space**).
- The number of customer arrivals at a checkout counter during a given minute**(time).**
- The number of machine breakdowns during a given day **(time).**
- The number of traffic accidents on a section of freeway during a given time period(**Space**).

In each example, **X represents the number of events that occur in a period of time or space during which an average of " μ " such events can be expected to occur.**
The only ***assumptions*** needed when one uses the ***Poisson distribution*** to model experiments such as these are that the counts or events occur **randomly and independently** of one another. The formula for the Poisson probability distribution, as well as its mean and variance, are given next.

THE POISSON PROBABILITY DISTRIBUTION

Let μ be the average number of times that an event occurs in a certain period of time or space. The probability of k occurrences of this event is

$$P(x = k) = \frac{\mu^k e^{-\mu}}{k!}$$

for values of $k = 0, 1, 2, 3, \ldots$. The mean and standard deviation of the Poisson random variable x are

Mean: μ Standard deviation: $\sigma = \sqrt{\mu}$

The symbol $e = 2.71828\ldots$ is evaluated using your scientific calculator, which should have a function such as e^x. For each value of k, you can obtain the individual probabilities for the Poisson random variable, just as you did for the binomial random variable.

EXAMPLE The average number of traffic accidents on a certain section of highway is two per week. Assume that the number of accidents follows a Poisson distribution with $\mu = 2$.

1. Find the probability of no accidents on this section of highway during a 1-week period.
2. Find the probability of at most three accidents on this section of highway during a 2-week period.

Solution

1. The average number of accidents per week is $\mu = 2$. Therefore, the probability of no accidents on this section of highway during a given week is

$$P(x = 0) = p(0) = \frac{2^0 e^{-2}}{0!} = e^{-2} = .135335$$

2. During a 2-week period, the average number of accidents on this section of highway is 2(2) = 4. The probability of at most three accidents during a 2-week period is

$$P(x \leq 3) = p(0) + p(1) + p(2) + p(3)$$

where

$$P(x \leq 3) = p(0) + p(1) + p(2) + p(3)$$

where

$$p(0) = \frac{4^0 e^{-4}}{0!} = .018316 \qquad p(2) = \frac{4^2 e^{-4}}{2!} = .146525$$

$$p(1) = \frac{4^1 e^{-4}}{1!} = .073263 \qquad p(3) = \frac{4^3 e^{-4}}{3!} = .195367$$

Therefore,

$$P(x \leq 3) = .018316 + .073263 + .146525 + .195367 = .433471$$

Alternatively, you can use cumulative Poisson ***tables*** :

NEED TO KNOW...

How to Use Table 2 to Calculate Poisson Probabilities

1. Find the necessary value of μ. Isolate the appropriate column in Table 2.
2. Table 2 gives $P(x \leq k)$ in the row marked k. Rewrite the probability you need so that it is in this form.
   - List the values of x in your event.
   - From the list, write the event as either the difference of two probabilities:

   $$P(x \leq a) - P(x \leq b) \text{ for } a > b$$

   or the complement of the event:

   $$1 - P(x \leq a)$$

   or just the event itself:

   $$P(x \leq a) \text{ or } P(x < a - 1)$$

EXAMPLE Refer to Example where we calculated probabilities for a Poisson distribution with $\mu = 2$ and $\mu = 4$. Use the cumulative Poisson table to find the probabilities of these events:

1. No accidents during a 1-week period.
2. At most three accidents during a 2-week period.

Solution

A portion of Table in Appendix is shown in Figure

FIGURE
Portion of Table
Appendix

	μ				
k	2.0	2.5	3.0	3.5	4.0
0	.135	.082	.055	.033	.018
1	.406	.287	.199	.136	.092
2	.677	.544	.423	.321	.238
3	.857	.758	.647	.537	.433
4	.947	.891	.815	.725	.629
5	.983	.958	.916	.858	.785
6	.995	.986	.966	.935	.889
7	.999	.996	.988	.973	.949
8	1.000	.999	.996	.990	.979
9		1.000	.999	.997	.992
10			1.000	.999	.997
11				1.000	.999
12					1.000

1. From Example , the average number of accidents in a 1-week period is $\mu = 2.0$. Therefore, the probability of no accidents in a 1-week period can be read directly from Table 2 in the column marked "2.0" as $P(x = 0) = p(0) = .135$.
2. The average number of accidents in a 2-week period is $2(2) = 4$. Therefore, the probability of at most three accidents in a 2-week period is found in Table 2, indexing $\mu = 4.0$ and $k = 3$ as $P(x \le 3) = .433$.

Both of these probabilities match the calculations done in Example , correct to three decimal places.

Continuous Random Variables

Continuous random variables are random quantities that are measured on a ***continuous scale***. They can usually take on any value over some interval, which distinguishes them from discrete random variables, which can take on only a sequence of values, usually integers.

Probability distribution of a continuous random variable :
describe the probability distribution of a continuous random variable by giving its density function. A **density function** is a function ***f(x)*** which satisfies the following two properties:

1. $f(x) \geq 0$ for all x.
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

1- The ***first*** condition says that the **density function** is always nonnegative, so the graph of the density function always lies on or above ***the x-axis***.
2- The ***second*** condition ensures that the area under the density curve is **" 1 "**.

The probability that the random variable takes on a value :
the probability that the random variable takes on a value between **a and b** is the area under the curve between a and b. More precisely, if X is a random variable with density function *f (x)* and **a < b**, then :

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

Example : Suppose the income of people in a community can be approximated by a continuous distribution with density :

$$f(x) = \begin{cases} 2x^{-2} & \text{if } x \geq 2 \\ 0 & \text{if } x < 2 \end{cases}$$

a) Find the probability that a randomly chosen person has an income between \$30; 000 and \$50; 000.
b) Find the probability that a randomly chosen person has an income of at least \$60; 000.

Solution:

a) Let X be the income of a randomly chosen person. The probability that a randomly chosen person has an income between \$30; 000 and \$50; 000 is :

$$P(3 \leq X \leq 5) = \int_3^5 f(x)\, dx = \int_3^5 2x^{-2}\, dx = -2x^{-1}\Big|_{x=3}^{x=5} = -\frac{2}{5} - \left(-\frac{2}{3}\right) = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}.$$

b) The probability that a randomly chosen person has an income of at least \$60; 000 is :

$$P(X \geq 6) = \int_6^\infty f(x)\, dx = \int_6^\infty 2x^{-2}\, dx = \lim_{n\to\infty} \int_6^n 2x^{-2}\, dx$$
$$= \lim_{n\to\infty} -2x^{-1}\Big|_{x=6}^{x=n} = \lim_{n\to\infty} \left(-\frac{2}{n} + \frac{2}{6}\right) = \frac{1}{3}.$$

Expected value and standard deviation for continuous random variables

The procedure for finding expected values and standard deviations for random variables of ***continuous random variables*** is similar to the procedure used to calculate expected values and standard deviations for ***discrete*** random variables. **The differences are that sums in the formula for discrete random variables get replaced by *integrals*** (which are the continuous analogs of sums), while **probabilities** in the formula for discrete random variables get replaced by ***densities***. More precisely, if **X** is a random variable with density ***f(x)***, then the expected value of **X** is given by :

Expected value of X

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx,$$

while the ***variance*** is given by :

$$\mathrm{Var}(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx.$$

OR

$$\mathrm{Var}(X) = E[X^2] - \mu^2 = \left(\int_{-\infty}^{\infty} x^2 f(x)\, dx\right) - \mu^2.$$

As in the case of discrete random variables, **the standard deviation of X is the square root of the variance of X**. (**standard deviation** = $\sqrt{\mathrm{Var}\ (X)}$) .

Example : Suppose a train arrives shortly after 1:00 PM each day, and that the number of minutes after 1:00 that the train arrives can be modeled as a continuous random variable with density given :

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and standard deviation of the number of minutes after 1:00 that the train arrives.

Solution: Let X be the number of minutes after 1:00 that the train arrives. The mean (or, equivalently, the expected value) of **X** is given by:

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x \cdot 2(1-x)\, dx = \int_0^1 2x - 2x^2\, dx = \left(x^2 - \frac{2x^3}{3}\right)\Bigg|_{x=0}^{x=1} = \frac{1}{3}.$$

Also, we have

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2 \cdot 2(1-x) = \int_0^1 2x^2 - 2x^3\, dx = \left(\frac{2x^3}{3} - \frac{2x^4}{4}\right)\Bigg|_{x=0}^{x=1} = \frac{2}{3} - \frac{2}{4} = \frac{1}{6}.$$

Therefore,

$$\mathrm{Var}(X) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18},$$

and the standard deviation is $\sqrt{1/18} \approx 0.24$.

Special Continuous Distributions

As was the case with discrete random variables, when we gave special attention to the geometric, binomial, and Poisson distributions, some continuous distributions occur repeatedly in applications. Probably the three most important continuous distributions are the ***uniform*** distribution, the ***exponential*** distribution, and the ***normal*** distribution.

Uniform Distribution:

If **a** < **b**, then we say a random variable X has the uniform distribution on [a; b] if :

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Example : If X has the uniform distribution on $[2, 5]$, calculate $P(X > 4)$.

Solution: The density of X is given by

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } 2 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$P(X \geq 4) = \int_4^\infty f(x)\, dx = \int_4^5 \frac{1}{3}\, dx = \frac{x}{3}\Big|_{x=4}^{x=5} = \frac{5}{3} - \frac{4}{3} = \frac{1}{3}.$$

The Normal Probability Distribution :

The formula or **P**robability **D**ensity **F**unction (PDF) that generates this distribution is shown next.

NORMAL PROBABILITY DISTRIBUTION

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \qquad -\infty < x < \infty$$

The symbols e and π are mathematical constants given approximately by 2.7183 and 3.1416, respectively; μ and σ $(\sigma > 0)$ are parameters that represent the population mean and standard deviation, respectively.

EXAMPLE

The following data, reproduced from Table of Example . give the frequency distribution of the daily commuting times (in minutes) from home to work for all 25 employees of a company.

Daily Commuting Time (minutes)	Number of Employees
0 to less than 10	4
10 to less than 20	9
20 to less than 30	6
30 to less than 40	4
40 to less than 50	2

Calculate the variance and standard deviation.

Solution All four steps needed to calculate the variance and standard deviation for grouped data are shown after Table .

Table

Daily Commuting Time (minutes)	f	m	mf	m^2f
0 to less than 10	4	5	20	100
10 to less than 20	9	15	135	2025
20 to less than 30	6	25	150	3750
30 to less than 40	4	35	140	4900
40 to less than 50	2	45	90	4050
	$N = 25$		$\Sigma mf = 535$	$\Sigma m^2f = 14{,}825$

Step 1. *Calculate the value of* Σmf.

To calculate the value of Σmf, first find the midpoint m of each class (see the third column in Table) and then multiply the corresponding class midpoints and class frequencies (see the fourth column). The value of Σmf is obtained by adding these products. Thus,

$$\Sigma mf = 535$$

Step 2. *Find the value of* Σm^2f.

To find the value of Σm^2f, square each m value and multiply this squared value of m by the corresponding frequency (see the fifth column in Table). The sum of these products (that is, the sum of the fifth column) gives Σm^2f. Hence,

$$\Sigma m^2f = 14{,}825$$

Step 3. *Calculate the variance.*

Because the data set includes all 25 employees of the company, it represents the population. Therefore, we use the formula for the population variance:

$$\sigma^2 = \frac{\Sigma m^2f - \frac{(\Sigma mf)^2}{N}}{N} = \frac{14{,}825 - \frac{(535)^2}{25}}{25} = \frac{3376}{25} = \mathbf{135.04}$$

Step 4. *Calculate the standard deviation.*

To obtain the standard deviation, take the (positive) square root of the variance.

$$\sigma = \sqrt{\sigma^2} = \sqrt{135.04} = \mathbf{11.62\ minutes}$$

Thus, the standard deviation of the daily commuting times for these employees is 11.62 minutes.

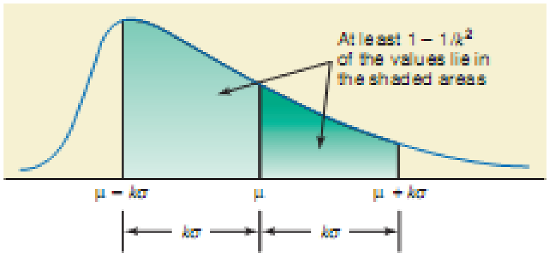
Chebyshev's Theorem

Chebyshev's theorem gives a lower bound for the area under a curve between two points that are on opposite sides of the mean and at the same distance from the mean.

Definition

Chebyshev's Theorem For any number k greater than 1, at least $(1 - 1/k^2)$ of the data values lie within k standard deviations of the mean.

Figure illustrates Chebyshev's theorem.



Chebyshev's theorem.

Thus, for example, if $k = 2$, then

$$1 - \frac{1}{k^2} = 1 - \frac{1}{(2)^2} = 1 - \frac{1}{4} = 1 - .25 = .75 \text{ or } 75\%$$

Therefore, according to Chebyshev's theorem, at least .75 or 75% of the values of a data set lie within two standard deviations of the mean. This is shown in Figure on the next page.

If $k = 3$, then,

$$1 - \frac{1}{k^2} = 1 - \frac{1}{(3)^2} = 1 - \frac{1}{9} = 1 - .11 = .89 \text{ or } 89\% \text{ approximately}$$

EXAMPLE

The average systolic blood pressure for 4000 women who were screened for high blood pressure was found to be 187 with a standard deviation of 22. Using Chebyshev's theorem, find at least what percentage of women in this group have a systolic blood pressure between 143 and 231.

Solution Let μ and σ be the mean and the standard deviation, respectively, of the systolic blood pressures of these women. Then, from the given information,

$$\mu = 187 \quad \text{and} \quad \sigma = 22$$

To find the percentage of women whose systolic blood pressures are between 143 and 231, the first step is to determine k. As shown below, each of the two points, 143 and 231, is 44 units away from the mean.

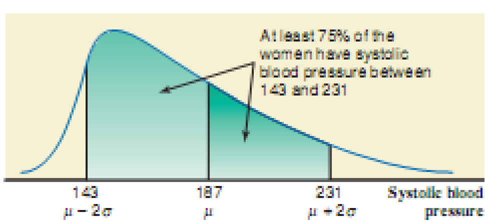
$$|\leftarrow 143 - 187 = -44 \rightarrow | \leftarrow 231 - 187 = 44 \rightarrow|$$

143 $\qquad$ $\mu = 187$ $\qquad$ 231

The value of k is obtained by dividing the distance between the mean and each point by the standard deviation. Thus,

$$k = 44/22 = 2$$

$$1 - \frac{1}{k^2} = 1 - \frac{1}{(2)^2} = 1 - \frac{1}{4} = 1 - .25 = .75 \text{ or } \mathbf{75\%}$$



Percentage of women with systolic blood pressure between 143 and 231.

Empirical Rule

Whereas Chebyshev's theorem is applicable to any kind of distribution, the **empirical rule** applies only to a specific type of distribution called a *bell-shaped distribution*, as shown in Figure where it is called a *normal curve*. In this section, only the following three rules for the curve are given.

Empirical Rule For a bell-shaped distribution, approximately

1. 68% of the observations lie within one standard deviation of the mean
2. 95% of the observations lie within two standard deviations of the mean
3. 99.7% of the observations lie within three standard deviations of the mean

Figure illustrates the empirical rule. Again, the empirical rule applies to population data as well as to sample data.

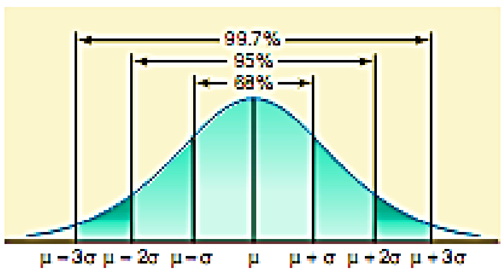


Illustration of the empirical rule.

■ EXAMPLE

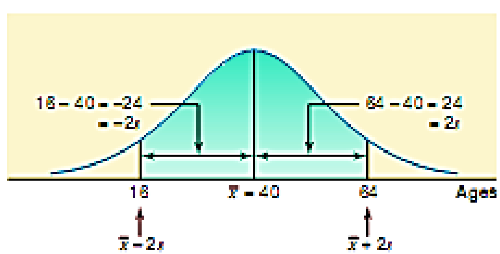
The age distribution of a sample of 5000 persons is bell-shaped with a mean of 40 years and a standard deviation of 12 years. Determine the approximate percentage of people who are 16 to 64 years old.

Solution We use the empirical rule to find the required percentage because the distribution of ages follows a bell-shaped curve. From the given information, for this distribution,

$$\bar{x} = 40 \text{ years} \quad \text{and} \quad s = 12 \text{ years}$$

Each of the two points, 16 and 64, is 24 units away from the mean. Dividing 24 by 12, we convert the distance between each of the two points and the mean in terms of standard deviations. Thus, the distance between 16 and 40 and between 40 and 64 is each equal to $2s$.

Figure Percentage of people who are 16 to 64 years old.



Consequently, as shown in Figure the area from 16 to 64 is the area from $\bar{x} - 2s$ to $\bar{x} + 2s$.

Because the area within two standard deviations of the mean is approximately 95% for a bell-shaped curve, approximately **95%** of the people in the sample are 16 to 64 years old. ■

SEQUENCES AND INFINITE SERIES

(Sequences and Series)

In this we'll be taking a look at sequences and (infinite) series. Actually, this section will deal almost exclusively with series. However, we also need to understand some of the basics of sequences in order to properly deal with series

Sequences

Let's start off this section with a discussion of just what a sequence is. A ***sequence*** is nothing more than a list of numbers written in a specific order. The list may or may not have an infinite number of terms in them although we will be dealing exclusively with infinite sequences in this section. General sequence terms are denoted as follows,

$$\begin{aligned} &a_1 - \text{first term} \\ &a_2 - \text{second term} \\ &\quad\vdots \\ &a_n - n^{th} \text{ term} \\ &a_{n+1} - (n+1)^{st} \text{ term} \\ &\quad\vdots \end{aligned}$$

Because we will be dealing with infinite sequences each term in the sequence will be followed by another term as noted above. In the notation above we need to be very careful with the subscripts. The subscript of $n+1$ denotes the next term in the sequence and NOT one plus the n^{th} term! In other words,

$$a_{n+1} \neq a_n + 1$$

so be very careful when writing subscripts to make sure that the "+1" doesn't migrate out of the subscript! This is an easy mistake to make when you first start dealing with this kind of thing.

There is a variety of ways of denoting a sequence. Each of the following are equivalent ways of denoting a sequence.

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

In the second and third notations above a_n is usually given by a formula.

Example Write down the first few terms of each of the following sequences.

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

Solution

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}=\left\{\underbrace{2}_{n=1},\underbrace{\frac{3}{4}}_{n=2},\underbrace{\frac{4}{9}}_{n=3},\underbrace{\frac{5}{16}}_{n=4},\underbrace{\frac{6}{25}}_{n=5},\ldots\right\}$$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

This one is similar to the first one. The main difference is that this sequence doesn't start at $n=1$.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}=\left\{-1,\frac{1}{2},-\frac{1}{4},\frac{1}{8},-\frac{1}{16},\ldots\right\}$$

Note that the terms in this sequence alternate in signs. Sequences of this kind are sometimes called alternating sequences.

***Working* Definition of Limit**

1. We say that

$$\lim_{n\to\infty} a_n = L$$

if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.

2. We say that

$$\lim_{n\to\infty} a_n = \infty$$

if we can make a_n as large as we want for all sufficiently large n. Again, in other words, the value of the a_n's get larger and larger without bound as n approaches infinity.

3. We say that

$$\lim_{n\to\infty} a_n = -\infty$$

if we can make a_n as large and negative as we want for all sufficiently large n. Again, in other words, the value of the a_n's are negative and get larger and larger without bound as n approaches infinity.

The working definitions of the various sequence limits are nice in that they help us to visualize what the limit actually is. Just like with limits of functions however, there is also a precise definition for each of these limits. Let's give those before proceeding

***Precise* Definition of Limit**

1. We say that $\lim_{n\to\infty} a_n = L$ if for every number $\varepsilon > 0$ there is an integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N$$

2. We say that $\lim_{n\to\infty} a_n = \infty$ if for every number $M > 0$ there is an integer N such that

$$a_n > M \qquad \text{whenever} \qquad n > N$$

3. We say that $\lim_{n\to\infty} a_n = -\infty$ if for every number $M < 0$ there is an integer N such that

$$a_n < M \qquad \text{whenever} \qquad n > N$$

We won't be using the precise definition often, but it will show up occasionally.

Note that both definitions tell us that in order for a limit to exist and have a finite value all the sequence terms must be getting closer and closer to that finite value as n increases.

Now that we have the definitions of the limit of sequences out of the way we have a bit of terminology that we need to look at. If $\lim_{n\to\infty} a_n$ exists and is finite we say that the sequence is **convergent**. If $\lim_{n\to\infty} a_n$ doesn't exist or is infinite we say the sequence **diverges**. Note that sometimes we will say the sequence **diverges to** ∞ if $\lim_{n\to\infty} a_n = \infty$ and if $\lim_{n\to\infty} a_n = -\infty$ we will sometimes say that the sequence **diverges to** $-\infty$.

Properties

1. $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

2. $\lim\limits_{n\to\infty} c a_n = c \lim\limits_{n\to\infty} a_n$

3. $\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$

4. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided $\lim\limits_{n\to\infty} b_n \neq 0$

5. $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ provided $a_n \geq 0$

Testing the sequences for convergence or divergent:

- Method of n[th] term test.

Example Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

(b) $\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$

Solution

(a) $\left\{\frac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

To do a limit in this form all we need to do is factor from the numerator and denominator the largest power of n, cancel and then take the limit.

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2} = \lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)} = \lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5} = \frac{3}{5}$$

So the sequence converges and its limit is $\frac{3}{5}$.

(b) $\left\{ \dfrac{e^{2n}}{n} \right\}_{n=1}^{\infty}$

We will need to be careful with this one. We will need to use L'Hospital's Rule on this sequence. The problem is that L'Hospital's Rule only works on functions and not on sequences. Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x) = \frac{e^{2x}}{x}$$

and note that,

$$f(n) = \frac{e^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty} \frac{e^{2n}}{n} = \lim_{x\to\infty} \frac{e^{2x}}{x} = \lim_{x\to\infty} \frac{2e^{2x}}{1} = \infty$$

So, the sequence in this part diverges (to ∞).

Test of monotone for the sequences :

In the previous section we introduced the concept of a sequence and talked about limits of sequences and the idea of convergence and divergence for a sequence. In this section we want to take a quick look at some ideas involving sequences.

Let's start off with some terminology and definitions.

Given any sequence $\{a_n\}$ we have the following.

1. We call the sequence **increasing** if $a_n < a_{n+1}$ for every n.
2. We call the sequence **decreasing** if $a_n > a_{n+1}$ for every n.
3. If $\{a_n\}$ is an increasing sequence or $\{a_n\}$ is a decreasing sequence we call it **monotonic**.
4. If there exists a number m such that $m \le a_n$ for every n we say the sequence is **bounded below**. The number m is sometimes called a **lower bound** for the sequence.
5. If there exists a number M such that $a_n \le M$ for every n we say the sequence is **bounded above**. The number M is sometimes called an **upper bound** for the sequence.
6. If the sequence is both bounded below and bounded above we call the sequence **bounded**.

Note that in order for a sequence to be increasing or decreasing it must be increasing/decreasing for every n. In other words, a sequence that increases for three terms and then decreases for the rest of the terms is ***NOT*** a decreasing sequence! Also note that a monotonic sequence must always increase or it must always decrease.

Example Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

(c) $\left\{\dfrac{2}{n^2}\right\}_{n=5}^{\infty}$ (H.W.)

Solution

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) because,

$$-n^2 > -(n+1)^2$$

for every n.

Also, since the sequence terms will be either zero or negative this sequence is bounded above. We can use any positive number or zero as the bound, M, however, it's standard to choose the smallest possible bound if we can and it's a nice number. So, we'll choose $M = 0$ since,

$$-n^2 \le 0 \qquad \text{for every } n$$

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

The sequence terms in this sequence alternate between 1 and -1 and so the sequence is neither an increasing sequence or a decreasing sequence. Since the sequence is neither an increasing nor decreasing sequence it is not a monotonic sequence.

The sequence is bounded however since it is bounded above by 1 and bounded below by -1.

Again, we can note that this sequence is also divergent.

Series :

In this section we will introduce The topic that is ***infinite series***. So just what is an infinite series? Well, let's start with a sequence ,

$\{a_n\}_{n=1}^{\infty}$ (note the $n=1$ is for convenience, it can be anything) and define the following,

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$s_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

The s_n are called **partial sums** and notice that they will form a sequence, $\{s_n\}_{n=1}^{\infty}$. Also recall that the Σ is used to represent this summation and called a variety of names. The most common names are : **series notation, summation notation,** and **sigma notation.**

Now back to series. We want to take a look at the limit of the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$. Notationally we'll define,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

We will call $\sum_{i=1}^{\infty} a_i$ an **infinite series** and note that the series "starts" at $i=1$ because that is where our original sequence, $\{a_n\}_{n=1}^{\infty}$, started. Had our original sequence started at 2 then our infinite series would also have started at 2. The infinite series will start at the same value that the sequence of terms (as opposed to the sequence of partial sums) starts.

If the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$, is convergent and its limit is finite then we also call the infinite series, $\sum_{i=1}^{\infty} a_i$ **convergent** and if the sequence of partial sums is divergent then the infinite series is also called **divergent**.

Note that sometimes it is convenient to write the infinite series as,

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Now, in $\sum_{i=1}^{\infty} a_i$ the i is called the **index of summation** or just **index** for short and note that the letter we use to represent the index does not matter. So for example the following series are all the same. The only difference is the letter we've used for the index.

$$\sum_{i=0}^{\infty} \frac{3}{i^2+1} = \sum_{k=0}^{\infty} \frac{3}{k^2+1} = \sum_{n=0}^{\infty} \frac{3}{n^2+1} \quad etc.$$

Properties

If $\sum a_n$ and $\sum b_n$ are both convergent series then,

6. $\sum ca_n$, where c is any number, is also convergent and

$$\sum ca_n = c \sum a_n$$

7. $\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$ is also convergent and,

$$\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n = \sum_{n=k}^{\infty} (a_n \pm b_n).$$

Testing methods for series convergence or divergence

1- Alternating Series Test :

The ***alternating series test*** that we looked at for series is convergence that have required ***all the terms*** in the series be ***positive***. Of course there are many series out there that have negative terms in them and so we now need to start looking at tests for these kinds of series.

The test that we are going to look into in this section will be a test for alternating series. An **alternating series** is any series, $\sum a_n$, for which the series terms can be written in one of the following two forms.

$$a_n = (-1)^n b_n \qquad b_n \geq 0$$
$$a_n = (-1)^{n+1} b_n \qquad b_n \geq 0$$

There are many other ways to deal with the alternating sign, but they can all be written as one of the two forms above. For instance,

$$(-1)^{n+2} = (-1)^n (-1)^2 = (-1)^n$$
$$(-1)^{n-1} = (-1)^{n+1} (-1)^{-2} = (-1)^{n+1}$$

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. Then if,

1. $\lim_{n\to\infty} b_n = 0$ and,
2. $\{b_n\}$ is a decreasing sequence

the series $\sum a_n$ is convergent.

Example Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \qquad b_n = \frac{1}{n}$$

Now, all that we need to do is run through the two conditions in the test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Both conditions are met and so by the Alternating Series Test the series must converge.

Example Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

Solution

The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,

$$\cos(n\pi) = (-1)^n$$

and so the series is really,

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \Rightarrow \qquad b_n = \frac{1}{\sqrt{n}}$$

Checking the two condition gives,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = b_{n+1}$$

The two conditions of the test are met and so by the Alternating Series Test the series is convergent.

(H.W.) Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

2-Ratio Test :

In this section we are going to take a look at a test that we can use to see **if a series is absolutely convergent or not**. Recall that if a series is absolutely convergent then we will also know that it's convergent and so we will often use it to simply determine the convergence of a series.

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Example Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$$

Solution

With this first example let's be a little careful and make sure that we have everything down correctly. Here are the series terms a_n.

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Recall that to compute a_{n+1} all that we need to do is substitute $n+1$ for all the n's in a_n.

$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$

Now, to define L we will use,

$$L = \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right|$$

since this will be a little easier when dealing with fractions as we've got here. So.

$$\begin{aligned}L &= \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}\;\frac{4^{2n+1}(n+1)}{(-10)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{-10(n+1)}{4^2(n+2)}\right| \\ &= \frac{10}{16}\lim_{n\to\infty}\frac{n+1}{n+2} \\ &= \frac{10}{16} < 1\end{aligned}$$

So, $L < 1$ and so by the Ratio Test the series converges absolutely and hence will converge.

Example Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{n!}{5^n}$$

Solution

Now that we've worked one in detail we won't go into quite the detail with the rest of these. Here is the limit.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!}{5^{n+1}} \frac{5^n}{n!} \right| = \lim_{n\to\infty} \frac{(n+1)!}{5\ n!}$$

In order to do this limit we will need to eliminate the factorials. We simply can't do the limit with the factorials in it. To eliminate the factorials we will recall from our discussion on factorials above that we can always "strip out" terms from a factorial. If we do that with the numerator (in this case because it's the larger of the two) we get,

$$L = \lim_{n\to\infty} \frac{(n+1)\ n!}{5\ n!}$$

at which point we can cancel the $n!$ for the numerator an denominator to get,

$$L = \lim_{n\to\infty} \frac{(n+1)}{5} = \infty > 1$$

So, by the Ratio Test this series diverges.

(H.W.) Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1}\, n}$$

Example Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Solution

Let's first get L.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)^2+1} \frac{n^2+1}{(-1)^n} \right| = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = 1$$

So, as implied earlier we get $L = 1$ which means the ratio test is no good for determining the convergence of this series. We will need to resort to another test for this series. This series is an alternating series and so let's check the two conditions from that test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

$$b_n = \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = b_{n+1}$$

The two conditions are met and so by the Alternating Series Test this series is convergent. We'll leave it to you to verify this series is also absolutely convergent.

3-Root Test :

This is the last test for series convergence that we're going to be looking at. As with the Ratio Test this test will also tell whether a series is absolutely convergent or not rather than simple convergence.

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

4. if $L < 1$ the series is absolutely convergent (and hence convergent).
5. if $L > 1$ the series is divergent.
6. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

As with the ratio test, if we get $L = 1$ the root test will tell us nothing and we'll need to use another test to determine the convergence of the series. Also note that if $L = 1$ in the Ratio Test then the Root Test will also give $L = 1$.

We will also need the following fact in some of these problems.

Fact

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Example Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

Solution

There really isn't much to these problems other than computing the limit and then using the root test. Here is the limit for this problem.

$$L = \lim_{n\to\infty} \left| \frac{n^n}{3^{1+2n}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} = \infty > 1$$

So, by the Root Test this series is divergent.

Example Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$$

Solution

Again, there isn't too much to this series.

$$L=\lim_{n\to\infty}\left|\left(\frac{5n-3n^3}{7n^3+2}\right)^n\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{5n-3n^3}{7n^3+2}\right|=\left|\frac{-3}{7}\right|=\frac{3}{7}<1$$

Therefore, by the Root Test this series converges absolutely and hence converges.

Note that we had to keep the absolute value bars on the fraction until we'd taken the limit to get the sign correct.

(H.W.) Determine if the following series is convergent or divergent.

$$\sum_{n=3}^{\infty}\frac{(-12)^n}{n}$$

Power Series :

We've spent quite a bit of time talking about series now and with only a couple of exceptions we've spent most of that time talking about how to determine if a series will converge or not. It's now time to start looking at some specific kinds of series.

In this section we are going to start talking about ***power series***. A ***power series about a***, or just ***power series***, is any series that can be written in the form :

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where " **a** " and " $\mathbf{c_n}$ " are numbers. The " $\mathbf{c_n}$**'s** " are often called ***the coefficients of the series***. The first thing to notice about a power series is that it ***is a function of x***. That is different from any other kind of series that we've looked at to this point. In all the prior sections we've only allowed ***numbers*** in the series and now we are allowing ***variables*** to be in the series as well. This will not change how things work however. Everything that we know about series still holds.

In the discussion of power series **convergence** is still a major question that we'll be dealing with. The difference is that the **convergence** of the series will now depend upon ***the values of x*** that we put into the series. *A power series may converge for some values of x and not for other values of x.*

Before we get too far into power series there is some terminology that we need to get out of the way.

First, as we will see in our examples, we will be able to show that there is a number R so that the power series will converge for, $|x-a| < R$ and will diverge for $|x-a| > R$. This number is called the **radius of convergence** for the series. Note that the series may or may not converge if $|x-a| = R$. What happens at these points will not change the radius of convergence.

Secondly, the interval of all x's, including the endpoints if need be, for which the power series converges is called the **interval of convergence** of the series.

These two concepts are fairly closely tied together. If we know that the radius of convergence of a power series is R then we have the following.

$a-R < x < a+R$	power series converges
$x < a-R$ and $x > a+R$	power series diverges

The interval of convergence must then contain the interval $a-R < x < a+R$ since we know that the power series will converge for these values. We also know that the interval of convergence can't contain x's in the ranges $x < a-R$ and $x > a+R$ since we know the power series diverges for these value of x. Therefore, to completely identify the interval of convergence all that we have to do is determine if the power series will converge for $x = a-R$ or $x = a+R$. If the power series converges for one or both of these values then we'll need to include those in the interval of convergence.

Before getting into some examples let's take a quick look at the convergence of a power series for the case of $x = a$. In this case the power series becomes,

$$\sum_{n=0}^{\infty} c_n (a-a)^n = \sum_{n=0}^{\infty} c_n (0)^n = c_0 (0)^0 + \sum_{n=1}^{\infty} c_n (0)^n = c_0 + \sum_{n=1}^{\infty} 0 = c_0 + 0 = c_0$$

and so the power series converges. Note that we had to strip out the first term since it was the only non-zero term in the series.

It is important to note that no matter what else is happening in the power series we are guaranteed to get convergence for $x = a$. The series may not converge for any other value of x, but it will always converge for $x = a$.

Example Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n$$

we know that this power series will converge for x = - 3, but that's it at this point.

*(1) To determine the remainder of the x's for which we'll get convergence we can ***use any of the tests that we've discussed*** to this point

*(2) After application of the test that we choose to work with we will arrive at ***condition(s)* on x** that we can use to determine which values of x for which the power series will ***converge*** and which values of x for which the power series will ***diverge***.

*(3) From this we can get the ***radius of convergence*** and most of the ***interval of convergence.***

1. With all that said, the best tests to use here are almost always the ***ratio*** or ***root test***. Most of the power series that we'll be looking at are set up for one or the other. In this case we'll use the ***ratio test***.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (n+1)(x+3)^{n+1}}{4^{n+1}} \frac{4^n}{(-1)^n (n)(x+3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-(n+1)(x+3)}{4n} \right|$$

2.

$$L = |x+3| \lim_{n\to\infty} \frac{n+1}{4n}$$

$$= \frac{1}{4}|x+3|$$

So, the ratio test tells us that if $L < 1$ the series will converge, if $L > 1$ the series will diverge, and if $L = 1$ we don't know what will happen. So, we have,

$$\frac{1}{4}|x+3| < 1 \quad \Rightarrow \quad |x+3| < 4 \qquad \text{series converges}$$

$$\frac{1}{4}|x+3| > 1 \quad \Rightarrow \quad |x+3| > 4 \qquad \text{series diverges}$$

3. We'll deal with the $L = 1$ case in a bit. Notice that we now have the radius of convergence for this power series. These are exactly the conditions required for the radius of convergence. The radius of convergence for this power series is $R = 4$.

Now, let's get the interval of convergence. We'll get most (if not all) of the interval by solving the first inequality from above.

$$-4 < x+3 < 4$$

$$-7 < x < 1$$

So, most of the interval of validity is given by $-7 < x < 1$. All we need to do is determine if the power series will converge or diverge at the endpoints of this interval. Note that these values of x will correspond to the value of x that will give $L = 1$.

The way to determine convergence at these points is to simply plug them into the original power series and see if the series converges or diverges using any test necessary.

$x = -7$:

In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-1)^n 4^n$$

$$= \sum_{n=1}^{\infty} (-1)^n (-1)^n n \qquad (-1)^n (-1)^n = (-1)^{2n} = 1$$

$$= \sum_{n=1}^{\infty} n$$

This series is divergent by the Divergence Test since $\lim_{n\to\infty} n = \infty \neq 0$.

$x = 1$:

In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(4)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This series is also divergent by the Divergence Test since $\lim_{n\to\infty} (-1)^n n$ doesn't exist.

So, in this case the power series will not converge for either endpoint. The interval of convergence is then,

$$-7 < x < 1$$

Example Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Solution

We'll start this example with the ratio test as we have for the previous ones.

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)n!\,(2x+1)}{n!}\right| \\ &= |2x+1|\lim_{n\to\infty}(n+1) \end{aligned}$$

At this point we need to be careful. The limit is infinite, but there is that term with the x's in front of the limit. We'll have $L=\infty>1$ provided $x\neq -\frac{1}{2}$. So, this power series will only converge if $x=-\frac{1}{2}$.

In this case we say the radius of convergence is $R=0$ and the interval of convergence is $x=-\frac{1}{2}$, and yes we really did mean interval of convergence even though it's only a point.

Example Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty}\frac{(x-6)^n}{n^n}$$

Solution

In this example the root test seems more appropriate. So,

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(x-6)^n}{n^n}\right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\left|\frac{x-6}{n}\right| \\ &= |x-6|\lim_{n\to\infty}\frac{1}{n} \\ &= 0 \end{aligned}$$

So, since $L=0<1$ regardless of the value of x this power series will converge for every x.

In these cases we say that the radius of convergence is $R=\infty$ and interval of convergence is $-\infty<x<\infty$.

So, let's summarize the last two examples. If the power series only converges for $x=a$ then the radius of convergence is $R=0$ and the interval of convergence is $x=a$. Likewise if the power series converges for every x the radius of convergence is $R=\infty$ and interval of convergence is $-\infty<x<\infty$.

(H.W.) Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$$

(H.W.) Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{(-3)^n}$$

Taylor Series :

In the previous section we started looking at writing down a power series representation of a function. The problem with the approach in that section is that everything came down to needing to be able to relate the function in some way to :

$$\frac{1}{1-x}$$

and while there are many functions out there that can be related to this function there are many more that simply can't be related to this.

So, without taking anything away from the process we looked at in the previous section, what we need to do is come up with ***a more general method for writing a power series representation for a function***.

1

So, for the time being, let's make two assumptions. First, let's assume that the function $f(x)$ does in fact have a power series representation about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

Next, we will need to assume that the function, $f(x)$, has derivatives of every order and that we can in fact find them all.

3 Now that we've assumed that a power series representation exists we need to determine what the coefficients, c_n, are. This is easier than it might at first appear to be. Let's first just evaluate everything at $x = a$. This gives,

$$f(a) = c_0$$

So, all the terms except the first are zero and we now know what c_0 is. Unfortunately, there isn't any other value of x that we can plug into the function that will allow us to quickly find any of the other coefficients. However, if we take the derivative of the function (and its power series) then plug in $x = a$ we get,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1$$

and we now know c_1.

Let's continue with this idea and find the second derivative.

$$f''(x) = 2c_2 + 3(2)c_3(x-a) + 4(3)c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2$$

So, it looks like,

$$c_2 = \frac{f''(a)}{2}$$

Using the third derivative gives, $f'''(x) = 3(2)c_3 + 4(3)(2)c_4(x-a) + \cdots$

$$f'''(a) = 3(2)c_3$$

$$c_3 = \frac{f'''(a)}{3(2)}$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 4(3)(2)c_4 + 5(4)(3)(2)c_5(x-a)\cdots$$

$$f^{(4)}(a) = 4(3)(2)c_4 \qquad \Rightarrow \qquad c_4 = \frac{f^{(4)}(a)}{4(3)(2)}$$

Hopefully by this time you've seen the pattern here. It looks like, in general, we've got the following formula for the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This even works for $n=0$ if you recall that $0! = 1$ and define $f^{(0)}(x) = f(x)$.

So, provided a power series representation for the function $f(x)$ about $x = a$ exists the **Taylor Series for** $f(x)$ **about** $x = a$ is,

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

If we use $a = 0$, so we are talking about the Taylor Series about $x = 0$, we call the series a **Maclaurin Series** for $f(x)$ or,

Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$
$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Example Find the Taylor Series for $f(x) = \mathbf{e}^x$ about $x = 0$.

Solution
This is actually one of the easier Taylor Series that we'll be asked to compute. To find the Taylor Series for a function we will need to determine a general formula for $f^{(n)}(a)$. This is one of the few functions where this is easy to do right from the start.

To get a formula for $f^{(n)}(0)$ all we need to do is recognize that,

$$f^{(n)}(x) = \mathbf{e}^x \qquad n = 0,1,2,3,\ldots$$

and so,

$$f^{(n)}(0) = \mathbf{e}^0 = 1 \qquad n = 0,1,2,3,\ldots$$

Therefore, the Taylor series for $f(x) = \mathbf{e}^x$ about $x=0$ is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example Find the Taylor Series for $f(x) = \mathbf{e}^{-x}$ about $x = 0$.

Solution

As with the first example we'll need to get a formula for $f^{(n)}(0)$. However, unlike the first one we've got a little more work to do. Let's first take some derivatives and evaluate them at $x=0$.

$$\begin{aligned}
f^{(0)}(x) &= \mathbf{e}^{-x} & f^{(0)}(0) &= 1 \\
f^{(1)}(x) &= -\mathbf{e}^{-x} & f^{(1)}(0) &= -1 \\
f^{(2)}(x) &= \mathbf{e}^{-x} & f^{(2)}(0) &= 1 \\
f^{(3)}(x) &= -\mathbf{e}^{-x} & f^{(3)}(0) &= -1 \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n \mathbf{e}^{-x} & f^{(n)}(0) &= (-1)^n \qquad n = 0,1,2,3
\end{aligned}$$

After a couple of computations we were able to get general formulas for both $f^{(n)}(x)$ and $f^{(n)}(0)$. We often won't be able to get a general formula for $f^{(n)}(x)$ so don't get too excited about getting that formula. Also, as we will see it won't always be easy to get a general formula for $f^{(n)}(a)$.
So, in this case we've got general formulas so all we need to do is plug these into the Taylor Series formula and be done with the problem.

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Example Find the Taylor Series for $f(x)=x^4\mathbf{e}^{-3x^2}$ about $x=0$.

Solution

For this example we will take advantage of the fact that we already have a Taylor Series for $\mathbf{e}^x$ about $x=0$. In this example, unlike the previous example, doing this directly would be significantly longer and more difficult.

$$x^4\mathbf{e}^{-3x^2}=x^4\sum_{n=0}^{\infty}\frac{\left(-3x^2\right)^n}{n!}$$
$$=x^4\sum_{n=0}^{\infty}\frac{(-3)^n x^{2n}}{n!}$$
$$=\sum_{n=0}^{\infty}\frac{(-3)^n x^{2n+4}}{n!}$$

To this point we've only looked at Taylor Series about $x=0$ (also known as Maclaurin Series) so let's take a look at a Taylor Series that isn't about $x=0$. Also, we'll pick on the exponential function one more time since it makes some of the work easier. This will be the final Taylor Series for exponentials in this section.

Example Find the Taylor Series for $f(x)=\mathbf{e}^{-x}$ about $x=-4$.

Solution

Finding a general formula for $f^{(n)}(-4)$ is fairly simple.

$$f^{(n)}(x)=(-1)^n\mathbf{e}^{-x} \qquad f^{(n)}(-4)=(-1)^n\mathbf{e}^{4}$$

The Taylor Series is then,

$$\mathbf{e}^{-x}=\sum_{n=0}^{\infty}\frac{(-1)^n\mathbf{e}^4}{n!}(x+4)^n$$

Example Find the Taylor Series for $f(x)=\cos(x)$ about $x=0$.

Solution

First we'll need to take some derivatives of the function and evaluate them at $x=0$.

$$\begin{aligned}
f^{(0)}(x) &= \cos x & f^{(0)}(0) &= 1 \\
f^{(1)}(x) &= -\sin x & f^{(1)}(0) &= 0 \\
f^{(2)}(x) &= -\cos x & f^{(2)}(0) &= -1 \\
f^{(3)}(x) &= \sin x & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \\
f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= 0 \\
f^{(6)}(x) &= -\cos x & f^{(6)}(0) &= -1 \\
&\vdots & &\vdots
\end{aligned}$$

In this example, unlike the previous ones, there is not an easy formula for either the general derivative or the evaluation of the derivative. However, there is a clear pattern to the evaluation So, let's plug what we've got into the Taylor series and see what we get,

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\
&= \underbrace{1}_{n=0} + \underbrace{0}_{n=1} - \underbrace{\frac{1}{2!}x^2}_{n=2} + \underbrace{0}_{n=3} + \underbrace{\frac{1}{4!}x^4}_{n=4} + \underbrace{0}_{n=5} - \underbrace{\frac{1}{6!}x^6}_{n=6} + \cdots
\end{aligned}$$

So, we only pick up terms with even powers on the x's. This doesn't really help us to get a general formula for the Taylor Series. However, let's drop the zeroes and "renumber" the terms as follows to see what we can get.

$$\cos x = \underbrace{1}_{n=0} - \underbrace{\frac{1}{2!}x^2}_{n=1} + \underbrace{\frac{1}{4!}x^4}_{n=2} - \underbrace{\frac{1}{6!}x^6}_{n=3} + \cdots$$

By renumbering the terms as we did we can actually come up with a general formula for the Taylor Series and here it is.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

(H. W.) Find the Taylor Series for $f(x)=\sin(x)$ about $x=0$.

Example Find the Taylor Series for $f(x)=\ln(x)$ about $x=2$.

Solution

Here are the first few derivatives and the evaluations.

$$f^{(0)}(x)=\ln(x) \qquad f^{(0)}(2)=\ln 2$$
$$f^{(1)}(x)=\frac{1}{x} \qquad f^{(1)}(2)=\frac{1}{2}$$
$$f^{(2)}(x)=-\frac{1}{x^2} \qquad f^{(2)}(2)=-\frac{1}{2^2}$$
$$f^{(3)}(x)=\frac{2}{x^3} \qquad f^{(3)}(2)=\frac{2}{2^3}$$
$$f^{(4)}(x)=-\frac{2(3)}{x^4} \qquad f^{(4)}(2)=-\frac{2(3)}{2^4}$$
$$f^{(5)}(x)=\frac{2(3)(4)}{x^5} \qquad f^{(5)}(2)=\frac{2(3)(4)}{2^5}$$
$$\vdots \qquad \vdots$$
$$f^{(n)}(x)=\frac{(-1)^{n+1}(n-1)!}{x^n} \qquad f^{(n)}(2)=\frac{(-1)^{n+1}(n-1)!}{2^n} \qquad n=1,2,3,\ldots$$

Note that while we got a general formula here it doesn't work for $n=0$. This will happen on occasion so don't worry about it when it does.

Note that while we got a general formula here it doesn't work for $n=0$. This will happen on occasion so don't worry about it when it does.

In order to plug this into the Taylor Series formula we'll need to strip out the $n=0$ term first.

$$\begin{aligned}\ln(x)&=\sum_{n=0}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n\\&=f(2)+\sum_{n=1}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n\\&=\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!}{n!\;2^n}(x-2)^n\\&=\ln(2)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\,2^n}(x-2)^n\end{aligned}$$

Notice that we simplified the factorials in this case. You should always simplify them if there are more than one and it's possible to simplify them.

(H.W.) Find the Taylor Series for $f(x)=\frac{1}{x^2}$ about $x=-1$.

(H.W.) Find the Taylor Series for $f(x)=x^3-10x^2+6$ about $x=3$.